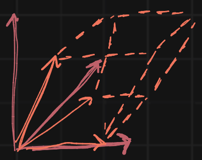
ELECTRODYNAMICS

why scalar triple products don't care about order

Vector Analysis

Scalar Triple Product

$A \cdot (B \times C)$: Volume of parallelepiped



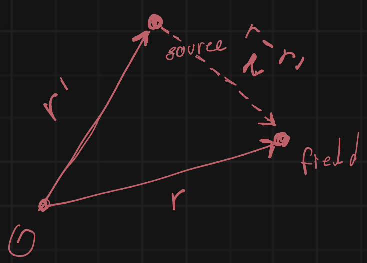
then $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$ [Alphabetical]

or $A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

Vector triple product

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

Position Vectors



$\mathscr{r} = \vec{r} - \vec{r}'$

$|\mathscr{r}| = |\vec{r} - \vec{r}'|$

$\hat{\mathscr{r}} = \dfrac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$

$\mathscr{r} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$

$\hat{\mathscr{r}} = \dfrac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x') + (y - y') + (z - z')}}$

Vector transformation

Rotation matrix $\begin{bmatrix} \bar{A}_x \\ \bar{A}_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$

explanation

$A_x = A\cos\theta$

$A_y = A\sin\theta$

$\begin{bmatrix} \cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha \\ \sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha \end{bmatrix}$

$\bar{A}_x = A\cos(\theta - \alpha) = \boxed{A\cos\theta}\cos\alpha + \boxed{A\sin\theta\sin\alpha}$

$\bar{A}_y = A\sin(\theta - \alpha) = \boxed{A\sin\theta}\cos\alpha - \boxed{A\cos\theta\sin\alpha}$

$\bar{A}_x = A_x\cos\alpha + A_y\sin\alpha$

$\bar{A}_y = -A_x\sin\alpha + A_y\cos\alpha$

$A_x \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} + A_y \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$

rotation matrix!

$\begin{array}{cc} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} & \begin{bmatrix} A_x \\ A_y \end{bmatrix} \\ \hat{i} \quad\quad \hat{j} & \end{array}$

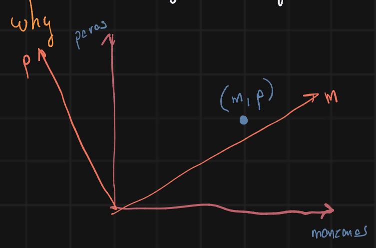
Generalization

$$\begin{bmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\bar{A} = \sum_{j=1}^{3} R_{ij} A_j$$

what is really a vector

then, a vector is any set of 3 components that transforms as a displacement when you change coordinates.



physical quantities like these CAN'T change.

a second rank tensor is a quantity with 9 components and transforms with 2 factors of R.

what is a tensor

Basic definition of a tensor

Mathematical object that has m dimensions at rank n, with m^n components.

Amount of unit vectors to describe force

Therefore,

scalar	$m = 3$ dimensions, $n = 0$ rank	$3^0 = 1$ component	Rank 0
vector	$m = 3$ dimensions, $n = 1$ rank	$3^1 = 3$ components	Rank 1
(sort of) matrix	$m = 3$, $n = 2$	$3^2 = 9$ components	Rank 2

$$\bar{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}$$

generally, a tensor has n-th rank, 3^n components and n factors of R.

A scalar has $n = 0$, then

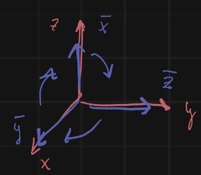
$R = \{0\}$, you can't rotate a scalar

$n = 1$ $R = \{1\}$, one R.

...

Problem 1.9

Find transformation matrix R for a rotation of 120°.



$\frac{360}{120} = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 1.10

① How components change on TRANSLATION $(\bar{x} = x,\ \bar{y} = y - a,\ \bar{z} = z)$ $\bar{A} \xrightarrow{\text{TRANSLATION}} A$

② How components change on INVERSION $(\bar{x} = -x,\ \bar{y} = -y,\ \bar{z} = -z)$ $\bar{A} \xrightarrow{\text{INVERSION}} -A$

③ Behavior of cross product INVERSION $(A \times B) \xrightarrow{\text{INVERSION}} (-A) \times (-B) = (A \times B)$

So, if $C = (A \times B)$

$C \xrightarrow{\text{INVERSION}} C$

result pseudovector

pseudovectors

Cross product of pseudovectors

If A, B are pseudovectors $(A \to A,\ B \to B)$, $A \times B = A \times B$

result is a pseudovector

If A is a pseudovector and B a vector, $A \times B = A \times -B$

result is a vector.

pseudo scalar

1. Triple scalar product under INVERSION

$$\underbrace{A \cdot (B \times C)}_{D} \xrightarrow{\text{INVERSION}} -A \cdot [(-B) \times (-C)] = -A \cdot (B \times C)$$

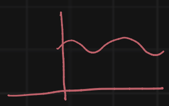
$$D \xrightarrow{\text{INVERSION}} -D$$

pseudo scalar

Differential Calculus

$f(x)$, function of one variable, then $\frac{df}{dx}$ tells us how rapidly $f(x)$ varies with x changed by a tiny dx.

This is an ordinary derivative.



$\frac{df}{dx}$ small

$\frac{df}{dx}$ big

a gradient is basically a derivative with more variables

Gradient:

Derivatives with multiple variables. Partial derivatives!

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \underbrace{(\Delta T)}_{\text{Gradient}} \cdot dl$$

$$\Delta T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \qquad \text{(vector)}$$

Geometric interpretation

$$dT = \Delta T \cdot dl = |dl|\,|\Delta T| \cos\theta$$

if we fix $|dl|$, ΔT is higher when $\theta = 0$.

dT is greates in the same direction as ΔT

ΔT points in the direction of maximum increase of function T.

$|\nabla T|$ gives the slope along its max direction.

If $\Delta T = 0$, $dT = 0$. This is a stationary point. (max, min, pass, shoulder)

Similar to derivatives $= 0$.

Problem 1.13 c

General formula for $\nabla(r^n)$.

$$\frac{\partial}{\partial x}(r^n) = n r^{n-1} \frac{r}{dx} = n r^{n-1} \left(\frac{1}{2}\frac{1}{r} 2r\, x\right)$$

Del operator (nabla)

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

del operator can act the same as a vector

If we know a vector $\vec{A}$ can:

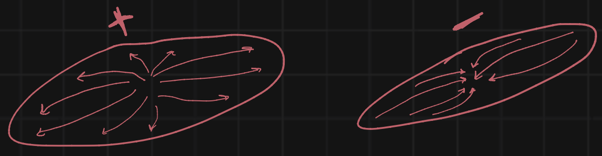
Del can too $\longrightarrow$

	$\vec{A}$	∇
scalar	$\vec{A}a$	∇T
dot product	$\vec{A} \cdot \vec{B}$	$\nabla \cdot \vec{v}$
cross product	$\vec{A} \times \vec{B}$	$\nabla \times \vec{v}$

Divergence

$$\nabla \cdot \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Geometric interpretation

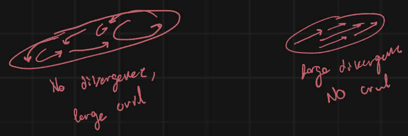


How much the vector spreads out.

The Curl

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix} \quad \text{(vector)}$$

Geometric interpretation



Derivative Rules

Ordinary derivative rules and vector derivative equivalent

There are 2 ways to make a scalar : fg and $\vec{A} \cdot \vec{B}$

2 ways to make a vector : $f\vec{A}$ and $\vec{A} \times \vec{B}$

Gradient takes a scalar
$\nabla(fg)$
$\nabla(\vec{A} \cdot \vec{B})$

(i) $\nabla(fg) = f\nabla g + g\nabla f,$

(ii) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A},$

Divergence takes a vector
$\nabla \cdot (f\vec{A})$
$\nabla \cdot (\vec{A} \times \vec{B})$

(iii) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f),$

(iv) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$

Curl takes a vector
$\nabla \times (f\vec{A})$
$\nabla \times (\vec{A} \times \vec{B})$

(v) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f),$

(vi) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$

Second Derivatives (Laplacian)

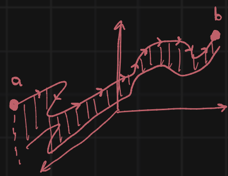
Gradient — Is a vector :
$\nabla \cdot (\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$
$\nabla \times (\nabla T) =$ Always 0.

Divergence — Is a scalar :
$\nabla(\nabla \cdot \vec{v}) =$ gradient of the divergence

Curl — Is a vector :
$\nabla \cdot (\nabla \times \vec{v}) =$ Always 0.
$\nabla \times (\nabla \times \vec{v}) =$ Combo $\nabla(\nabla \cdot \vec{v}) - \nabla^2 v$

Integral Calculus

Line Integrals

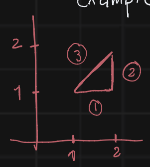


$$\int_a^b \vec{v} \cdot d\vec{l}$$

$\vec{v}$: Vector function
$d\vec{l}$: Displacement vector
Path from a to b.

If a = b $\oint \vec{v} \cdot d\vec{l}$

Example



$\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$

① $dy = 0 = dz$
$y = 1$, $x = 1-2$ $\int_1^2 1^2\,dx = 1$
$d\vec{l} = dx\,\hat{x}$
$\vec{v} \cdot d\vec{l} = y^2\,dx$

surface and line integrals are ~~very similar~~ the same thing!

Surface Integrals

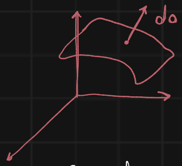
$$\int_S \vec{v} \cdot d\vec{a}$$

$\vec{v}$: Vector function

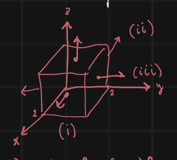
$d\vec{a}$: Infinitesimal patch area, $\perp$ to S. (ambiguous sign)

Over surface S.

If S is closed (balloon) $\oint_S \vec{v} \cdot d\vec{a}$



Example



$\vec{v} = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$

(i) $x = 2$

x=cte $\therefore$ $d\vec{a} = dy\,dz\ \hat{x}$

Direcciones en las que hay cambio (no ctes)

Volume Integrals

$$\int_V T\, d\tau$$

T: Scalar function

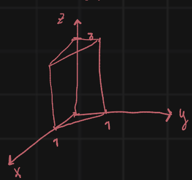
$d\tau$: Infinitesimal volume element (Cartesian: $dx\,dy\,dz$)

ocasionally, vector functions

$$\int \vec{T}\, d\tau = \sum_{i=1}^{3} \hat{e}_i \int V_i\, d\tau$$

Example:

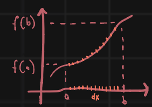
Volume Integral of $T = xyz^2$ of



$$\int T\, d\tau = \int_0^3 z^2 \left[\int_0^1 y \left[\int_0^{1-y} x\, dx \right] dy \right] dz = \frac{3}{8}$$

Fundamental Theorem of Calculus

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$



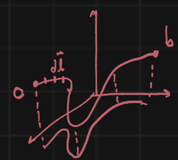
We have 3 derivatives in vector calculus, each with its theorem:

Fundamental theorem for Gradients

$$\int_a^b (\nabla T) \cdot d\vec{l} = T(b) - T(a)$$

T: Scalar function

$d\vec{l}$: Small displacement



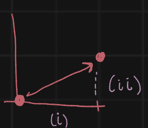
- Independent of the path.
- $\oint (\nabla T) \cdot d\vec{l} = 0$, since $a = b$, $T(a) - T(a) = 0$

$$\left[\ \frac{df}{dx} \longrightarrow \nabla F \cdot d\vec{l} = \frac{dF}{dx}\hat{x} + \frac{dF}{dy}\hat{y} + \frac{dF}{dz}\hat{z} \cdot dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\ \right]$$

Example

$T = xy^2$ $\quad \nabla T = y^2\,\hat{x} + 2xy\,\hat{y}$ $\quad d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

From $(0,0,0)$ — $(2,1,0)$



(i) $y = 0$, $0 < x < 2$, $d\vec{l} = dx\,\hat{x}$ $\quad \int \nabla T \cdot d\vec{l} = 0$

(ii) $x = 2$, $0 < y < 1$, $d\vec{l} = dy\,\hat{y}$ $\quad \int_0^1 2xy\, dy = xy^2 \Big|_0^1 = x = 2$

$T(b) - T(0) = 2 - 0 = 2$

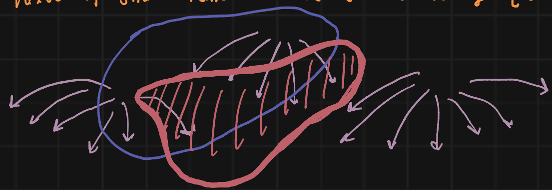
Fundamental Theory for Divergences [Gauss's, Green's, Divergence Theorem]

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

[The integral of a derivative [divergence] over a region [volume] is equal to the value of that function at the boundary [surface].]

Geometrical

$\vec{v}$

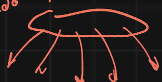


$\int$ faucets within volume $=$ $\int$ flow out through the surface

$\vec{v}$: Cómo se comporta el líquido.

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

Buscando "spreading out" de los vectores. [Buscando llaves]

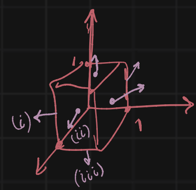


El flujo por cada pequeño punto de la superficie.

Example: Divergence Theorem

$\vec{v} = y^2 \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$

$\nabla \cdot \vec{v} = 2(x+y)$

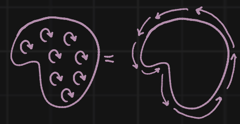


$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

$$\int_V 2(x+y)\, d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\, dx\, dy\, dz = 2$$

$\oint_S \vec{v} \cdot d\vec{a} =$ (i) $y = 0 = dy$, $x, z \in [0,1]$, $dx\,dz = -d\vec{a}\,\hat{y}$ $\leftarrow$ $-\int_0^1\int_0^1 2x\cancel{y} + z^2 \, dy\, dz$ ($\cancel{y} = 0$)

Fundamental Theorem for curls [Stokes']



$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \underbrace{\oint_P \vec{v} \cdot d\vec{l}}_{\text{circulation}}$$

[The integral of a derivative [curl] over a region [patch of surface] is equal to the value of that function at the boundary [surface].]

Geometrical

Total Amount of Curl in Area $=$ Flow on the edge (perimeter)

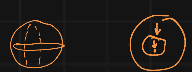
[sign given by $d\vec{a}$, $\oint$ right hand rule:

Very similar to gradient theorem!

- $\int (\nabla \times \vec{v}) \cdot d\vec{a}$ depends on P, not S.
- $\oint (\nabla \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface.

why?

[Closed surface: Compact and no boundary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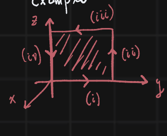
Example



$\vec{v} = 2xz + 3y^2\,\hat{y} + 4yz^2\,\hat{z}$

if $d\vec{a} = dy\,dz\,\hat{x}$, ↺

if $d\vec{a} = -dy\,dz\,\hat{x}$, ↻ $(x=0)$

① $\int (\nabla \times \vec{v}) \cdot da = \int_0^1\int_0^1 4z^2\, dy\, dz = \frac{4}{3}$

② i) $x = 0$, $z = 0$, $0 < y < 1$

$\vec{v} \cdot d\vec{l} = 3y^2\, dy$

$\int_0^1 \vec{v} \cdot d\vec{l} = 1$

ii) $x = 0$ $y = 1$, $d\vec{l} = dz$, $0 < z < 1$

$\int_0^1 \vec{v} \cdot d\vec{l} = \int_0^1 4yz^2 = \int_0^1 4z^2 = \frac{4}{3}$

iii) $d\vec{l} = dy$

$1 < y < 0$

Integration by Parts

$$\int_a^b \frac{d}{dx}(fg)\,dx = \int_a^b \frac{d}{dx}(f)\,g + \int_a^b f\,\frac{d}{dx}(g) = fg\Big|_a^b$$

You can transfer the derivative with a minus and a boundary term.

$$\int_a^b u\,\frac{d}{dx}(v)\,dx = uv\Big|_a^b - \int_a^b v\,\frac{d}{dx}(u)\,dx$$

We can exploit this for vector calculus: Integrand f, derivative $\nabla\cdot\vec{A}$.

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

$$\int \nabla\cdot(f\vec{A})\,d\tau = \int f(\nabla\cdot\vec{A})\,d\tau + \int \vec{A}\cdot(\nabla f)\,d\tau = \oint f\vec{A}\cdot d\vec{a}$$

$$\int f(\nabla\cdot\vec{A})\,d\tau = \oint f\vec{A}\cdot d\vec{a} - \int \vec{A}\cdot(\nabla f)\,d\tau$$

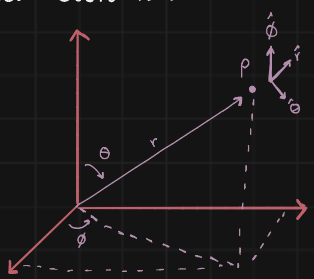
$$\int_S (\nabla\times f\vec{A})\cdot d\vec{a} = \oint_P f\vec{A}\cdot d\vec{l} = \int f(\nabla\times\vec{A}) - \int \vec{A}\times(\nabla f)$$

$$(\nabla\times\vec{A}f) = f\,\nabla\times\vec{A} - \vec{A}\times\nabla f$$

$$\nabla\times A\cdot\vec{B} = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

Curvilinear Coordinates

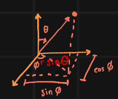
Spherical Coordinates

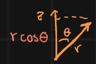


$x = r\sin\theta\cos\phi$

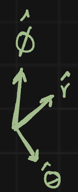
$y = r\sin\theta\sin\theta$

$z = r\cos\theta$



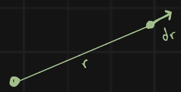


$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}},\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}.\end{aligned}$$

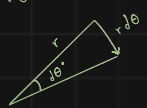


Changes direction with P.

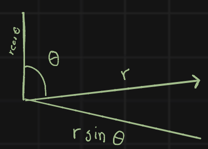
$dl_r = dr$

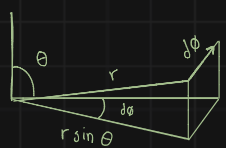


$dl_\theta = r\,d\theta$



$dl_\phi = r\sin\theta\,d\phi$





$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

For $d\vec{a}$, one must check the surface.

r doesn't change $d\vec{a} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

θ is constant $d\vec{a} = r\sin\theta\,dr\,d\phi\,\hat{\theta}$

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}. \tag{1.71}$$

Curl:

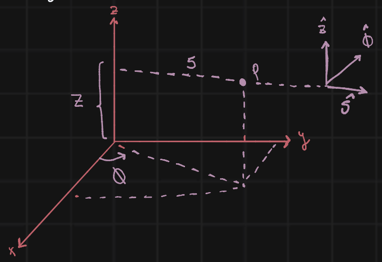
$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}. \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}. \tag{1.73}$$

Cylindrical Coordinates



$dl_s = ds$
$dl_\phi = s\, d\phi$
$dl_z = dz$

$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$
$d\tau = s\; ds\, d\phi\, dz$

$$\left.\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}\right\}$$

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{1.79}$$

Divergence:

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{1.80}$$

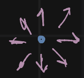
Curl:

$$\nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{1.82}$$

Dirac Delta Function

A weird introduction: We have $\vec{v} = \frac{1}{r^2}\hat{r}$ , with $\nabla\cdot\vec{v} = 0$

but a sphere with radius R, $\int \vec{v}\cdot d\vec{a} = 4\pi$. WTF?

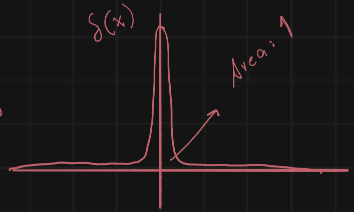
The problem is $r = 0$.

$\nabla\cdot\vec{v} = 0$ everywhere except $r = 0$, so all must come from 0.

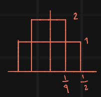
$\nabla\cdot\vec{v}$ banishes everywhere but in a point (!)

(similar to the density of a point particle)

One dimensional Dirac Delta



$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

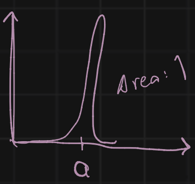
[$\delta(x)$ is a distribution. Similar to: ]

then, if

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0) \qquad [\,f(x)\,\delta(x) = f(0)\,\delta(x)\,]$$

sine its only defined at $x = 0$.

We can move the point $x = a$



$$\delta(x-a)\begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

Example:

$$\boxed{\int_0^3} x^3\,\delta(x - \boxed{2})\,dx = 2^3\,(1)$$

inside the domain

3D Delta Function

$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$, its 0 everywhere, but blows up at $(0,0,0)$

$$\int_{\text{all space}} \delta^3(\vec{r})\,d\tau = 1$$

Finally, the weird introduction: $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})$

Potentials

If $\nabla \times \vec{F} = 0$ everywhere, $\longleftrightarrow$ $\vec{F} = -\nabla V.$

$\vec{F}$ can be written as the gradient of scalar potential.

Theorem 1:

Curl-less (irrotacional)

$\nabla \times \vec{F} = 0$

- $\oint \vec{F} \cdot d\vec{l} = 0$ any closed loop.
- $\int_a^b \vec{F} \cdot d\vec{l}$ independent of path.
- $\vec{F} = -\nabla V$ has a scalar function

Theorem 2:

Divergence-less (Selenoidal)

$\nabla \cdot \vec{F} = 0$

- $\oint \vec{F} \cdot d\vec{a} = 0$ for any close surface.
- $\int \vec{F} \cdot d\vec{a}$ independent of surface.
- $\vec{F}$ is the curl of some $\vec{F} = \nabla \times \vec{A}$

Always: $\vec{F} = -\nabla \vec{V} + \nabla \times \vec{A}$

Electrostatics

Electric field

Superposition: Interaction between charges is unaffected by the presence of others.
[experimental !]

Coulombs Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$$F = F_1 + F_2 + F_3 + \dots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \dots\right)$$

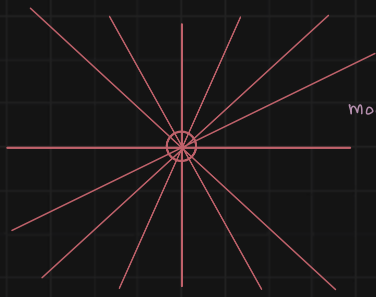
$$F = Q\vec{E},$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$$

for continuos distribuiton $\quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}} \quad dq \sim \lambda\, d\vec{l} \sim \sigma\, d\vec{a} \sim \rho\, d\vec{\tau}$

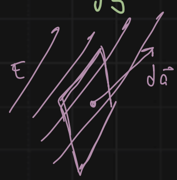
Divergene and Curl of electrostatic fields

Integrals are stupid, develop tricks.

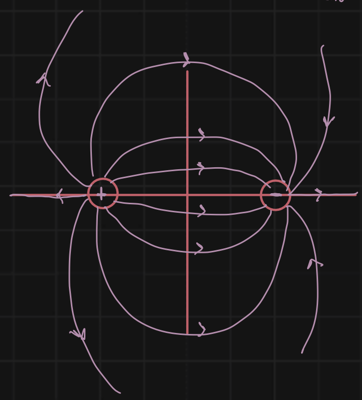


magnitude: density of lines

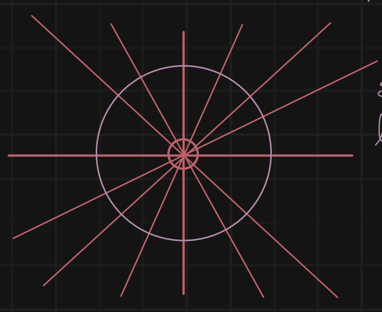
$$\Phi \equiv \int_S \vec{E}\cdot d\vec{a}$$



CAN'T · intersect
· end mid air



In the case of a point charge, flux through spherical surface:



same amount of lines go through

no matter the R

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}q$$

by superposition:

Gauss's Law

$$\oint_{\text{Any } S}\vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$$

The flux doesn't depend on the chosen surface because of $\frac{1}{r^2}$!

Similar $\frac{1}{r^2}$ forces obey Gauss's Law (classical gravity).

Gauss's Law (differential) and divergence of E.

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$$

$$\frac{1}{4\pi\varepsilon_0}\int \underbrace{\nabla \cdot \left(\frac{\hat{r}}{r^2}\right)}_{4\pi\,\delta^3(\vec{r})} \rho(\vec{r}')\, d\tau' = \frac{1}{\varepsilon_0}\rho(r)$$

Applications of Gauss's Law

Surface parallel to $d\vec{a}$!! so that $\vec{E} \cdot d\vec{a} = E\,da$

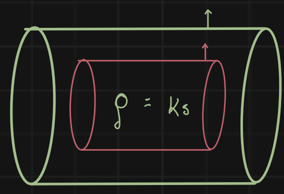
① Spherical

$r > R$

$$\int_S \vec{E} \cdot d\vec{a} = \int E\,da = EA = \frac{Q_{enc}}{\varepsilon_0}, \quad \text{for sphere:} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

field outside of sphere behaves as a point charge

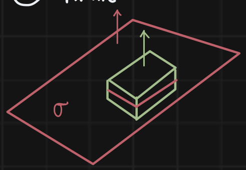
② Cilindrical



$$Q_{en} = \int \rho\, d\tau = \frac{2}{3}\pi k l^3$$

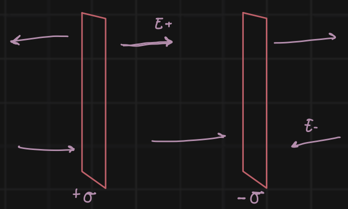
$$E = \frac{1}{3\varepsilon_0} k s^2 \hat{s}$$

③ Plane



$$\int \vec{E} \cdot d\vec{a} = 2AE = \frac{1}{\varepsilon_0}\sigma A \qquad E = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

By superposition, we can use Gauss in scenarios with multiple symetric shapes



Curl of E

$$\nabla \times \vec{E} = 0$$

No curl !

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

or

for $r_a = r_b$ (closed path)

$$\underbrace{\oint \vec{E} \cdot d\vec{l} = 0 \qquad \int \nabla \times \vec{E} \cdot d\vec{a} = 0}_{\text{stokes}}$$

Electric Potential

$\vec{E}$ is a very special field $(\nabla \times \vec{E} = 0)$ now we can get a scalar function $(\vec{E} = -\nabla V)$.

Because line integral is independent from the path :

$$V(\vec{r}) = -\int_{O}^{r} \vec{E} \cdot d\vec{l}$$

reference

Potential Difference

$$V(\vec{b}) - V(\vec{a}) = -\int_{O}^{a} \vec{E} \cdot d\vec{l} + \int_{O}^{b} \vec{E} \cdot d\vec{l} = -\int_{a}^{b} \underbrace{\vec{E}}_{} \cdot d\vec{l} = \int_{a}^{b} \underbrace{(\nabla V)}_{} \cdot d\vec{l}$$

$-\vec{E} = \nabla V$

Some comments

i) Potential $\neq$ Potential Energy

ii) Get $\vec{E}$ if you already have V !

iii) Changing reference point $V'(\vec{r}) = -\int_{O'}^{O} \vec{E} \cdot d\vec{l} - \int_{O}^{r} \vec{E} \cdot d\vec{l} = \boxed{K} + V(r)$ it just adds a constant

(Analogous to altitude: Above sea level ? Above California ?)

Natural place : $V(O) = 0$, $O = \infty$ [Except for ∞ charge distributions]

iv) Potential obeys superposition !

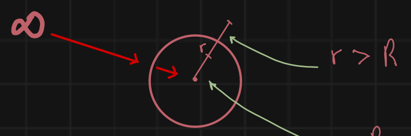
$\vec{F} = \vec{F}_1 + \vec{F}_2 \Rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2 \Rightarrow \vec{V} = \vec{V}_1 + \vec{V}_2$

v) Volts

V of spherical shell

Ex. V of a spherical shell when:

Reference at ∞



$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

$$V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{l} - \cancel{\int_{R}^{r} \vec{E} \cdot d\vec{l}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R}$$

Even if $\vec{E} = 0$, V is not necessarily 0.

Poisson's and Laplace's Equations

If $\vec{E} = -\nabla V$, $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$, $\nabla \times \vec{E} = 0$:

- $\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$

Poisson's Equation $$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

with $\rho = 0$:

Laplace's Equation $$\nabla^2 V = 0$$

- Curl Law

$$\nabla \times \vec{E} = \nabla \times (-\nabla V) = 0$$

Since it guarantees that $\vec{E} = -\nabla V$

tic

Potential of Localized Charge Distribuiton

To calculate V out of ρ, we invert Poisson :

$$V(\vec{r}) = -\int \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

used to make potential of positive charge positive

why we use infinity !

In general: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\mathscr{r}}$

Superposition: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum \frac{q_i}{\mathscr{r}_i}$

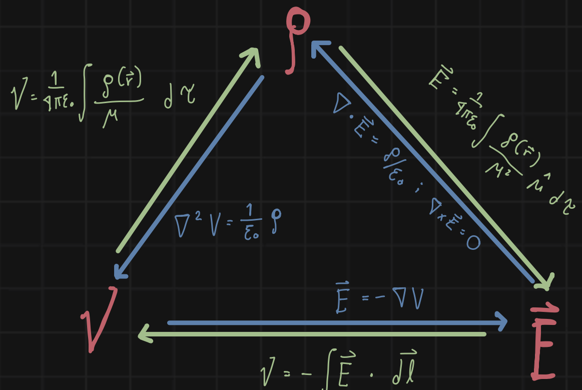
if $O = \infty$:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{\mathscr{r}}$$

relationship triangle

We generally want $\vec{E}$ and use V as an intermidiate step (excepting symetry)

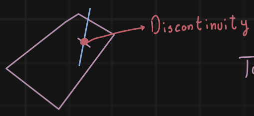
$\rho, V, \vec{E}$ are very important



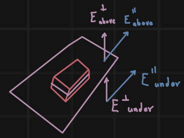
Boundary Conditions

The electric field always undergoes a discontinuity when crossing a surface σ.

discontinuity at surface



To find the amount of discontinuity



for the lid: $\oint_b \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0}$

$L \to 0$ $\quad \int_{arriba} \vec{E}\cdot d\vec{a} + \int_{abajo} \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\underbrace{\boxed{\int_A \sigma\, dA}}_{}^{q}$ (σ cte por área muy pequeña)

Componente perpendicular

$\int E_{\perp arriba}\, da + \int (-E_{\perp abajo})\, da = \frac{1}{\varepsilon_0}\sigma \int da$

$E_{\perp arriba}\cancel{A} - E_{\perp abajo}\cancel{A} = \frac{\sigma}{\varepsilon_0}\cancel{A}$

$$E_{\perp arriba} - E_{\perp abajo} = \frac{\sigma}{\varepsilon_0}$$ Es discontinua (!)

Área pequeña para paralela:

$= 0$

$\oint \vec{E}\cdot d\vec{l} = 0$

$L \to 0$

$\int_{arriba} \vec{E}\cdot d\vec{l} + \int_{abajo} \vec{E}\cdot d\vec{l} = 0$

Componente perpendicular

$\int E_{arriba}\, dl - \int_{abajo} (-E)\, dl = 0$

$$E_{\parallel arriba} - E_{\parallel abajo} = 0$$ Es continua (!)

Se pueden simplificar:

final boundary expression

$$\vec{E}_{arriba} - \vec{E}_{abajo} = \frac{\sigma}{\varepsilon_0}\hat{u}_n$$

Electric potential is continuos:

$V_a - V_b = -\int_a^b \vec{E}\cdot d\vec{l}$

$V_a = V_b$

a = out
b = in

continuity of potential

but:

$\nabla V_a - \nabla V_b = -\frac{1}{\varepsilon_0}\sigma\hat{n}$ or $\nabla V_a\cdot\hat{n} - \nabla V_b\cdot\hat{n} = -\frac{1}{\varepsilon_0}\sigma = \frac{\partial V_a}{\partial n} - \frac{\partial V_b}{\partial n} = -\frac{1}{\varepsilon_0}\sigma$

Work and Energy

Work to move a charge

Definition for work

if $\vec{F} = Q\vec{E}$, and we ought to find the minumum force to move a charge, then $\vec{F} = -Q\vec{E}$

Ex. $\uparrow -q$ $\downarrow q$

Work $W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\left[V(a) - V(b)\right]$

Path independent

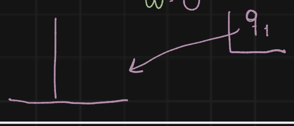
Now for $b = \infty$ $\quad W = Q\,V(\vec{r})$ and $V = WQ$ work per charge

potential energy

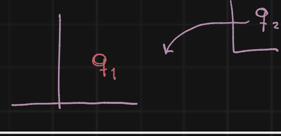
Energy of a point charge distribution

Trabajo para traer q_1

$W = 0$



Trabajo para carga q_2.

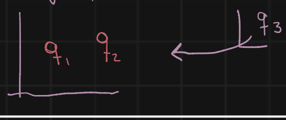


(Misma fuerza, sentido contrario)

$$W = \int_\infty^{P_2} \vec{F}\cdot d\vec{l} = \boxed{-}q_2 \boxed{\int_\infty^{P_2} \vec{E}\cdot d\vec{l}} = q_2 V_{(1)}(P_2)$$

Trabajo para traer q_3. $\quad W = \int_\infty^{P_3} \vec{F}\cdot d\vec{l} = -q_3 \int_\infty^{P_3} \vec{E}\cdot d\vec{l}$

$$= q_3 V_{(1,2)}(P_3)$$



q_4: $\quad q_4 V_{(1,2,3)}(P_4)$

Trabajo total para n cargas:

$$W = 0 + q_2 V_{(1)}(P_2) + q_3 V_{(1,2)}(P_3) + \dots$$

$$W = Kq_2\left(\frac{q_1}{r_{1,2}}\right) + Kq_3\left(\frac{q_1}{r_{1,3}} + \frac{q_2}{r_{2,3}}\right) + \dots$$

	1	2	3	4
1	X			
2		X		
3			X	
4				X

Energy for discrete charge

$$W = \underbrace{\frac{1}{2}}_{\text{Account for pairs}} \sum_i q_i V_{\text{por carga}}(P_i) \qquad \text{Discreta}$$

Energy of continuous distribution

Energy for continuous charge

$$W = \frac{1}{2}\int V_{\text{todo } \rho}\, dq \qquad \text{Continua} \quad \rho$$

$dq = \rho\, d\tau$

Rewrite this result:

$$W = \frac{1}{2}\int V\rho\, d\tau, \qquad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{E} = -\nabla V$$

$$W = \frac{\varepsilon_0}{2}\int V(\nabla\cdot\vec{E})\, d\tau$$

Integration by parts!

$$= \frac{\varepsilon_0}{2}\left[\int \nabla\cdot V\vec{E}\, d\tau - \int \vec{E}\cdot\nabla V\, d\tau\right] = \frac{\varepsilon_0}{2}\left[\oint V\vec{E}\cdot d\vec{a} - \int \vec{E}\cdot(-\vec{E})\, d\tau\right]$$

From $W = \frac{1}{2}\int V\rho\, d\tau$, if we integrate from a large region it won't matter ($\rho = 0$).

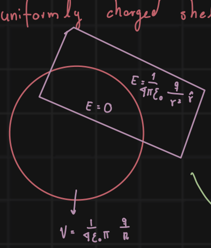
$$\frac{\varepsilon_0}{2}\left[\oint V\vec{E}\cdot d\vec{a} + \int E^2\, d\tau\right] = W$$

	$2r$	10
$A = 4\pi r^2$	$= 4(A)$	$= 100\,(A)$
$V = \frac{4}{3}\pi r^3$	$= 8(V)$	$= 1{,}000\,(V)$

Over all space $\quad W = \frac{\varepsilon_0}{2}\int E^2\, d\tau$

Example:

W of a uniformly charged shell of q, R.



① $W = \frac{1}{2}\int \sigma V \, dA$ → $\frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ $= \frac{q}{R 8\pi\varepsilon_0}\int \sigma \, dA = \frac{q^2}{R 8\pi\varepsilon_0}$

② $W_T = \frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int \frac{q^2}{r^4}\left[r^2 \sin\theta\right] dr\, d\theta\, d\phi$

$= \frac{1}{32\pi^2\varepsilon_0} q^2 4\pi \int_R^{\infty} \frac{1}{r^2} dr = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$

all space

Some comments

i) $W = \frac{1}{2}\sum q_i V_i$ **VS** $W = \frac{\varepsilon_0}{2}\int_{\text{all}} E^2 \, d\tau$ → whole story

useful for discrete

since: Energy of a point charge is ∞, and $\frac{1}{2}\sum V_i q_i$ ignores that part.

where is the energy ?

ii) Where is the energy stored ?

$W = \frac{1}{2}\int \rho V \, d\tau$ **VS** $W = \frac{\varepsilon_0}{2}\int_{\text{all}} E^2 \, d\tau$

over a charge distribution | over the field

Field or charge

correct: relativity, radiation | for now

does it follow superposition ?

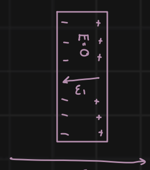
iii) **NO** superposition principle

$W_T \neq W_1 + W_2$ $\quad W_T = \frac{\varepsilon_0}{2}\int E^2 \, d\tau = \frac{\varepsilon_0}{2}\int (E_1 + E_2)^2 \, d\tau = W_1 + W_2 + \boxed{\varepsilon_0 \int E_1 \cdot E_2 \, d\tau}$

Conductors

Basic properties

i) $E = 0$ inside a conductor

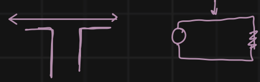


$\mathcal{E}_1$ cancels out $\mathcal{E}_0$.

ii) $\rho = 0$ inside a conductor $\quad$ Gauss's Law $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ if $\vec{E} = 0$, $\rho = 0$.

iii) Excess charge on the surface.
MINIMUM POTENTIAL ENERGY

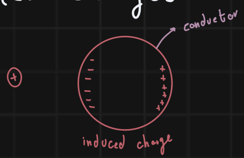
iv) Equipotential $\quad V(a) - V(b) = 0$:



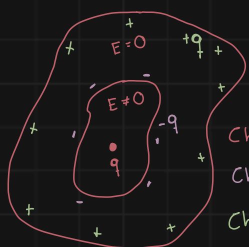
v) $E \perp$ surface, just outside

Inducted charges

induced charge



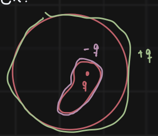
and



All fields are canceled at the boundaries.

Charge q inside
Charge $-q$ induced inside
Charge $+q$ induced outside

Ex.

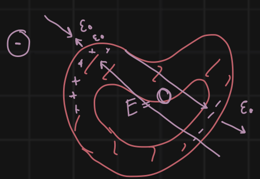


Field outside of the sphere?

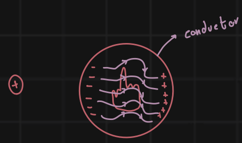
Since the conductor as a whole has charge q viewed from outside, it "hides" the cavity:

$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

E inside a hollow conductor no charge



Any hollow conductor will give $E = 0$ (no matter the shape)



No charge on the surface
(Faraday cage)

Surface charge and force on a conductor

E on a conductor

field inside = 0, boundary conditions : $\vec{E}_{outside} = \frac{\sigma}{\varepsilon_0}\hat{n}$

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

electrostatic force

force on any surface charge

$\vec{f} = \sigma \vec{E}$ since $\vec{E}$ is discontinuous $\vec{f} = \sigma \vec{E}_{avg} = \frac{1}{2}\sigma\left(\vec{E}_{above} + \vec{E}_{below}\right)$

averaging is mostly to remove the contribution of itself

electrostatic force on conductor

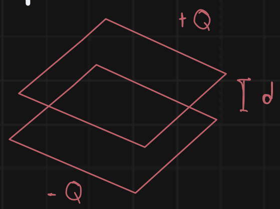
force of a conductor

$$E_{avg} = \frac{1}{2}\Big(0 + \underbrace{\frac{\sigma}{\varepsilon_0}\hat{n}}_{E \text{ on a conductor}}\Big) = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$\vec{f}_{conductor} = \frac{1}{2\varepsilon_0}\sigma^2\hat{n}$$

force its always positive since σ^2

Capacitors



$V = V_+ - V_- = -\int_+^- \vec{E}\cdot d\vec{l}$

Always positive minus negative

capacitance

Since $\vec{E}$ is proportional to Q

$$C = \frac{Q}{V}$$

Constant of proportionality is the capacitance

charge up capacitor

To charge a capacitor, you go against the $\vec{E}$, then the work to reach Q: $= \frac{1}{2}\frac{Q^2}{C}$

$$W = \frac{1}{2}CV^2$$

V is the final potential

Potentials

Laplace's Equation

need for Laplace's equation

Intro

We want to find $\vec{E}$

→ We started with Coulomb's Law [TOO HARD except for dummy exercises]

→ We went on wit Gauss's Law (symmetry) [TOO HARD except for dummy exercises]

→ Use the POTENTIAL

$$\cancel{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\imath}\rho(\vec{r})\, d\tau'}$$ STILL TOO HARD

Therefore, we may use Poisson's Equation: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

However, very often we want to analyze a region where $\rho = 0$, getting

Laplace's Equation $\nabla^2 V = 0$ [3D: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$]

(if $\rho = 0$ EVERYWHERE, $V = 0$)

Its solutions are called harmonic functions

Laplace's Equation 1D

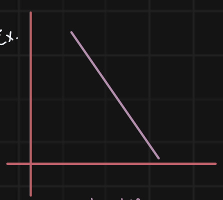
one dimension and properties

$$\frac{\partial^2 V}{\partial x^2} = 0$$

General solution: $V(x) = mx + b$

It's always a straight line!

Ex.



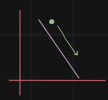
boring

Which means:

1. $V(x)$ is the average $V(x+a)$, $V(x-a)$ for any a.

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

2. $V(x)$ has NO local maxima nor minima (extreme values at endpoints)



Intuition:

Since second derivative $= 0$, no extrema. [This is not enterely true (x^4 for example) but it offers an easy way to remember]

Laplace's Equation 2D

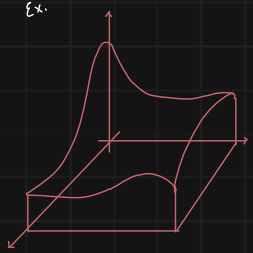
two dimensions some properties

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Impossible to find a general solution.

However, we can deduce common properties of the solutions.

Ex.



Just as 1D:

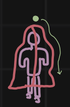
1. V at point x, y is the average of those around

$$V(x,y) = \frac{1}{2\pi R}\oint V\, dl$$

2. NO local mimima or maxima; extrema at boundaries

Viz:

A ball placed on top will roll to the edge.



No valleys or pockets or dents.

Laplace Equation 3D

three dimensions
same properties

No general solution nor graphic example! (That would be 4D)

Just as 1D and 2D:

(1) V at any point $\vec{r}$ is the average of V around a sphere radius R centered at $\vec{r}$.

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V \, da$$

(2) NO local minima nor maxima; extreme values at boundaries.

Boundary conditions and Uniqueness Theorem

what is a uniqueness theorem

Laplace's eq. is not enough to determine V. We need BOUNDARY CONDITIONS.

1D: $ax+b$ — we need just 2 conditions

Not all works. ie: 2 derivatives would be redundant

2D, 3D: partial DE — we need rules for boundary conditions

The proof that a set of boundary conditions will work is a uniqueness theorem. There are many.

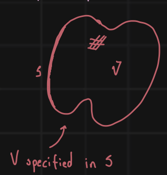
first uniqueness theorem

1st uniqueness theorem

The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface S.

In other words, if you find an eq. that satisfies Laplace's and has correct value at boundaries, it's right!

quick proof



$\nabla^2 V_1 = 0$
$\nabla^2 V_2 = 0$ Two solutions $V_3 = V_2 - V_1$

$$\nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$$

max and min must be 0

hence $V_1 = V_2$

same argument for Poisson!

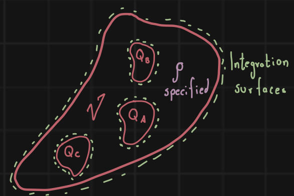
second uniqueness theorem

2nd uniqueness theorem

In a volume $\mathcal{V}$ surrounded by conductors and containing specified ρ, $\vec{E}$ is uniquely determined if the TOTAL CHARGE on each conductor is given.

the proof is a mess

trust me on this one

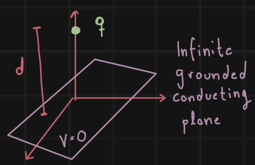


Method of images

basic images problem

Intro

We have



What is V above the plane?

Since q will induce a charge on the surface of the conductor, V_{TOTAL} is part q, part induced

Our problem is:

Solve Poisson's eq. at $z>0$, with:

$V = 0$ at $z=0$

$V \to 0$ very far

1st uniqueness theorem guarantees a single solution to these conditions

We use

Where:

$$\boxed{V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}\right]}$$

We get the same problem!

$V = 0$ at $z=0$

$V \to 0$ very far

If it satisfies Poisson in the region and assumes correct boundaries, it's right!

get induced surface charge

Induced surface charge

Since $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$, $\quad \sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$ (negative)

Total induced charge $Q = \int \sigma \, da \qquad Q = -q$ (induced charge on the plane)

Force and Energy

We know q is attracted to the plane by $-q$

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

energies are not the same!

Be careful! $\quad W_{\text{two point charges}} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d} \qquad W_{\text{charge and conducting plane}} = \frac{1}{2}\left[-\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}\right]$

Getting a charge to a conductor ($V=0$) costs nothing, however bringing two charges, work is done on both.

use cases for images method

Other image problem

Any stationary charge distribution near a grounded conducting plane can introduce an image! or find the equations

(Useful to remember: image charge is always opposite sign to ensure $V=0$ at plane)

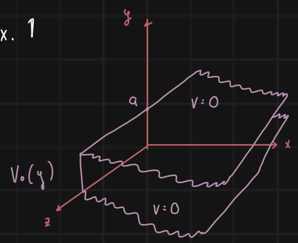
Separation of Variables

Used when potential (V) or charge density (σ) is specified on the boundaries and we want the potential in the interior.

Cartesian Coordinates

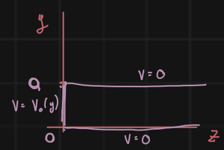
example

Ex. 1



Two infinite conducting plates lie parallel to xz, at $y=0$, $y=a$.
Left end $x=0$, closed off insulated strip, with $V = V_0(y)$
Find potential at slot

This is a 2D problem



We can see boundary conditions:

1) $V(y=0) = 0$
2) $V(y=a) = 0$
3) $V(x=0) = V_0(y)$
4) $V \to 0$ as $x \to \infty$

Therefore, $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Proposed: $V = X(x)Y(y)$

Plug in and multiplying by V:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

This is essentially: $\underbrace{f(x) + g(y) = 0}_{\text{MUST be constant}}$

$$f(x) = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1 \qquad g(y) = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2$$

$$X'' = X(k^2) \qquad Y'' = Y(-k^2)$$

$$X(x) = Ae^{kx} + Be^{-kx} \qquad Y(y) = C\sin(ky) + D\cos(ky)$$

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$

Now we must use boundary conditions

$A = 0$ (4), $D = 0$ (2), $ka = 0$ (1)

$$V(x,y) = Ce^{-kx}\sin ky, \quad k = \frac{n\pi}{a}$$

Family of solutions for n

Laplace's equation is linear!

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \dots = \alpha_1(0) + \alpha_2(0) + \dots = 0$$

for α_n constants

Hence, we can add them all

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

Last boundary condition: $V(0,y) = \underbrace{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right)}_{\text{Fourier Series}} = V_0(y)$

fourier's trick

By Fourier's Trick (Dirichlet's Theorem):

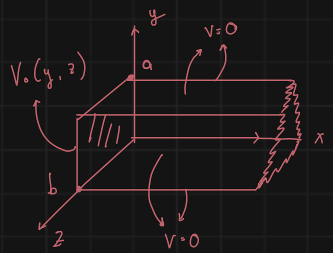
$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right) dy}_{\begin{cases} 0 \,, & n \neq n' \\ \frac{a}{2} \,, & n = n' \end{cases}} = \int_0^a V_0(y) \sin\left(\frac{n'\pi}{a}y\right) dy$$

We get

$$C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi}{a}y\right) dy$$

Overview of another problem

Overview 3D Laplace's eq.



Laplace's equation: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Conditions: (6)

1) $V(z=0)=0$
2) $V(z=b)=0$
3) $V \to 0\,,\quad x \to \infty$
4) $V(x=0) = V_0(y,z)$
5) $V(y=0)=0$
6) $V(y=a)=0$

Spherical coordinates

In spherical system, Laplace's equation: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$

azimuthal symetry

Azimuthal symetry (V independent from ϕ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

After plugin it in: $\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) \qquad \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)$

general solutions

General solutions: $\boxed{R(r) = A r^{\ell} + \frac{B}{r^{\ell+1}}}$

Angular equation is weirder

$$\Theta(\theta) = \underbrace{P_\ell(\cos\theta)}_{\text{Legendre Polynomials}}$$

Where $P_\ell(x)$ is defined by Rodrigues formula

$$P_\ell(x) = \frac{\ell}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

1st Legendre polynomials
$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = (3x^2-1)\frac{1}{2}$

Minimal separable solution to Laplace's eq in spherical:

$$V(r,\theta) = \left(A r^\ell + \frac{B}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

general solution spherical coordinates

General solution (each ℓ is a different solution):

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

B_ℓ terms: blow up at the origin
A_ℓ terms: don't go to zero at infinity

Example (should check more, this is very confusing)

A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find V inside / outside.

We could use $V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_0}{\mathscr{r}}\, da$ but separation of variables is easier.

#must check

For interior: $r \le R$ $\quad V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ $\qquad$ For exterior: $r > R$ $\quad V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

We know V is constant at R: $\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$

And V is discontinuous at R: $\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta)$

...

Multipole expansion

Approximate Potentials at large distances

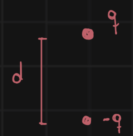
V at large distances

If you're very far, a charge distribuiton looks like a point charge with potential approx. $V \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

But what if $q = 0$? $\quad V = ?$

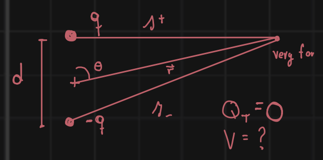
what is a physical dipole

(physical) electric dipole: Two equal, opposite charges separated by d.



Intro

What is the potential at points very far?



By law of cosines $\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta$

Thus $\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} = \frac{d}{r^2}\cos\theta$

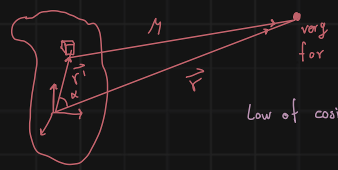
$$V(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\frac{q\,d\cos\theta}{r^2}$$ Falls off at $\frac{1}{r^2}$

Potential behaviour:

potential when $q=0$ from a very large distance (discrete)

Systematic expansion for any charge distribuiton in powers of $\frac{1}{r}$

Use discrete to approximate continuos



We know $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}\, d\tau$

Law of cosines $\quad \mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos(\alpha) = r^2\left(1 + \frac{r'^2}{r^2} - \frac{2r'\cos\alpha}{r}\right)$

$\mathscr{r} = r\sqrt{1+\varepsilon}$ $\qquad \varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

binomial expansion:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \dots\right]$$

or $$\frac{1}{\mathscr{r}} = \frac{1}{r}\Bigg[\underbrace{1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos\alpha - 1}{2}\right)}_{\text{Legendre Polynomials !}} + \dots\Bigg]$$

surprinsigly: $\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)P_n(\cos\alpha)$, r = constant

potential charge distribuiton at large distances

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec{r}')\,d\tau'$$

or $$= \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{1}{r}\int (1)(1)\rho(\vec{r}')\,d\tau}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int r'(\cos\alpha)\rho(\vec{r}')\,d\tau}_{\text{dipole}} + \dots\right]$$

...

Monopole and dipole terms

all higher terms vanish

Monopole term: $V_{mono} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

but if $Q = \int \rho \, d\tau = 0$

Dipole term: $V_{di} = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\cancel{\frac{1}{r}\int (1)(1)\rho(\vec{r}')\,d\tau}}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int r' \,(\cos\alpha)\,\rho(\vec{r}')\,d\tau}_{\text{dipole}} + \cancel{\dots} \right]$

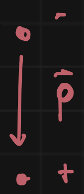
$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int \underbrace{r'\,(\cos\alpha)}\,\rho(\vec{r}')\,d\tau$

$r'\cos\alpha = \vec{r}' \cdot \hat{r}$

Then: $V_{di} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{r}\cdot \boxed{\int \vec{r}'\rho(\vec{r}')\,d\tau} \rightarrow \vec{p}$

what is dipolar moment

Dipolar moment: $\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\,d\tau$ or discrete $\vec{p} = \sum_{\infty}^{n} q_i \vec{r}'_i$



$\vec{p} = q\vec{d}$

Kinda like regular momentum $\vec{p} = m\vec{v}$

Finally $V_{di} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

perfect dipole: Make a dipole a point. ie make d very small $(d \to 0)$ or go very far $(r \to \infty)$

also $q \to \infty$!

pure dipole

pure dipole: $q \to \infty$, $d \to 0$ centered at origin

Origin of coordinates in multipole expansion

Multipole expansion uses r (distance from the origin)

so moving the origin alters the expansion

multipole expansion cases independent of coordinates

EXCEPT:

- Monopole moment never changes (independent of coordinates)

  monopole moment: $Q = \int \rho \, d\tau$

- If the total charge is 0, dipole moment is independent of the origin.

$\vec{p} = \int \vec{r}'\rho(\vec{r})\,d\tau' \xrightarrow[\text{move origin}]{} \int (\vec{r}' - \vec{a})\,\rho(\vec{r})\,d\tau' = \int \vec{r}'\rho(\vec{r}')\,d\tau - \int \vec{a}\,\rho(\vec{r}')\,d\tau = \vec{p} - \underset{Q=0}{Q\vec{a}} = \vec{p}$

Electric field of a dipole

If $\vec{p}$ at origin and pointing to $\hat{z}$, the potential is $V_{dip} = \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

placing limits on $\vec{p}$ to get on expression for $\vec{E}$

So, $\vec{E} = -\nabla V \rightarrow$

$$E_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Electric fields in matter

Polarization

Dielectrics

what is a dielectric

Most matter is either a conductor or dielectric.

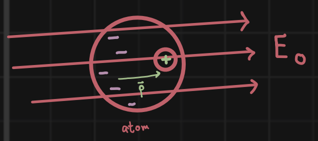
Dielectric : All charges are attached to specific atoms or molecules.

Induced Dipoles

dipole moment on atoms

Atom have positively charged core and negatively charged electron cloud that interacts with electric fields.

after it reaches equilibrium, the atom is polarized.



The atom has a small dipole moment, going $\parallel$ to E_0.

Typically : $\vec{p} = \alpha \vec{E}$ if its not too strong !

α: atomic polarizability

atomic polarizability

Atoms can be approximated with relative ease

However, molecules are very difficult. They polarize differently depending on direction.

$$\vec{p} = \alpha_{\perp} E_{\perp} + \alpha_{\parallel} E_{\parallel}$$

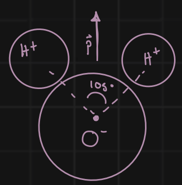
Then
$$p_x = \alpha_{xx} E_x + \dots$$
$$p_y = \alpha_{yx} E_x + \dots$$
$$p_z = \alpha_{zx} E_z + \dots$$
Polarizability tensor

Alignment of polar molecules

torque on polar molecules

Some molecules come already with dipole moments

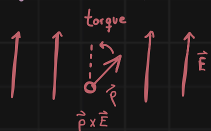
H_2O



When placed on an electric field, they experience

torque $\vec{N} = \vec{p} \times \vec{E}$ in uniform field by dipole

Trying to get $\vec{p}$ parallel to $\vec{E}$.



Polarization

Basically, whe a dielectric is placed in an electric field, it

General effect of field on dielectric

- has induced dipoles inside atoms
- if polar, has a torque going towards the field.

All of this create a lot of mini-dipoles pointing to the field.

What is polarization

Polarization $P \equiv$ dipole moment per unit volume

There are some other effects (stretching) but these don't matter for now.

Field of Polorized Object

Bound Charges

field on a polarized object

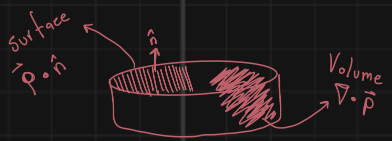
We have a polorized object (tons of mini dipoles) and $\vec{P}$ is given.

We want the field

We can add up all the tiny dipoles and get V.

$$\vec{p} = \vec{P}(\vec{r})\, d\tau' \qquad \nabla\left(\tfrac{1}{\mathscr{r}}\right) = \tfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{integration by parts}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' = \frac{1}{4\pi\varepsilon_0}\int_V P\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right) d\tau' = \frac{1}{4\pi\varepsilon_0}\left[\int_V \nabla'\cdot\frac{\vec{P}}{\mathscr{r}}\, d\tau' - \int_V \frac{1}{\mathscr{r}}(\nabla'\cdot\vec{P})\, d\tau'\right]$$



divergence

$$V = \frac{1}{4\pi\varepsilon_0}\oint \underbrace{\frac{1}{\mathscr{r}}\vec{P}\cdot d\vec{a}}_{\sigma_b = \vec{P}\cdot\hat{n}} - \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{\mathscr{r}}\underbrace{(\nabla'\cdot\vec{P})}_{\rho_b = -\nabla\cdot\vec{P}}\, d\tau'$$

surface charge — volume charge

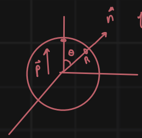
bound charges

So, we get

$$V = \frac{1}{4\pi\varepsilon_0}\left[\oint \frac{\sigma_b}{\mathscr{r}}\, da' + \int_V \frac{\rho_b}{\mathscr{r}}\, d\tau'\right]$$

We can get the bound charges to calculate V (and $E = -\nabla V$)

Ex.



Uniformly polarized sphere

$\rho_b = 0$

$\sigma = P\cdot\hat{n} = P\cos\theta$

Field produced by:

$$\vec{E} = -\nabla V = \begin{cases} -\frac{P}{3\varepsilon_0}\hat{z} & r < R \\ \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} & R \geq R \end{cases}$$

Physical Interpretation of bound charges (or wtf is σ_b and ρ_b ?)

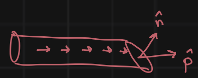
what bound charge really is

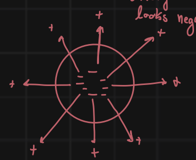
Suppose we have

are the same

bound charge (bc it cannot be removed lol)

$\sigma_b = \vec{P}\cdot\hat{n}$





$$\int \rho_b\, d\tau = -\oint_S \vec{P}\cdot\hat{n}\, da = -\int_V (\nabla\cdot\vec{P})\, d\tau$$

same charge inside and at surface — divergence theorem

or

2 spheres with opposite polarization

one on top of the other.

We're using the macroscopic $\vec{E}$ in this section. The actual microscopic $\vec{E}$ is impossible (and useless)

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau', \qquad \vec{E}_b = -\nabla V$$

Electric Displacement

Gauss's Law in Dielectrics

We know the field due to polarization (from σ_b, ρ_b), now we add the field due to everything else (free charge [ρ_f])

(ions, electrons, whatever; any charge that's NOT polarization)

what is free charge

Therefore

$$\rho = \rho_b + \rho_f$$

Gauss's Law

$$\varepsilon_0 (\nabla \cdot \vec{E}_T) = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

$$= \nabla \cdot \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\downarrow} = \rho_f$$

$\vec{D}$ is just a way to make Gauss's Law simple for dielectrics but $\vec{D} \neq \vec{E}$

electric displacement definition

Electric Displacement $\qquad \vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$

now $\qquad \nabla \cdot \vec{D} = \rho_f \qquad$ or $\qquad \oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}$

Total enclosed free charge

$\vec{D}$ allows us to write Gauss' Law in terms of free charge alone

Notes:

Polarization drops to 0 outside, so its derivative is a delta function.

We can think of the material having a thick edge, where the polarization slowly goes to 0, then there's no σ_b.

A Deceptive Parallel

$\vec{E}$ is very different from $\vec{D}$

don't confuse $\vec{E}$ and $\vec{D}$

The divergence alone is not enough to develop a vector field.

We also need the curl, and

$\nabla \times \vec{E} = 0 \qquad$ but $\qquad \nabla \times \vec{D} = \varepsilon_0 (\nabla \times \vec{P}) + (\nabla \times \vec{P}) = \nabla \times \vec{P}$

you cannot assume it banishes

Practically, $\vec{D}$ works similarly to Gauss's Law, look for symetry!

Boundary Conditions

boundary conditions in $\vec{D}$

Normal boundary cond. just in $\vec{D}$ now

perpendicular

$$\vec{D}_{above} - \vec{D}_{below} = \sigma_f$$

parallel

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

Linear Dielectrics

Susceptibility, Permittivity, Dielectric constant

We now that for many substances, $\qquad \vec{P} = \varepsilon_0 \chi_e \vec{E}$

polarization is proportional to the field

Materials that obey are called linear dialectrics.

susceptibility

χ_e : Electric susceptivity

Getting the field is really fucking hard, so:

We have $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (\chi_e + 1) \vec{E} = \varepsilon \vec{E}$

permittivity

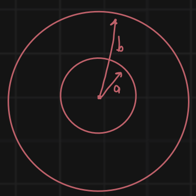
$$\varepsilon = \varepsilon_0 (1 + \chi_e) \quad : \quad \text{Permittivity}$$

And

dielectric constant

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e \quad : \quad \text{Dielectric constant}$$

Ex.



Metal sphere ($R=a$) with charge Q
Linear dielectric ($R=b$) permittivity ε
Find Potential at the center

Calculating $\vec{P}$ or $\vec{E}$ is dumb (how??)

① Therefore, we get $\vec{D}$ first (symmetry)

$$\oint \vec{D} \cdot d\vec{a} = Q_{free}$$

$$D(4\pi r^2) = Q$$

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r} \quad \text{for } r > a$$

$\vec{E} = 0$ inside $r < a$ (conductor)

② $\vec{D} = \varepsilon \vec{E}$

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi r^2 \varepsilon} \hat{r} & a < r < b \\ \dfrac{Q}{4\pi r^2 \varepsilon_0} \hat{r} & r > b \end{cases}$$

③ Therefore,

$$V_{center} = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{b} \frac{Q}{4\pi r^2 \varepsilon_0} dr - \int_{b}^{a} \frac{Q}{4\pi r^2 \varepsilon} dr - \int_{a}^{0} 0 \, dr = \frac{Q}{4\pi} \left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b} \right)$$

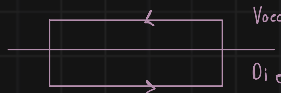
Polarization $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \dfrac{Q \varepsilon_0 \chi_e}{4\pi r^2 \varepsilon} \hat{r}$

and $\rho_b = -\nabla \cdot \vec{P} = 0$

curl of $\vec{D}$

Curl of $\vec{D}$

It cannot vanish ($\nabla \times \vec{P} = 0$) except if the entire space is a dielectric material
be:



different ε and, hence: $\oint \vec{P} \cdot d\vec{l} \neq 0$

isotropic
If a medium has properties (susceptibility) equal in all directions.
[Sometimes linear dielectrics ($\vec{E} = \alpha \vec{P}$) have different factors (α) depending on direction]

Boundary Value Problems with Linear Dielectrics

Here
$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} \vec{D} \right) = -\left(\frac{\chi_e}{1 + \chi_e} \right) \rho_f$$

Unless free charge is already embedded, $\rho = 0$ all charge at surface.

It obeys Laplace's Eq. !! $\quad \varepsilon_{above} E^{\perp}_{above} - \varepsilon_{below} E^{\perp}_{below} = \sigma_f$

$$\varepsilon_{above} \frac{\partial V_{above}}{\partial n} - \varepsilon_{below} \frac{\partial V_{below}}{\partial n} = \sigma_f$$

Magnetostatics

Lorentz Force Law

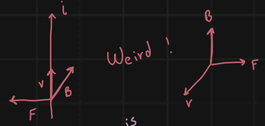
Magnetic Fields

magnetic fields

Up to now, we wanted $\vec{F}$ some charge excerts on other, statically (electrostatics)

Now we shall study charges in motion

Magnetic Force ($\vec{B}$)
force by moving charges



force is perpendicular to current

moving charges, then, produce $\vec{E}$ and $\vec{B}$.

Magnetic Forces

Lorentz Law: $\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$

charge, velocity, magnetic field

fundamental axiom!

force with $\vec{E}$ and $\vec{B}$ $\quad \vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]$

work done by $\vec{B}$

Magnetic Forces do no work.

$(dW = F_{mag} \cdot dl = Q(\vec{v} \times \vec{B}) \cdot d\vec{l} = 0)$

It can't alter the speed, only the direction.

Currents

what is current

Current in a wire is the charge per unit time.

The electrons are the ones that move, in the direction opposite to the electric current.

(Benjamin Franklin was fucking dumb and arbitrarily named them)

segment current density

$$\vec{I} = \lambda \vec{v}$$

λ refers only to moving charges.

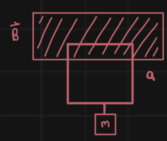
If e^- and p^+ both move, $\vec{I} = \lambda_+ \vec{v}_+ + \lambda_- \vec{v}_-$

Then, force on a segment $\quad \vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B}) \lambda\, dl = \int (\vec{I} \times \vec{B})\, dl$

$\vec{I}$ and $d\vec{l}$ in some direction : $\vec{F}_{mag} = \int I(d\vec{l} \times \vec{B})$

Constant I : $\vec{F}_{mag} = I \int (d\vec{l} \times \vec{B})$

Example

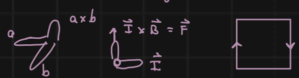


Uniform magnetic field $\vec{B}$ ● into page

For what current I would $\vec{F}_{mag}$ balance the $\vec{F}_g$?

1st, the force must go ↑

then:



$\vec{I}$ must go ⟶

Now $F_{mag} = \int I(d\vec{l} \times \vec{B}) = IB\int dl = IBa$

(since vertical cancels and only 1 horizontal is in the field)

So, $IBa = mg$ $\quad$ To balance weight

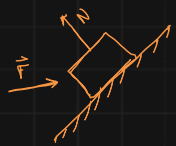
$$\boxed{I = \frac{mg}{Ba}}$$

If we were to increase the current, the weight would go up

But $\vec{B}$ does no work!

Therefore, the battery does the work and $\vec{B}$ just redirects it.

Think of

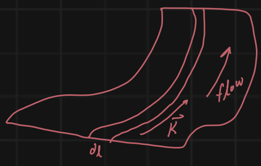


$\vec{B}$ is similar to $\vec{N}$

it has ↑ component that lifts the box and a ← component that has to be overcome.

it is doing no active work, however it redirects the efforts of active agent!

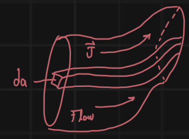
surface current density



$$\vec{K} = \frac{d\vec{I}}{dl_\perp} = \sigma \vec{v} \quad \text{and} \quad \vec{F}_{mag} = \int (\vec{K} \times \vec{B})\, da$$

volume current density

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \rho \vec{v} \quad \text{and} \quad \vec{F}_{mag} = \int (\vec{J} \times \vec{B})\, d\tau$$



Since

$$I = \int_S J\, da_\perp \ , \quad \vec{I} = \int_S \vec{J} \cdot d\vec{a} \ , \quad \oint_S \vec{J} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{J})\, d\tau \underset{\text{whats inside}}{=} -\frac{d}{dt}\int_V \rho\, dt = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Continuity Equation: $\nabla \cdot \vec{J} = \frac{\partial \rho}{\partial t}$ (magnetostatics $\nabla \cdot \vec{J} = 0$)

The Biot-Savart Law

Steady Currents

Steady charges : Constant electric field : electrostatics

Steady currents : Constant magnetic field : magnetostatics

what is a steady current

Steady Current : $\frac{\partial \rho}{\partial t} = 0$, $\frac{\partial \vec{J}}{\partial t} = 0$

No charge, no pilling up. I constant

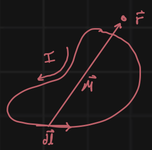
Therefore, the continuity eq: $\nabla \cdot \vec{J} = 0$

Magnetic field of steady current

Biot-Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

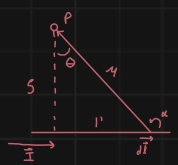
Coulomb's law but magnetic



μ_0 : Permeability of free space

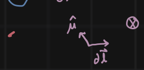
$\vec{\mathscr{r}}$: Vector from source to point $\vec{r}$

Example



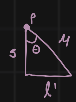
Find the magnetic field a distance s from a long straight wire with steady current.

① $d\vec{l} \times \hat{\mathscr{r}}$



and $d\vec{l} \times \hat{\mathscr{r}} = dl \cos\theta$

②



$\cos\theta = \frac{s}{\mathscr{r}}$

$\tan\theta = \frac{l'}{s}$

$\cancel{\sin\theta = \frac{l'}{\mathscr{r}}}$

$\mathscr{r} = \frac{s}{\cos\theta} \Rightarrow \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$

$l' = \tan\theta\, s \Rightarrow dl' = \frac{1}{\cos^2\theta}\, s\, d\theta$

③ Substituting

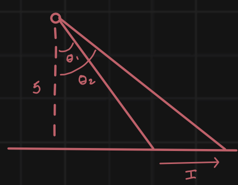
$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int \frac{dl \cos\theta}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{1}{\cos^2\theta}\, s\, d\theta\right)\cos\theta = \frac{\mu_0 I}{4\pi s}\int \cos\theta\, d\theta = \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)}$$

magnetic field of an infinite wire

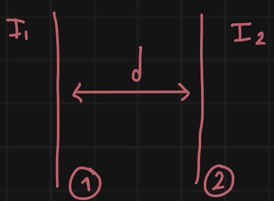
Infinite wire : $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Inversely proportional to the distance, just like $\vec{E}$!



Now, to get the force between 2 wires :



Field at ② due ① $B = \frac{\mu_0 I_1}{2\pi d}$

$$\vec{F}_{mag} = I\int d\vec{l} \times \vec{B} = IB\int dl = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl \quad \text{(Lorentz Law)}$$

infinite !

But at unit length: $f = \frac{I_2 I_1 \mu_0}{2\pi d}$

if $I_1 \uparrow \downarrow I_2$ $I_1(-I_2)$ repell

if $I_1 \uparrow \uparrow I_2$ $(I_1)(I_2)$ attract !

Biot-Savart

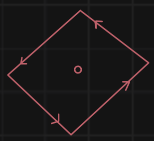
volume and surface

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da' \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

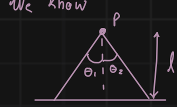
* A point charge doesn't constitute a steady current, therefore Biot-Savart doesn't work.
* Superposition applies to magnetic fields

Exercises

Find magnetic field at the center of a square.

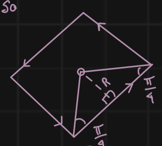


We know



$$\frac{\mu_0 I}{4\pi l}(\sin\theta_2 - \sin\theta_1)$$

So



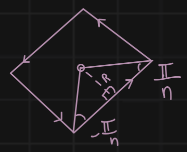
4 sides

$$= 4\left(\frac{\mu_0 I}{4\pi l}(\sin\theta_2 - \sin\theta_1)\right)$$

contribution per side

$$= \frac{\mu_0 I}{\pi R}\left(\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right)$$

For a n sided polygon

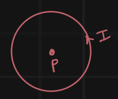


sides

$$n\left(\frac{\mu_0 I}{4\pi s}\left(\sin\left(\frac{\pi}{n}\right) - \sin\left(-\frac{\pi}{n}\right)\right)\right) = \frac{n\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)$$

contribution per side)

For a circle

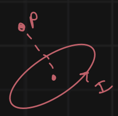


when $n \to \infty$

$$\lim_{n\to\infty}\left(\frac{n\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)\right) = \frac{\mu_0 I}{2R}$$

B for a point of center of a circle

Also, for

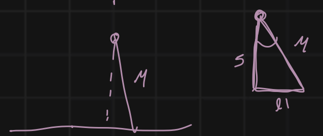


Too lazy to compute

$$B = \frac{\mu_0 I}{2\pi R}$$

B for a point over a circle

random useful doodles



$\sin\theta = \frac{l'}{\mathscr{r}}$

$\cos\theta = \frac{s}{\mathscr{r}}$

$\tan\theta = \frac{s}{l'}$

$\mathscr{r} = \frac{s}{\cos\theta} \rightsquigarrow \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$

$l' = s\tan\theta$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int d\vec{l} \times \hat{\mathscr{r}}\left(\frac{1}{\mathscr{r}^2}\right)$$

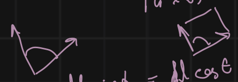
$\frac{1}{\mathscr{r}^2}$ dl'

$dl' = s\sec^2\theta\, d\theta$

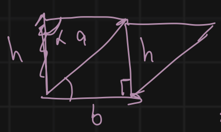
$= s\frac{1}{\cos^2\theta}d\theta$

$$B = \frac{\mu_0}{4\pi} I \int \frac{1}{\mathscr{r}^2}\cos\theta\, dl$$

$|\vec{a}\times\vec{b}| = ab\sin\theta$



$dl\sin\alpha = dl\cos\theta$



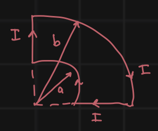
$|\vec{b}|\sin\theta\, a$

$\cos\alpha\, a$

$\sin\theta = \frac{h}{a}$

$\cos h = \frac{h}{a}$

Magnetic field of

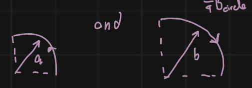


We have here

↑ and ←

which contribute nothing

But



and

$\frac{1}{4}B_{circle}$

is a quarter of a circle

Therefore

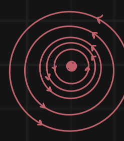
$$B = \frac{1}{4}\left(\frac{\mu_0 I}{2a} - \frac{\mu_0 I}{2b}\right)$$

$$= \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The Divergence and Curl of B

intuitive curl of $\vec{B}$

Straight Line Currents (or a visual approach to the curl of $\vec{B}$)



Magnetic field of infinite straight wire

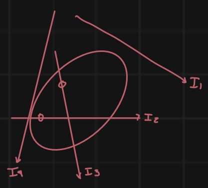
you can already see $\nabla\times\vec{B}\neq 0$
$\nabla\cdot\vec{B}=0$?

Has $\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl = \frac{\mu_0 I}{2\pi s}\oint dl = \frac{\mu_0 I}{2\pi s}(2\pi s) = \mu_0 I$

or, with cylindrical

$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s}\, s\, d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$

If we now have



$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$ where $I_{enc} = \int \vec{J}\cdot d\vec{a}$

by Stoke's $\int(\nabla\times\vec{B})\cdot d\vec{a} = \mu_0 I_{enc} = \mu_0\int\vec{J}\cdot d\vec{a}$

Ampere's Law

$$\nabla\times\vec{B} = \mu_0\vec{J}$$

We may observe how this only holds by straight line currents, however it gives an easy to remember intuition.

For a rigurous result, take $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}'(\vec{r})\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$ (Integration over prime coordinates, divergence and curl over unprimed)

$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \qquad \nabla\cdot\vec{B} = 0$$

no point sources
begin and end closed loop

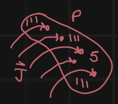
$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int\nabla\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

Ampere's Law (or Gauss's Law magnetic equivalence [literally])

$$\nabla\times\vec{B} = \mu_0\vec{J}$$

current enclosed by Amperian loop

$I_{enc} = \mu_0\int\vec{I}\cdot d\vec{a}$

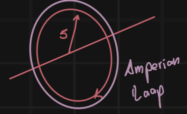


and, integral form

$$\oint(\nabla\times\vec{B})\cdot d\vec{a} = \mu_0 I_{enc}$$

Examples

Magnetic field with steady I, a distance s from a long straight wire



$\oint\vec{B}\cdot d\vec{l} = B\oint d\vec{l} = B2\pi s = \mu_0 I$

$|\vec{B}|$ is constant around a loop of radius s

$$B = \frac{\mu_0 I}{2\pi s}$$

Work on understanding more examples!

- Ampere's Law only holds for steady currents
- Symetric configurations :
  - → Infinite straight lines
  - → Infinite planes
  - → Infinite Selenoids
  - → Toroids ⚠ Weird one

Electric VS Magnetic fields

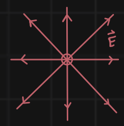
Electrostatics

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{Gauss's Law}$$

$$\nabla \times \vec{E} = 0$$

boundary condition $\vec{E} \to 0$ far away

Maxwell's equations for electrostatics



Diverge away from a charge

Begin on positive and end on negative

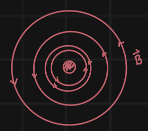
Magnetostatics

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{Ampere's Law}$$

boundary condition $\vec{B} \to 0$ far away

Maxwell's equations for magnetostatics



Curls around a current

Closed loops or infinity ($\nabla \cdot \vec{B} = 0$)

Force Law : $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$

Magnetic fields originate from moving charges and have no monopoles.

Typically, electric forces are much greater than magnetic ones (has to do with the values of ε_0 and μ_0)

Work out some problems ⚠

Magnetic Vector Potential

The Vector Potential

Since $\nabla \cdot \vec{B} = 0$, we have a vector potential $\vec{B} = \nabla \times \vec{A}$

Now, Ampere's Law $\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2(\vec{A}) = \mu_0 \vec{J}$

since $\nabla \cdot \vec{A} = 0$ odd to prove, but $\nabla \cdot A = 0$ can always be a solution, we just set it as a condition

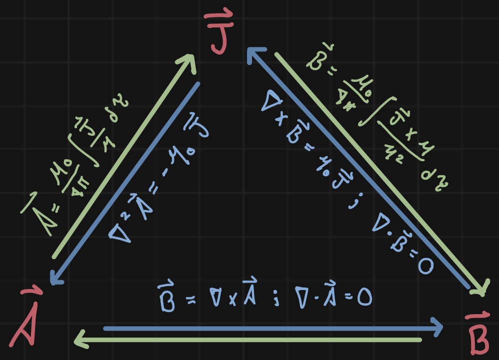
Ampere's Law in terms of potential : $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

(in Cartesian coordinates : 3 Poisson equations)

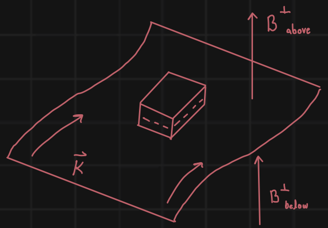
Assuming $J \to 0$ at ∞ : $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} \, d\tau'$ and $\vec{A} = \frac{\mu_0}{4\pi} I \int \frac{1}{\mathscr{r}} \, d\vec{l}$ $\quad A = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{\mathscr{r}} \, da'$

Typically the direction of $\vec{A}$ is the direction of the current.
(Of course, if it extends to infinity we can't even use the expression)

Boundary Conditions



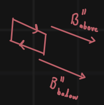
Just like $\vec{E}$ was discontinuous at surface **charge**, $\vec{B}$ is discontinuous at surface **current**



but now $\oint \vec{B} \cdot d\vec{a} = 0$ makes the lids continuous (contrary to $\vec{E}$)

$B^{\perp}_{above} = B^{\perp}_{below}$

and $\oint \vec{B} \cdot d\vec{l} = (B^{\parallel}_{above} - B^{\parallel}_{below})\, l = \mu_0 I_{enc} = \mu_0 K l$



$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$

Can be combined in $\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$

Also, $\vec{A}$ is continuous across boundary $\vec{A}_{above} = \vec{A}_{below}$

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} = \Phi$$

but the derivative inherits discontinuity

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

Multipole Expansion of Vector Potential

To approximate the vector potential of a current distribuiton at distant points by series of $\frac{1}{r}$.

same as electric potential :

$$\frac{1}{r} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

therefore the potential can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint\frac{1}{\mathscr{r}}\,d\vec{l} = \frac{\mu_0 I}{4\pi}\left[\underbrace{\overset{\text{no magnetic monopoles} \to 0}{\cancel{\frac{1}{r}\oint d\vec{l}'}}}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, dl'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) dl'}_{\text{quadrupole}} + \dots\right]$$

magnetic dipole (dominant)

Dipole is the dominant term

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint r'\cos\alpha\, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint(\hat{r}\cdot r')\,d\vec{l}'$$

Magnetic Dipole Moment : $\vec{m} \equiv I\int d\vec{a} = I\vec{a}$

Then

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{m \times \hat{r}}{r^2}$$

The magnetic dipole moment is always independent of the choice of origin (since the monopole is always 0)

Magnetic Fields in Matter

Magnetization

Diamagnets, Paramagnets, Ferromagnets

If you could inspect a magnetic material really close, you'd find tiny little charges in motion !

When a magnetic field is applied the otherwise random currents align ! Become magnetized.

Unlike $\vec{E}$ which is almost always in equal direction,

some materials are magnetized :

Paramagnets : Parallel to $\vec{B}$

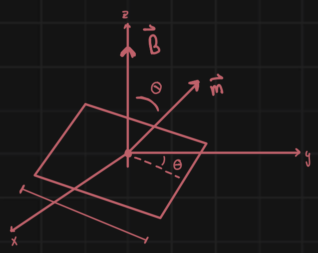
Diamagnets : Opposite to $\vec{B}$

Ferromagnets : Retain their magnetization after $\vec{B}$ has been removed

[Para : Parallel]

Torques and Forces on Magnetic Dipoles

torque and force similar to electric



For paramagnetism only (parallel)

$$\vec{N} = m B \sin\theta \hat{x} = \vec{m} \times \vec{B}$$

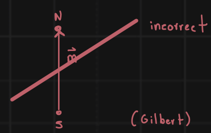
[Analogous to $\vec{N} = \vec{p} \times \vec{E}$]

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$

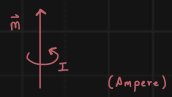
[Analogous to $\vec{F} = \nabla(\vec{p} \cdot \vec{E})$]

Because of these similarities,

people thought a magnetic dipole was :



but really it is :



magnetostatics is not directly analogous to electrostatics (no such thing as magnetic monopoles) !

Effect of a Magnetic Field on Atomic Orbits

Every electron has a magnetic dipole (spin)

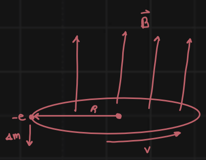
and revolves around a nucleus which creates a (almost) steady current

Like any dipole, it experiences a torque but very small.

However it has a more significant effect:

SPEEDS UP OR SLOWS DOWN the electron

electrons change speed and induce a Δm opposite to $\vec{B}$



A change in velocity means a change to the dipole moment !

Δm is opposite to $\vec{B}$: Diamagnetism

Usually orbits are random, but with a $\vec{B}$ they all line up and its orbits produce a Δm opposite to $\vec{B}$ (usually weak compared with paramagnetism)

this makes them diamagnetic !

Paramagnetism: Usually extra electron to produce enough torque (Pauli couples twin electrons)

Diamagnetism: Usually even electrons to cancel paramagnetism

Magnetization

Inside a $\vec{B}$ matter magnetizes (mini tiny dipoles aligned)

1) Paramagnetism: Torque aligns dipoles to $\vec{B}$
2) Diamagnetism: Orbital speed altered to make dipoles opposite to $\vec{B}$

Both weak af

$\vec{M}$: Magnetization

Magnetic dipole moment per unit volume

Field of Magnetized Object

Bound Currents

We have $\vec{M}$, what is $\vec{B}$?

We know $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ now for whole volume $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$

We expand (product rule):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \underbrace{\frac{1}{\mathscr{r}}[\nabla' \times \vec{M}(\vec{r}\,')]}_{\vec{J}_b = \nabla \times \vec{M}} d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} \underbrace{[\vec{M}(\vec{r}\,') \times d\vec{a}\,']}_{\vec{K}_b = M \times \hat{n}}$$

Finally

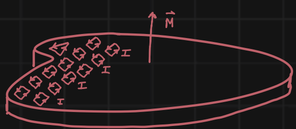
$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} J_b(\vec{r}\,')\, d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} K_b(\vec{r}\,')\, da'$$

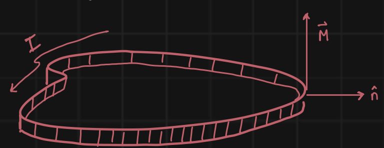
Instead of integrating for tiny currents, we get bound currents ($\vec{J}$ and $\vec{k}$)

Similar to electric case:

	$\vec{E}$	$\vec{B}$
Surface	$\rho_b = -\nabla \cdot \vec{P}$	$\vec{J}_b = \nabla \times \vec{M}$
Volume	$\sigma_b = \vec{P} \times \hat{n}$	$\vec{K}_b = \vec{M} \times \hat{n}$

Physical Interpretation of Bound Charges



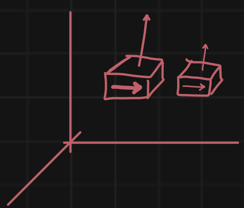


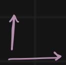
$K_b = \vec{M} \times \hat{n}$

(no current top or bottom because the curl cancels it)

all internal currents cancel, but at the edge no adjacent loop is the to cancel

For volume:





Derivatives with perpendicular contribuitons

$$\vec{J}_b = \nabla \times \vec{M}$$

Follows $\nabla \cdot \vec{J} = 0$ ✓

Magnetic Field Inside Matter

The microscopic field is tiny and irrelevant

we focus on the macroscopic ($\vec{M}$ smooth)

Auxiliary Field H

Ampère's Law in Magnetized Materials

We know the bound current effects,

now : $$\vec{J} = \underbrace{\vec{J}_b}_{\text{bound current}} + \underbrace{\vec{J}_f}_{\text{free current}}$$

bound current [Because of magnetization]

free current [actual transport of charge]

Therefore :

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J} = \vec{J}_b + \vec{J}_f = \vec{J}_f + (\nabla \times \vec{M})$$

or :

$$\nabla \times \underbrace{\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)} = \vec{J}_f$$

$$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

which now is :

$$\nabla \times \vec{H} = \vec{J}_f \qquad \text{or} \qquad \oint \vec{H} \cdot d\vec{l} = I_{f_{enc}}$$

Use similar to Amperian loops

$\vec{H}$ is analogous to $\vec{D}$

$\vec{H}$ lets us write Ampere's Law in terms of free current alone (which we control)