

# Constrained Interpolation Methods

by Manuel Diaz, NTU, 2015.08.10.

```
Quit[];
```

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## Proposed Interpolation Functions

```
In[9]:= f0[ξ_] := a + b ξ + c ξ^2
```

```
f1[ξ_] := a + b ξ + c ξ^2 + d ξ^3
```

```
f2[ξ_] := 
$$\frac{a + b \xi + c \xi^2}{1 + e \xi}$$

```

```
f3[ξ_] := 
$$-\frac{c}{a} \Delta x \operatorname{Log}\left[\xi - \frac{\Delta x}{2} \left(\frac{2}{a} - 1\right)\right] + -\frac{d}{b} \Delta x \operatorname{Log}\left[\xi - \frac{\Delta x}{2} \left(\frac{2}{b} - 1\right)\right]$$

```

Just Testing

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## Interpolating Polynomial f0[x]

constrains

$$f0'[-\Delta x / 2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$f0'[\Delta x / 2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c \Delta x == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c \Delta x == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_0[\xi] d\xi\right] == f_i$$

$$a + \frac{c \Delta x^2}{12} == f_i$$

## solving as a system of equations

$$\text{coefs} = \text{Solve}\left[\begin{aligned} f_0'[\Delta x/2] &== \frac{f_{i+1} - f_i}{\Delta x} \ \&\& f_0'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \ \&\& + \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_0[\xi] d\xi == f_i, \{a, b, c\} \end{aligned}\right]$$

$$\left\{\left\{a \rightarrow \frac{1}{24} (-f_{-1+i} + 26 f_i - f_{1+i}), b \rightarrow -\frac{f_{-1+i} - f_{1+i}}{2 \Delta x}, c \rightarrow -\frac{-f_{-1+i} + 2 f_i - f_{1+i}}{2 \Delta x^2}\right\}\right\}$$

$$f_0[-\Delta x/2] /. \text{coefs} // \text{Expand}$$

$$f_0[\Delta x/2] /. \text{coefs} // \text{Expand}$$

$$\left\{\frac{f_{-1+i}}{3} + \frac{5 f_i}{6} - \frac{f_{1+i}}{6}\right\}$$

$$\left\{-\frac{1}{6} f_{-1+i} + \frac{5 f_i}{6} + \frac{f_{1+i}}{3}\right\}$$

$$\text{coefs} = \text{Solve}\left[\begin{aligned} f_0'[\Delta x/2] &== \frac{d_{i+1/2}}{\Delta x} \ \&\& f_0'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x} \ \&\& + \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_0[\xi] d\xi == f_i, \{a, b, c\} \end{aligned}\right]$$

$$\left\{\left\{a \rightarrow \frac{1}{24} \left(d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i} + 24 f_i\right), b \rightarrow -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2 \Delta x}, c \rightarrow -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2 \Delta x^2}\right\}\right\}$$

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## Interpolating Polynomial f1[x]

### constrains

$$f_1'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$f_1'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c \Delta x + \frac{3 d \Delta x^2}{4} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c \Delta x + \frac{3 d \Delta x^2}{4} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_1[\xi] d\xi\right] == f_i$$

$$a + \frac{c \Delta x^2}{12} == f_i$$

## solving as a system of equations

$$\text{coefs} = \text{Solve}\left[\begin{aligned} f_1'[\Delta x/2] &== \frac{f_{i+1} - f_i}{\Delta x} \ \&\& f_1'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \ \&\& + \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_1[\xi] d\xi == f_i, \{a, c, d\} \end{aligned}\right]$$

$$\left\{\left\{a \rightarrow \frac{1}{24} (-f_{-1+i} + 26 f_i - f_{1+i}), c \rightarrow -\frac{-f_{-1+i} + 2 f_i - f_{1+i}}{2 \Delta x^2}, d \rightarrow -\frac{2 (2 b \Delta x + f_{-1+i} - f_{1+i})}{3 \Delta x^3}\right\}\right\}$$

`f1[-Δx/2] /. coefs // Expand`

`f1[Δx/2] /. coefs // Expand`

$$\left\{-\frac{b \Delta x}{3} + \frac{f_{-1+i}}{6} + \frac{5 f_i}{6}\right\}$$

$$\left\{\frac{b \Delta x}{3} + \frac{5 f_i}{6} + \frac{f_{1+i}}{6}\right\}$$

What is b?

To find out we will assume and extra constrain:

$$\text{bcoef} = \text{Solve}\left[f_1'[0] == \frac{f_{i+1} - f_{i-1}}{2 \Delta x}, b\right]$$

$$\left\{\left\{b \rightarrow \frac{-f_{-1+i} + f_{1+i}}{2 \Delta x}\right\}\right\}$$

`f1[-Δx/2] /. coefs /. bcoef // Expand`

`f1[Δx/2] /. coefs /. bcoef // Expand`

$$\left\{\left\{\frac{f_{-1+i}}{3} + \frac{5 f_i}{6} - \frac{f_{1+i}}{6}\right\}\right\}$$

$$\left\{\left\{-\frac{1}{6} f_{-1+i} + \frac{5 f_i}{6} + \frac{f_{1+i}}{3}\right\}\right\}$$

NOTE: this is exactly the result from using polynomial `f0[x]`!

`Clear[d];`

$$\text{coefs} = \text{Solve}\left[\begin{aligned} f_1'[\Delta x/2] &== \frac{\delta_{i+1/2}}{\Delta x} \ \&\& f_1'[-\Delta x/2] == \frac{\delta_{i-1/2}}{\Delta x} \ \&\& + \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f_1[\xi] d\xi == f_i, \{a, c, d\} \end{aligned}\right]$$

$$\left\{\left\{a \rightarrow \frac{1}{24} \left(24 f_i + \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right), c \rightarrow -\frac{\delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}}{2 \Delta x^2}, d \rightarrow -\frac{2 \left(2 b \Delta x - \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right)}{3 \Delta x^3}\right\}\right\}$$

Therefore, it is obvious that we can choose 'b' as a free parameter to control the slope of the approximation!

# Non-polynomial Interpolation function f2[x]

## constraints

$$f2'[-\Delta x / 2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$f2'[\Delta x / 2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$\frac{b - c \Delta x}{1 - \frac{e \Delta x}{2}} - \frac{e \left( a - \frac{b \Delta x}{2} + \frac{c \Delta x^2}{4} \right)}{\left( 1 - \frac{e \Delta x}{2} \right)^2} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{b + c \Delta x}{1 + \frac{e \Delta x}{2}} - \frac{e \left( a + \frac{b \Delta x}{2} + \frac{c \Delta x^2}{4} \right)}{\left( 1 + \frac{e \Delta x}{2} \right)^2} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify} \left[ \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} f2[\xi] d\xi \right] == f_i$$

$$\text{ConditionalExpression} \left[ \frac{e (-c + b e) \Delta x + 2 (c + e (-b + a e)) \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right]}{e^3 \Delta x} == f_i, \right.$$

$$\left. \text{Re} \left[ \frac{1}{e \Delta x} \right] > \frac{1}{2} \mid \mid \text{Re} \left[ \frac{1}{e \Delta x} \right] < -\frac{1}{2} \mid \mid \frac{1}{e \Delta x} \notin \text{Reals} \right]$$

## Solve system

$$\text{coefs} = \text{Solve} \left[ f2'[\Delta x / 2] == \frac{f_{i+1} - f_i}{\Delta x} \ \&\& \ f2'[-\Delta x / 2] == \frac{f_i - f_{i-1}}{\Delta x} \ \&\& + \right.$$

$$\left. \frac{1}{e^3 \Delta x} \left( e (-c + b e) \Delta x + 2 (c + e (-b + a e)) \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) == f_i, \{a, b, c\} \right] // \text{Simplify}$$

$$\left\{ \left\{ a \rightarrow \frac{1}{32 e^3 \Delta x^3} \left( (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} + \right. \right. \right.$$

$$\left. \left( -2 e \Delta x (16 - 24 e^2 \Delta x^2 + e^4 \Delta x^4) + 4 (-4 + e^2 \Delta x^2)^2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i + \right.$$

$$\left. \left( -4 + e^2 \Delta x^2 \right)^2 \left( e \Delta x - 2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right\},$$

$$b \rightarrow -\frac{1}{16 e^2 \Delta x^3} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} - \right.$$

$$2 \left( -8 e \Delta x + 6 e^3 \Delta x^3 + (-4 + e^2 \Delta x^2)^2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i +$$

$$\left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \text{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right\},$$

$$c \rightarrow \frac{1}{8 \Delta x^2} \left( (-2 + e \Delta x)^2 f_{-1+i} - 2 (4 + e^2 \Delta x^2) f_i + (2 + e \Delta x)^2 f_{1+i} \right) \right\}$$

for example notice that

$$\frac{(-2 + e \Delta x)^2 f_{-1+i} - 2 (4 + e^2 \Delta x^2) f_i + (2 + e \Delta x)^2 f_{1+i}}{8 \Delta x^2} /. e \rightarrow 0 // \text{Simplify}$$

$$\frac{f_{-1+i} - 2 f_i + f_{1+i}}{2 \Delta x^2}$$

**f2[-Δx / 2] /. coeffs**

**f2[+Δx / 2] /. coeffs**

$$\left\{ \frac{1}{1 - \frac{e \Delta x}{2}} \left( \frac{1}{32} \left( (-2 + e \Delta x)^2 f_{-1+i} - 2 (4 + e^2 \Delta x^2) f_i + (2 + e \Delta x)^2 f_{1+i} \right) + \right. \right. \\ \left. \frac{1}{(32 e^3 \Delta x^3)} \left( (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} + \right. \right. \\ \left. \left( -2 e \Delta x (16 - 24 e^2 \Delta x^2 + e^4 \Delta x^4) + 4 (-4 + e^2 \Delta x^2)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i + \right. \\ \left. \left. (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right) + \right. \\ \left. \frac{1}{(32 e^2 \Delta x^2)} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} - \right. \right. \\ \left. \left. 2 \left( -8 e \Delta x + 6 e^3 \Delta x^3 + (-4 + e^2 \Delta x^2)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i + \right. \right. \\ \left. \left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right) \right) \right\} \\ \left\{ \frac{1}{1 + \frac{e \Delta x}{2}} \left( \frac{1}{32} \left( (-2 + e \Delta x)^2 f_{-1+i} - 2 (4 + e^2 \Delta x^2) f_i + (2 + e \Delta x)^2 f_{1+i} \right) + \right. \right. \\ \left. \frac{1}{(32 e^3 \Delta x^3)} \left( (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} + \right. \right. \\ \left. \left( -2 e \Delta x (16 - 24 e^2 \Delta x^2 + e^4 \Delta x^4) + 4 (-4 + e^2 \Delta x^2)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i + \right. \\ \left. \left. (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right) - \right. \\ \left. \frac{1}{(32 e^2 \Delta x^2)} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{-1+i} - \right. \right. \\ \left. \left. 2 \left( -8 e \Delta x + 6 e^3 \Delta x^3 + (-4 + e^2 \Delta x^2)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_i + \right. \right. \\ \left. \left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \operatorname{ArcTanh} \left[ \frac{e \Delta x}{2} \right] \right) f_{1+i} \right) \right) \right\}$$

which is consistent with previous results. Re-compute with d-slopes variables:

$$\begin{aligned}
\text{coefs} = & \text{Solve}\left[f_2'[\Delta x / 2] == \frac{d_{i+1/2}}{\Delta x} \ \&\& \ f_2'[-\Delta x / 2] == \frac{d_{i-1/2}}{\Delta x} \ \&\& + \right. \\
& \left. \frac{1}{e^3 \Delta x} \left( e (-c + b e) \Delta x + 2 (c + e (-b + a e)) \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) = f_i, \{a, b, c\} \right] // \text{Simplify} \\
\{ \{ a \rightarrow & \frac{1}{32 e^3 \Delta x^3} \left( -(-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} + \right. \\
& \left. (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 32 e^3 \Delta x^3 f_i \right), \\
b \rightarrow & \frac{1}{16 e^2 \Delta x^3} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} - \right. \\
& \left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 16 e^3 \Delta x^3 f_i \right), \\
c \rightarrow & \left. \frac{-(-2 + e \Delta x)^2 d_{-\frac{1}{2}+i} + (2 + e \Delta x)^2 d_{\frac{1}{2}+i}}{8 \Delta x^2} \right\} \}
\end{aligned}$$

**f2[-Δx / 2] /. coefs**

**f2[+Δx / 2] /. coefs**

$$\begin{aligned}
& \left\{ \frac{1}{1 - \frac{e \Delta x}{2}} \left( \frac{1}{32} \left( -(-2 + e \Delta x)^2 d_{-\frac{1}{2}+i} + (2 + e \Delta x)^2 d_{\frac{1}{2}+i} \right) - \right. \right. \\
& \frac{1}{32 e^2 \Delta x^2} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} - \right. \\
& \left. \left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 16 e^3 \Delta x^3 f_i \right) + \right. \\
& \left. \frac{1}{32 e^3 \Delta x^3} \left( -(-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} + \right. \right. \\
& \left. \left. (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 32 e^3 \Delta x^3 f_i \right) \right) \} \\
& \left\{ \frac{1}{1 + \frac{e \Delta x}{2}} \left( \frac{1}{32} \left( -(-2 + e \Delta x)^2 d_{-\frac{1}{2}+i} + (2 + e \Delta x)^2 d_{\frac{1}{2}+i} \right) + \right. \right. \\
& \frac{1}{32 e^2 \Delta x^2} \left( (-2 + e \Delta x)^2 \left( -2 e \Delta x + (2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} - \right. \\
& \left. \left. (2 + e \Delta x)^2 \left( -2 e \Delta x + (-2 + e \Delta x)^2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 16 e^3 \Delta x^3 f_i \right) + \right. \\
& \left. \frac{1}{32 e^3 \Delta x^3} \left( -(-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{-\frac{1}{2}+i} + \right. \right. \\
& \left. \left. (-4 + e^2 \Delta x^2)^2 \left( e \Delta x - 2 \text{ArcTanh}\left[\frac{e \Delta x}{2}\right] \right) d_{\frac{1}{2}+i} + 32 e^3 \Delta x^3 f_i \right) \right) \}
\end{aligned}$$

How to define variable e?

What's the derivative of ArcTanh[x]?

```
ArcTanh'[  $\frac{e^{\Delta x}}{2}$  ] // Simplify
```

$$\frac{4}{4 - e^{2\Delta x^2}}$$

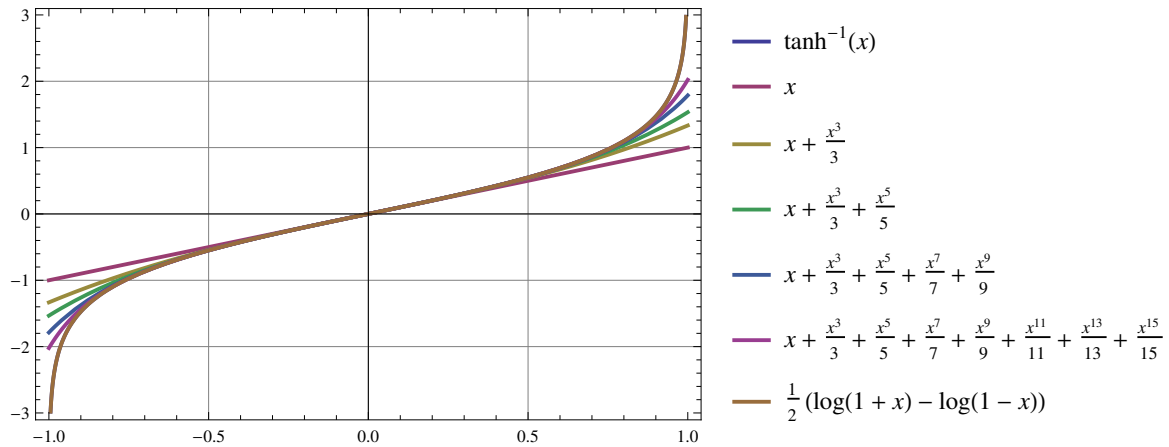
can we approximate  $\text{Tanh}^{-1}[x]$  some how?

```
Series[ArcTanh[x], {x, 0, 24}]
```

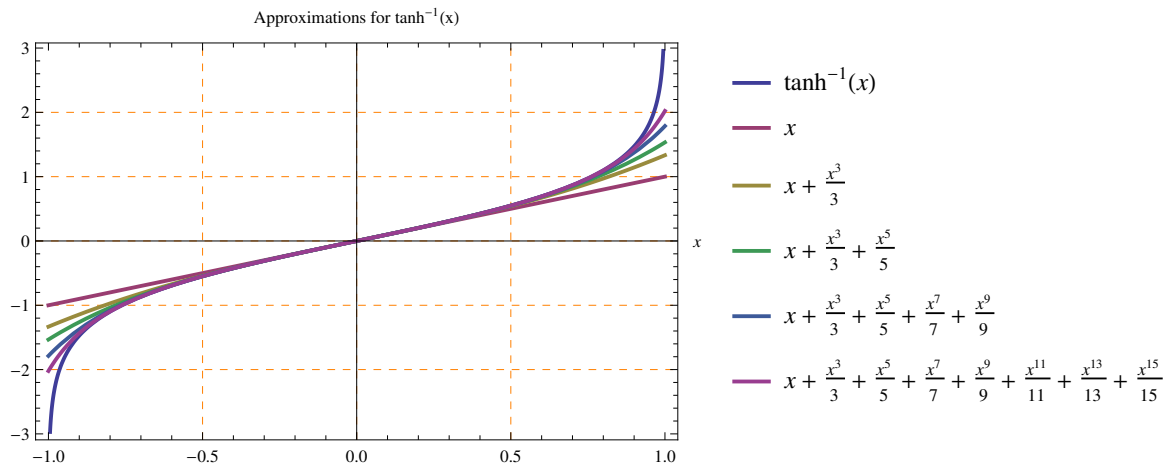
$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + \frac{x^{21}}{21} + \frac{x^{23}}{23} + O[x]^{25}$$

```
Plot[ { ArcTanh[x], x, x +  $\frac{x^3}{3}$ , x +  $\frac{x^3}{3} + \frac{x^5}{5}$ , x +  $\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9}$ ,  
x +  $\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15}$ ,  $\frac{1}{2} (\text{Log}[1+x] - \text{Log}[1-x])$  }, {x, -1, 1},
```

```
PlotLegends -> "Expressions", PlotStyle -> Thick, Frame -> True, GridLines -> Automatic ]
```



```
Plot[{ArcTanh[x], x, x +  $\frac{x^3}{3}$ , x +  $\frac{x^3}{3}$  +  $\frac{x^5}{5}$ ,
      x +  $\frac{x^3}{3}$  +  $\frac{x^5}{5}$  +  $\frac{x^7}{7}$  +  $\frac{x^9}{9}$ , x +  $\frac{x^3}{3}$  +  $\frac{x^5}{5}$  +  $\frac{x^7}{7}$  +  $\frac{x^9}{9}$  +  $\frac{x^{11}}{11}$  +  $\frac{x^{13}}{13}$  +  $\frac{x^{15}}{15}$ }, {x, -1, 1},
PlotLegends -> "Expressions", PlotStyle -> Thick, Frame -> True,
GridLines -> Automatic, GridLinesStyle -> Directive[Orange, Dashed],
AxesLabel -> {x, "Approximations for  $\tanh^{-1}(x)$ "}]
```



```
Export["ArcTanh.pdf", %]
```

```
ArcTanh.pdf
```

Yes! It can be approximated very well, and as long as  $\Delta x \rightarrow 0$ . ( Which is always a desirable feature. ;D )

## Non-polynomial Interpolation function f3[x] (incomplete)

constrains

$$f3'[-\Delta x / 2] = \frac{f_i - f_{i-1}}{\Delta x} // \text{Simplify}$$

$$f3'[+\Delta x / 2] = \frac{f_{i+1} - f_i}{\Delta x} // \text{Simplify}$$

$$c + d = \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{c - b \quad c + d - a \quad d}{(-1 + a) \quad (-1 + b)} = \frac{-f_i + f_{1+i}}{\Delta x}$$

solving for c and d



```
cnd = Solve[{f3'[-Δx/2] == δ1 && f3'[Δx/2] == δ2}, {c, d}] // Simplify
```

$$\left\{ \left\{ c \rightarrow \frac{(-1+a)(\delta_1 + (-1+b)\delta_2)}{a-b}, d \rightarrow -\frac{(-1+b)(\delta_1 + (-1+a)\delta_2)}{a-b} \right\} \right\}$$

yes, but in the paper we found that

$$d == -\frac{(-1+b)(\delta_1 + (-1+a)\delta_2)}{a-b} /. \text{Solve}\left[c == \frac{(-1+a)(\delta_1 + (-1+b)\delta_2)}{a-b}, \delta_2\right] // \text{Simplify}$$

$$\{c + d == \delta_1\}$$

symmetry condition  $\delta_1 = -\delta_2$  when  $r'[0] = 0$ .

```
f3'[0] == 0 /. cnd /. δ1 → -δ2 // Simplify
```

$$\left\{ \frac{(a(-1+b) - b)\delta_2}{(-2+a)(-2+b)} == 0 \right\}$$

```
Solve[f3'[0] == 0, {b}] /. δ1 → -δ2 // Simplify
```

$$\left\{ \left\{ b \rightarrow \frac{2c + 2d - ad}{c} \right\} \right\}$$

therefore

$$b == \frac{a}{a-1};$$

lets examing f3[x] with detail:

```
f3[ξ]
```

$$-\frac{c \Delta x \operatorname{Log}\left[-\frac{1}{2}\left(-1 + \frac{2}{a}\right) \Delta x + \xi\right]}{a} - \frac{d \Delta x \operatorname{Log}\left[-\frac{1}{2}\left(-1 + \frac{2}{b}\right) \Delta x + \xi\right]}{b}$$

$$\text{Assuming}\left[\{\operatorname{Re}[a] < 1, \operatorname{Re}[\Delta x] > 0\}, \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \frac{c \Delta x \operatorname{Log}\left[-\frac{1}{2}\left(-1 + \frac{2}{a}\right) \Delta x + \xi\right]}{a} d\xi\right]$$

$$\frac{1}{2a^2} c \Delta x \left( 2a \operatorname{ArcTanh}[1-a] - 2 \operatorname{Log}[2-a] - (-2+a) \operatorname{Log}[-2+a] - 2 \operatorname{Log}[-1+a] + a \left( -2 + \operatorname{Log}\left[-\frac{1}{a}\right] + 2 \operatorname{Log}[(-1+a)\Delta x] \right) \right)$$

$$\text{Assuming}\left[\{\operatorname{Re}[b] < 1, \operatorname{Re}[\Delta x] > 0\}, \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} \frac{d \Delta x \operatorname{Log}\left[-\frac{1}{2}\left(-1 + \frac{2}{b}\right) \Delta x + \xi\right]}{b} d\xi\right]$$

$$\frac{1}{2b^2} d \Delta x \left( 2b \operatorname{ArcTanh}[1-b] - 2 \operatorname{Log}[2-b] - (-2+b) \operatorname{Log}[-2+b] - 2 \operatorname{Log}[-1+b] + b \left( -2 + \operatorname{Log}\left[-\frac{1}{b}\right] + 2 \operatorname{Log}[(-1+b)\Delta x] \right) \right)$$

recall that for any given 'x' value:

```


$$\left\{ \frac{1}{2} (\text{Log}[1 + x] - \text{Log}[1 - x]) == \text{ArcTanh}[x] \right\} /. x \rightarrow 0.656$$

{True}

```

## MCV3

### constrains

```

In[19]:= f1[-1] == fL
          f1[1] == fR

Out[19]= a - b + c - d == fL

Out[20]= a + b + c + d == fR

In[17]:= f1'[-1] == f'L
          f1'[1] == f'R

Out[17]= b - 2 c + 3 d == f'L

Out[18]= b + 2 c + 3 d == f'R

```

### solving as a system of equations

```

In[22]:= coeffs = Solve[
          f1[-1] == fL && f1[1] == fR && f1'[-1] == f'L && f1'[1] == f'R, {a, b, c, d}] // Simplify

Out[22]=  $\left\{ \left\{ a \rightarrow \frac{1}{4} (2 f_L + 2 f_R + f'_L - f'_R), \right. \right.$ 
           $b \rightarrow \frac{1}{4} (-3 f_L + 3 f_R - f'_L - f'_R), c \rightarrow \frac{1}{4} (-f'_L + f'_R), d \rightarrow \frac{1}{4} (f_L - f_R + f'_L + f'_R) \left. \right\} \}$ 

In[37]:= f1[ξ] /. coeffs

Out[37]=  $\left\{ \frac{1}{4} \xi (-3 f_L + 3 f_R - f'_L - f'_R) + \right.$ 
           $\left. \frac{1}{4} (2 f_L + 2 f_R + f'_L - f'_R) + \frac{1}{4} \xi^2 (-f'_L + f'_R) + \frac{1}{4} \xi^3 (f_L - f_R + f'_L + f'_R) \right\}$ 

In[41]:= f1'[ξ] /. coeffs

Out[41]=  $\left\{ \frac{1}{4} (-3 f_L + 3 f_R - f'_L - f'_R) + \frac{1}{2} \xi (-f'_L + f'_R) + \frac{3}{4} \xi^2 (f_L - f_R + f'_L + f'_R) \right\}$ 

```

```
In[45]:= f1'[-1] /. coefs // Simplify
          f1'[0] /. coefs // Simplify
          f1'[+1] /. coefs // Simplify
```

```
Out[45]= {f'_L}
```

```
Out[46]= {1/4 (-3 f_L + 3 f_R - f'_L - f'_R)}
```

```
Out[47]= {f'_R}
```

## CIP-CSL3

Notice that the information to be evolved are the cell boundary values!

### constrains

```
In[48]:= f1[-Δx/2] == f_{i-1/2}
          f1[Δx/2] == f_{i+1/2}
```

```
Out[48]= a - b Δx/2 + c Δx^2/4 - d Δx^3/8 == f_{-1/2+i}
```

```
Out[49]= a + b Δx/2 + c Δx^2/4 + d Δx^3/8 == f_{1/2+i}
```

```
Simplify[1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ] == f_i
```

```
a + c Δx^2/12 == f_i
```

```
In[60]:= f1'[0] == dd_i
```

```
Out[60]= b == dd_i
```

### solving as a system of equations

```
In[61]:= coefs = Solve[f1[Δx/2] == f_{i+1/2} && f1[-Δx/2] == f_{i-1/2} &&
```

```
1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ == f_i && f1'[0] == dd_i, {a, b, c, d}] // Expand
```

```
Out[61]= {{a -> -1/4 f_{-1/2+i} + 3/2 f_i - 1/4 f_{1/2+i}, b -> dd_i,
           c -> (3 f_{-1/2+i}/Δx^2 - 6 f_i/Δx^2 + 3 f_{1/2+i}/Δx^2), d -> (-4 dd_i/Δx^2 - 4 f_{-1/2+i}/Δx^3 + 4 f_{1/2+i}/Δx^3)}}
```

```
f1[x - xi-1/2] /. Coefs (*x \in [xi-1/2, xi+1/2] *)
```

$$\text{Out[63]} = \left\{ -\frac{1}{4} f_{-\frac{1}{2}+i} + \frac{3 f_i}{2} - \frac{1}{4} f_{\frac{1}{2}+i} + d d_i \left( x - x_{-\frac{1}{2}+i} \right) + \left( \frac{3 f_{-\frac{1}{2}+i}}{\Delta x^2} - \frac{6 f_i}{\Delta x^2} + \frac{3 f_{\frac{1}{2}+i}}{\Delta x^2} \right) \left( x - x_{-\frac{1}{2}+i} \right)^2 + \left( -\frac{4 d d_i}{\Delta x^2} - \frac{4 f_{-\frac{1}{2}+i}}{\Delta x^3} + \frac{4 f_{\frac{1}{2}+i}}{\Delta x^3} \right) \left( x - x_{-\frac{1}{2}+i} \right)^3 \right\}$$

is  $f_i$  the cell center value?

```
In[65]:= a +  $\frac{c \Delta x^2}{12}$  /. Coefs // Simplify
```

```
Out[65]= { fi }
```

yes!