1D steady-state heat conduction (Laplace equation)

Laplace equation for heat conduction:

$$-k\nabla^2 T = 0$$

After integration:

$$-\int_\Omega k\ \nabla \cdot \nabla T = 0$$

After applying div theorem:

$$-\oint_\Gamma k\ \nabla T \cdot \mathbf{n}\ \mathrm{d}\Gamma = 0$$

FVM discretisation

$$-\sum_{f=1}^{f=N_f} \int_{\Gamma_f} k\ \nabla T \cdot \mathbf{n}_f\ \mathrm{d}\Gamma_f = 0$$

After numerical quadrature:

$$-\sum_{f=1}^{f=N_f} k_f \underbrace{\left[\sum_{g=1}^{g=N_g} \alpha_g \nabla T(\mathbf{x}_{f,g}) \cdot \mathbf{n}_f\right]}_{\int_{\Gamma_f} \nabla T \cdot \mathbf{n}_f} \Gamma_f = 0$$

For 1D above equation transform to ($N_g = 1, \alpha_g = 1$):

$$-\sum_{f=1}^{f=N_f} k_f \left[\nabla T(\mathbf{x}_f) \cdot \mathbf{n}_f\right] \Gamma_f = 0 \qquad \text{(A3)}$$

Interpolation scheme for ∇T

Temperature gradient at face is obtained multiplying each neighboring node in stencil with corresponding weight:

$$\frac{\partial T}{\partial x}(\tilde{x}) = \sum_{n=1}^{n=N_n} c_{x,n}(\tilde{\mathbf{x}}) T_n \qquad \text{(A4)}$$

- $\tilde{\mathbf{x}}$ is the field point location
- T_N is temperature at the neighbor cell centre
- $\mathbf{x}_n$ is location of cell centre

For 1D problem, scalar product of temperature gradient and outward pointing normal vector reduces to:

$$\nabla T \cdot \mathbf{n}_f = \frac{\partial T}{\partial x} n_{f,x}$$

Using eq. (A4) above equation is transformed to:

$$\nabla T \cdot \mathbf{n}_f = \left[\sum_{n=1}^{n=N_n} c_{x,n}(\tilde{\mathbf{x}}) T_n \right] n_{f,x} \tag{A5}$$

Finally, discretised Laplace equation in 1D is obtained substituting (A5) to (A3)

$$-\sum_{f=1}^{f=N_f} k_f \left[\sum_{n=1}^{n=N_n} c_{x,n}(\tilde{\mathbf{x}}) T_n \right] n_{f,x} \Gamma_f = 0 \tag{A6}$$

Weight function

Exponential kernel (radially symmetric exponential function):

$$w(\mathbf{x}_n, \mathbf{x}, k) = \frac{e^{-\left(\frac{d}{d_m}\right)^2 k^2} - e^{-k^2}}{-e^{-k^2}} \quad \text{or like this} \quad w = \frac{e^{-\left(\frac{d}{c}\right)^2} - e^{-\left(\frac{d_m}{c}\right)^2}}{-e^{-\left(\frac{d_m}{c}\right)^2}}$$

$d = ||\mathbf{x}_n - \mathbf{x}||$

$d_m = 2 \max(|||\mathbf{x}_f - \mathbf{x}_n||) = 2\, r_s$

$k = 6$ is shape parameter

$k = d_m/c$ where d_m is smoothing length $d_m = r_s$ and $c = d_m/s_x$ where s_x is shape parameter of the kernel (some constant)

Local Regression Estimators

Truncated Taylor expansion using N_p terms:

$$\tilde{T}(x) = T(\tilde{x}) + \frac{\partial T}{\partial x}(\tilde{x})(x - \tilde{x}) \ \ldots$$

$$\mathbf{q}^T(\mathbf{x} - \tilde{\mathbf{x}}) = [1, \ \ (x - \tilde{x}), \ \ldots]$$

$$\tilde{\mathbf{a}}^T(\tilde{\mathbf{x}}) = [T(\tilde{x}), \ \ \frac{\partial T}{\partial x}(\tilde{x}), \ \ldots]$$

$$\mathcal{R} = \frac{1}{2} \sum_{n=1}^{N=N_n} w(\mathbf{x}_n - \tilde{\mathbf{x}})[\tilde{T}(\mathbf{x}_n) - T_n]^2$$

$$= \frac{1}{2} \sum_{n=1}^{N=N_n} w(\mathbf{x}_n - \tilde{\mathbf{x}})[\mathbf{q}^T(\mathbf{x}_n - \tilde{\mathbf{x}})\tilde{\mathbf{a}}(\tilde{\mathbf{x}}) - T_n]^2$$

$$\frac{\partial \mathcal{R}}{\partial a} = \sum_{n=1}^{N=N_n} w(\mathbf{x}_n - \tilde{\mathbf{x}})[\mathbf{q}^T(\mathbf{x} - \tilde{\mathbf{x}})\tilde{\mathbf{a}}(\tilde{\mathbf{x}}) - T_n] = 0$$

$$w(\mathbf{x}_n - \tilde{\mathbf{x}}) = \text{n-th column of } \mathbf{W}$$

$$\mathbf{q}(\mathbf{x}_n - \tilde{\mathbf{x}}) = \text{n-th column of } \mathbf{Q}$$

$$0 = \mathbf{W}[\mathbf{Q}^T\tilde{\mathbf{a}} - \mathbf{T}_n]$$

$$\mathbf{W}\mathbf{Q}^T\tilde{\mathbf{a}} = \mathbf{W}\mathbf{T}_n \quad /\mathbf{Q}$$

$$\mathbf{Q}\mathbf{W}\mathbf{Q}^T\tilde{\mathbf{a}} = \mathbf{Q}\mathbf{W}\mathbf{T}_n$$

$$\tilde{\mathbf{M}} = \mathbf{Q}\mathbf{W}\mathbf{Q}^T$$

$$\tilde{\mathbf{A}} = \tilde{\mathbf{M}}^{-1}\mathbf{Q}\mathbf{W}$$

$$\tilde{\mathbf{a}} = \tilde{\mathbf{A}}\mathbf{T}_n$$

$$\begin{bmatrix} T(\tilde{\mathbf{x}}) \\ \frac{\partial T}{\partial x}(\tilde{\mathbf{x}}) \\ \frac{\partial T}{\partial y}(\tilde{\mathbf{x}}) \\ \vdots \\ \frac{\partial^2 T}{\partial y^2}(\tilde{\mathbf{x}}) \end{bmatrix} = \tilde{\mathbf{A}}\mathbf{T}_n$$

On boundary:

$$\tilde{T}(x) = T(\tilde{x}) + \frac{\partial T}{\partial x}(\tilde{x})\,\cancel{(x - \tilde{x})}^{0} + \ \ldots = T(\tilde{x})$$

$$\mathbf{q}^T(\mathbf{x} - \tilde{\mathbf{x}}) = [1, \ \ 0, \ldots]$$

$$\tilde{\mathbf{a}}^T(\tilde{\mathbf{x}}) = [T(\tilde{x}), \ \ \frac{\partial T}{\partial x}(\tilde{x}), \ldots]$$

For $N_p = 2$ (same is for $N_p > 2$)

$$w(\mathbf{x}_n - \tilde{\mathbf{x}}) = \text{n-th column of } \mathbf{W}$$

$$\mathbf{q}(\mathbf{x}_n - \tilde{\mathbf{x}}) = \text{n-th column of } \mathbf{Q}$$

$$W = \begin{bmatrix} w(\mathbf{x}_n - \tilde{\mathbf{x}}) & 0 \\ 0 & w(\mathbf{x}_n - \tilde{\mathbf{x}}) \end{bmatrix} = \begin{bmatrix} w(\mathbf{x}_n - \tilde{\mathbf{x}}) & 0 \\ 0 & w(0) \end{bmatrix} = \begin{bmatrix} w(\mathbf{x}_n - \tilde{\mathbf{x}}) & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ \mathbf{x}_n - \tilde{\mathbf{x}} & 0 \end{bmatrix}$$

$$\begin{bmatrix} T(\tilde{\mathbf{x}}) \\ \frac{\partial T}{\partial x}(\tilde{\mathbf{x}}) \end{bmatrix} = (\mathbf{Q}\mathbf{W}\mathbf{Q}^T)^{-1}\mathbf{Q}\mathbf{W}\begin{bmatrix} T_0 \\ T_b \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} T_0 \\ T_b \end{bmatrix}$$

$$\frac{\partial T}{\partial x}(\tilde{\mathbf{x}}) = \underbrace{T_0 * A_{21}}_{\text{diag coeff}} + \underbrace{T_b * A_{22}}_{\text{source vector}}$$

Solution procedure

1. step: Loop over interior cells

$$\sum_{f=w,e} k_f n_{fx} \Gamma_f \sum_{n=1}^{n=N_n} \mathbf{c}_{x,n}$$

2. step: Loop over boundary faces
3. Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ system of equations

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Example 1

1D rod with constant cross-section area with prescribed temperature at end points. Without volume source term.

INPUT:
Cross-section area $\Gamma_f = 10$
Diffusion coefficient $k_f = 10$
Overall rod length $L = 10$
Number of CVs = 10
$\delta x = 1$
$T_A = 0$
$T_B = 10$
Outward pointing normal at east face is $n_e = 1$
Outward pointing normal at west face is $n_e = -1$

Example 2

1D rod with constant cross-section area with prescribed zero temperature at end points.

Volume source term is calculated using **MMS (Method of Manufactured Solutions)**

Expected solution:

$$T = \sin\left(2\pi \frac{x^2}{100}\right)$$

The source term is then found by substituting the manufactured expression for T into the governing equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{1}{625}\pi\left(\pi x^2 \sin\left(\frac{\pi x^2}{50}\right) - 25\cos\left(\frac{\pi x^2}{50}\right)\right)$$

Analytical integration for each cell:

$$\int_a^b -\frac{1}{625}\pi\left(\pi x^2 \sin\left(\frac{\pi x^2}{50}\right) - 25\cos\left(\frac{\pi x^2}{50}\right)\right)\mathrm{d}x = \frac{1}{25}\left(\pi b\cos\left(\frac{\pi b^2}{50}\right) - \pi a\cos\left(\frac{\pi a^2}{50}\right)\right)$$

INPUT:
Cross-section area $\Gamma_f = 10$
Diffusion coefficient $k_f = 10$
Overall rod length $L = 10$
Number of CVs = 10
$\delta x = 1$
$T_A = 0$

$T_B = 10$
Outward pointing normal at east face is $n_e = 1$
Outward pointing normal at west face is $n_e = -1$