Constrained Interpolation Methods

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In[128]:=
```
Quit[];
```

Proposed Interpolation Functions

In[1]:= $f0[\xi\_] := a + b\,\xi + c\,\xi^2$

$f1[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3$

$$f2[\xi\_] := \frac{a + b\,\xi + c\,\xi^2}{1 + e\,\xi}$$

$$f3[\xi\_] := -\frac{c}{a}\,\Delta x\,\mathrm{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{a} - 1\right)\right] + -\frac{d}{b}\,\Delta x\,\mathrm{Log}\left[\xi - \frac{\Delta x}{2}\left(\frac{2}{b} - 1\right)\right]$$

Just Testing

Interpolating Polynomial f0[x]

constrains

$$f0'[-\Delta x / 2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$f0'[\Delta x / 2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c\,\Delta x == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c\,\Delta x == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\, d\xi\right] == f_i$$

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

$$\text{coefs} = \text{Solve}\left[\text{f0}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}\ \&\&\ \text{f0}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}\ \&\&\ + \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\, d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i} + 26\, f_i - f_{1+i}\right),\ b \to -\frac{f_{-1+i} - f_{1+i}}{2\,\Delta x},\ c \to -\frac{-f_{-1+i} + 2\, f_i - f_{1+i}}{2\,\Delta x^2}\right\}\right\}$$

```
f0[-Δx / 2] /. coefs // Expand
f0[Δx / 2] /. coefs // Expand
```

$$\left\{\frac{f_{-1+i}}{3} + \frac{5\, f_i}{6} - \frac{f_{1+i}}{6}\right\}$$

$$\left\{-\frac{1}{6} f_{-1+i} + \frac{5\, f_i}{6} + \frac{f_{1+i}}{3}\right\}$$

$$\text{coefs} = \text{Solve}\left[\text{f0}'[\Delta x/2] == \frac{d_{i+1/2}}{\Delta x}\ \&\&\ \text{f0}'[-\Delta x/2] == \frac{d_{i-1/2}}{\Delta x}\ \&\&\ + \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \text{f0}[\xi]\, d\xi == f_i,\ \{a, b, c\}\right]$$

$$\left\{\left\{a \to \frac{1}{24}\left(d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i} + 24\, f_i\right),\ b \to -\frac{-d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x},\ c \to -\frac{d_{-\frac{1}{2}+i} - d_{\frac{1}{2}+i}}{2\,\Delta x^2}\right\}\right\}$$

Interpolating Polynomial f1[x]

constrains

$$\text{f1}'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$\text{f1}'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$b - c\,\Delta x + \frac{3\, d\,\Delta x^2}{4} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$b + c\,\Delta x + \frac{3\, d\,\Delta x^2}{4} == \frac{-f_i + f_{1+i}}{\Delta x}$$

```
Simplify[1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ] == f_i
```

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

solving as a system of equations

```
coefs = Solve[
  f1'[Δx / 2] == (f_{i+1} - f_i)/Δx && f1'[-Δx / 2] == (f_i - f_{i-1})/Δx && + 1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ == f_i, {a, c, d}]
```

$$\left\{\left\{a \to \frac{1}{24}\left(-f_{-1+i} + 26\, f_i - f_{1+i}\right),\ c \to -\frac{-f_{-1+i} + 2\, f_i - f_{1+i}}{2\,\Delta x^2},\ d \to -\frac{2\,(2\, b\,\Delta x + f_{-1+i} - f_{1+i})}{3\,\Delta x^3}\right\}\right\}$$

```
f1[-Δx / 2] /. coefs // Expand
f1[Δx / 2] /. coefs // Expand
```

$$\left\{-\frac{b\,\Delta x}{3} + \frac{f_{-1+i}}{6} + \frac{5\, f_i}{6}\right\}$$

$$\left\{\frac{b\,\Delta x}{3} + \frac{5\, f_i}{6} + \frac{f_{1+i}}{6}\right\}$$

What is b?
To find out we will assuming and extra constrain:

```
bcoef = Solve[f1'[0] == (f_{i+1} - f_{i-1})/(2 Δx), b]
```

$$\left\{\left\{b \to \frac{-f_{-1+i} + f_{1+i}}{2\,\Delta x}\right\}\right\}$$

```
f1[-Δx / 2] /. coefs /. bcoef // Expand
f1[Δx / 2] /. coefs /. bcoef // Expand
```

$$\left\{\left\{\frac{f_{-1+i}}{3} + \frac{5\, f_i}{6} - \frac{f_{1+i}}{6}\right\}\right\}$$

$$\left\{\left\{-\frac{1}{6} f_{-1+i} + \frac{5\, f_i}{6} + \frac{f_{1+i}}{3}\right\}\right\}$$

NOTE: this is exactly the result from using polynomial f0[x]!

```
Clear[d];
```

```
coefs = Solve[
  f1'[Δx / 2] == δ_{i+1/2}/Δx && f1'[-Δx / 2] == δ_{i-1/2}/Δx && + 1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ == f_i, {a, c, d}]
```

$$\left\{\left\{a \to \frac{1}{24}\left(24\, f_i + \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right),\ c \to -\frac{\delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}}{2\,\Delta x^2},\ d \to -\frac{2\left(2\, b\,\Delta x - \delta_{-\frac{1}{2}+i} - \delta_{\frac{1}{2}+i}\right)}{3\,\Delta x^3}\right\}\right\}$$

Therefore, it is obvious that we can choose ‘b’ as a free parameter to control the slope of the approximation!

Non-polynomial Interpolation function f2[x]

constrains

$$f2'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}$$

$$f2'[+\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}$$

$$\frac{b - c\,\Delta x}{1 - \frac{e\,\Delta x}{2}} - \frac{e\left(a - \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 - \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{b + c\,\Delta x}{1 + \frac{e\,\Delta x}{2}} - \frac{e\left(a + \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4}\right)}{\left(1 + \frac{e\,\Delta x}{2}\right)^2} == \frac{-f_i + f_{1+i}}{\Delta x}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f2[\xi]\, d\xi\right] == f_i$$

$$\text{ConditionalExpression}\left[\frac{e\,(-c + b\,e)\,\Delta x + 2\,(c + e\,(-b + a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]}{e^3\,\Delta x} == f_i,\right.$$
$$\left.\text{Re}\left[\frac{1}{e\,\Delta x}\right] > \frac{1}{2}\ \|\ \text{Re}\left[\frac{1}{e\,\Delta x}\right] < -\frac{1}{2}\ \|\ \frac{1}{e\,\Delta x} \notin \text{Reals}\right]$$

Solve system

$$\text{coefs} = \text{Solve}\left[f2'[\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x}\ \&\&\ f2'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x}\ \&\&\right.$$
$$\left.\frac{1}{e^3\,\Delta x}\left(e\,(-c + b\,e)\,\Delta x + 2\,(c + e\,(-b + a\,e))\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) == f_i,\ \{a, b, c\}\right]\ //\ \text{Simplify}$$

$$\left\{\left\{a \to \frac{1}{32\,e^3\,\Delta x^3}\left(\left(-4 + e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} +\right.\right.\right.$$
$$\left(-2\,e\,\Delta x\left(16 - 24\,e^2\,\Delta x^2 + e^4\,\Delta x^4\right) + 4\left(-4 + e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i +$$
$$\left.\left(-4 + e^2\,\Delta x^2\right)^2\left(e\,\Delta x - 2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right),$$
$$b \to -\frac{1}{16\,e^2\,\Delta x^3}\left(\left(-2 + e\,\Delta x\right)^2\left(-2\,e\,\Delta x + \left(2 + e\,\Delta x\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{-1+i} -\right.$$
$$2\left(-8\,e\,\Delta x + 6\,e^3\,\Delta x^3 + \left(-4 + e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_i +$$
$$\left.\left(2 + e\,\Delta x\right)^2\left(-2\,e\,\Delta x + \left(-2 + e\,\Delta x\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right) f_{1+i}\right),$$
$$\left.\left.c \to \frac{1}{8\,\Delta x^2}\left(\left(-2 + e\,\Delta x\right)^2 f_{-1+i} - 2\left(4 + e^2\,\Delta x^2\right) f_i + \left(2 + e\,\Delta x\right)^2 f_{1+i}\right)\right\}\right\}$$

for example notice that

```
((-2 + e Δx)^2 f_{-1+i} - 2 (4 + e^2 Δx^2) f_i + (2 + e Δx)^2 f_{1+i}) / (8 Δx^2) /. e → 0 // Simplify
```

$$\frac{f_{-1+i}-2\,f_i+f_{1+i}}{2\,\Delta x^2}$$

```
f2[-Δx / 2] /. coefs
f2[+Δx / 2] /. coefs
```

$$\left\{\frac{1}{1-\frac{e\,\Delta x}{2}}\left(\frac{1}{32}\left((-2+e\,\Delta x)^2\,f_{-1+i}-2\left(4+e^2\,\Delta x^2\right)f_i+(2+e\,\Delta x)^2\,f_{1+i}\right)+\right.\right.$$
$$1\Big/\left(32\,e^3\,\Delta x^3\right)\left(\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x-2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{-1+i}+\right.$$
$$\left(-2\,e\,\Delta x\left(16-24\,e^2\,\Delta x^2+e^4\,\Delta x^4\right)+4\left(-4+e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_i+$$
$$\left.\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x-2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{1+i}\right)+$$
$$1\Big/\left(32\,e^2\,\Delta x^2\right)\left((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x+(2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{-1+i}-\right.$$
$$2\left(-8\,e\,\Delta x+6\,e^3\,\Delta x^3+\left(-4+e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_i+$$
$$\left.\left.\left.(2+e\,\Delta x)^2\left(-2\,e\,\Delta x+(-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{1+i}\right)\right)\right\}$$

$$\left\{\frac{1}{1+\frac{e\,\Delta x}{2}}\left(\frac{1}{32}\left((-2+e\,\Delta x)^2\,f_{-1+i}-2\left(4+e^2\,\Delta x^2\right)f_i+(2+e\,\Delta x)^2\,f_{1+i}\right)+\right.\right.$$
$$1\Big/\left(32\,e^3\,\Delta x^3\right)\left(\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x-2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{-1+i}+\right.$$
$$\left(-2\,e\,\Delta x\left(16-24\,e^2\,\Delta x^2+e^4\,\Delta x^4\right)+4\left(-4+e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_i+$$
$$\left.\left(-4+e^2\,\Delta x^2\right)^2\left(e\,\Delta x-2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{1+i}\right)-$$
$$1\Big/\left(32\,e^2\,\Delta x^2\right)\left((-2+e\,\Delta x)^2\left(-2\,e\,\Delta x+(2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{-1+i}-\right.$$
$$2\left(-8\,e\,\Delta x+6\,e^3\,\Delta x^3+\left(-4+e^2\,\Delta x^2\right)^2\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_i+$$
$$\left.\left.\left.(2+e\,\Delta x)^2\left(-2\,e\,\Delta x+(-2+e\,\Delta x)^2\,\text{ArcTanh}\left[\frac{e\,\Delta x}{2}\right]\right)f_{1+i}\right)\right)\right\}$$

which is consisten with previous results. Re-compute with d-slopes variables:

```
coefs = Solve[f2'[Δx / 2] == d_{i+1/2}/Δx && f2'[-Δx / 2] == d_{i-1/2}/Δx && +
    1/(e^3 Δx) (e (-c + b e) Δx + 2 (c + e (-b + a e)) ArcTanh[e Δx/2]) == f_i, {a, b, c}] // Simplify
```

$$\left\{\left\{a \to \frac{1}{32\, e^3\, \Delta x^3}\left(-\left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\, e^3\, \Delta x^3\, f_i\right),\right.\right.$$

$$b \to \frac{1}{16\, e^2\, \Delta x^3}\left((-2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (-2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\, e^3\, \Delta x^3\, f_i\right),$$

$$\left.\left. c \to \frac{-(-2 + e\, \Delta x)^2\, d_{-\frac{1}{2}+i} + (2 + e\, \Delta x)^2\, d_{\frac{1}{2}+i}}{8\, \Delta x^2}\right\}\right\}$$

```
f2[-Δx / 2] /. coefs
f2[+Δx / 2] /. coefs
```

$$\left\{\frac{1}{1 - \frac{e\, \Delta x}{2}}\left(\frac{1}{32}\left(-(-2 + e\, \Delta x)^2\, d_{-\frac{1}{2}+i} + (2 + e\, \Delta x)^2\, d_{\frac{1}{2}+i}\right) - \frac{1}{32\, e^2\, \Delta x^2}\left((-2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (-2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\, e^3\, \Delta x^3\, f_i\right) + \frac{1}{32\, e^3\, \Delta x^3}\left(-\left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\, e^3\, \Delta x^3\, f_i\right)\right)\right\}$$

$$\left\{\frac{1}{1 + \frac{e\, \Delta x}{2}}\left(\frac{1}{32}\left(-(-2 + e\, \Delta x)^2\, d_{-\frac{1}{2}+i} + (2 + e\, \Delta x)^2\, d_{\frac{1}{2}+i}\right) + \frac{1}{32\, e^2\, \Delta x^2}\left((-2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} - (2 + e\, \Delta x)^2 \left(-2\, e\, \Delta x + (-2 + e\, \Delta x)^2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 16\, e^3\, \Delta x^3\, f_i\right) + \frac{1}{32\, e^3\, \Delta x^3}\left(-\left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{-\frac{1}{2}+i} + \left(-4 + e^2\, \Delta x^2\right)^2 \left(e\, \Delta x - 2\, \text{ArcTanh}\left[\frac{e\, \Delta x}{2}\right]\right) d_{\frac{1}{2}+i} + 32\, e^3\, \Delta x^3\, f_i\right)\right)\right\}$$

How to define variable e?

What’s the derivative of ArcTanh[x]?

```
ArcTanh'[e Δx/2] // Simplify
```

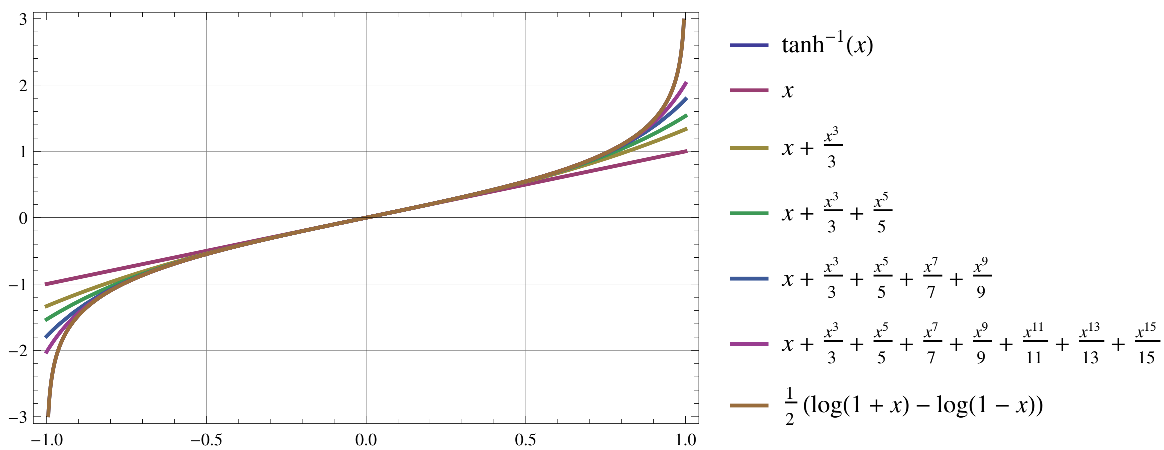
$$\frac{4}{4 - e^2 \, \Delta x^2}$$

can we approximate Tanh^{-1}[x] some how?

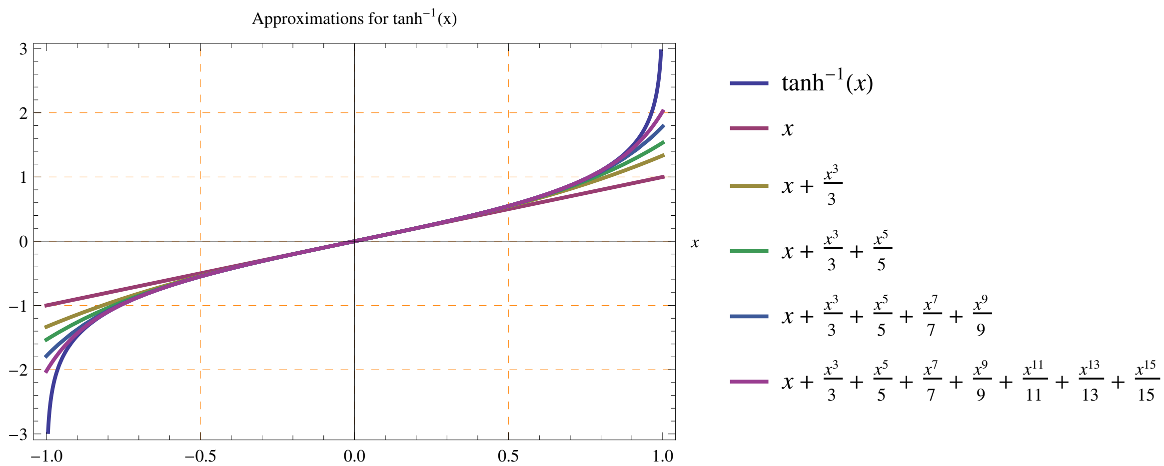
```
Series[ArcTanh[x], {x, 0, 24}]
```

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + \frac{x^{21}}{21} + \frac{x^{23}}{23} + O[x]^{25}$$

```
Plot[{ArcTanh[x], x, x + x^3/3, x + x^3/3 + x^5/5, x + x^3/3 + x^5/5 + x^7/7 + x^9/9,
  x + x^3/3 + x^5/5 + x^7/7 + x^9/9 + x^11/11 + x^13/13 + x^15/15, 1/2 (Log[1 + x] - Log[1 - x]) }, {x, -1, 1},
 PlotLegends → "Expressions", PlotStyle → Thick, Frame → True, GridLines → Automatic]
```



```
Plot[{ArcTanh[x], x, x + x^3/3, x + x^3/3 + x^5/5,
  x + x^3/3 + x^5/5 + x^7/7 + x^9/9, x + x^3/3 + x^5/5 + x^7/7 + x^9/9 + x^11/11 + x^13/13 + x^15/15}, {x, -1, 1},
 PlotLegends → "Expressions", PlotStyle → Thick, Frame → True,
 GridLines → Automatic, GridLinesStyle → Directive[Orange, Dashed],
 AxesLabel → {x, "Approximations for tanh^-1(x)"}]
```




```
Export["ArcTanh.pdf", %]
```

ArcTanh.pdf

Yes! It can be approximated very well, and as long as Δx → 0. (Which is always a desirable feature. ;D)

Non-polynomial Interpolation function f3[x] (incomplete)

constrains

$$f3'[-\Delta x/2] == \frac{f_i - f_{i-1}}{\Delta x} \text{ // Simplify}$$

$$f3'[+\Delta x/2] == \frac{f_{i+1} - f_i}{\Delta x} \text{ // Simplify}$$

$$c + d == \frac{-f_{-1+i} + f_i}{\Delta x}$$

$$\frac{c - b\,c + d - a\,d}{(-1 + a)\,(-1 + b)} == \frac{-f_i + f_{1+i}}{\Delta x}$$

solving for c and d

```
cnd = Solve[{f3'[-Δx / 2] == δ1 && f3'[+Δx / 2] == δ2}, {c, d}] // Simplify
```

$$\left\{\left\{c \to \frac{(-1+a)\,(\delta 1 + (-1+b)\,\delta 2)}{a-b},\ d \to -\frac{(-1+b)\,(\delta 1 + (-1+a)\,\delta 2)}{a-b}\right\}\right\}$$

yes, but in the paper we found that

$$d == -\frac{(-1+b)\,(\delta 1 + (-1+a)\,\delta 2)}{a-b}\ /.\ \mathrm{Solve}\left[c == \frac{(-1+a)\,(\delta 1 + (-1+b)\,\delta 2)}{a-b},\ \delta 2\right]\ //\ \mathrm{Simplify}$$

$$\{c + d == \delta 1\}$$

symmetry condition $\delta 1 = -\delta 2$ when r'[0] = 0.

```
f3'[0] == 0 /. cnd /. δ1 → -δ2 // Simplify
```

$$\left\{\frac{(a\,(-1+b) - b)\,\delta 2}{(-2+a)\,(-2+b)} == 0\right\}$$

```
Solve[f3'[0] == 0, {b}] /. δ1 → -δ2 // Simplify
```

$$\left\{\left\{b \to \frac{2\,c + 2\,d - a\,d}{c}\right\}\right\}$$

therefore

$$b == \frac{a}{a-1};$$

lets examing f3[x] with detail:

```
f3[ξ]
```

$$-\frac{c\,\Delta x\,\mathrm{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a} - \frac{d\,\Delta x\,\mathrm{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}$$

$$\mathrm{Assuming}\left[\{\mathrm{Re}[a] < 1,\ \mathrm{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{c\,\Delta x\,\mathrm{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{a}\right)\Delta x + \xi\right]}{a}\,d\xi\right]$$

$$\frac{1}{2\,a^2} c\,\Delta x\,\Big(2\,a\,\mathrm{ArcTanh}[1-a] - 2\,\mathrm{Log}[2-a] - (-2+a)\,\mathrm{Log}[-2+a] - 2\,\mathrm{Log}[-1+a] + a\left(-2 + \mathrm{Log}\left[-\frac{1}{a}\right] + 2\,\mathrm{Log}[(-1+a)\,\Delta x]\right)\Big)$$

$$\mathrm{Assuming}\left[\{\mathrm{Re}[b] < 1,\ \mathrm{Re}[\Delta x] > 0\},\ \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \frac{d\,\Delta x\,\mathrm{Log}\left[-\frac{1}{2}\left(-1+\frac{2}{b}\right)\Delta x + \xi\right]}{b}\,d\xi\right]$$

$$\frac{1}{2\,b^2} d\,\Delta x\,\Big(2\,b\,\mathrm{ArcTanh}[1-b] - 2\,\mathrm{Log}[2-b] - (-2+b)\,\mathrm{Log}[-2+b] - 2\,\mathrm{Log}[-1+b] + b\left(-2 + \mathrm{Log}\left[-\frac{1}{b}\right] + 2\,\mathrm{Log}[(-1+b)\,\Delta x]\right)\Big)$$

recall that for any given ‘x’ value:

```
{1/2 (Log[1 + x] - Log[1 - x]) == ArcTanh[x]} /. x → 0.656
```

$\{\text{True}\}$

MCV3

constrains

```
f1[-1] == f_L
f1[1] == f_R
```

$a - b + c - d == f_L$

$a + b + c + d == f_R$

```
f1'[-1] == f'_L
f1'[1] == f'_R
```

$b - 2\,c + 3\,d == f'_L$

$b + 2\,c + 3\,d == f'_R$

solving as a system of equations

```
coefs = Solve[
   f1[-1] == f_L && f1[1] == f_R && f1'[-1] == f'_L && f1'[1] == f'_R, {a, b, c, d}] // Simplify
```

$$\left\{\left\{a \to \frac{1}{4}\,(2\,f_L + 2\,f_R + f'_L - f'_R),\; b \to \frac{1}{4}\,(-3\,f_L + 3\,f_R - f'_L - f'_R),\; c \to \frac{1}{4}\,(-f'_L + f'_R),\; d \to \frac{1}{4}\,(f_L - f_R + f'_L + f'_R)\right\}\right\}$$

```
f1[ξ] /. coefs
```

$$\left\{\frac{1}{4}\,\xi\,(-3\,f_L + 3\,f_R - f'_L - f'_R) + \frac{1}{4}\,(2\,f_L + 2\,f_R + f'_L - f'_R) + \frac{1}{4}\,\xi^2\,(-f'_L + f'_R) + \frac{1}{4}\,\xi^3\,(f_L - f_R + f'_L + f'_R)\right\}$$

```
f1'[ξ] /. coefs
```

$$\left\{\frac{1}{4}\,(-3\,f_L + 3\,f_R - f'_L - f'_R) + \frac{1}{2}\,\xi\,(-f'_L + f'_R) + \frac{3}{4}\,\xi^2\,(f_L - f_R + f'_L + f'_R)\right\}$$

```
f1'[-1] /. coefs // Simplify
f1'[ 0 ] /. coefs // Simplify
f1'[+1] /. coefs // Simplify
```

$\{f'_L\}$

$\left\{\frac{1}{4}(-3\,f_L + 3\,f_R - f'_L - f'_R)\right\}$

$\{f'_R\}$

MCV3-UPCC

primary lagrange interpolation at solutions points ξ = {-1,0,1}:

In[29]:= $p[\xi\_] := \frac{\xi}{2}(\xi - 1)\,f_L - (\xi + 1)(\xi - 1)\,f_C + \frac{\xi}{2}(\xi + 1)\,f_R;$

In[31]:= `p'[0]`

Out[31]= $-\frac{f_L}{2} + \frac{f_R}{2}$

In[32]:= `p''[0]`

Out[32]= $-2\,f_C + f_L + f_R$

now we consider a new approximation function

In[1]:= $f4[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3 + e\,\xi^4$

constrains

In[68]:= $f4[-1] == P_L$

$f4[0] == f_C$

$f4[1] == P_R$

Out[68]= $a - b + c - d + e == P_L$

Out[69]= $a == f_C$

Out[70]= $a + b + c + d + e == P_R$

In[59]:= `f4'[0] == p'[0]`

`f4''[0] == p''[0]`

Out[59]= $b == -\frac{f_L}{2} + \frac{f_R}{2}$

Out[60]= $2\,c == -2\,f_C + f_L + f_R$

solving as a system of equations

In[71]:=
```
coefs =
  Solve[f4[-1] == P_L && f4[0] == f_C && f4[1] == P_R && f4'[0] == p'[0] && f4''[0] == p''[0],
    {a, b, c, d, e}] // Simplify
```

Out[71]= $$\left\{\left\{a \to f_C,\ b \to \frac{1}{2}(-f_L + f_R),\ c \to \frac{1}{2}(-2 f_C + f_L + f_R),\ d \to \frac{1}{2}(f_L - f_R - P_L + P_R),\ e \to \frac{1}{2}(-f_L - f_R + P_L + P_R)\right\}\right\}$$

In[72]:=
```
f4[ξ] /. coefs
```

Out[72]= $$\left\{f_C + \frac{1}{2}\xi(-f_L + f_R) + \frac{1}{2}\xi^2(-2 f_C + f_L + f_R) + \frac{1}{2}\xi^3(f_L - f_R - P_L + P_R) + \frac{1}{2}\xi^4(-f_L - f_R + P_L + P_R)\right\}$$

In[73]:=
```
f4'[ξ] /. coefs
```

Out[73]= $$\left\{\frac{1}{2}(-f_L + f_R) + \xi(-2 f_C + f_L + f_R) + \frac{3}{2}\xi^2(f_L - f_R - P_L + P_R) + 2\xi^3(-f_L - f_R + P_L + P_R)\right\}$$

In[74]:=
```
f4'[-1] /. coefs // Expand
f4'[ 0 ] /. coefs // Expand
f4'[+1] /. coefs // Expand
```

Out[74]= $$\left\{2 f_C + 2 f_L - \frac{7 P_L}{2} - \frac{P_R}{2}\right\}$$

Out[75]= $$\left\{-\frac{f_L}{2} + \frac{f_R}{2}\right\}$$

Out[76]= $$\left\{-2 f_C - 2 f_R + \frac{P_L}{2} + \frac{7 P_R}{2}\right\}$$

MCV3-CPCC

primary lagrange interpolation at solutions points ξ = {-sqrt(3)/2,0,sqrt(3)/2}:

In[77]:= $$\xi 1 = -\frac{\sqrt{3}}{2};\ \xi 2 = 0;\ \xi 3 = \frac{\sqrt{3}}{2};$$

In[78]:= $$p[\xi\_] := \frac{(\xi - \xi 2)(\xi - \xi 3)}{(\xi 1 - \xi 2)(\xi 1 - \xi 3)} f_L + \frac{(\xi - \xi 1)(\xi - \xi 3)}{(\xi 2 - \xi 1)(\xi 2 - \xi 3)} f_C + \frac{(\xi - \xi 1)(\xi - \xi 2)}{(\xi 3 - \xi 1)(\xi 3 - \xi 2)} f_R;$$

In[79]:=
```
p'[0]
```

Out[79]= $$-\frac{f_L}{\sqrt{3}} + \frac{f_R}{\sqrt{3}}$$

In[80]:= `p''[0]`

Out[80]= $-\frac{8 f_C}{3}+\frac{4 f_L}{3}+\frac{4 f_R}{3}$

now we consider a new approximation function

In[1]:= $\texttt{f4}[\xi\_] := a + b\,\xi + c\,\xi^2 + d\,\xi^3 + e\,\xi^4$

constrains

In[81]:= $\texttt{f4}[-1] == P_L$

$\texttt{f4}[0] == f_C$

$\texttt{f4}[1] == P_R$

Out[81]= $a-b+c-d+e == P_L$

Out[82]= $a == f_C$

Out[83]= $a+b+c+d+e == P_R$

In[84]:= `f4'[0] == p'[0]`

`f4''[0] == p''[0]`

Out[84]= $b == -\frac{f_L}{\sqrt{3}}+\frac{f_R}{\sqrt{3}}$

Out[85]= $2\,c == -\frac{8 f_C}{3}+\frac{4 f_L}{3}+\frac{4 f_R}{3}$

solving as a system of equations

In[86]:= coefs =
 $\texttt{Solve}[\texttt{f4}[-1] == P_L \,\&\&\, \texttt{f4}[0] == f_C \,\&\&\, \texttt{f4}[1] == P_R \,\&\&\, \texttt{f4}'[0] == \texttt{p}'[0] \,\&\&\, \texttt{f4}''[0] == \texttt{p}''[0],$
 $\{a, b, c, d, e\}]$ `// Simplify`

Out[86]= $\left\{\left\{a\to f_C,\ b\to\frac{-f_L+f_R}{\sqrt{3}},\ c\to-\frac{2}{3}\,(2 f_C-f_L-f_R),\right.\right.$
$\left.\left.d\to\frac{1}{6}\left(2\sqrt{3}\,f_L-2\sqrt{3}\,f_R-3 P_L+3 P_R\right),\ e\to\frac{1}{6}\,(2 f_C-4 f_L-4 f_R+3 P_L+3 P_R)\right\}\right\}$

In[87]:= $\texttt{f4}[\xi]$ `/. coefs`

Out[87]= $\left\{f_C-\frac{2}{3}\,\xi^2\,(2 f_C-f_L-f_R)+\frac{\xi\,(-f_L+f_R)}{\sqrt{3}}+\right.$
$\left.\frac{1}{6}\,\xi^3\left(2\sqrt{3}\,f_L-2\sqrt{3}\,f_R-3 P_L+3 P_R\right)+\frac{1}{6}\,\xi^4\,(2 f_C-4 f_L-4 f_R+3 P_L+3 P_R)\right\}$

In[88]:= `f4'[ξ] /. coefs`

Out[88]= $\left\{-\frac{4}{3}\xi\,(2\,f_C - f_L - f_R) + \frac{-f_L + f_R}{\sqrt{3}} + \frac{1}{2}\xi^2\left(2\sqrt{3}\,f_L - 2\sqrt{3}\,f_R - 3\,P_L + 3\,P_R\right) + \frac{2}{3}\xi^3\,(2\,f_C - 4\,f_L - 4\,f_R + 3\,P_L + 3\,P_R)\right\}$

In[125]:= `f4'[-√3 / 2] /. coefs // Expand`

`f4'[0] /. coefs // Expand`

`f4'[√3 / 2] /. coefs // Expand`

Out[125]= $\left\{\frac{5\,f_C}{2\sqrt{3}} + \frac{3\sqrt{3}\,f_L}{4} - \frac{f_R}{4\sqrt{3}} - \frac{9\,P_L}{8} - \frac{3\sqrt{3}\,P_L}{4} + \frac{9\,P_R}{8} - \frac{3\sqrt{3}\,P_R}{4}\right\}$

Out[126]= $\left\{-\frac{f_L}{\sqrt{3}} + \frac{f_R}{\sqrt{3}}\right\}$

Out[127]= $\left\{-\frac{5\,f_C}{2\sqrt{3}} + \frac{f_L}{4\sqrt{3}} - \frac{3\sqrt{3}\,f_R}{4} - \frac{9\,P_L}{8} + \frac{3\sqrt{3}\,P_L}{4} + \frac{9\,P_R}{8} + \frac{3\sqrt{3}\,P_R}{4}\right\}$

CIP-CSL3

Notice that the information to be evolved are the cell boundary values!

constrains

`f1[-Δx / 2] == f_{i-1/2}`

`f1[Δx / 2] == f_{i+1/2}`

$$a - \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4} - \frac{d\,\Delta x^3}{8} == f_{-\frac{1}{2}+i}$$

$$a + \frac{b\,\Delta x}{2} + \frac{c\,\Delta x^2}{4} + \frac{d\,\Delta x^3}{8} == f_{\frac{1}{2}+i}$$

$$\text{Simplify}\left[\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f1[\xi]\,d\xi\right] == f_i$$

$$a + \frac{c\,\Delta x^2}{12} == f_i$$

`f1'[0] == dd_i`

$b == dd_i$

solving as a system of equations

```
coefs = Solve[f1[Δx / 2] == f_{i+1/2} && f1[-Δx / 2] == f_{i-1/2} &&
    1/Δx ∫_{-Δx/2}^{Δx/2} f1[ξ] dξ == f_i && f1'[0] == dd_i, {a, b, c, d}] // Expand
```

$$\left\{\left\{a \to -\frac{1}{4} f_{-\frac{1}{2}+i} + \frac{3 f_i}{2} - \frac{1}{4} f_{\frac{1}{2}+i},\ b \to dd_i,\ c \to \frac{3 f_{-\frac{1}{2}+i}}{\Delta x^2} - \frac{6 f_i}{\Delta x^2} + \frac{3 f_{\frac{1}{2}+i}}{\Delta x^2},\ d \to -\frac{4 dd_i}{\Delta x^2} - \frac{4 f_{-\frac{1}{2}+i}}{\Delta x^3} + \frac{4 f_{\frac{1}{2}+i}}{\Delta x^3}\right\}\right\}$$

```
f1[x - x_{i-1/2}] /. coefs (*x \in [x_{i-1/2},x_{i+1/2}] *)
```

$$\left\{-\frac{1}{4} f_{-\frac{1}{2}+i} + \frac{3 f_i}{2} - \frac{1}{4} f_{\frac{1}{2}+i} + dd_i \left(x - x_{-\frac{1}{2}+i}\right) + \left(\frac{3 f_{-\frac{1}{2}+i}}{\Delta x^2} - \frac{6 f_i}{\Delta x^2} + \frac{3 f_{\frac{1}{2}+i}}{\Delta x^2}\right)\left(x - x_{-\frac{1}{2}+i}\right)^2 + \left(-\frac{4 dd_i}{\Delta x^2} - \frac{4 f_{-\frac{1}{2}+i}}{\Delta x^3} + \frac{4 f_{\frac{1}{2}+i}}{\Delta x^3}\right)\left(x - x_{-\frac{1}{2}+i}\right)^3\right\}$$

is f_i the cell center value?

```
a + c Δx^2/12 /. coefs // Simplify
```

$$\{f_i\}$$

yes!

CIP-CSL3 (Yabe (2001) paper version)

Notice that the information to be evolved are the cell boundary values!

Left side constrains

In[13]:=
```
f1[-Δx] == f_{i-1}
f1[0] == f_i
```

Out[13]= $a - b\,\Delta x + c\,\Delta x^2 - d\,\Delta x^3 == f_{-1+i}$

Out[14]= $a == f_i$

In[15]:=
```
Simplify[∫_{-Δx}^{0} f1[ξ] dξ] == ρ_{i-1/2}
```

Out[15]= $\frac{1}{12} \Delta x\,(12\,a + \Delta x\,(-6\,b + \Delta x\,(4\,c - 3\,d\,\Delta x))) == \rho_{-\frac{1}{2}+i}$

In[16]:=
```
f1'[-Δx / 2] == s_{i-1/2}
```

Out[16]= $b - c\,\Delta x + \frac{3\,d\,\Delta x^2}{4} == s_{-\frac{1}{2}+i}$

Solving as a system of equations

In[18]:=
```
coefs = Solve[f1[ 0 ] == f_i && f1[-Δx] == f_{i-1} &&
   ∫_{-Δx}^0 f1[ξ] dξ == ρ_{i-1/2} && f1'[-Δx / 2] == s_{i-1/2}, {a, b, c, d}] // Expand
```

Out[18]= $\left\{\left\{a \to f_i,\ b \to \frac{6\, f_i}{\Delta x} - 2\, s_{-\frac{1}{2}+i} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^2},\right.\right.$
$\left.\left. c \to -\frac{3\, f_{-1+i}}{\Delta x^2} + \frac{9\, f_i}{\Delta x^2} - \frac{6\, s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^3},\ d \to -\frac{4\, f_{-1+i}}{\Delta x^3} + \frac{4\, f_i}{\Delta x^3} - \frac{4\, s_{-\frac{1}{2}+i}}{\Delta x^2}\right\}\right\}$

In[19]:=
```
f1[x - x_i] /. coefs (* x \in [x_{i-1},x_i] *)
```

Out[19]= $\left\{f_i + \left(-\frac{4\, f_{-1+i}}{\Delta x^3} + \frac{4\, f_i}{\Delta x^3} - \frac{4\, s_{-\frac{1}{2}+i}}{\Delta x^2}\right)(x - x_i)^3 +\right.$
$\left.(x - x_i)^2 \left(-\frac{3\, f_{-1+i}}{\Delta x^2} + \frac{9\, f_i}{\Delta x^2} - \frac{6\, s_{-\frac{1}{2}+i}}{\Delta x} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^3}\right) + (x - x_i)\left(\frac{6\, f_i}{\Delta x} - 2\, s_{-\frac{1}{2}+i} - \frac{6\, \rho_{-\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$

evaluating the time integral of u * f1(x_i-ut) yields

In[22]:=
```
∫_{-Δt}^0 u f1[-u t] dt
```

Out[22]= $a\, u\, \Delta t + \frac{1}{2} b\, u^2\, \Delta t^2 + \frac{1}{3} c\, u^3\, \Delta t^3 + \frac{1}{4} d\, u^4\, \Delta t^4$

NOTE: u Δt indicates the direction of the advecting wave.

$d_{1\pm1/2}$ are here free parameters. Note that we can define it by using the resulting interpolation functions evaluating at Δx/2 and/or -Δx/2. e.g. :

In[21]:=
```
f1[-Δx / 2] /. coefs // Expand
```

Out[21]= $\left\{-\frac{1}{4} f_{-1+i} - \frac{f_i}{4} + \frac{3\, \rho_{-\frac{1}{2}+i}}{2\, \Delta x}\right\}$

this is in agreement with equation (23) in Yabe [2001].

Right side constrains

In[5]:=
```
f1[ 0 ] == f_i
f1[Δx] == f_{i+1}
```

Out[5]= $a == f_i$

Out[6]= $a + b\, \Delta x + c\, \Delta x^2 + d\, \Delta x^3 == f_{1+i}$

In[7]:= `Simplify[∫_0^Δx f1[ξ] dξ] == ρ_{i+1/2}`

$$\text{Out[7]=}\quad \frac{1}{12}\Delta x\,(12\,a+\Delta x\,(6\,b+\Delta x\,(4\,c+3\,d\,\Delta x)))==\rho_{\frac{1}{2}+i}$$

In[8]:= `f1'[Δx / 2] == s_{i+1/2}`

$$\text{Out[8]=}\quad b+c\,\Delta x+\frac{3\,d\,\Delta x^2}{4}==s_{\frac{1}{2}+i}$$

Solving as a system of equations

In[9]:= `coefs = Solve[f1[ 0 ] == f_i && f1[Δx] == f_{i+1} && ∫_0^Δx f1[ξ] dξ == ρ_{i+1/2} && f1'[Δx / 2] == s_{i+1/2}, {a, b, c, d}] // Expand`

$$\text{Out[9]=}\quad \left\{\left\{a\to f_i,\ b\to -\frac{6\,f_i}{\Delta x}-2\,s_{\frac{1}{2}+i}+\frac{6\,\rho_{\frac{1}{2}+i}}{\Delta x^2},\right.\right.$$
$$\left.\left. c\to \frac{9\,f_i}{\Delta x^2}-\frac{3\,f_{1+i}}{\Delta x^2}+\frac{6\,s_{\frac{1}{2}+i}}{\Delta x}-\frac{6\,\rho_{\frac{1}{2}+i}}{\Delta x^3},\ d\to -\frac{4\,f_i}{\Delta x^3}+\frac{4\,f_{1+i}}{\Delta x^3}-\frac{4\,s_{\frac{1}{2}+i}}{\Delta x^2}\right\}\right\}$$

In[10]:= `f1[x - x_{i-1}] /. coefs (* x \in [x_i,x_{i+1}] *)`

$$\text{Out[10]=}\quad \left\{f_i+\left(-\frac{4\,f_i}{\Delta x^3}+\frac{4\,f_{1+i}}{\Delta x^3}-\frac{4\,s_{\frac{1}{2}+i}}{\Delta x^2}\right)(x-x_{-1+i})^3+\right.$$
$$\left.(x-x_{-1+i})^2\left(\frac{9\,f_i}{\Delta x^2}-\frac{3\,f_{1+i}}{\Delta x^2}+\frac{6\,s_{\frac{1}{2}+i}}{\Delta x}-\frac{6\,\rho_{\frac{1}{2}+i}}{\Delta x^3}\right)+(x-x_{-1+i})\left(-\frac{6\,f_i}{\Delta x}-2\,s_{\frac{1}{2}+i}+\frac{6\,\rho_{\frac{1}{2}+i}}{\Delta x^2}\right)\right\}$$

evaluating the time integral of u * f1(x_i+ut) yields

In[24]:= `∫_0^Δt u f1[u t] dt`

$$\text{Out[24]=}\quad a\,u\,\Delta t+\frac{1}{2}\,b\,u^2\,\Delta t^2+\frac{1}{3}\,c\,u^3\,\Delta t^3+\frac{1}{4}\,d\,u^4\,\Delta t^4$$

NOTE: u Δt indicates the direction of the advecting wave.

In[12]:= `f1[Δx / 2] /. coefs // Expand`

$$\text{Out[12]=}\quad \left\{-\frac{f_i}{4}-\frac{f_{1+i}}{4}+\frac{3\,\rho_{\frac{1}{2}+i}}{2\,\Delta x}\right\}$$