CIP—CSL3 scheme implementation in MATLAB

Oleg Kravchenko[1] & Manuel Diaz*

[1] Department of Higher Mathematics, Bauman Moscow State Technical University, Moscow, Russia.
* Institute of Applied Mechanics, National Taiwan University, Taiwan R.O.C.

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Abstract

Several versions of CIP—CSL3 scheme.

1 First version of CIP—CSL3

The outline of the computational procedure was introduced in Dr. F. Xiao's papers Xiao and Yabe (2001), Xiao and Akio (2003).

1.1 Mathematical background

The i–th interpolation polynomial is constructed over upwind stencils. Left–bias and right–bias components are the following

$$F_i^L(x) = \sum_{k=0}^{3} c_{ki}^L (x - x_i)^k, \quad x \in [x_{i-1},\, x_i] \tag{1}$$ {eq:01}

and

$$F_i^R(x) = \sum_{k=0}^{3} c_{ki}^R (x - x_i)^k, \quad x \in [x_i,\, x_{i+1}]. \tag{2}$$ {eq:02}

A construction over the left–side stencil of grid i with $x \in [x_{i-1},\, x_i]$ implies the case of $u \geqslant 0$ and the right–side stencil the case of $u < 0$. Constrains for left–bias

$$F_i^L(x_i) = f^n(x_i), \tag{3}$$ {eq:03}

$$F_i^L(x_{i-1}) = f^n(x_{i-1}), \tag{4}$$ {eq:04}

$$\int\limits_{x_{i-1}}^{x_i} F_i^L(x)\, dx = \rho_{i-\frac{1}{2}}^n, \tag{5}$$ {eq:05}

$$\left.\frac{dF_i^L(x)}{dx}\right|_{x=x_{i-\frac{1}{2}}} = d_{i-\frac{1}{2}}^n, \tag{6}$$ {eq:06}

and right–bias, respectively.

$$F_i^R(x_i) = f^n(x_i), \qquad (7) \quad \{\texttt{eq:07}\}$$

$$F_i^R(x_{i+1}) = f^n(x_{i+1}), \qquad (8) \quad \{\texttt{eq:08}\}$$

$$\int\limits_{x_i}^{x_{i+1}} F_i^R(x)\, dx = \rho^n_{i+\frac{1}{2}}, \qquad (9) \quad \{\texttt{eq:09}\}$$

$$\left.\frac{dF_i^R(x)}{dx}\right|_{x=x_{i+\frac{1}{2}}} = d^n_{i+\frac{1}{2}}. \qquad (10) \quad \{\texttt{eq:10}\}$$

Consequently, unknown coefficients c^L_{ki} could be found from the system (3)—(6) and c^R_{ki} from (7)—(10) with the usage of someone symbolic mathematics toolbox, for example, Wolfram Mathematica. Define $h := \Delta x_{i-\frac{1}{2}}$ and $\tau := \triangle t$. Then, one could find coefficients of polynomial (1)

$$\begin{cases} c^L_{0i} &= f^n_i, \\ c^L_{1i} &= -\dfrac{6}{h^2}\rho^n_{i-\frac{1}{2}} + \dfrac{6}{h} f^n_i - 2d^n_{i-\frac{1}{2}}, \\ c^L_{2i} &= -\dfrac{6}{h^3}\rho^n_{i-\frac{1}{2}} + \dfrac{3}{h^2}(3f^n_i - f^n_{i-1}) - \dfrac{6}{h} d^n_{i-\frac{1}{2}}, \\ c^L_{3i} &= \dfrac{4}{h^3}(f^n_i - f^n_{i-1}) - \dfrac{4}{h^2} d^n_{i-\frac{1}{2}}, \end{cases} \qquad (11) \quad \{\texttt{eq:11}\}$$

and coefficients of polynomial (2) analogously,

$$\begin{cases} c^R_{0i} &= f^n_i, \\ c^R_{1i} &= \dfrac{6}{h^2}\rho^n_{i+\frac{1}{2}} - \dfrac{6}{h} f^n_i - 2d^n_{i+\frac{1}{2}}, \\ c^R_{2i} &= -\dfrac{6}{h^3}\rho^n_{i+\frac{1}{2}} + \dfrac{3}{h^2}(3f^n_i - f^n_{i-1}) + \dfrac{6}{h} d^n_{i+\frac{1}{2}}, \\ c^R_{3i} &= -\dfrac{4}{h^3}(f^n_i - f^n_{i+1}) - \dfrac{4}{h^2} d^n_{i+\frac{1}{2}}. \end{cases} \qquad (12) \quad \{\texttt{eq:12}\}$$

1.2 Algorithm

1. Compute $d^n_{i-\frac{1}{2}}$ from $f^n_{i-\frac{1}{2}}$ values using any slope approximation

   $$f^n_{i-\frac{1}{2}} = \frac{3}{2h}\rho^n_{i-\frac{1}{2}} - \frac{1}{4}(f^n_i + f^n_{i-1}).$$

   Hyman's approximation:

   $$d^n_{i-\frac{1}{2}} = \frac{1}{12h}(-f^n_{i+\frac{3}{2}} + 8f^n_{i+\frac{1}{2}} - 8f^n_{i-\frac{3}{2}} + f^n_{i-\frac{5}{2}}).$$

   UNO approximation:

   $$d^n_{i-\frac{1}{2}} = \operatorname{minmod}(S^+_{i-\frac{1}{2}},\, S^-_{i-\frac{1}{2}}).$$

   CW (Collela&Woodward) approximation of $d_{i-\frac{1}{2}}$:

   $$d^n_{i-\frac{1}{2}} = \begin{cases} \dfrac{1}{h}\min(|\delta f^n_{i-\frac{1}{2}}|, 2|f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}|, 2|f^n_{i-\frac{1}{2}} - f^n_{i-\frac{3}{2}}|)\operatorname{sgn}(\delta f^n_{i-\frac{1}{2}}), & \text{otherwise} \\ 0, \text{ if } (f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}})(f^n_{i-\frac{1}{2}} - f^n_{i-\frac{3}{2}}) \leqslant 0. \end{cases}$$

   Simplified approximation of $d_{i-\frac{1}{2}}$ via minmod function:

   $$\hat{d}^n_{i-\frac{1}{2}} = \operatorname{minmod}(\tilde{d}^n_{i-\frac{1}{2}},\, 2S_i,\, 2S_{i-1})$$

with $S_i = \dfrac{1}{h}\left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right)$ and $\tilde{d}_{i-\frac{1}{2}} = \dfrac{1}{2h}(f^n_{i+1} - f^n_{i-1})$. Finally,

$$d^n_{i-\frac{1}{2}} = \beta \hat{d}^n_{i-\frac{1}{2}}.$$

2. Determine the cubic polynomial (1),(2) in terms of f_i, $f^n_{i-1}, d^n_{i-1/2}, \rho^n_{i-1/2}$ by (11),(12).
3. Calculate the semi–Lagrangian solution of f

$$\tilde{f}_i = \begin{cases} F^L_i(x_i - u\tau), & u \geqslant 0, \\ F^R_i(x_i - u\tau), & u < 0. \end{cases}$$

4. Correct f according to velocity divergence

$$f^{n+1}_i = \tilde{f}_i - \frac{\tau}{2h}\tilde{f}_i(u_{i+1} - u_{i-1}).$$

5. Compute the flux

$$g_i = -\min(0,\xi)\left(f^n_i + \sum_{k=1}^{3}\frac{1}{k+1}c^R_{ki}\xi^k\right) - \max(0,\xi)\left(f^n_i + \sum_{k=1}^{3}\frac{1}{k+1}c^L_{ki}\xi^k\right).$$

6. Predict the cell-integrated average ρ using the exactly conservative formulation

$$\rho^{n+1}_{i-\frac{1}{2}} = \rho^n_{i-\frac{1}{2}} - (g_i - g_{i-1}).$$

Notice, that condition (5) have been defined in Xiao and Yabe (2001) and in Xiao and Akio (2003) it have been introduced in the following form

$$\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} F^L_i(x)\,dx = \bar{f}^n_{i-\frac{1}{2}}.$$

So, one could find that

$$\rho^n_{i-\frac{1}{2}} = h\bar{f}^n_{i-\frac{1}{2}}.$$

2 Second version of CIP-CSL3

The second version of CIP-CSL3 numerical scheme is devoted to the different type of discretization with respect to *point–values* (PV) and *volume–integrated averages* (VIA) Ii and Xiao (2007), Ii and Xiao (2009). This novel ideology of spatial discretization led to evolution of the abbreviation, such as VSIAM, CIP/MM-FVM, MCV.

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