Constrained Interpolation Methods

by Manuel Diaz, NTU, 2015.08.10.

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Quit[];
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Proposed Interpolation Functions

$$f0[\xi\_] := a + b\,\xi + c\,\xi^2$$
$$f1[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3$$
$$f2[\xi\_] := \frac{a + b\,\xi + c\,\xi^2}{1 + e\,\xi}$$
$$f3[\xi\_] := -\frac{c}{a}\,\Delta x\,\mathrm{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{a} - 1\right)\right] + -\frac{d}{b}\,\Delta x\,\mathrm{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{b} - 1\right)\right]$$

Just Testing

Interpolating Polynomial f0[x]

constrains

$$f0'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$
$$f0'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$
$$b - c\,\Delta x == \frac{-f_{-1+i} + f_i}{\Delta x}$$
$$b + c\,\Delta x == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\mathrm{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f0[\xi]\,d\xi\right] == f_i$$
$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

$$\texttt{coefs = Solve}\left[\texttt{f0}'[\Delta x/2] == \frac{f_{i+1}-f_i}{\Delta x} \;\&\&\; \texttt{f0}'[-\Delta x/2] == \frac{f_i-f_{i-1}}{\Delta x} \;\&\&\; +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \texttt{f0}[\xi]\,d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i}+26 f_i - f_{1+i}\right),\ b \to -\frac{f_{-1+i}-f_{1+i}}{2\,\Delta x},\ c \to -\frac{-f_{-1+i}+2 f_i - f_{1+i}}{2\,\Delta x^2}\right\}\right\}$$

$$\texttt{coefs = Solve}\left[\texttt{f0}'[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x} \;\&\&\; \texttt{f0}'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \;\&\&\; +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \texttt{f0}[\xi]\,d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i} + 24 f_i\right),\ b \to -\frac{-d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x},\ c \to -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x^2}\right\}\right\}$$

```
f0[-Δx / 2] /. coefs // Expand
f0[Δx / 2] /. coefs // Expand
```

$$\left\{\frac{f_{-1+i}}{3} + \frac{5 f_i}{6} - \frac{f_{1+i}}{6}\right\}$$

$$\left\{-\frac{1}{6} f_{-1+i} + \frac{5 f_i}{6} + \frac{f_{1+i}}{3}\right\}$$

Interpolating Polynomial f1[x]

constrains

$$\texttt{f1}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$\texttt{f1}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\texttt{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \texttt{f1}[\xi]\,d\xi\right] == f_i$$

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

$$\texttt{coefs = Solve}\left[\texttt{f1}'[\Delta x/2] == \frac{f_{i+1}-f_i}{\Delta x} \;\&\&\; \texttt{f1}'[-\Delta x/2] == \frac{f_i-f_{i-1}}{\Delta x} \;\&\&\; +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\texttt{f1}[\xi]\,d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i}+26 f_i - f_{1+i}\right),\ b \to -\frac{3\,d\,\Delta x^3 + 2 f_{-1+i} - 2 f_{1+i}}{4\,\Delta x},\ c \to -\frac{-f_{-1+i}+2 f_i - f_{1+i}}{2\,\Delta x^2}\right\}\right\}$$

$$\texttt{coefs = Solve}\left[\texttt{f1}'[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x} \;\&\&\; \texttt{f1}'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \;\&\&\; +\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\texttt{f1}[\xi]\,d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i} + 24 f_i\right),\ b \to -\frac{3\,d\,\Delta x^3 - 2\,d_{-\frac{1}{2}+i} - 2\,d_{\frac{1}{2}+i}}{4\,\Delta x},\ c \to -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x^2}\right\}\right\}$$

```
f1[-Δx / 2] /. coefs // Expand
f1[Δx / 2] /. coefs // Expand
```

$$\left\{\frac{d\,\Delta x^3}{4} + \frac{f_{-1+i}}{3} + \frac{5 f_i}{6} - \frac{f_{1+i}}{6}\right\}$$

$$\left\{-\frac{d\,\Delta x^3}{4} - \frac{f_{-1+i}}{6} + \frac{5 f_i}{6} + \frac{f_{1+i}}{3}\right\}$$

Non-polynomial Interpolation function f2[x]

constrains

$$\texttt{f2}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$\texttt{f2}'[+\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$\frac{b - c\,\Delta x}{1 - \frac{e\,\Delta x}{2}} - \frac{e\left(a - \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 - \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{b + c\,\Delta x}{1 + \frac{e\,\Delta x}{2}} - \frac{e\left(a + \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 + \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f2[\xi]\, d\xi\right] == f_i$$

$$\text{ConditionalExpression}\left[\frac{e\,(-c+b\,e)\,\Delta x + 2\,(c+e\,(-b+a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]}{e^3\,\Delta x} == f_i,\ \text{Re}\left[\frac{1}{e\,\Delta x}\right] > \frac{1}{2} \,||\, \text{Re}\left[\frac{1}{e\,\Delta x}\right] < -\frac{1}{2} \,||\, \frac{1}{e\,\Delta x} \notin \text{Reals}\right]$$

Solve system

$$\text{coefs} = \text{Solve}\left[f2'[\Delta x/2] == \frac{f_{i+1}-f_i}{\Delta x}\ \&\&\ f2'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}\ \&\&\ \frac{1}{e^3\,\Delta x}\left(e\,(-c+b\,e)\,\Delta x + 2\,(c+e\,(-b+a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) == f_i,\ \{a, b, c\}\right]\ //\ \text{Simplify}$$

$$\left\{\left\{a \to \frac{1}{32\,e^3\,\Delta x^3}\left(\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} + \left(-2\,e\,\Delta x\,\left(16 - 24\,e^2\,\Delta x^2 + e^4\,\Delta x^4\right) + 4\,\left(-4+e^2\,\Delta x^2\right)^2 \text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + \left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right),\right.$$

$$b \to -\frac{1}{16\,e^2\,\Delta x^3}\left((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} - 2\left(-8\,e\,\Delta x + 6\,e^3\,\Delta x^3 + \left(-4+e^2\,\Delta x^2\right)^2 \text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i + (2+e\,\Delta x)^2\left(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right),$$

$$\left. c \to \frac{1}{8\,\Delta x^2}\left((-2+e\,\Delta x)^2 f_{-1+i} - 2\left(4+e^2\,\Delta x^2\right) f_i + (2+e\,\Delta x)^2 f_{1+i}\right)\right\}\right\}$$

for example notice that

$$\frac{(-2+e\,\Delta x)^2 f_{-1+i} - 2\left(4+e^2\,\Delta x^2\right) f_i + (2+e\,\Delta x)^2 f_{1+i}}{8\,\Delta x^2}\ /.\ e \to 0\ //\ \text{Simplify}$$

$$\frac{f_{-1+i} - 2\,f_i + f_{1+i}}{2\,\Delta x^2}$$

which is consisten with previous resoults. Re-compute with d-slopes variables:

$$\texttt{coefs = Solve}\Big[\texttt{f2'}[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x} \;\&\&\; \texttt{f2'}[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \;\&\&\; + \frac{1}{e^3\,\Delta x}\Big(e\,(-c+b\,e)\,\Delta x + 2\,(c+e\,(-b+a\,e))\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big) == f_i,\ \{a, b, c\}\Big]\ \texttt{// Simplify}$$

$$\Big\{\Big\{a \to \frac{1}{32\,e^3\,\Delta x^3}\Big(-\big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} + \big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big),$$
$$b \to \frac{1}{16\,e^2\,\Delta x^3}\Big((-2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big),$$
$$c \to \frac{-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}}{8\,\Delta x^2}\Big\}\Big\}$$

```
f2[-Δx / 2] /. coefs
f2[+Δx / 2] /. coefs
```

$$\Big\{\frac{1}{1-\frac{e\,\Delta x}{2}}\Big(\frac{1}{32}\Big(-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}\Big) - \frac{1}{32\,e^2\,\Delta x^2}\Big((-2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big) + \frac{1}{32\,e^3\,\Delta x^3}\Big(-\big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} + \big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big)\Big)\Big\}$$

$$\Big\{\frac{1}{1+\frac{e\,\Delta x}{2}}\Big(\frac{1}{32}\Big(-(-2+e\,\Delta x)^2\,d_{-\frac{1}{2}+i} + (2+e\,\Delta x)^2\,d_{\frac{1}{2}+i}\Big) + \frac{1}{32\,e^2\,\Delta x^2}\Big((-2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} - (2+e\,\Delta x)^2\,\Big(-2\,e\,\Delta x + (-2+e\,\Delta x)^2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 16\,e^3\,\Delta x^3\,f_i\Big) + \frac{1}{32\,e^3\,\Delta x^3}\Big(-\big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{-\frac{1}{2}+i} + \big(-4+e^2\,\Delta x^2\big)^2\,\Big(e\,\Delta x - 2\,\text{ArcTanh}\Big[\frac{e\,\Delta x}{2}\Big]\Big)\,d_{\frac{1}{2}+i} + 32\,e^3\,\Delta x^3\,f_i\Big)\Big)\Big\}$$

How to define variable e?

What’s the derivative of ArcTanh[x]?

$$\text{ArcTanh}'\left[\frac{e\,\Delta x}{2}\right]\ //\ \text{Simplify}$$

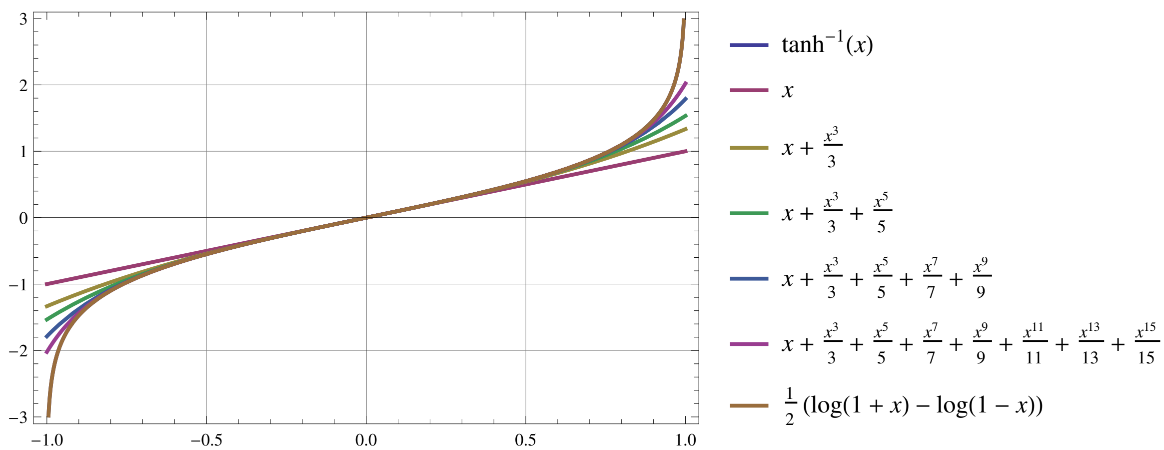
$$\frac{4}{4 - e^2\,\Delta x^2}$$

can we approximate Tanh^{-1}[x] some how?

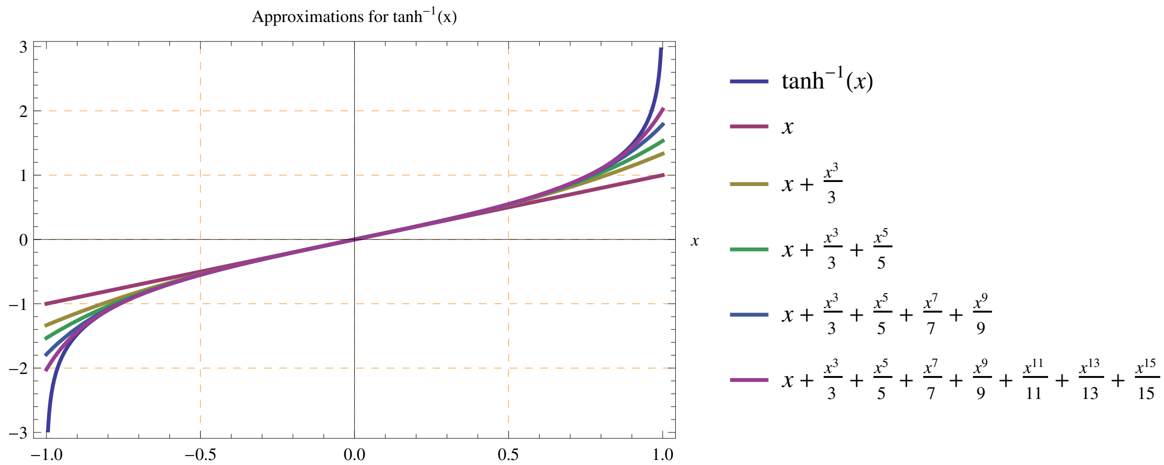
```
Series[ArcTanh[x], {x, 0, 24}]
```

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + \frac{x^{21}}{21} + \frac{x^{23}}{23} + O[x]^{25}$$

$$\text{Plot}\Big[\Big\{\text{ArcTanh}[x],\ x,\ x + \frac{x^3}{3},\ x + \frac{x^3}{3} + \frac{x^5}{5},\ x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9},$$
$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15},\ \frac{1}{2}\,(\text{Log}[1+x] - \text{Log}[1-x])\Big\},\ \{x, -1, 1\},$$
$$\text{PlotLegends} \to \text{"Expressions"},\ \text{PlotStyle} \to \text{Thick},\ \text{Frame} \to \text{True},\ \text{GridLines} \to \text{Automatic}\Big]$$



```
Plot[{ArcTanh[x], x, x + x^3/3, x + x^3/3 + x^5/5,
  x + x^3/3 + x^5/5 + x^7/7 + x^9/9, x + x^3/3 + x^5/5 + x^7/7 + x^9/9 + x^11/11 + x^13/13 + x^15/15}, {x, -1, 1},
 PlotLegends → "Expressions", PlotStyle → Thick, Frame → True,
 GridLines → Automatic, GridLinesStyle → Directive[Orange, Dashed],
 AxesLabel → {x, "Approximations for tanh^-1(x)"}]
```




```
Export["ArcTanh.pdf", %]
```

ArcTanh.pdf

Yes! It can be approximated very well, and as long as $\Delta x \to 0$. (Which is always a desirable feature. ;D)

Non-polynomial Interpolation function f3[x] (incomplete)

constrains

$$f3'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \text{ // Simplify}$$

$$f3'[+\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x} \text{ // Simplify}$$

$$c + d == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{c - b\,c + d - a\,d}{(-1 + a)\,(-1 + b)} == \frac{-f_i + f_{1+i}}{\Delta x}$$

solving for c and d

```
cnd = Solve[{f3'[-Δx / 2] == δ1 && f3'[+Δx / 2] == δ2}, {c, d}] // Simplify
```

$$\left\{\left\{c \to \frac{(-1+a)\ (\delta 1 + (-1+b)\ \delta 2)}{a-b},\ d \to -\frac{(-1+b)\ (\delta 1 + (-1+a)\ \delta 2)}{a-b}\right\}\right\}$$

yes, but in the paper we found that

$$d == -\frac{(-1+b)\ (\delta 1 + (-1+a)\ \delta 2)}{a-b}\ /.\ \text{Solve}\left[c == \frac{(-1+a)\ (\delta 1 + (-1+b)\ \delta 2)}{a-b},\ \delta 2\right]\ //\ \text{Simplify}$$

$$\{c + d == \delta 1\}$$

symmetry condition $\delta 1 = -\delta 2$ when r'[0] = 0.

```
f3'[0] == 0 /. cnd /. δ1 → -δ2 // Simplify
```

$$\left\{\frac{(a\ (-1+b) - b)\ \delta 2}{(-2+a)\ (-2+b)} == 0\right\}$$

```
Solve[f3'[0] == 0, {b}] /. δ1 → -δ2 // Simplify
```

$$\left\{\left\{b \to \frac{2\,c + 2\,d - a\,d}{c}\right\}\right\}$$

therefore

$$b == \frac{a}{a-1};$$

lets examing f3[x] with detail:

```
f3[ξ]
```

$$-\frac{c\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a} - \frac{d\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}$$

$$\text{Assuming}\left[\{\text{Re}[a] < 1,\ \text{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{c\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a}\, d\xi\right]$$

$$\frac{1}{2\,a^2} c\ \Delta x\ \Big(2\,a\ \text{ArcTanh}[1-a] - 2\ \text{Log}[2-a] - (-2+a)\ \text{Log}[-2+a] - 2\ \text{Log}[-1+a] + a\left(-2 + \text{Log}\left[-\frac{1}{a}\right] + 2\ \text{Log}[(-1+a)\ \Delta x]\right)\Big)$$

$$\text{Assuming}\left[\{\text{Re}[b] < 1,\ \text{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{d\ \Delta x\ \text{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}\, d\xi\right]$$

$$\frac{1}{2\,b^2} d\ \Delta x\ \Big(2\,b\ \text{ArcTanh}[1-b] - 2\ \text{Log}[2-b] - (-2+b)\ \text{Log}[-2+b] - 2\ \text{Log}[-1+b] + b\left(-2 + \text{Log}\left[-\frac{1}{b}\right] + 2\ \text{Log}[(-1+b)\ \Delta x]\right)\Big)$$

recall that for any given ‘x’ value:

```
{1/2 (Log[1 + x] - Log[1 - x]) == ArcTanh[x]} /. x → 0.656
```

```
{True}
```