Object Orientation: A Mathematical Perspective

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Science, defined

Science is the effort to reveal the kinds and laws of nature to the faculties of independent reason.

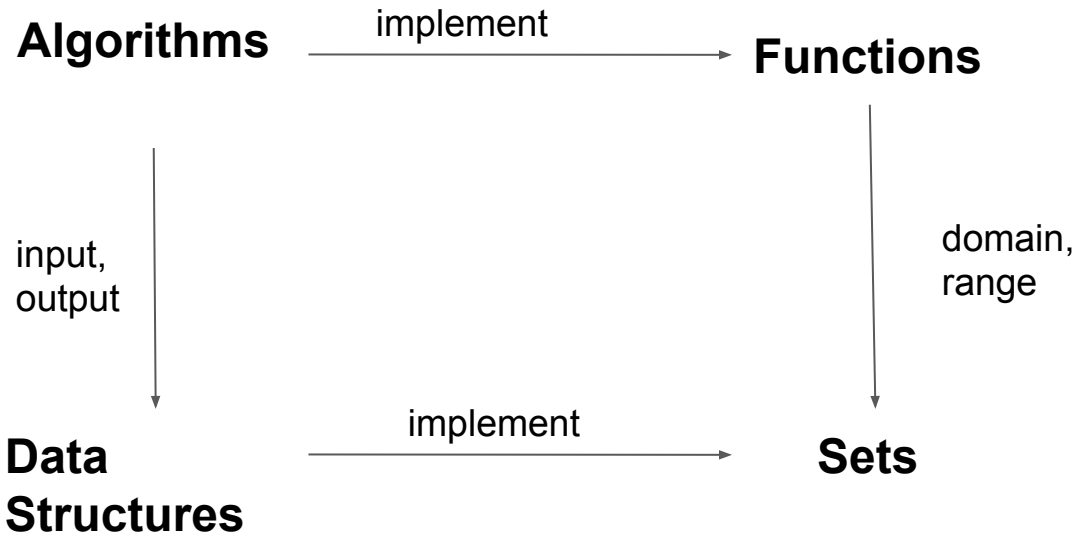
Natural Kinds

categories in immediate perception	**categories in nature**

● little lights in the sky
● big lights in the sky
  ○ sun
  ○ moon
● *terra*
	● stars
	● planets

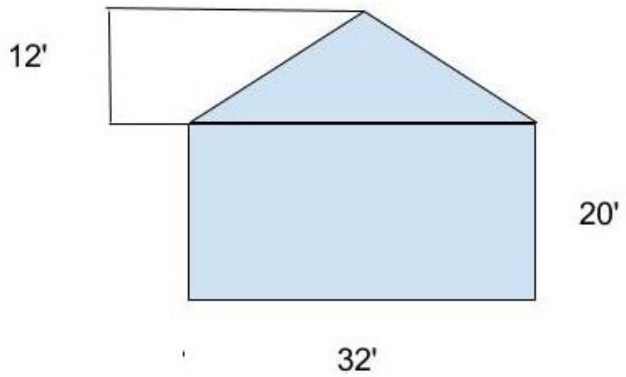
● big swimming animals
  ○ sharks
  ○ whales
● little flying animals
  ○ bats
  ○ sparrows
  ○ moths
	● fish
	● birds
	● mammals
	● insects

CS Ontology

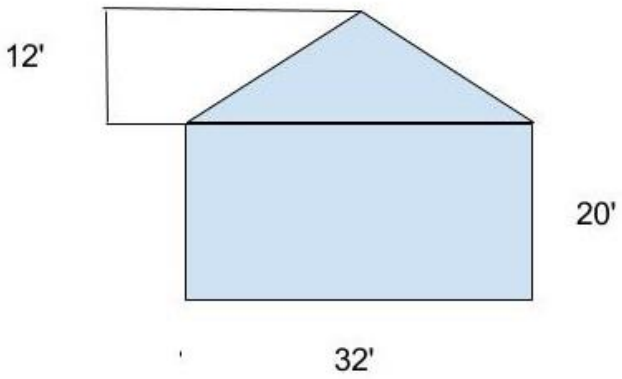

Algorithms
implement
Functions
input,
output
domain,
range
implement
Data
Structures
Sets

A 5th grade math problem

The North and South walls of a barn each consist of a 32' by 20' rectangle and an isosceles triangle with base 32' and height 12'. The east and west walls of the barn are 50' by 20' rectangles (not shown in the figure), which are perpendicular to the North and South walls. What is the volume of the interior barn?



Some required mathematical knowledge

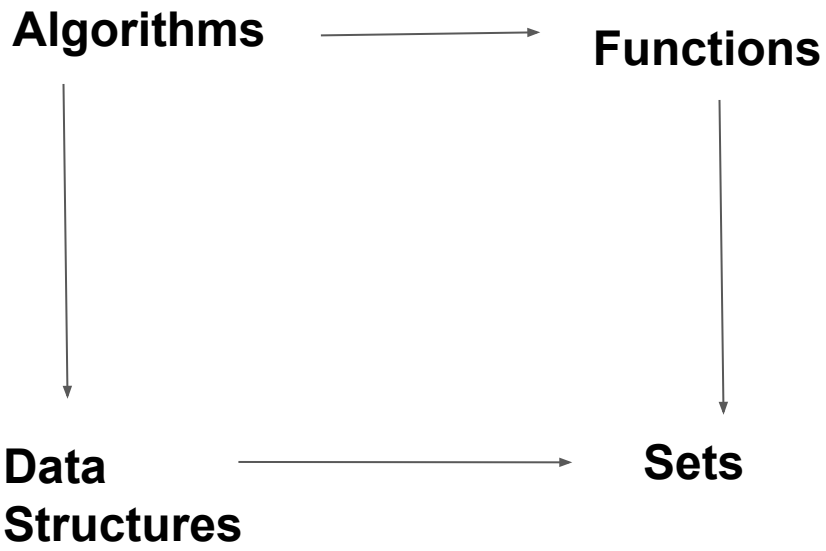


Barn is 50' deep (not shown in figure)

1. If T is a triangle, then
   $area(T) = \frac{1}{2}\, base(T) height(T)$.

2. If R is a rectangle, then
   $area(R) = width(R) \cdot height(R)$.

3. The volume of a prism is the area of one of its faces times the distance between the two faces.



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The concept *area*

Question: if *area* is a function, what is its domain?

The concept *plane figure*

Fifth grader: A *plane figure* is a rectangle, a triangle, a circle, or, presumably, one of another kind we will talk about later (maybe 6th grade).

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Engineer: A *plane figure* is a polygon, circle, ellipse, the region enclosed by the graph of a continuous closed curve, or, presumably, a set of another kind they talk in other disciplines (maybe the math department).

Mathematician: A *plane figure* is a Lebesgue measurable subset of $\mathbf{R}^2$ (and its area is its Lebesgue measure).

The concept of *set*

Mathematician: A *set* is...

1. the empty set,
2. the set of finite ordinals,
3. the power set of a given set,
4. the union of a given set of sets,
5. the image of a known set under a function that is first order definable in terms of set membership,
6. the smallest member of a known set under an unspecified well ordering, or
7. **a set supposed to exist for other reasons, presumably talked about by my colleagues (maybe set theorists).**

The concept of *set*

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6. the smallest member of a known set under an unspecified well ordering, or
7. **the least upper bound of the ordinals that can be proven to exist by 1-6, or**
8. **presumably others we'll get to later (maybe next year).**

???

Fifth grader: A *plane figure* is a rectangle, a triangle, a circle, or, presumably, one of another kind we will talk about later (maybe 6th grade).

The area of a triangle T is $\frac{1}{2}\ base(T)\cdot height(T)$.

The area of a rectangle R is $width(R)\cdot height(R)$

The area of a circle C is $\pi\cdot radius^2(C)$

For other figures, we'll cross that bridge when we come to it (maybe 6th grade).

Polymorphism

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The volume of a prism is the area of one of its faces times the distance between the two faces.

Inheritance

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The concept *set* refers to a...

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The concept *set* refers to a...*class*!

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Coincidence?

"We chose the terms 'class' and 'objects' of classes for our new Simula. The notion of subclass was especially appealing to us, since we had seen many cases of objects belonging to different classes having common properties. It would be useful to collect the common properties in a separate class, say C to be specialised differently for different purposes, possibly in different programs. The solution came with the idea of class prefixing: using C as a prefix to another class, the latter would be taken to be a subclass of C inheriting all properties of C."

-- Ole Johan Dahl, 2001

New paradigm? or just syntactic sugar?

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- in a sense, yes, by Church's Thesis

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- in a sense, no:

```
class Animal {public: virtual int GetNumberOfLegs() = 4; };

class Duck : public Animal { public: int GetNumberOfLegs() { return 2; } };

// Theorem 1: Every animal has a positive, even  number of legs
// (also true in the C equivalent)
```

New paradigm? or just syntactic sugar?

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- in a sense, yes, by Church's Thesis
- in a sense, no.

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class Animal {public: virtual int GetNumberOfLegs() = 4; };

class Duck : public Animal { public: int GetNumberOfLegs() { return 2; } };

// Theorem 1: Every animal has a positive, even number of legs
// (also true in the C equivalent)

class Fish : public Animal {public: int GetNumberOfLegs() { return 0; } };

// Theorem 2: Theorem 1 is false.
// (But it is true in every extension of the C program equivalent to lines 1 and 2.)
```

Other OO jargon (ironically, not inherited)

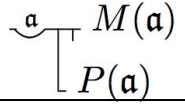
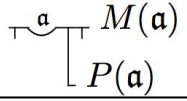
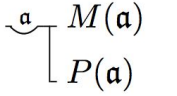
OO terminology	**Standard terminology**
object	data
method	function
private/public/friend	visibility
`x.a`	`a(x)`

Messy beginnings

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal.

-- Isaac Newton : *Principia Mathematica,* 1687

Messy beginnings

English	**Modern Notation**	**Original Notation (Frege: *Begriffsschrift*, 1879)**
Every person is mortal	$\forall x.(Px \rightarrow Mx)$	
Some person is mortal	$\exists x.(Px \ \wedge \ Mx)$	
No person is mortal	$\forall x.(Px \rightarrow \neg Mx)$	

Structures in SequenceL

A ***point*** is a structure with *x* and *y* coordinates, which are floating point numbers.

```
Point ::= (x:float, y:float);
```

Functions in SequenceL

The distance between points is given by the usual formula.

```
distance: Point*Point -> float

distance(A,B) := sqrt((A.x - B.x)^2 + (A.y - B.y)^2)
```

Classes and virtual functions in SequenceL

Figure is a class, on which the *area* function is defined.

```
class Figure requires area:Figure -> float
```

Type instantiating a class

A ***triangle*** is a figure with attributes *A*, *B*, and *C*, whose values are points.

```
Triangle:Figure ::= (A:Point,B:Point, C:Point)
```

Polymorphic function

The area of a triangle is given by Heron's formula

```
area: Triangle -> float

area(T) := sqrt(s*(s-a)*(s-b)*(s-c))
where
     a:= distance(T.B,T.C),
     b:= distance(T.A,T.C),
     c:= distance(T.A,T.B),
     s:= (a+b+c)/2
```

Polymorphic function

A rectangle is a figure defined by its upper left corner and lower right corner, both of which are points.

```
Rectangle:Figure ::= (upLeft:Point, lowRight:Point);
```

The area of a rectangle is its width times its height.

```
area: Rectangle -> float

area(R):= width*height
where
   width := R.upLeft.y - R.lowRight.y,
   height:= R.lowRight.x - R.upLeft.x
```