Constrained Interpolation Methods

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In[43]:= `Quit[];`

Proposed Interpolation Functions

In[1]:= $f0[\xi\_] := a + b\,\xi + c\,\xi^2$

$f1[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3$

$f2[\xi\_] := \frac{a + b\,\xi + c\,\xi^2}{1 + e\,\xi}$

$f3[\xi\_] := -\frac{c}{a}\,\Delta x\,\text{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{a} - 1\right)\right] + -\frac{d}{b}\,\Delta x\,\text{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{b} - 1\right)\right]$

Just Testing

Interpolating Polynomial f0[x]

constrains

$f0'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$

$f0'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$

$b - c\,\Delta x == \frac{-f_{-1+i} + f_i}{\Delta x}$

$b + c\,\Delta x == \frac{-f_i + f_{1+i}}{\Delta x}$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\,d\xi\right] == f_i$$

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

$$\text{coefs} = \text{Solve}\left[\text{f0}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x} \,\&\&\, \text{f0}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \,\&\&\, + \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\,d\xi == f_i, \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i} + 26 f_i - f_{1+i}\right),\; b \to -\frac{f_{-1+i} - f_{1+i}}{2\,\Delta x},\; c \to -\frac{-f_{-1+i} + 2 f_i - f_{1+i}}{2\,\Delta x^2}\right\}\right\}$$

```
f0[-Δx / 2] /. coefs // Expand
f0[Δx / 2] /. coefs // Expand
```

$$\left\{\frac{f_{-1+i}}{3} + \frac{5 f_i}{6} - \frac{f_{1+i}}{6}\right\}$$

$$\left\{-\frac{1}{6} f_{-1+i} + \frac{5 f_i}{6} + \frac{f_{1+i}}{3}\right\}$$

$$\text{coefs} = \text{Solve}\left[\text{f0}'[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x} \,\&\&\, \text{f0}'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \,\&\&\, + \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\,d\xi == f_i, \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i} + 24 f_i\right),\; b \to -\frac{-d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x},\; c \to -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x^2}\right\}\right\}$$

Interpolating Polynomial f1[x]

constrains

$$\text{f1}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$\text{f1}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f1}[\xi]\,d\xi\right] == f_i$$

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

$$\text{coefs} = \text{Solve}\left[\text{f1}'[\Delta x/2] == \frac{f_{i+1}-f_i}{\Delta x}\ \&\&\ \text{f1}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}\ \&\&\ +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\text{f1}[\xi]\,d\xi == f_i,\ \{a, c, d\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i} + 26\, f_i - f_{1+i}\right),\ c \to -\frac{-f_{-1+i} + 2\, f_i - f_{1+i}}{2\,\Delta x^2},\ d \to -\frac{2\,(2\, b\,\Delta x + f_{-1+i} - f_{1+i})}{3\,\Delta x^3}\right\}\right\}$$

```
f1[-Δx / 2] /. coefs // Expand
f1[Δx / 2] /. coefs // Expand
```

$$\left\{-\frac{b\,\Delta x}{3} + \frac{f_{-1+i}}{6} + \frac{5\, f_i}{6}\right\}$$

$$\left\{\frac{b\,\Delta x}{3} + \frac{5\, f_i}{6} + \frac{f_{1+i}}{6}\right\}$$

What is b?
To find out we will assuming and extra constrain:

$$\text{bcoef} = \text{Solve}\left[\text{f1}'[0] == \frac{f_{i+1} - f_{i-1}}{2\,\Delta x},\ b\right]$$

$$\left\{\left\{b \to \frac{-f_{-1+i} + f_{1+i}}{2\,\Delta x}\right\}\right\}$$

```
f1[-Δx / 2] /. coefs /. bcoef // Expand
f1[Δx / 2] /. coefs /. bcoef // Expand
```

$$\left\{\left\{\frac{f_{-1+i}}{3} + \frac{5\, f_i}{6} - \frac{f_{1+i}}{6}\right\}\right\}$$

$$\left\{\left\{-\frac{1}{6} f_{-1+i} + \frac{5\, f_i}{6} + \frac{f_{1+i}}{3}\right\}\right\}$$

NOTE: this is exactly the result from using polynomial f0[x]!

```
Clear[d];
```

$$\text{coefs} = \text{Solve}\left[\text{f1}'[\Delta x/2] == \frac{\delta_{i+1/2}}{\Delta x}\ \&\&\ \text{f1}'[-\Delta x/2] == \frac{\delta_{i-1/2}}{\Delta x}\ \&\&\ +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\text{f1}[\xi]\,d\xi == f_i,\ \{a, c, d\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(24\, f_i + \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right),\ c \to -\frac{\delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}}{2\,\Delta x^2},\ d \to -\frac{2\left(2\, b\,\Delta x - \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right)}{3\,\Delta x^3}\right\}\right\}$$

Therefore, it is obvious that we can choose ‘b’ as a free parameter to control the slope of the approximation!

Non-polynomial Interpolation function f2[x]

constrains

$$\text{f2}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$\text{f2}'[+\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$\frac{b - c\,\Delta x}{1 - \frac{e\,\Delta x}{2}} - \frac{e\left(a - \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 - \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{b + c\,\Delta x}{1 + \frac{e\,\Delta x}{2}} - \frac{e\left(a + \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 + \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f2}[\xi]\, d\xi\right] == f_i$$

$$\text{ConditionalExpression}\left[\frac{e\,(-c + b\,e)\,\Delta x + 2\,(c + e\,(-b + a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]}{e^3\,\Delta x} == f_i,\ \text{Re}\left[\frac{1}{e\,\Delta x}\right] > \frac{1}{2} \,||\, \text{Re}\left[\frac{1}{e\,\Delta x}\right] < -\frac{1}{2} \,||\, \frac{1}{e\,\Delta x} \notin \text{Reals}\right]$$

Solve system

$$\text{coefs} = \text{Solve}\left[\text{f2}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x} \,\&\&\, \text{f2}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \,\&\&\, \frac{1}{e^3\,\Delta x}\left(e\,(-c + b\,e)\,\Delta x + 2\,(c + e\,(-b + a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) == f_i,\ \{a, b, c\}\right] \,//\, \text{Simplify}$$

$$\Bigg\{\Bigg\{a \to \frac{1}{32\,e^3\,\Delta x^3}\Bigg(\left(-4 + e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} + \left(-2\,e\,\Delta x\left(16 - 24\,e^2\,\Delta x^2 + e^4\,\Delta x^4\right) + 4\left(-4 + e^2\,\Delta x^2\right)^2 \text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + \left(-4 + e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\Bigg),$$

$$b \to -\frac{1}{16\,e^2\,\Delta x^3}\Bigg(\left(-2 + e\,\Delta x\right)^2\left(-2\,e\,\Delta x + (2 + e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} - 2\left(-8\,e\,\Delta x + 6\,e^3\,\Delta x^3 + \left(-4 + e^2\,\Delta x^2\right)^2 \text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + (2 + e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2 + e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\Bigg),$$

$$c \to \frac{1}{8\,\Delta x^2}\left((-2 + e\,\Delta x)^2 f_{-1+i} - 2\left(4 + e^2\,\Delta x^2\right) f_i + (2 + e\,\Delta x)^2 f_{1+i}\right)\Bigg\}\Bigg\}$$

for example notice that

$$\frac{(-2+e\,\Delta x)^2 f_{-1+i} - 2\left(4+e^2\Delta x^2\right) f_i + (2+e\,\Delta x)^2 f_{1+i}}{8\,\Delta x^2} \text{ /. } e \to 0 \text{ // Simplify}$$

$$\frac{f_{-1+i} - 2 f_i + f_{1+i}}{2\,\Delta x^2}$$

```
f2[-Δx / 2] /. coefs
f2[+Δx / 2] /. coefs
```

$$\left\{\frac{1}{1-\frac{e\,\Delta x}{2}}\left(\frac{1}{32}\left((-2+e\,\Delta x)^2 f_{-1+i} - 2\left(4+e^2\Delta x^2\right) f_i + (2+e\,\Delta x)^2 f_{1+i}\right) + 1\Big/\left(32\,e^3\,\Delta x^3\right)\left(\left(-4+e^2\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} + \left(-2\,e\,\Delta x\left(16-24\,e^2\Delta x^2+e^4\Delta x^4\right) + 4\left(-4+e^2\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + \left(-4+e^2\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right) + 1\Big/\left(32\,e^2\,\Delta x^2\right)\left((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} - 2\left(-8\,e\,\Delta x + 6\,e^3\Delta x^3 + \left(-4+e^2\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right)\right)\right\}$$

$$\left\{\frac{1}{1+\frac{e\,\Delta x}{2}}\left(\frac{1}{32}\left((-2+e\,\Delta x)^2 f_{-1+i} - 2\left(4+e^2\Delta x^2\right) f_i + (2+e\,\Delta x)^2 f_{1+i}\right) + 1\Big/\left(32\,e^3\,\Delta x^3\right)\left(\left(-4+e^2\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} + \left(-2\,e\,\Delta x\left(16-24\,e^2\Delta x^2+e^4\Delta x^4\right) + 4\left(-4+e^2\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + \left(-4+e^2\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right) - 1\Big/\left(32\,e^2\,\Delta x^2\right)\left((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} - 2\left(-8\,e\,\Delta x + 6\,e^3\Delta x^3 + \left(-4+e^2\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right)\right)\right\}$$

which is consisten with previous results. Re-compute with d-slopes variables:

$$\text{coefs} = \text{Solve}\left[f2'[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x} \;\&\&\; f2'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \;\&\&+ \frac{1}{e^3\,\Delta x}\left(e\,(-c+b\,e)\,\Delta x + 2\,(c+e\,(-b+a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) == f_i,\ \{a, b, c\}\right] \text{// Simplify}$$

$$\Big\{\Big\{a \to \frac{1}{32\,e^3\,\Delta x^3}\Big(-\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big),$$

$$b \to \frac{1}{16\,e^2\,\Delta x^3}\Big((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big),$$

$$c \to \frac{-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}}{8\,\Delta x^2}\Big\}\Big\}$$

f2[-Δx / 2] /. coefs
f2[+Δx / 2] /. coefs

$$\Big\{\frac{1}{1-\frac{e\,\Delta x}{2}}\Big(\frac{1}{32}\left(-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}\right) - \frac{1}{32\,e^2\,\Delta x^2}\Big((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big) + \frac{1}{32\,e^3\,\Delta x^3}\Big(-\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big)\Big)\Big\}$$

$$\Big\{\frac{1}{1+\frac{e\,\Delta x}{2}}\Big(\frac{1}{32}\left(-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}\right) + \frac{1}{32\,e^2\,\Delta x^2}\Big((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big) + \frac{1}{32\,e^3\,\Delta x^3}\Big(-\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big)\Big)\Big\}$$

How to define variable e?

What’s the derivative of ArcTanh[x]?

```
ArcTanh'[e Δx/2] // Simplify
```

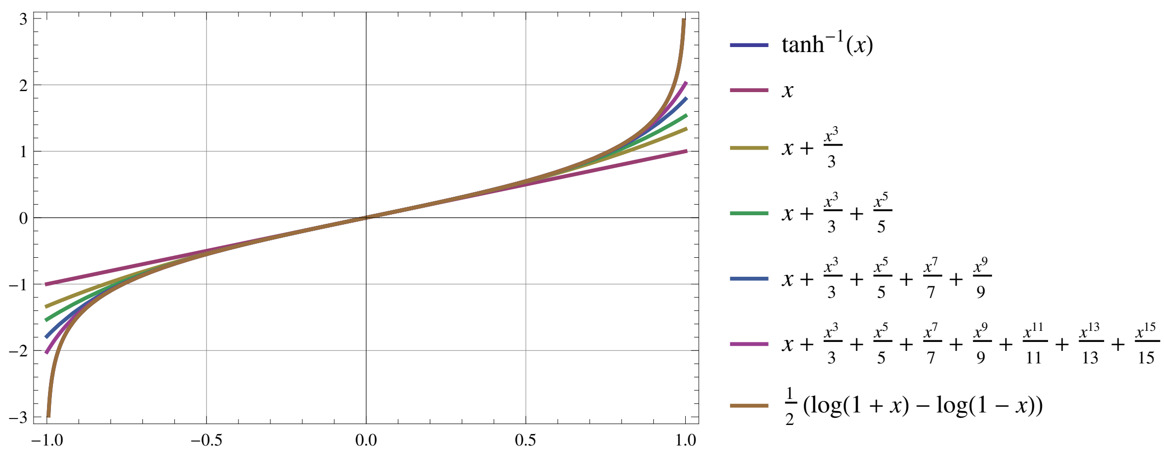
$$\frac{4}{4 - e^2\, \Delta x^2}$$

can we approximate Tanh^{-1}[x] some how?

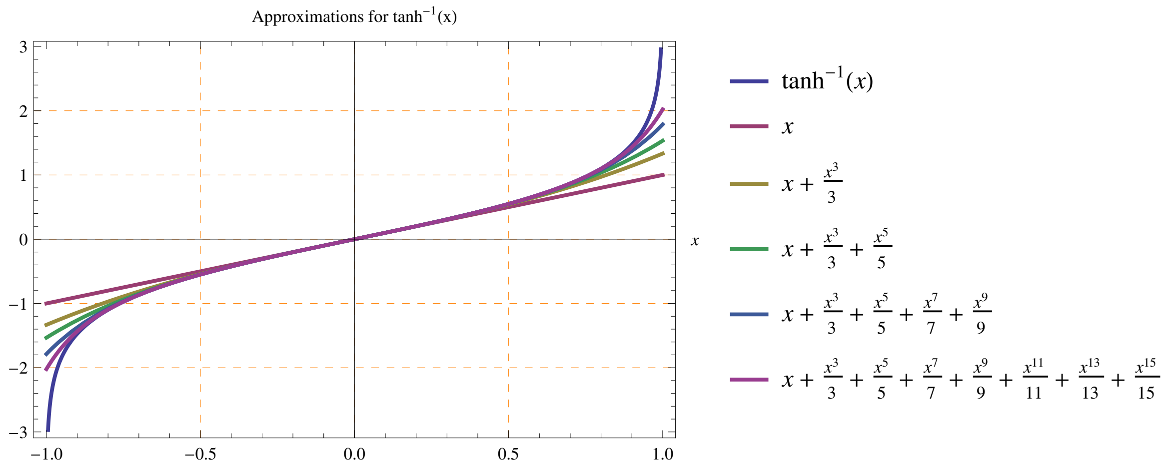
```
Series[ArcTanh[x], {x, 0, 24}]
```

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + \frac{x^{21}}{21} + \frac{x^{23}}{23} + O[x]^{25}$$

```
Plot[{ArcTanh[x], x, x + x^3/3, x + x^3/3 + x^5/5, x + x^3/3 + x^5/5 + x^7/7 + x^9/9,
  x + x^3/3 + x^5/5 + x^7/7 + x^9/9 + x^11/11 + x^13/13 + x^15/15, 1/2 (Log[1 + x] - Log[1 - x])}, {x, -1, 1},
 PlotLegends → "Expressions", PlotStyle → Thick, Frame → True, GridLines → Automatic]
```



```
Plot[{ArcTanh[x], x, x + x^3/3, x + x^3/3 + x^5/5,
  x + x^3/3 + x^5/5 + x^7/7 + x^9/9, x + x^3/3 + x^5/5 + x^7/7 + x^9/9 + x^11/11 + x^13/13 + x^15/15}, {x, -1, 1},
 PlotLegends → "Expressions", PlotStyle → Thick, Frame → True,
 GridLines → Automatic, GridLinesStyle → Directive[Orange, Dashed],
 AxesLabel → {x, "Approximations for tanh^-1(x)"}]
```




```
Export["ArcTanh.pdf", %]
```

ArcTanh.pdf

Yes! It can be approximated very well, and as long as $\Delta x \to 0$. (Which is always a desirable feature. ;D)

Non-polynomial Interpolation function f3[x] (incomplete)

constrains

$$f3'[-\Delta x / 2] == \frac{f_i - f_{i-1}}{\Delta x} \text{ // Simplify}$$

$$f3'[+\Delta x / 2] == \frac{f_{i+1} - f_i}{\Delta x} \text{ // Simplify}$$

$$c + d == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{c - b\,c + d - a\,d}{(-1 + a)\,(-1 + b)} == \frac{-f_i + f_{1+i}}{\Delta x}$$

solving for c and d

```
cnd = Solve[{f3'[-Δx / 2] == δ1 && f3'[+Δx / 2] == δ2}, {c, d}] // Simplify
```

$$\left\{\left\{c \to \frac{(-1+a)\ (\delta 1 + (-1+b)\ \delta 2)}{a-b},\ d \to -\frac{(-1+b)\ (\delta 1 + (-1+a)\ \delta 2)}{a-b}\right\}\right\}$$

yes, but in the paper we found that

$$d == -\frac{(-1+b)\ (\delta 1 + (-1+a)\ \delta 2)}{a-b}\ /.\ \text{Solve}\left[c == \frac{(-1+a)\ (\delta 1 + (-1+b)\ \delta 2)}{a-b},\ \delta 2\right]\ //\ \text{Simplify}$$

$$\{c + d == \delta 1\}$$

symmetry condition $\delta 1 = -\delta 2$ when r'[0] = 0.

```
f3'[0] == 0 /. cnd /. δ1 → -δ2 // Simplify
```

$$\left\{\frac{(a\ (-1+b) - b)\ \delta 2}{(-2+a)\ (-2+b)} == 0\right\}$$

```
Solve[f3'[0] == 0, {b}] /. δ1 → -δ2 // Simplify
```

$$\left\{\left\{b \to \frac{2\ c + 2\ d - a\ d}{c}\right\}\right\}$$

therefore

$$b == \frac{a}{a-1};$$

lets examing f3[x] with detail:

```
f3[ξ]
```

$$-\frac{c\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a} - \frac{d\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}$$

$$\text{Assuming}\left[\{\text{Re}[a] < 1,\ \text{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{c\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a}\, d\xi\right]$$

$$\frac{1}{2\ a^2} c\ \Delta x\ \Big(2\ a\ \text{ArcTanh}[1-a] - 2\ \text{Log}[2-a] - (-2+a)\ \text{Log}[-2+a] - 2\ \text{Log}[-1+a] + a\ \left(-2 + \text{Log}\left[-\frac{1}{a}\right] + 2\ \text{Log}[(-1+a)\ \Delta x]\right)\Big)$$

$$\text{Assuming}\left[\{\text{Re}[b] < 1,\ \text{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{d\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}\, d\xi\right]$$

$$\frac{1}{2\ b^2} d\ \Delta x\ \Big(2\ b\ \text{ArcTanh}[1-b] - 2\ \text{Log}[2-b] - (-2+b)\ \text{Log}[-2+b] - 2\ \text{Log}[-1+b] + b\ \left(-2 + \text{Log}\left[-\frac{1}{b}\right] + 2\ \text{Log}[(-1+b)\ \Delta x]\right)\Big)$$

recall that for any given ‘x’ value:

$\left\{\frac{1}{2}(\text{Log}[1+x]-\text{Log}[1-x]) == \text{ArcTanh}[x]\right\}$ /. x → 0.656

{True}

MCV3

constrains

```
f1[-1] == f_L
f1[1] == f_R
```

$a - b + c - d == f_L$

$a + b + c + d == f_R$

```
f1'[-1] == f'_L
f1'[1] == f'_R
```

$b - 2c + 3d == f'_L$

$b + 2c + 3d == f'_R$

solving as a system of equations

```
coefs = Solve[
   f1[-1] == f_L && f1[1] == f_R && f1'[-1] == f'_L && f1'[1] == f'_R, {a, b, c, d}] // Simplify
```

$$\left\{\left\{a \to \frac{1}{4}(2 f_L + 2 f_R + f'_L - f'_R),\; b \to \frac{1}{4}(-3 f_L + 3 f_R - f'_L - f'_R),\; c \to \frac{1}{4}(-f'_L + f'_R),\; d \to \frac{1}{4}(f_L - f_R + f'_L + f'_R)\right\}\right\}$$

```
f1[ξ] /. coefs
```

$$\left\{\frac{1}{4}\xi(-3 f_L + 3 f_R - f'_L - f'_R) + \frac{1}{4}(2 f_L + 2 f_R + f'_L - f'_R) + \frac{1}{4}\xi^2(-f'_L + f'_R) + \frac{1}{4}\xi^3(f_L - f_R + f'_L + f'_R)\right\}$$

```
f1'[ξ] /. coefs
```

$$\left\{\frac{1}{4}(-3 f_L + 3 f_R - f'_L - f'_R) + \frac{1}{2}\xi(-f'_L + f'_R) + \frac{3}{4}\xi^2(f_L - f_R + f'_L + f'_R)\right\}$$

```
f1'[-1] /. coefs // Simplify
f1'[ 0 ] /. coefs // Simplify
f1'[+1] /. coefs // Simplify
```

$\{f'_L\}$

$\left\{\frac{1}{4}(-3 f_L + 3 f_R - f'_L - f'_R)\right\}$

$\{f'_R\}$

CIP-CSL3

Notice that the information to be evolved are the cell boundary values!

constrains

```
f1[-Δx / 2] == f_{i-1/2}
f1[Δx / 2] == f_{i+1/2}
```

$a - \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4} - \frac{d\,\Delta x^3}{8} == f_{-\frac{1}{2}+i}$

$a + \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4} + \frac{d\,\Delta x^3}{8} == f_{\frac{1}{2}+i}$

$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f1[\xi]\,d\xi\right] == f_i$

$a + \frac{c\,\Delta x^2}{12} == f_i$

$f1'[0] == dd_i$

$b == dd_i$

solving as a system of equations

$\text{coefs} = \text{Solve}\left[f1[\Delta x/2] == f_{i+1/2}\ \&\&\ f1[-\Delta x/2] == f_{i-1/2}\ \&\&\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f1[\xi]\,d\xi == f_i\ \&\&\ f1'[0] == dd_i,\ \{a, b, c, d\}\right]\ //\ \text{Expand}$

$\left\{\left\{a \to -\frac{1}{4} f_{-\frac{1}{2}+i} + \frac{3 f_i}{2} - \frac{1}{4} f_{\frac{1}{2}+i},\ b \to dd_i,\ c \to \frac{3 f_{-\frac{1}{2}+i}}{\Delta x^2} - \frac{6 f_i}{\Delta x^2} + \frac{3 f_{\frac{1}{2}+i}}{\Delta x^2},\ d \to -\frac{4\,dd_i}{\Delta x^2} - \frac{4 f_{-\frac{1}{2}+i}}{\Delta x^3} + \frac{4 f_{\frac{1}{2}+i}}{\Delta x^3}\right\}\right\}$

```
f1[x - x_{i-1/2}] /. coefs (*x \in [x_{i-1/2},x_{i+1/2}] *)
```

$$\left\{-\frac{1}{4} f_{-\frac{1}{2}+i} + \frac{3 f_i}{2} - \frac{1}{4} f_{\frac{1}{2}+i} + dd_i \left(x - x_{-\frac{1}{2}+i}\right) + \left(\frac{3 f_{-\frac{1}{2}+i}}{\Delta x^2} - \frac{6 f_i}{\Delta x^2} + \frac{3 f_{\frac{1}{2}+i}}{\Delta x^2}\right) \left(x - x_{-\frac{1}{2}+i}\right)^2 + \left(-\frac{4 dd_i}{\Delta x^2} - \frac{4 f_{-\frac{1}{2}+i}}{\Delta x^3} + \frac{4 f_{\frac{1}{2}+i}}{\Delta x^3}\right) \left(x - x_{-\frac{1}{2}+i}\right)^3\right\}$$

is f_i the cell center value?

$$a + \frac{c\, \Delta x^2}{12} \text{ /. coefs // Simplify}$$

$\{f_i\}$

yes!

CIP-CSL3 (Yabe (2001) paper version)

Notice that the information to be evolved are the cell boundary values!

Left side constrains

In[5]:= `f1[ 0 ] == f_i`
`f1[Δx] == f_{i+1}`

Out[5]= $a == f_i$

Out[6]= $a + b\, \Delta x + c\, \Delta x^2 + d\, \Delta x^3 == f_{1+i}$

In[7]:= $\text{Simplify}\left[\int_0^{\Delta x} \text{f1}[\xi]\, d\xi\right] == \rho_{i+1/2}$

Out[7]= $\frac{1}{12} \Delta x\, (12\, a + \Delta x\, (6\, b + \Delta x\, (4\, c + 3\, d\, \Delta x))) == \rho_{\frac{1}{2}+i}$

In[8]:= `f1'[Δx / 2]  ==  s_{i+1/2}`

Out[8]= $b + c\, \Delta x + \frac{3\, d\, \Delta x^2}{4} == s_{\frac{1}{2}+i}$

Solving as a system of equations

In[9]:= $\text{coefs} = \text{Solve}\left[\text{f1}[0] == f_i \;\&\&\; \text{f1}[\Delta x] == f_{i+1} \;\&\&\; \int_0^{\Delta x} \text{f1}[\xi]\, d\xi == \rho_{i+1/2} \;\&\&\; \text{f1}'[\Delta x/2] == s_{i+1/2},\ \{a, b, c, d\}\right] \text{ // Expand}$

Out[9]= $$\left\{\left\{a \to f_i,\ b \to -\frac{6 f_i}{\Delta x} - 2 s_{\frac{1}{2}+i} + \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^2},\ c \to \frac{9 f_i}{\Delta x^2} - \frac{3 f_{1+i}}{\Delta x^2} + \frac{6 s_{\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^3},\ d \to -\frac{4 f_i}{\Delta x^3} + \frac{4 f_{1+i}}{\Delta x^3} - \frac{4 s_{\frac{1}{2}+i}}{\Delta x^2}\right\}\right\}$$

```
f1[x - x_{i-1}] /. coefs (* x \in [x_i, x_{i+1}] *)
```

Out[10]= $$\left\{f_i + \left(-\frac{4 f_i}{\Delta x^3} + \frac{4 f_{1+i}}{\Delta x^3} - \frac{4 s_{\frac{1}{2}+i}}{\Delta x^2}\right)(x - x_{-1+i})^3 + (x - x_{-1+i})^2 \left(\frac{9 f_i}{\Delta x^2} - \frac{3 f_{1+i}}{\Delta x^2} + \frac{6 s_{\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^3}\right) + (x - x_{-1+i})\left(-\frac{6 f_i}{\Delta x} - 2 s_{\frac{1}{2}+i} + \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$$

Right side constrains

In[23]:= $f1[-\Delta x] == f_{i-1}$

$f1[0] == f_i$

Out[23]= $a - b\,\Delta x + c\,\Delta x^2 - d\,\Delta x^3 == f_{-1+i}$

Out[24]= $a == f_i$

In[25]:= $\text{Simplify}\left[\int_{-\Delta x}^{0} f1[\xi]\, d\xi\right] == \rho_{i-1/2}$

Out[25]= $\frac{1}{12}\Delta x\,(12 a + \Delta x\,(-6 b + \Delta x\,(4 c - 3 d\,\Delta x))) == \rho_{-\frac{1}{2}+i}$

In[26]:= $f1'[-\Delta x/2] == s_{i-1/2}$

Out[26]= $b - c\,\Delta x + \frac{3 d\,\Delta x^2}{4} == s_{-\frac{1}{2}+i}$

Solving as a system of equations

In[28]:= $\text{coefs} = \text{Solve}\left[f1[0] == f_i \;\&\&\; f1[-\Delta x] == f_{i-1} \;\&\&\; \int_{-\Delta x}^{0} f1[\xi]\, d\xi == \rho_{i-1/2} \;\&\&\; f1'[-\Delta x/2] == s_{i-1/2}, \{a, b, c, d\}\right] // \text{Expand}$

Out[28]= $$\left\{\left\{a \to f_i,\; b \to \frac{6 f_i}{\Delta x} - 2 s_{-\frac{1}{2}+i} - \frac{6 \rho_{-\frac{1}{2}+i}}{\Delta x^2},\; c \to -\frac{3 f_{-1+i}}{\Delta x^2} + \frac{9 f_i}{\Delta x^2} - \frac{6 s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{-\frac{1}{2}+i}}{\Delta x^3},\; d \to -\frac{4 f_{-1+i}}{\Delta x^3} + \frac{4 f_i}{\Delta x^3} - \frac{4 s_{-\frac{1}{2}+i}}{\Delta x^2}\right\}\right\}$$

```
In[29]:= f1[x - x_i] /. coefs (* x \in [x_{i-1}, x_i] *)
```

Out[29]= $$\left\{f_i + \left(-\frac{4 f_{-1+i}}{\Delta x^3} + \frac{4 f_i}{\Delta x^3} - \frac{4 s_{-\frac{1}{2}+i}}{\Delta x^2}\right)(x - x_i)^3 + (x - x_i)^2 \left(-\frac{3 f_{-1+i}}{\Delta x^2} + \frac{9 f_i}{\Delta x^2} - \frac{6 s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{-\frac{1}{2}+i}}{\Delta x^3}\right) + (x - x_i)\left(\frac{6 f_i}{\Delta x} - 2 s_{-\frac{1}{2}+i} - \frac{6 \rho_{-\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$$

evaluating the time integral of u * f1(x_i-ut) yields

In[33]:= $\int_0^{\Delta t} u\, f1[-u\, t]\, dt$

Out[33]= $a\, u\, \Delta t - \frac{1}{2} b\, u^2\, \Delta t^2 + \frac{1}{3} c\, u^3\, \Delta t^3 - \frac{1}{4} d\, u^4\, \Delta t^4$

NOTE: u Δt indicates the direction of the advecting wave.

$d_{1\pm1/2}$ are here free parameters. Note that we can define it by using the resulting interpolation functions evaluating at Δx/2 and/or -Δx/2. e.g. :

In[39]:= `f1[-Δx / 2] /. coefs // Expand`

Out[39]= $\left\{-\frac{1}{4} f_{-1+i} - \frac{f_i}{4} + \frac{3\, \rho_{-\frac{1}{2}+i}}{2\, \Delta x}\right\}$

this is in agreement with equation (23) in Yabe [2001].