Cubic Interpolation Polynomials

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In[46]:= `Quit[];`

Proposed Interpolation Functions

In[1]:= $f1[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3$

$f2[\xi\_] := \dfrac{a + b\,\xi + c\,\xi^2}{1 + e\,\xi}$

CIP - original formulation

constrains

In[3]:= $f1[0] == f_i$

$f1[\Delta x] == f_{i+1}$

Out[3]= $a == f_i$

Out[4]= $a + b\,\Delta x + c\,\Delta x^2 + d\,\Delta x^3 == f_{1+i}$

In[5]:= $f1'[0] == df_i$

$f1'[\Delta x] == df_{i+1}$

Out[5]= $b == df_i$

Out[6]= $b + 2\,c\,\Delta x + 3\,d\,\Delta x^2 == df_{1+i}$

In[7]:= $\text{Simplify}\left[\dfrac{1}{\Delta x}\int_0^{\Delta x} f1[\xi]\, d\xi\right] == f_{i+1/2}$

Out[7]= $a + \dfrac{1}{12}\Delta x\,(6\,b + \Delta x\,(4\,c + 3\,d\,\Delta x)) == f_{\frac{1}{2}+i}$

solving as a system of equations

to right side formulation

In[8]:=
```
coefs = Solve[f1[0] == f_i && f1[Δx] == f_{i+1} && f1'[0] == df_i && f1'[Δx] == df_{i+1}, {a, b, c, d}]
```

Out[8]= $\left\{\left\{a \to f_i,\ b \to df_i,\ c \to -\frac{2\,\Delta x\, df_i + \Delta x\, df_{1+i} + 3\, f_i - 3\, f_{1+i}}{\Delta x^2},\ d \to -\frac{-\Delta x\, df_i - \Delta x\, df_{1+i} - 2\, f_i + 2\, f_{1+i}}{\Delta x^3}\right\}\right\}$

to tleft side formulation

In[30]:=
```
coefs =
 Solve[f1[0] == f_i && f1[-Δx] == f_{i-1} && f1'[0] == df_i && f1'[-Δx] == df_{i-1}, {a, b, c, d}]
```

Out[30]= $\left\{\left\{a \to f_i,\ b \to df_i,\ c \to -\frac{-\Delta x\, df_{-1+i} - 2\,\Delta x\, df_i - 3\, f_{-1+i} + 3\, f_i}{\Delta x^2},\ d \to -\frac{-\Delta x\, df_{-1+i} - \Delta x\, df_i - 2\, f_{-1+i} + 2\, f_i}{\Delta x^3}\right\}\right\}$

In[13]:=
```
f1[x] // Expand
```

Out[13]= $a + b\,x + c\,x^2 + d\,x^3$

In[14]:=
```
f1'[x] // Expand
```

Out[14]= $b + 2\,c\,x + 3\,d\,x^2$

In[11]:=
```
f1[x] /. coefs // Expand
```

Out[11]= $\left\{x\, df_i + \frac{x^3\, df_i}{\Delta x^2} - \frac{2\,x^2\, df_i}{\Delta x} + \frac{x^3\, df_{1+i}}{\Delta x^2} - \frac{x^2\, df_{1+i}}{\Delta x} + f_i + \frac{2\,x^3\, f_i}{\Delta x^3} - \frac{3\,x^2\, f_i}{\Delta x^2} - \frac{2\,x^3\, f_{1+i}}{\Delta x^3} + \frac{3\,x^2\, f_{1+i}}{\Delta x^2}\right\}$

In[12]:=
```
f1'[x] /. coefs // Expand
```

Out[12]= $\left\{df_i + \frac{3\,x^2\, df_i}{\Delta x^2} - \frac{4\,x\, df_i}{\Delta x} + \frac{3\,x^2\, df_{1+i}}{\Delta x^2} - \frac{2\,x\, df_{1+i}}{\Delta x} + \frac{6\,x^2\, f_i}{\Delta x^3} - \frac{6\,x\, f_i}{\Delta x^2} - \frac{6\,x^2\, f_{1+i}}{\Delta x^3} + \frac{6\,x\, f_{1+i}}{\Delta x^2}\right\}$

RCIP - original formulation

constrains

In[15]:=
```
f2[0] == f_i
f2[Δx] == f_{i+1}
```

Out[15]= $a == f_i$

Out[16]= $\frac{a + b\,\Delta x + c\,\Delta x^2}{1 + e\,\Delta x} == f_{1+i}$

In[17]:=
```
f2'[0] == df_i
f2'[Δx] == df_{i+1}
```

Out[17]= $b - a\,e == df_i$

Out[18]= $\frac{b + 2\,c\,\Delta x}{1 + e\,\Delta x} - \frac{e\,(a + b\,\Delta x + c\,\Delta x^2)}{(1 + e\,\Delta x)^2} == df_{1+i}$

In[19]:= $\text{Simplify}\left[\frac{1}{\Delta x}\int_0^{\Delta x} f2[\xi]\, d\xi\right] == f_{i+1/2}$

Out[19]= $\text{ConditionalExpression}\Big[\frac{1}{2\, e^3\, \Delta x}\left(e\, \Delta x\, (2\, b\, e + c\, (-2 + e\, \Delta x)) + 2\, (c + e\, (-b + a\, e))\, \text{Log}[1 + e\, \Delta x]\right) == f_{\frac{1}{2}+i},$
$\text{Re}\left[\frac{1}{e\, \Delta x}\right] \geq 0 \,||\, \text{Re}\left[\frac{1}{e\, \Delta x}\right] < -1 \,||\, \frac{1}{e\, \Delta x} \notin \text{Reals}\Big]$

solving as a system of equations

to right side formulation

In[33]:=
```
coefs = Solve[
   f2[0] == f_i && f2[Δx] == f_{i+1} && f2'[0] == df_i && f2'[Δx] == df_{i+1}, {a, b, c, e}] // Simplify
```

Out[33]= $\Big\{\Big\{a \to f_i,\ b \to (-f_i\, (\Delta x\, df_{1+i} + 2\, f_i - 2\, f_{1+i}) + \Delta x\, df_i\, (\Delta x\, df_{1+i} - f_{1+i}))\, /\, (\Delta x\, (\Delta x\, df_{1+i} + f_i - f_{1+i})),$
$c \to \frac{-\Delta x^2\, df_i\, df_{1+i} + (f_i - f_{1+i})^2}{\Delta x^2\, (\Delta x\, df_{1+i} + f_i - f_{1+i})},\ e \to -\frac{\Delta x\, df_i + \Delta x\, df_{1+i} + 2\, f_i - 2\, f_{1+i}}{\Delta x\, (\Delta x\, df_{1+i} + f_i - f_{1+i})}\Big\}\Big\}$

to left side formulation

In[34]:=
```
coefs =
  Solve[f2[0] == f_i && f2[-Δx] == f_{i-1} && f2'[0] == df_i && f2'[-Δx] == df_{i-1}, {a, b, c, e}] //
   Simplify
```

Out[34]= $\Big\{\Big\{a \to f_i,$
$b \to (\Delta x\, df_i\, f_{-1+i} + 2\, (f_{-1+i} - f_i)\, f_i + \Delta x\, df_{-1+i}\, (\Delta x\, df_i + f_i))\, /\, (\Delta x\, (\Delta x\, df_{-1+i} + f_{-1+i} - f_i)),$
$c \to \frac{\Delta x^2\, df_{-1+i}\, df_i - (f_{-1+i} - f_i)^2}{\Delta x^2\, (\Delta x\, df_{-1+i} + f_{-1+i} - f_i)},\ e \to \frac{\Delta x\, df_{-1+i} + \Delta x\, df_i + 2\, f_{-1+i} - 2\, f_i}{\Delta x\, (\Delta x\, df_{-1+i} + f_{-1+i} - f_i)}\Big\}\Big\}$

In[25]:= **f2[x]**

Out[25]= $\frac{a + b\, x + c\, x^2}{1 + e\, x}$

In[26]:= **f2'[x]**

Out[26]= $\frac{b + 2\, c\, x}{1 + e\, x} - \frac{e\, (a + b\, x + c\, x^2)}{(1 + e\, x)^2}$

In[29]:= **f2[x] /. coefs // Simplify**

Out[29]= $\Big\{-\left(\Delta x^2\, (-x + \Delta x)\, df_{1+i}\, f_i - x\, \Delta x^2\, df_i\, ((x - \Delta x)\, df_{1+i} + f_{1+i}) + (f_i - f_{1+i})\, \left((x - \Delta x)^2\, f_i - x^2\, f_{1+i}\right)\right) /$
$(\Delta x\, (x\, \Delta x\, df_i + (x - \Delta x)\, \Delta x\, df_{1+i} + (2\, x - \Delta x)\, (f_i - f_{1+i})))\Big\}$

In[24]:= `f2'[x] /. coefs // Simplify`

Out[24]= $\Big\{\Big(x^2\,\Delta x^3\,df_i^2\,df_{1+i} + \Delta x\,df_i\,\big((x-\Delta x)^2\,\Delta x^2\,df_{1+i}^2 + 2\,\Delta x\,(x^2 - x\,\Delta x + \Delta x^2)\,df_{1+i}\,(f_i - f_{1+i}) - (x^2 - \Delta x^2)\,(f_i - f_{1+i})^2\big) - x\,\big((x - 2\,\Delta x)\,\Delta x\,df_{1+i} + 2\,(x - \Delta x)\,(f_i - f_{1+i})\big)\,(f_i - f_{1+i})^2\Big) \Big/ \Big(\Delta x\,(x\,\Delta x\,df_i + (x-\Delta x)\,\Delta x\,df_{1+i} + (2\,x - \Delta x)\,(f_i - f_{1+i}))^2\Big)\Big\}$

CIP-CSL3 - Original Formulation

Notice that the information to be evolved are the cell boundary values!

Left side constrains

$f1[-\Delta x] == f_{i-1}$

$f1[0] == f_i$

$a - b\,\Delta x + c\,\Delta x^2 - d\,\Delta x^3 == f_{-1+i}$

$a == f_i$

$\text{Simplify}\left[\int_{-\Delta x}^{0} f1[\xi]\,d\xi\right] == \rho_{i-1/2}$

$\frac{1}{12}\,\Delta x\,(12\,a + \Delta x\,(-6\,b + \Delta x\,(4\,c - 3\,d\,\Delta x))) == \rho_{-\frac{1}{2}+i}$

$f1'[-\Delta x/2] == s_{i-1/2}$

$b - c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == s_{-\frac{1}{2}+i}$

Solving as a system of equations

$\text{coefs} = \text{Solve}\left[f1[0] == f_i \;\&\&\; f1[-\Delta x] == f_{i-1} \;\&\&\; \int_{-\Delta x}^{0} f1[\xi]\,d\xi == \rho_{i-1/2} \;\&\&\; f1'[-\Delta x/2] == s_{i-1/2},\ \{a, b, c, d\}\right] \,//\, \text{Expand}$

$\Big\{\Big\{a \to f_i,\ b \to \frac{6\,f_i}{\Delta x} - 2\,s_{-\frac{1}{2}+i} - \frac{6\,\rho_{-\frac{1}{2}+i}}{\Delta x^2},\ c \to -\frac{3\,f_{-1+i}}{\Delta x^2} + \frac{9\,f_i}{\Delta x^2} - \frac{6\,s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6\,\rho_{-\frac{1}{2}+i}}{\Delta x^3},\ d \to -\frac{4\,f_{-1+i}}{\Delta x^3} + \frac{4\,f_i}{\Delta x^3} - \frac{4\,s_{-\frac{1}{2}+i}}{\Delta x^2}\Big\}\Big\}$

```
f1[x - x_i] /. coefs (* x \in [x_{i-1}, x_i] *)
```

$$\left\{f_i + \left(-\frac{4\, f_{-1+i}}{\Delta x^3} + \frac{4\, f_i}{\Delta x^3} - \frac{4\, s_{-\frac{1}{2}+i}}{\Delta x^2}\right)(x - x_i)^3 + (x - x_i)^2 \left(-\frac{3\, f_{-1+i}}{\Delta x^2} + \frac{9\, f_i}{\Delta x^2} - \frac{6\, s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^3}\right) + (x - x_i)\left(\frac{6\, f_i}{\Delta x} - 2\, s_{-\frac{1}{2}+i} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$$

evaluating the time integral of u * f1(x_i-ut) yields

$$\int_{-\Delta t}^{0} u\, f1[-u\, t]\, dt$$

$$a\, u\, \Delta t + \frac{1}{2} b\, u^2\, \Delta t^2 + \frac{1}{3} c\, u^3\, \Delta t^3 + \frac{1}{4} d\, u^4\, \Delta t^4$$

NOTE: u Δt indicates the direction of the advecting wave.

$d_{1\pm1/2}$ are here free parameters. Note that we can define it by using the resulting interpolation functions evaluating at Δx/2 and/or -Δx/2. e.g. :

```
f1[-Δx / 2] /. coefs // Expand
```

$$\left\{-\frac{1}{4} f_{-1+i} - \frac{f_i}{4} + \frac{3\, \rho_{-\frac{1}{2}+i}}{2\, \Delta x}\right\}$$

this is in agreement with equation (23) in Yabe [2001].

Right side constrains

```
f1[ 0 ] == f_i
f1[Δx] == f_{i+1}
```

$$a == f_i$$

$$a + b\, \Delta x + c\, \Delta x^2 + d\, \Delta x^3 == f_{1+i}$$

$$\text{Simplify}\left[\int_0^{\Delta x} f1[\xi]\, d\xi\right] == \rho_{i+1/2}$$

$$\frac{1}{12} \Delta x\, (12\, a + \Delta x\, (6\, b + \Delta x\, (4\, c + 3\, d\, \Delta x))) == \rho_{\frac{1}{2}+i}$$

```
f1'[Δx / 2] == s_{i+1/2}
```

$$b + c\, \Delta x + \frac{3\, d\, \Delta x^2}{4} == s_{\frac{1}{2}+i}$$

Solving as a system of equations

```
coefs = Solve[f1[ 0 ] == f_i && f1[Δx] == f_{i+1} &&
    ∫_0^Δx f1[ξ] dξ == ρ_{i+1/2} && f1'[Δx / 2] == s_{i+1/2}, {a, b, c, d}] // Expand
```

$$\left\{\left\{a \to f_i,\ b \to -\frac{6 f_i}{\Delta x} - 2 s_{\frac{1}{2}+i} + \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^2},\right.\right.$$
$$\left.\left. c \to \frac{9 f_i}{\Delta x^2} - \frac{3 f_{1+i}}{\Delta x^2} + \frac{6 s_{\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^3},\ d \to -\frac{4 f_i}{\Delta x^3} + \frac{4 f_{1+i}}{\Delta x^3} - \frac{4 s_{\frac{1}{2}+i}}{\Delta x^2}\right\}\right\}$$

```
f1[x - x_{i-1}] /. coefs (* x \in [x_i,x_{i+1}] *)
```

$$\left\{f_i + \left(-\frac{4 f_i}{\Delta x^3} + \frac{4 f_{1+i}}{\Delta x^3} - \frac{4 s_{\frac{1}{2}+i}}{\Delta x^2}\right) (x - x_{-1+i})^3 + \right.$$
$$\left.(x - x_{-1+i})^2 \left(\frac{9 f_i}{\Delta x^2} - \frac{3 f_{1+i}}{\Delta x^2} + \frac{6 s_{\frac{1}{2}+i}}{\Delta x} - \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^3}\right) + (x - x_{-1+i}) \left(-\frac{6 f_i}{\Delta x} - 2 s_{\frac{1}{2}+i} + \frac{6 \rho_{\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$$

evaluating the time integral of u * f1(x_i+ut) yields

```
∫_0^Δt u f1[u t] dt
```

$$a\,u\,\Delta t + \frac{1}{2} b\,u^2\,\Delta t^2 + \frac{1}{3} c\,u^3\,\Delta t^3 + \frac{1}{4} d\,u^4\,\Delta t^4$$

NOTE: u Δt indicates the direction of the advecting wave.

```
f1[Δx / 2] /. coefs // Expand
```

$$\left\{-\frac{f_i}{4} - \frac{f_{1+i}}{4} + \frac{3 \rho_{\frac{1}{2}+i}}{2 \Delta x}\right\}$$