Optics

Eugene Hetch

1 Brief History

what is light ?

2 Wave Motion

Particle ≠ Classical Particle (Localization)

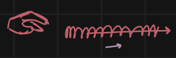
Particles interact via fields

A classical travelling wave is a self-sustained disturbance of a medium, which moves through space transporting energy and momentum.

Real waves are vast number of particles moving in concert.
Not continuous by themselves! (not even electromagnetic)

One Dimensional Waves

Longitudinal



Medium is displaced in the direction of the wave

Transverse

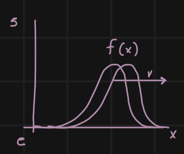
Medium is displaced perpendicular to the motion of the wave

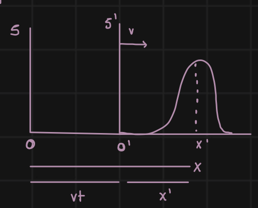
The disturbance advances, not the medium!

Disturbance $\overbrace{\Psi(x,t)}^{\text{wave function}} = \underbrace{f(x,t)}_{\text{wave shape}}$

By holding time constant, we get the profile:
$\Psi(x,t)|_{t=0} = f(x,0) = f(x)$ (photo of disturbance)

Easiest moving wave
Does not change its shape as it moves





$x' = x - vt$

$\Psi(x,t) = \overbrace{f}^{\text{shape}}\underbrace{(x - vt)}_{\text{moving across x axis}}$

It is 1 dimensional because the wave sweeps over points lying on a line

Example

$f(x) = ae^{-x^2}$ Gaussian Curve

$\Psi(x,t) = ae^{-(t - \frac{x}{v})^2}$

The Differential Wave Equation

Waves may take different forms, but all wave functions are solutions of the same differential wave equation.

We may get the 1D form of such wave eq (comparing to the info before):

2 independent variables

$$x' = x - vt$$

$$\psi = f(x,t) \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x'} \quad (1)$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial t} \Rightarrow \frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} \Rightarrow \frac{\partial \psi}{\partial t} = -v\frac{\partial f}{\partial x'}$$

$$\left.\right\} \frac{\partial \psi}{\partial t} = \pm v \frac{\partial \psi}{\partial x}$$

Similarly ... $$\underbrace{\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}_{\text{Wave equation for undamped systems}}$$

Harmonic Waves

Simplest wave form: profile is sine / cosine.

$$\psi(x,t)\big|_{t=0} = \psi(x) = A \sin kx = f(x)$$

A: max value, amplitude; k: propagation number (to kill units) $\left[\frac{rad}{m}\right]$

Transform into progressive wave: $\psi(x,t) = A \sin k(x-vt) = f(x-vt)$

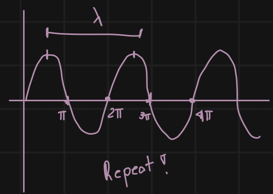
solution to DWE !

Wave length: Number of units of length per wave (λ)

$$\psi(x,t) = \psi(x \pm \lambda, t)$$

$$\sin[k(x-vt)] = \sin(k(x\pm\lambda) - vt) = \sin[\underbrace{k(x-vt)}_{\text{phase } (\varphi)} \pm 2\pi]$$

$$k\lambda = 2\pi \Rightarrow \underline{k = \frac{2\pi}{\lambda}}$$



period: Number of units of time per wave (τ)

$$\psi(x,t) = \psi(x, t \pm \tau)$$

$$\sin[k(x-vt)] = \sin[k(x - v(t\pm\tau)] = \sin[k(x-vt) \pm 2\pi]$$

$$\underline{kv\tau = 2\pi}$$

$$\frac{2\pi}{\lambda} v\tau = 2\pi \Rightarrow \tau = \frac{\lambda}{v}$$

Amount of time for a complete wave to pass a stationary observer

frecuency: Number of waves per units of time

$$f = \frac{1}{\tau}$$

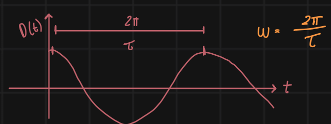
$$\lambda f = v$$

λ: lenght of wave; f: no. waves per sec; v: rate $\frac{m}{s}$

[Principia: To Find Velocity of Waves

angular frecuency (propagation num): how many radians per unit time

$$\omega = \frac{2\pi}{\tau} = 2\pi f$$



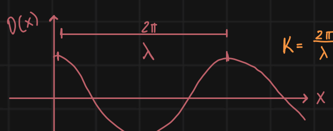
$$\omega = \frac{2\pi}{\tau}$$

$$D(t) = D\sin(\omega t + \phi) = D\sin(2\pi\tfrac{t}{\tau} + \phi)$$

Fraction of 2π

Wave number: how many radians per unit distance

$$k = \frac{2\pi}{\lambda}$$



$$k = \frac{2\pi}{\lambda}$$

$$D(x) = D\sin(kx + \phi) = D\sin(2\pi\tfrac{x}{\lambda} + \phi)$$

Fraction of 2π

This is a perfect wave $(-\infty$ to $\infty)$ "chromatic", it doesn't exist

"quasi chromatic" are very narrow almost perfect waves $(\cancel{\infty} - \infty)$

Phase and Phase Velocity

$\varphi = (kx - \omega t)$ $\qquad \varphi|_{t,x=0} = 0$ is only a special case

$$\psi(x,t) = A\sin(kx - \omega t)$$

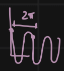
therefore

$$\psi(x,t) = A\sin(kx - \omega t + \underbrace{\varepsilon}_{\text{Initial phase}})$$

Surfaces of constant phase $kx - \omega t + \varepsilon = cte$:

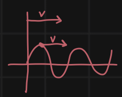
phase disturbance : $\varphi = kx - \omega t + \varepsilon$

Rate of phase with time $\left|\left(\frac{\partial\varphi}{\partial t}\right)_x\right| = \omega$ $\qquad$ For each cycle, φ changes ω



Rate of phase with distance $\left|\left(\frac{\partial\varphi}{\partial x}\right)_t\right| = k$

Speed of propagation of the condition of constant phase $\qquad \left(\frac{\partial x}{\partial t}\right)_\varphi = \frac{-\left(\frac{\partial\varphi}{\partial t}\right)_x}{\left(\frac{\partial\varphi}{\partial x}\right)_t}$



Phase velocity

$$\left(\frac{\partial x}{\partial t}\right)_\varphi = \pm\frac{\omega}{k} = \pm v$$

$$\frac{\omega}{k} = \frac{\frac{2\pi}{\tau}}{\frac{2\pi}{\lambda}} = \frac{\lambda}{\tau} = \lambda f = v$$

Since $\frac{\partial\varphi(x,t)}{\partial t} = 0$ and $\frac{\partial\varphi}{\partial t} = 0$ $\qquad$ because $\varphi = cte$

$$\pm v = \frac{-\left(\frac{\partial\varphi}{\partial t}\right)_x}{\left(\frac{\partial\varphi}{\partial x}\right)_t}$$

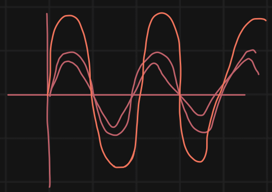
The Superposition Principle

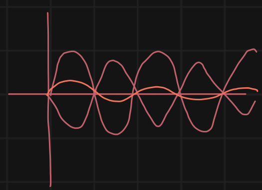
According to DWE, if ψ_1 and ψ_2 are both solutions, then $(\psi_1 + \psi_2)$ is also solution

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2\psi_2}{\partial t^2} \Rightarrow \frac{\partial^2}{\partial x^2}(\psi_1 + \psi_2) = \frac{1}{v^2}\frac{\partial^2}{\partial x^2}(\psi_1 + \psi_2)$$

When 2 separate waves arrive at the same place in space, the will add or substract without destroying the other wave !

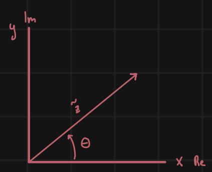
Interference





The Complex Representation

Sinusoidals are cringy, therfore we use $\qquad \tilde{z} = x + iy$



$\tilde{z} = x + iy = r(\cos\theta + i\sin\theta)$

Euler $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$\tilde{z} = re^{i\theta} = r\cos\theta + ir\sin\theta$

$|\tilde{z}| = r \qquad = (\tilde{z}\tilde{z}^*)^{1/2}$

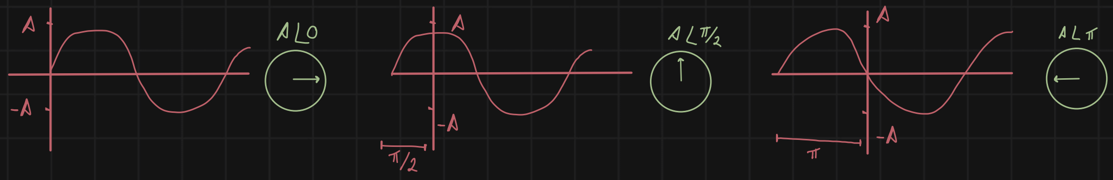
$\tilde{z}^* = (x + iy)^* = (x - iy)$

$$\psi(x,t) = Ae^{i(\omega t - kx + \varepsilon)} = Ae^{i\varphi}$$

where $\mathrm{Re}(Ae^{i\varphi})$ is the real wave

Be careful since we only want the real part, we are restricted on operations with real quantities.

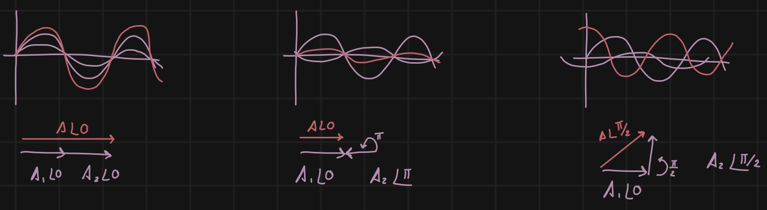
Phasors and Addition of Waves



Amplitude ∠ phase

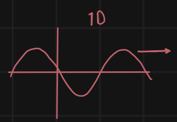
If you have 2 waves:

phasors tip to tail

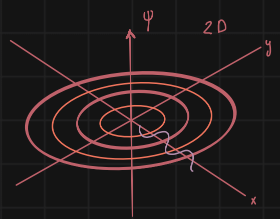


Since both rotate at ω, we can freeze at $t=0$

Plane Waves



It is 1 dimensional because the wave sweeps over points lying on a line



Wavefront: At any instant, a surface of constant phase.

Plane wave is the simplest 3D wave.

Mathematical derivation of plane waves:

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\qquad (\vec{r}-\vec{r}_0) = (x-x_0)\hat{\imath} + (y-y_0)\hat{\jmath} + (z-z_0)\hat{k}$

we set $\underline{(\vec{r}-\vec{r}_0)\cdot\vec{k} = 0}$ we get a plane.

plane is perpendicular to $\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k}$

the eq. becomes
$$k_x(x-x_0) + k_y(y-y_0) + k_z(z-z_0) = 0$$
$$k_x x + k_y y + k_z z = a$$
$$\vec{k}\cdot\vec{r} = a$$

Plane represents all points whose position vector have the same projection onto $\vec{k}$

Now we construct $\boxed{\psi(\vec{r}) = A\sin(\vec{k}\cdot\vec{r}) = A\cos(\vec{k}\cdot\vec{r}) = Ae^{i\vec{k}\cdot\vec{r}}}$

Homogeneous wave: Same strength planar waves

Spatial repetitive nature:
$$\psi(\vec{r}) = \psi\left(\vec{r} + \frac{\lambda\vec{k}}{k}\right)$$
$$Ae^{i\vec{k}\cdot\vec{r}} = Ae^{i\vec{k}\cdot(\vec{r}+\frac{\lambda\vec{k}}{k})} = Ae^{i\vec{k}\cdot\vec{r}}e^{i\lambda k}$$

To be true $e^{i\lambda k} = 1 = e^{i2\pi}$

so $\lambda k = 2\pi$ or $k = \frac{2\pi}{\lambda}$

Propagation vector: $\vec{k}$ with magnitude k (propagation number)

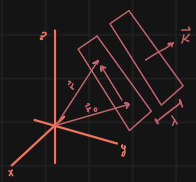
It is motionless, so we add time $\boxed{\Psi(\vec{r},t) = A e^{i(\vec{k}\cdot\vec{r} \mp \omega t)} \qquad \vec{k}\cdot\vec{r} = \frac{2\pi}{\lambda}\hat{k}\cdot\vec{r}}$

In cartesian coordinates $\Psi(x,y,z,t) = A e^{i(k_x x + k_y y + k_z z \mp \omega t)} = A e^{i[k(\alpha x + \beta y + \gamma z) \mp \omega t]}$

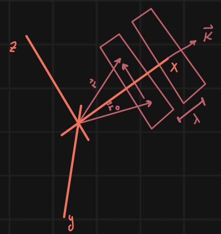
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

Surface joining all points of equal phase is the wave front

The velocity of a plane wave is equivalent to the propagation velocity of the wavefront



Aligned to x axis

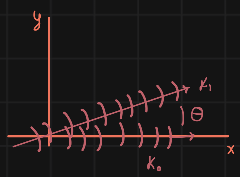


now $\vec{k}\cdot\vec{r} = kr_x = kx$

$\Psi(x,t) = A e^{i(kx \mp \omega t)}$ follows the 1D disturbance

$$\pm\frac{\omega}{k} = \pm v$$

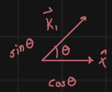
Example:



$$\xrightarrow[\vec{k}_0]{\hat{x}}$$

$$\psi_0 = A_0 \cos\left(\frac{2\pi}{\lambda}x - \omega t\right)$$

$$\psi_1 = A_1 \cos\left(\frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta) - \omega t\right)$$



example

$$\vec{E} = \left(100\ \frac{V}{m}\right)\hat{j}\ e^{i(kz+\omega t)}$$

a) $A = 100$

b) Wave travels to
$\vec{k}\cdot\vec{r} = kz$ parallel to z : $-\hat{z}$
$(kz + \omega t)$ negative dir

c) direction of $\vec{E}$
$\pm\hat{j}$
(time dependent)

d) $v = \frac{\omega}{k} = \frac{\lambda}{\tau} = \lambda f$
$f = \frac{v}{\lambda} = \frac{3E8}{500E-9}$

Any 3D wave can be expressed as a combo of plane waves

3D Differential Wave Equation

Waves with unchanging profile are special, we must find a general recipe for 3D waves:

We use $\psi = A e^{i[k(\alpha x + \beta y + \gamma z) \mp \omega t]}$

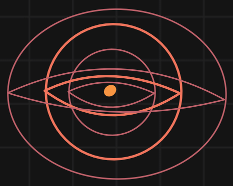
$$\frac{\partial^2\psi}{\partial x^2} = -\alpha^2 k^2 \psi \qquad \frac{\partial^2\psi}{\partial y^2} = -\beta^2 k^2 \psi \qquad \frac{\partial^2\psi}{\partial z^2} = -\gamma k^2 \psi \qquad \frac{\partial^2\psi}{\partial t^2} = -\omega^2 \psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y^2} + \frac{\partial\psi}{\partial z^2} = -k^2\psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial y^2} + \frac{\partial\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Spherical Waves



3D Laplacian

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

we only want spheres

i.e. $\psi(\vec{r}) = \psi(r, \theta, \phi) = \psi(r)$

therefore

$$\nabla^2 \psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right)$$

Which can also be written as:

$$\nabla^2 \psi(r) = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \underbrace{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\psi) \qquad = }_{\text{3D wave eq.}} \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$\frac{\partial^2}{\partial r^2}(r\,\psi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\,\psi) \quad \Bigg\} \text{ 1D wave eq. !}$$

The solution is

$$r\,\psi(r,t) = f(r - vt)$$

$$\psi(r,t) = \frac{f(r - vt)}{r}$$

Also has superposition

A special solution is

$$\psi(r,t) = C_1 \frac{f(r - vt)}{r} + C_2 \frac{g(r - vt)}{r}$$

is harmonic spherical wave

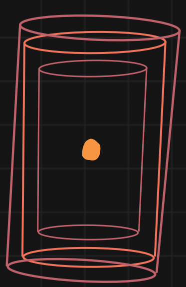
$$\psi(r,t) = \left(\frac{\mathcal{A}}{r}\right)\cos[k(r \mp vt)] = \left(\frac{\mathcal{A}}{r}\right) e^{ik(r \mp vt)}$$

$\mathcal{A}$ = source strength

Very far from the source a spherical wave will resemble a plane wave.

Cylindrical Waves

Infinite circular cylinder



Laplacian cylindrical coordinates

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

cylindrical symetry (circles, infinite in z)

$$\psi(\vec{r}) = \psi(r, \theta, z) = \psi(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \Bigg\} \text{ Bessel eq. (after some math)}$$

With solution

$$\psi(r,t) \approx \frac{\mathcal{A}}{\sqrt{r}} e^{ik(r \mp vt)} = \frac{\mathcal{A}}{\sqrt{r}}\cos[k(r \pm vt)]$$

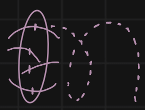
Twisted Light

You can twist light.

light
(not in the center)

Math is kinda weird, but in general they have a phase: $e^{-il\phi}$

basically the crests and trough occur at different angles



A cilyndrical wave with peaks lying on a spiral

3 Electromagnetic Theory, Photons and Light

All particles have wave properties and vice versa.

The quantum (discrete) nature of light can be sometimes ignored and be seen as simply an electromagnetic wave, even if it is wrong.

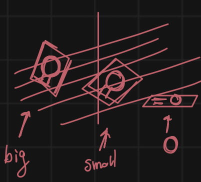
Basic Laws of Electromagnetic Theory

$$\vec{F}_m = q_0 \vec{E} + q_0 \vec{v} \times \vec{B}$$ } Depend both on $\vec{E}$ and $\vec{B}$

Faraday's Induction Law

Faraday is crazy OP and found out that when magnetic field is constant, the induced emf is proportional to rate of change of the perpendicular area

i.e.



the voltage between 2 terminals (on constant $\vec{B}$) depends on how fast $\vec{B}$ is moving and the area directly perpendicular

basically

$A_\perp = cte \qquad emf \propto A_\perp \frac{\Delta B}{\Delta t}$

$B = cte \qquad emf \propto B \frac{\Delta A_\perp}{\Delta t}$

This suggests flux of magnetic field

$$\Phi = B_\perp A = B A_\perp = BA\cos\theta$$

$\vec{B}$ varies in space

$$\Phi_m = \iint_A \vec{B} \cdot \underbrace{d\vec{s}}_{\text{outwar perpendicular to surface}}$$

the induced emf is then

$$\text{emf} = -\frac{d\Phi_M}{dt}$$

induced current   against   current flux

An emf is potential difference, which corresponds to work per unit charge, or $E \times$ dis

$$\text{emf} = \oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_A \vec{B}\cdot d\vec{s}$$

The Induction Law

$$\oint_C \vec{E}\cdot d\vec{l} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

A time varying magnetic field will have an associated electric field.

Gauss's Law - Electric

Relationship between flux and its sources

$$\oiint_A \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\iiint_V \rho \, dv$$

Kinda like water

and the

$E 4\pi r^2 = \Phi_E = \frac{q_\circ}{\varepsilon_0}$

ε_0 is for free space only

ε: How much field the medium will permit.

To avoid wasting time on different ε, we use

$$K_E = \frac{\varepsilon}{\varepsilon_0} \quad \Rightarrow \quad \varepsilon = K_E\,\varepsilon_0$$

Gauss's law - Magnetic

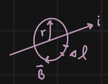
No monopoles, no sinks nor sources:

$$\Phi = \oiint_A \vec{B}\cdot d\vec{s} = 0$$

Ampère's Circuital Law

Initially thought as

along a circle



$\text{Force}_m = q_m B_{\parallel}$ ← imagine !

$\text{Work}_m = q_m B_{\parallel}(\Sigma \Delta l)$

$= q_m B_{wire} (2\pi r)_{circle} = q_w \frac{\mu_0 i}{\cancel{2\pi r}}(\cancel{2\pi r}) = \mu_0 i \cancel{q_m}$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \iint_A \vec{J}\cdot d\vec{s}$$

μ_0: Permeability of space

$\mu_0 = K_M \mu_0$

it's not complete !

$E = \frac{Q}{\varepsilon A} \Rightarrow \varepsilon\frac{\partial E}{\partial t} = \frac{i}{A}$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \iint_A \left(\vec{J} + \varepsilon\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{s}$$

Even if $\vec{J} = 0$, a $\vec{E}$ will have a $\vec{B}$

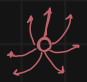
Maxwell Equations

$\varepsilon = \varepsilon_0 \qquad \mu = \mu_0 \qquad$ No charges, no currents: $\rho = 0, \ \vec{J} = 0.$

$$\boxed{\oint_C \vec{E} \cdot d\vec{l} = -\iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \qquad \oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t} \cdot d\vec{s}}$$

$$\boxed{\oiint_A \vec{B} \cdot d\vec{s} = 0 \qquad \oiint \vec{E} \cdot d\vec{s} = 0}$$

Interesting way to visualize divergence

+ div  If the flow is greater away than towards

A non zero divergence occurs only where there are charges !

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{E} \cdot d\vec{s} = \nabla \cdot \vec{E}$$

Take a point, surround it with small area and volume, divide by volume and shrink to a tiny spot.

$$\left(\oiint_A \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dv \right) \frac{1}{\Delta V} = \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad ; \quad \nabla \cdot \vec{B} = 0}$$

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_C \vec{E} \cdot d\vec{l} = \nabla \times \vec{E}$$

circulation per unit area

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

All the formular correspond to 8 differential eq. (in cartesian)

Electromagnetic Waves

Time varying $\vec{E}$ produces a perpendicular $\vec{B}$ and the other way around.

$\vec{E}$ and $\vec{B}$ are aspects of electromagnetic field (moving charges)

$$\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t} \qquad \nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{B}}{\partial t}$$

} Concise form of already known eq.

6 D.E. in cartesian

which obey the scalar wave D.E. if

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

which in free space is c !

Transverse Waves

The electromagnetic wave has no $\vec{E}$ in the direction of propagation: it is transverse

for a planar wave only to $+\hat{x}$, where $\vec{E} = \vec{E}(x,t)$

Gauss' Law

$$\frac{\partial E_x}{\partial x} + \cancel{\frac{\partial E_y}{\partial y}} + \cancel{\frac{\partial E_z}{\partial z}} = 0 \qquad \frac{\partial E_x}{\partial x} = 0$$

No component in direction of propagation

To describe the moment by moment direction, we use polarization.

$$\vec{E} = E_y(x,t)\,\hat{j}$$

Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$\vec{B}$ only has component in the $\hat{z}$ direction

Waves are almost always transverse !

Now, continuing the example for a harmonic wave

We know $\vec{E}$ only goes to $\hat{j}$

$$E_y(x,t) = E_{0y}\cos\left(\omega\left(t - \frac{x}{c}\right) + \varepsilon\right)$$

and, accordingly, the magnetic flux is

$$B_z(x,t) = -\int \frac{\partial E_y}{\partial x}\,dt = \frac{1}{c}E_{0y}\cos\left(\omega\left(t - \frac{x}{c}\right) + \varepsilon\right) = \frac{1}{c}E_y$$

$$E_y = cB_z \quad !$$

On ordinary materials then

$$E = vB \qquad \text{where} \qquad v = \frac{1}{\sqrt{\mu\varepsilon}}$$

Energy and Momentum

The Poynting Vector

We can prove that the energy density (radiant energy per unit volume)

for electric field $\qquad u_E = \frac{\varepsilon_0}{2}E^2$

for magnetic field $\qquad u_B = \frac{1}{2\mu_0}B^2$

since $E = cB$

$$u_E = u_B$$

$$u = u_E + u_B$$

The energy through space of an electromagnetic wave is shared equally between both magnetic and electric fields !

$$u = \varepsilon_0 E^2 = \frac{1}{\mu_0}B^2$$

Now we want the transport of energy per unit area (W/m^2)

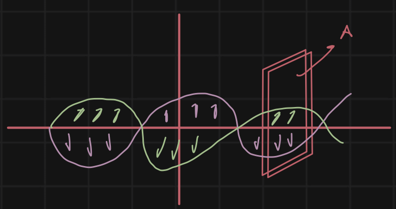
$$S = uc = \frac{1}{\mu_0}EB$$

Energy flows in the direction of propagation

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = c^2\varepsilon_0\,\vec{E}\times\vec{B}$$

Poynting Vector

Energy passing through a unit of area in 1 sec.



For a plane wave

squared

$$\vec{S} = c^2 \varepsilon_0 \vec{E}_0 \times \vec{B}_0 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

Averaging Harmonic Functions

changes very quickly from maxima to minima

So, we take the time average value over some T

$$\langle f(t) \rangle_T = \frac{1}{T} \int_{t-T/2}^{t+T/2} f(t)\, dt$$

Average of harmonic motion

$$\langle e^{i\omega t} \rangle_T = \left(\frac{\sin(\omega T/2)}{(\omega T/2)} \right) e^{i\omega t} = \operatorname{sinc}\left(\frac{\omega T}{2}\right) e^{i\omega t}$$

$$\left[\frac{\sin u}{u} = \operatorname{sinc} u \right.$$

With real and imaginary parts

Im: $\langle \sin \omega t \rangle_T = (\operatorname{sinc} u) \sin \omega t$

Re: $\langle \cos \omega t \rangle_T = (\operatorname{sinc} u) \cos \omega t$

The average of cosine is itself a cosine with amplitude sinc u. (amplitude drops to 0 fast)

Irradiance

Amount of light illuminating a surface

[Average energy per unit area per unit time]

We know the energy per area per time (Poynting), we take its average over T

$$\langle S \rangle_T = c^2 \varepsilon_0 |\vec{E}_0 \times \vec{B}_0| \underbrace{\langle \cos^2(\vec{k}\cdot\vec{r} - \omega t) \rangle}_{\frac{1}{2}} \quad [\text{for } T \ll \tau]$$

$$I \equiv \langle S \rangle_T = \frac{c\varepsilon_0}{2} E_0^2$$

Irradiance is proportional to the square amplitude of $\vec{E}$!

We can also write

$$I = \frac{c}{\mu_0} \langle B^2 \rangle_T = \varepsilon_0 c \langle E^2 \rangle_T$$

$$I = \varepsilon v \langle E^2 \rangle_T$$

Inverse Square Law

The irradiance from a point source is proportional to $1/r^2$