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DIFFERENCE SCHEME OF SECOND-ORDER OF ACCURACY ON THE MINIMAL PATTERN FOR HYPERBOLIC EQUATIONS*

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A method of constructing explicit difference schemes of second-order of accuracy on a minimal pattern is described. The method enables the required solution to be written uniquely in the class of piecewise continuous functions. As an example the two-dimensional non-stationary supersonic flow past an energy source is computed.

The construction of difference schemes for partial differential equations usually involves the replacement of the differential by difference operators and the choice of a method of discretizing the right-hand sides of the equations and the boundary conditions. After the solution of the difference problem has been found, it has to be interpolated up to a continuous solution. The choice of the method of interpolation is not usually connected with the discretization operations. From this point of view the finite element method is essentially different, in that the solution is sought here in the form of a piecewise polynomial or other types of function, specified in a continuum, the values of which are determined by a set of reference quantities at discrete nodes. In short, the operation of interpolation is absent in the finite element method.

We proposed in /1,2/, for hyperbolic equations of divergence type, a method for constructing families of explicit difference schemes of any order of accuracy on a non-increasing pattern. In the method of constructing the continuous solution, it is similar to the finite element method. But as distinct from the latter method, the difference equations obtained by our method are explicit. The solution on the chosen mesh is approximated by a class of piecewise linear (piecewise polynomial) functions, having discontinuities of the 1st kind in the domain considered. Initial equations in integral form /2/ are used to construct the difference equations. The approximating equations are constructed by the integro-interpolation method /3/. Since, for strictly divergence equations, their integration over the volume of the mesh cell can be reduced to integration over the cell boundary, it is sufficient to consider the behaviour of the functions and their derivatives only on the boundaries of the mesh cells.

The method of construction will be illustrated by the example of a second-order scheme for an equation of divergence type (the Cauchy problem):

$$\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}=0, \tag{1}$$

$$u=u(x,t),\ F=F(u),\quad -\infty<x<\infty,\ t>0,\quad u(x,0)=u_0(x).$$

We introduce into the domain the mesh $x_m=mh,\ m=0,\pm1,\dots,t_k,\ k=0,1,\dots$. The closed domain $x_{m-1}\leqslant x\leqslant x_{m+1},\ t_k\leqslant t\leqslant t_{k+1}$, will be called the cell of the difference scheme corresponding to indices m,k. The time step $\tau=t_{k+1}-t_k$ will be assumed to be non-uniform, and for brevity its index will be omitted. On integrating Eq.(1) over a cell, we obtain

$$J=\int\limits_{t_k}^{t_{k+1}}\int\limits_{x_{m-1}}^{x_{m+1}}\left[\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}\right]dx\,dt=0.$$

Reducing the integral over a cell to the integral over its boundary, we obtain the initial relations for discretization:

$$J=\int\limits_{x_{m-1}}^{x_{m+1}}[u(x,t_{k+1})-u(x,t_k)]dx+\int\limits_{t_k}^{t_{k+1}}[F(x_{m+1},t)-F(x_{m-1},t)]dt=0. \tag{2}$$

We shall use below the local numbering of the nodes, assigning local numbers 1,2,3 respectively to the nodes with coordinates (x_{m-1},t_k), (x_{m+1},t_k), (x_m,t_{k+1}). On Taylor-expanding the functions u,F up to second order terms in the neighborhoods of the nodes, we can write approximately

$$J\approx\int\limits_{x_{m-1}}^{x_{m+1}}[u_3+(x-x_m)u_{x_3}]dx-\int\limits_{x_{m-1}}^{x_m}[u_1+(x-x_{m-1})u_{x_1}]dx-$$

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$$\int_{x_m}^{x_{m+1}} [u_2+(x-x_{m+1})u_{x_2}]dx+\int_{t_k}^{t_{k+1}} [F_2+(t-t_k)F_{t_2}]dt-$$

$$\int_{t_k}^{t_{k+1}} [F_1+(t-t_k)F_{t_1}]dt.$$

After performing the integration, and dividing the result by the cell volume $2\tau h$, we obtain the difference analogue of Eq.(1), having second-order of approximation:

$$\frac{u_3-(u_1+u_2)/2}{\tau}+\frac{F_2-F_1}{2h}+\frac{h^2}{2\tau}\frac{u_{x_2}-u_{x_1}}{2h}+\frac{\tau}{2}\frac{F_{t_2}-F_{t_1}}{2h}=0; \tag{3}$$

here,

$$u_x=\frac{\partial u}{\partial x}, \qquad F_t=\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}=-\left(\frac{\partial F}{\partial u}\right)^2 u_x.$$

We shall regard u_x as an auxiliary independent unknown. A scheme using u_x as independent unknown was also constructed in /4/. When suitable smoothness conditions are satisfied, u_x is the solution of the Cauchy problem

$$\frac{\partial u_x}{\partial t}+\frac{\partial F_x}{\partial x}=0, \qquad u_x(x,0)=\frac{\partial u_0}{\partial x}, \qquad F_x(u,u_x)=\frac{\partial F}{\partial u}u_x. \tag{4}$$

Following the terminology of /5/, we shall refer to Eqs.(1), (4) as the continued system. For (4) we introduce the difference analogue

$$\frac{u_{x_3}-(u_{x_1}+u_{x_2})/2}{\tau}+\frac{F_{x_2}-F_{x_1}}{2h}=0, \tag{5}$$

which can be obtained in the same way as (3) by integrating Eq.(4) over a mesh cell and subsequently expanding the integrands in the neighborhoods of the nodes up to terms of zero order of smallness.

We shall solve the system of difference equations (3), (5) simultaneously at each time step. If the initial differential equations are linear, u_x can be found independently throughout the domain, and then used in Eqs.(3).

We can modify Eq.(5) by changing the method of evaluating the integral over the cell boundary $t=t_k$:

$$\int_{x_{m-1}}^{x_{m+1}} u_x(t_k,x)\,dx=u(t_k,x_{m+1})-u(t,x_{m-1}),$$

which leads to the difference relation

$$\frac{u_{x_3}-(u_2-u_1)/2h}{\tau}+\frac{F_{x_2}-F_{x_1}}{2h}=0.$$

The properties of the resulting difference solutions for large values of t were considered in /6/.

Our method of constructing the difference equations also indicates how the resulting discrete solution should be interpolated. In Fig.1 we show a fragment of the difference mesh in the neighborhood of node A. Knowing u and u_x at node A and the values of $u_t=-\partial Fu_x/\partial u$ calculated from them, we approximate the solution in the domain $BCDE$ by the polynomial

$$\tilde{u}(x,t)=u_A+(x-x_A)u_{x_A}+(t-t_A)u_{t_A},$$

which is identical on the cell boundaries adjacent to A with the representations of the integrands when obtaining the approximating system (3), (5). It can be seen from Fig.1 that the system of node neighborhoods of type $BCDE$ covers the entire domain. The continuous representation of the difference solution is thus uniquely obtained. Notice that, at points B, C, E of cell $LCEM$, this representation has discontinuities of the first kind.

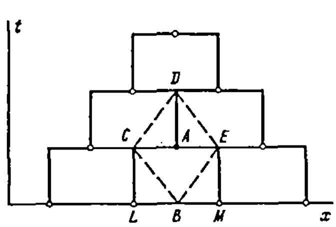

Fig.1

For the schemes constructed by the algorithm described in /1,2/, this unique property is possessed by the piecewise smooth representations of the solution with any order of accuracy. The same proof of uniqueness of the continuous form of the difference solution can be given for the case of multi-dimensional divergence equations of hyperbolic type. Let us emphasize that, for strictly divergence equations, only the representation of the solution along the cell boundaries is important.

As an illustration, consider the example of the numerical solution of the plane non-stationary problem on supersonic gas flow past an energy source. The solution is given in /7,8/ in the case of a weak source.

Below we make no assumption that the source power is small. We consider the case when a shock wave is formed in the domain in question under the action of heat liberation. Our algorithm can be used to complete discontinuous gas flows because the difference equations are conservative. No special devices were used to suppress oscillations of the difference solution in the neighborhood of a discontinuity.

A plane, initially homogeneous, gas flow is considered, with the following parameters: pressure p_0, density ρ_0, and Mach number M_0. At a certain instant, from which time is measured, a heat source is instantaneously switched on in the domain, its power having the space distribution

$$\varepsilon = \frac{Q}{\pi R^2}\frac{\rho}{\rho_0}\exp\left(-\frac{r^2}{R^2}\right), \tag{6}$$

where Q is a constant characterizing the total source power, ρ is the gas density, r is the distance to a fixed point which we shall call the source centre, and R is the characteristic radius.

The gas flow will be described by the system of equations

$$\frac{\partial f}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=H,$$

$$f=\begin{Vmatrix}\rho\\ \rho u\\ \rho v\\ E\end{Vmatrix},\quad F=\begin{Vmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{Vmatrix},\quad G=\begin{Vmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{Vmatrix},\quad H=\begin{Vmatrix}0\\ 0\\ 0\\ \varepsilon\end{Vmatrix},$$

where p is the gas pressure, ρ the density, u and v are the velocity components along the Cartesian axes x and y, and E is the total gas energy per unit volume.

The problem parameters are reduced to dimensionless form by normalizing p and E to p_0, ρ to ρ_0, u and v to $(p_0/\rho_0)^{1/2}$, time t to $R(\rho_0/p_0)^{1/2}$, and Q to $\pi R p_0 (p_0/\rho_0)^{1/2}$. All the quantities mentioned below must be regarded as dimensionless.

The origin of the Cartesian coordinate system is located at the source centre. The x axis is along the velocity of the undisturbed flow. The heat inflow is described by the dimensionless relation $\varepsilon=Q\rho\exp(-x^2-y^2)$. Obviously, the flow will remain symmetric about the x axis. In practice we are interested in the distribution of the gas parameters in the neighborhood of the source center, at which the values of $\exp(-x^2-y^2)$ are quite high.

In the computation the domain was bounded by the rectangle $a\leqslant x\leqslant b$, $0\leqslant y\leqslant c$, containing the source centre. The size of the domain was chosen so as to obtain boundary conditions for the difference equations, approximating the initial system (6), which introduce no significant disturbance into the parameter distributions close to the source.

To construct the difference equations approximating initial system (2), we introduce into the domain a mesh with node coordinates $x_i=a+ih_x$, $y_j=jh_y$, t_k, $i=0, 1, \ldots, I$, $j=0, 1, \ldots, J$, $k=0, 1, \ldots$, where h_x, h_y are the mesh steps in the x and y directions. The time step τ is chosen at each time layer and amounts to 0.8 or 0.9 of the limit permitted by the Courant-Friedrichs-Levy stability criterion.

The elementary mesh cell corresponding to the node (x_i, y_j, t_k) is the closed domain $x_{i-1}\leqslant x\leqslant x_{i+1}$, $y_{j-1}\leqslant y\leqslant y_{j+1}$, $t_k\leqslant t\leqslant t_{k+1}$. To obtain the difference relations, we integrate Eqs. (2) over the elementary cell:

$$\int_{t_k}^{t_{k+1}}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\left[\frac{\partial f}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}-H\right]dy\,dx\,dt=0. \tag{7}$$

Reducing (7) to an integral over the cell surface, we obtain

$$\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}[f(x,y,t_{k+1})-f(x,y,t_k)]\,dx\,dy+\int_{t_k}^{t_{k+1}}\int_{y_{j-1}}^{y_{j+1}}[F(x_{i+1},y,t)-$$

$$F(x_{i-1},y,t)]\,dy\,dt+\int_{t_k}^{t_{k+1}}\int_{x_{i-1}}^{x_{i+1}}[G(x,y_{j+1},t)-G(x,y_{j-1},t)]\,dx\,dt=\int_{t_k}^{t_{k+1}}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}H(x,y,t)\,dy\,dx\,dt.$$

In each cell, for typographical simplicity, we introduce local numbering of the nodes from 1 to 5 corresponding to nodes with coordinates

$$(x_{i-1},\ y_{j-1},\ t_k),\ (x_{i+1},\ y_{j-1},\ t_k),\ (x_{i+1},\ y_{j+1},\ t_k),\ (x_{i-1},\ y_{j+1},\ t_k),\ (x_i,\ y_j,\ t_{k+1})$$

respectively. We approximate the integrands in the equations by Taylor-expanding them in the neighborhoods of the nodes, retaining terms of first order in h_x, h_y, τ.

On performing the integration and dividing the result by the cell volume $4\tau h_x h_y$ we obtain on the left-hand side

$$\frac{f_5-0.25(f_1+f_2+f_3+f_4)}{\tau}+\frac{F_2+F_3-F_1-F_4}{4h_x}+\frac{G_3+G_4-G_1-G_2}{4h_y}+$$
$$\frac{h_x^2}{2\tau}\frac{f_{x_2}+f_{x_3}-f_{x_1}-f_{x_4}}{4h_x}+\frac{h_y^2}{2\tau}\frac{f_{y_3}+f_{y_4}-f_{y_1}-f_{y_2}}{4h_y}+$$
$$\frac{\tau}{2}\frac{F_{t_2}+F_{t_3}-F_{t_1}-F_{t_4}}{4h_x}+\frac{\tau}{2}\frac{G_{t_3}+G_{t_4}-G_{t_1}-G_{t_2}}{4h_y}+$$
$$\frac{h_y^2}{2}\frac{F_{y_2}+F_{y_4}-F_{y_1}-F_{y_3}}{4h_xh_y}+\frac{h_x^2}{2}\frac{G_{x_2}+G_{x_4}-G_{x_1}-G_{x_3}}{4h_xh_y}.$$

We have zero on the right-hand sides of the first three equations, while in the energy equation the right-hand side has the form (after dividing by $4\tau h_x h_y$)

$$Q\sum_{m=1}^{4}\left[\left(\rho_m+\frac{\tau}{2}\rho_{t_m}\right)A_mB_m+\rho_{x_m}C_mB_m+\rho_{y_m}A_mD_m\right],$$

$$A_m=\left|\int_{x_m}^{x_5}\exp(-x^2)\,dx\right|,\qquad B_m=\left|\int_{y_m}^{y_5}\exp(-y^2)\,dy\right|,$$

$$C_m=\frac{1}{2h_x}|\exp(-x_m^2)-\exp(-x_5^2)|-x_mA_m,$$

$$D_m=-\frac{1}{2h_y}|\exp(-y_m^2)-\exp(-y_5^2)|-y_mB_m,$$

where m is the local number of the node.

We shall use f_x and f_y as auxiliary independent unknowns, whose values will be found from the difference equations

$$\frac{f_{x_5}-0.25(f_{x_1}+f_{x_2}+f_{x_3}+f_{x_4})}{\tau}+\frac{F_{x_2}+F_{x_3}-F_{x_1}-F_{x_4}}{4h_x}+\frac{G_{x_3}+G_{x_4}-G_{x_1}-G_{x_2}}{4h_y}=\bar{H}_x,\tag{8}$$

$$\frac{f_{y_5}-0.25(f_{y_1}+f_{y_2}+f_{y_3}+f_{y_4})}{\tau}+\frac{F_{y_2}+F_{y_3}-F_{y_1}-F_{y_4}}{4h_x}+\frac{G_{y_3}+G_{y_4}-G_{y_1}-G_{y_2}}{4h_y}=\bar{H}_y.$$

Here,

$$F_x=\frac{\partial F}{\partial f}f_x,\qquad G_x=\frac{\partial G}{\partial f}f_x,\qquad F_y=\frac{\partial F}{\partial f}f_y,\qquad G_y=\frac{\partial G}{\partial f}f_y,$$
$$F_t=\frac{\partial F}{\partial f}(-F_x-G_y+H),\qquad G_t=\frac{\partial G}{\partial f}(-F_x-G_y+H)$$

and $H_x=H_y=H_t=0$ for the first three equations of each of systems (8), while for the energy equation

$$H_x=Q\rho_x\exp(-x^2-y^2)-2xQ\rho\exp(-x^2-y^2).$$

The same method is used to find H_y, H_t. The quantities $\bar{H}_x$, $\bar{H}_y$ on the right-hand side of Eqs. (8) are the integrals of H_x, H_y over an elementary cell, which are evaluated in the same way as for the basic equations, using the exact value of the integral of $\exp(-x^2-y^2)$. This approximation of the right-hand side is chosen because the function $\exp(-x^2-y^2)$ varies rapidly with x and y. It enables the total source power to be approximated with good accuracy.

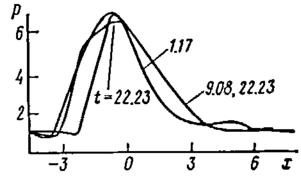

Fig.2

The following boundary conditions were used for the above difference equations. On the boundary $x=a$, $y=c$ the undisturbed flow parameters were strictly specified. The use of strict boundary conditions imposes a natural constraint on the size of the flow domain studied. If the Mach number of the undisturbed flow $M_0>1$, we choose a in such a way that the disturbances introduced by the energy source and travelling up-stream do not reach the boundary $x=a$ until after establishment of the flow. Since a priori estimation of the value of a satisfying this condition is difficult, we based the estimate on a number of test computations with different domain sizes.

When the boundary conditions are strictly specified, the boundary $y=c$ must in general be fairly remote in order for the disturbances appearing in the region of energy liberation not to reach it. In practice, the domain size along the y axis was chosen so that the disturbance introduced by the strict boundary conditions at $y=c$ did not influence the flow in the energy liberation region. The choice of c was based on trial computations.

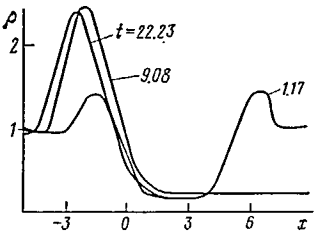

Fig.3

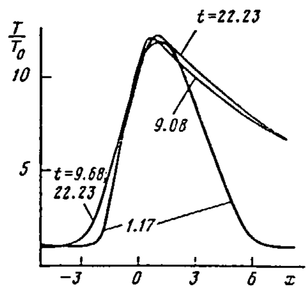

Fig.4

On the boundary $y=0$ we made use of the consequences of the flow symmetry about the x axis. For $x=b$ (down-stream from the source) we used "deflection" of the values of f_x and f_y at each time step and linear extrapolation of f at the last x layer. A series of computations showed that, with $b\geqslant 3R$, the solution in the basic domain $(r\leqslant 3R)$ is independent of the position of the upper boundary $y=b$.

Since the gas may be considerably heated in the energy liberation region, we used as its equation of state the thermodynamic functions of air, approximately described by the expressions /9/

$$e=\frac{m-1}{2}\frac{1}{\alpha},\qquad m=m_0+0.09(m_0-m_2)\ln\rho,$$

$$m_0=1+\frac{25.89\alpha+4}{10^3\alpha+1}+$$

$$\alpha(1-\alpha)\left(\frac{861}{3\cdot10^3\alpha^2+1}+\frac{2356}{9\cdot10^4\alpha^2+1}+\frac{41\cdot10^3}{12\cdot10^6\alpha^2+1}\right),$$

$$m_2=\frac{6002\alpha+4}{10^3\alpha+1},$$

where $\alpha=\rho/p$, and e is the specific internal energy.

No account was taken of properties such as thermal conductivity and radiation. Thermal conductivity can only have a significant influence at low subsonic speeds, and radiation, only at high temperatures.

We give the computational results for undisturbed flow Mach number $M_0=2$ and a dimensionless source power $Q=100$. In Figs.2–4 we show the distribution of air pressure, density, and temperature along the axis of symmetry at different instants. In Fig.5 we show the lines of equal density in the domain in the steady flow.

It can be seen that, in the neighborhood of the energy liberation zone, a domain of increased pressure and temperature is formed, in which the gas is rarefied. Under the action of heating, a shock wave is formed in this zone, leaving the domain down-stream, while moving up-stream to a distance of roughly $3R$.

The flow picture has a definite similarity with the case of supersonic flow past solid obstacles.

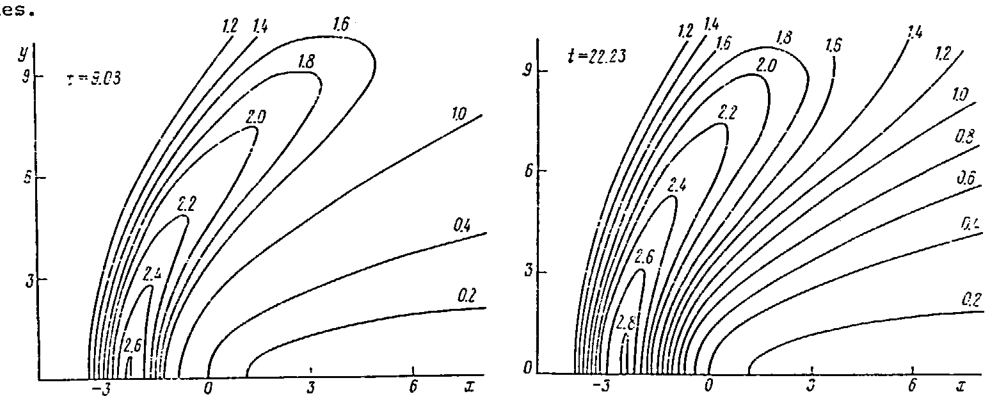

Fig.5

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