Calculus 3, Semester 2, Section 7.4, Week 15

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Last calculus lecture ever!

Boundary

Given a solid region $W \in \mathbb{R}^3$, δW is the positively oriented boundary surface, using the outward pointing normal vector if W has no holes. In the case that there are holes, the surface can be split.

Boundary Curve

A surface is **closed** if it has no boundary curves. Examples of closed surfaces are spheres and cubes. Conversely, a disk is not closed.

Ostrogradsky's Thm (Divergence Thm or Gauss' Thm)

The **Divergence Theorem**: Let D be a solid bounded region in $\mathbb{R}^3$ whose boundary δD consists of finitely many piecewise smooth, closed, orientable surfaces, oriented with normal vectors pointing away from D. Let $\vec{F}$ be a C' vector field on D. Then,

$$\oiint_{\delta D} \vec{F}\, d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV.$$

Notably, the hypotheses just say that it has a boundary that is not insane, and can normally be ignored.

Flux integrals over closed surfaces are denoted by $\oiint_S \vec{F} \cdot d\vec{S}$.

Let D be region $x^2 + y^2 + z^2 \leq 4$, and

$$\vec{F}(x, y, z) = (e^{yz} \cos z^2, e^{x^2 \sin z}, x^{47} y^{19} + z).$$

Compute $\iint_{\delta D} \vec{F} \cdot d\vec{S}$.

Notably, the divergence is simply one, so it's just the integral over the sphere of one, the volume. Hence, the answer is $\frac{32\pi}{3}$.

The left side of the Divergence Thm represents the flow out of E, while the right side adds up the microscopic expansion of each point in E.

Tips

Generally, if a function looks extreme you should think before integrating. If asked to integrate something like $\oint_C \vec{F} \cdot d\vec{s}$ or $\oiint_S \vec{F} \cdot d\vec{S}$ for a "totally bonkers" (-Rogness) vector field, then:

1. Convert $\oint_C \vec{F} \cdot d\vec{s}$ to $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$, given that $C = \delta S$ and the curl is nice.
2. Convert $\oiint_S \vec{F} \cdot d\vec{S}$ to $\iiint_D \operatorname{div}\vec{F}\, dV$ given that the divergence is nice.
3. Discover that's it's actually nice.

Generalized Stokes Thm

$$\int_{\Omega} d\omega = \int_{\delta\Omega} \omega,$$

given

- Ω : n-dimensional manifold
- $\delta\Omega$: $n-1$ dimensional boundary
- ω : differential form
- d : differential operator.

Called de Rham Cohomology. Every theorem is just Stokes in disguise. Feels like having Endgame spoiled.