πLAY

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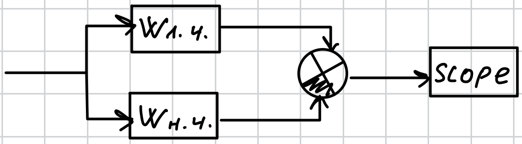
1. Попов ТАУ Нелинейные системы
2. Воронов ТАУ т.2
3. Ким Д.П. ТАУ т.2

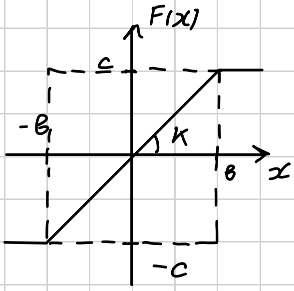
# Лекция №1

## Нелинейные системы

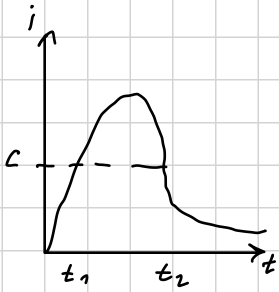
Нелинейная система – система, которая имеет хотя бы 1 нелинейный элемент

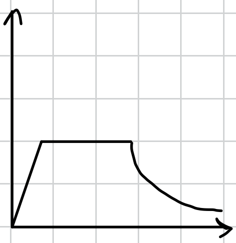



Если ошибка мала, то можно работать с линейной, иначе – с нелинейной




$$F(x) = \begin{cases} x > b: F(x) = c \\ x < b: F(x) = -c \\ x \in [-b; b]: F(x) = k \cdot x \end{cases}$$

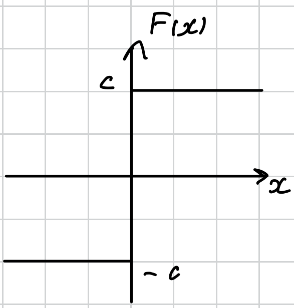


Нелинейности

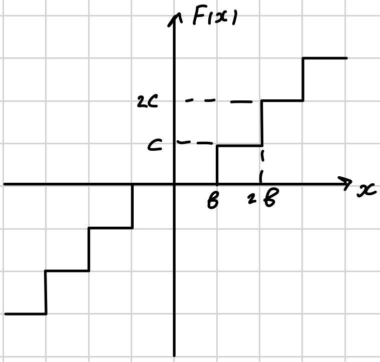
Статические
если есть статические нелинейность

Динамические
если есть динамические нелинейности
(вызваны Д нелинейными У, описывающими поведение системы)



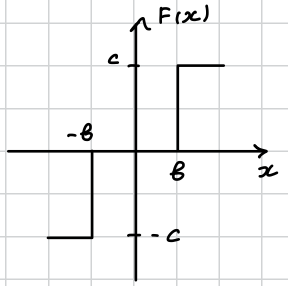

$$F(x) = \begin{cases} c, & x \geq 0 \\ -c, & x < 0 \end{cases}$$

Нелинейности бывают симметричные и не симметричные

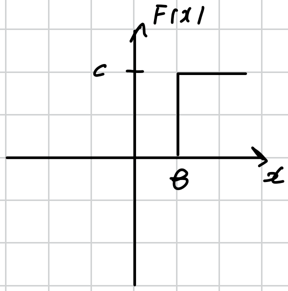



округление до целого

$$F(x) = round\left(\frac{x}{b}\right)\cdot c$$



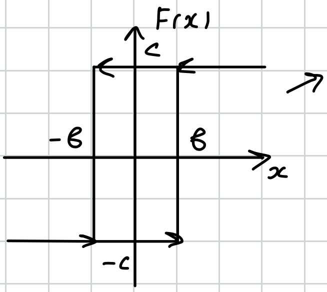

$$F(x) = \begin{cases} 0, & x \in [-b; b] \\ c, & x > b \\ -c, & x < -b \end{cases}$$




$$F(x) = \begin{cases} c, & x > b \\ 0, & x \le b \end{cases}$$

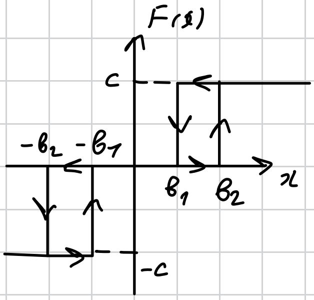
Нелинейность

Однозначная　　　Неоднозначная

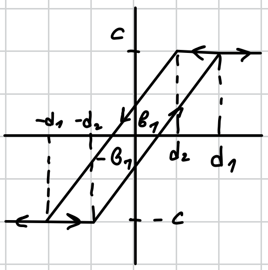



Неоднозначная

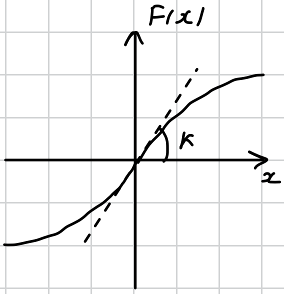
$$F(x) = \begin{cases} c, & x > b;\ x \in [-b; b],\ \dot{x} < 0 \\ -c, & x < -b;\ x \in [-b; b],\ \dot{x} > 0 \end{cases}$$



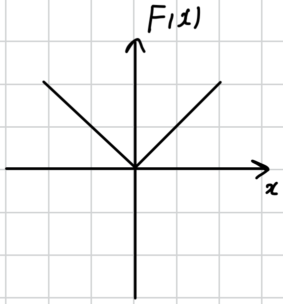

$$F(x) = \begin{cases} c, & x > b_2;\ x \in [b_1; b_2],\ \dot{x} < 0 \\ 0, & x \in [b_1; b_2],\ \dot{x} > 0;\ x \in [-b_1; b_1];\ x \in [-b_2; -b_1],\ \dot{x} < 0 \\ -c, & x < -b_2;\ x \in [-b_2; -b_1],\ \dot{x} > 0 \end{cases}$$



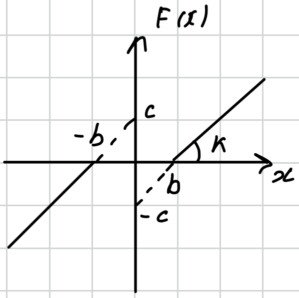

$$F(x) = \begin{cases} c, & x > d_1;\ x \in [d_2; d_1],\ \dot{x} < 0 \\ \frac{c}{d_2 + b_1}x + \frac{c\,b_1}{d_2 + b_1}, & x \in [-d_1, d_2],\ \dot{x} < 0 \\ -c, & x < d_1;\ x \in [-d_1; -d_2],\ \dot{x} > 0 \\ \frac{c}{d_2 + b_1}x - \frac{c\,b_1}{d_2 + b_1}, & x \in [-d_2; d_1],\ \dot{x} > 0 \end{cases}$$



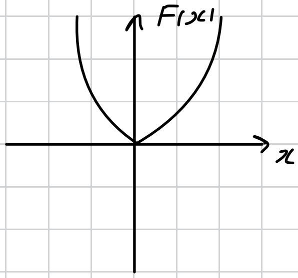
$F(x)=kx,\ x\to 0$



$F(x)=|x|$

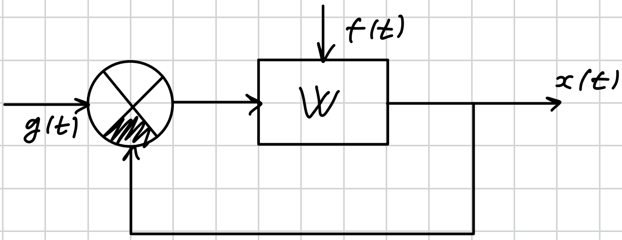


$$F(x)=\begin{cases}0, & x\in[-b;b]\\ kx-c, & x>b\\ kx+c, & x<-b\end{cases}$$



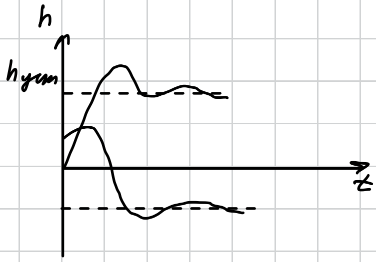
$F(x)=kx^2$

## Принципиальные отличия линейных и нелинейных систем




$x(t)=x_{пер}(t)+x_{уст}(t)$

переходная




$x^{(n)}+a_{n-1}x^{(n-1)}+\ldots+a_1\dot{x}+x(x+5)=-6g(t)$

$g(t)=1(t)$

$x^2+5x=-6$

$x^2+5x+6=0$

$x_1=-3,\ x_2=-2$

$D(p)x(t)=B(p)g(t)+M(p)f(t)$

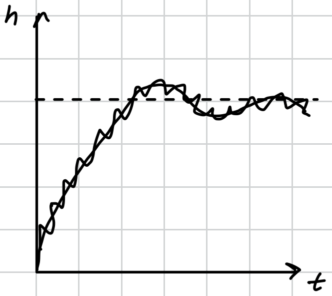
**Для нелинейных систем принцип суперпозиции не работает**

$x_1(t)$ при $f(t)=0$

$x_2(t)$ при $f(t)=0$

$x(t)=x_1(t)+x_2(t)$

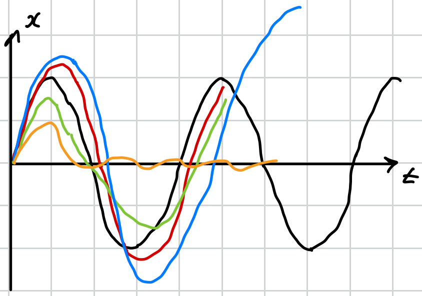
В нелинейных системах возможно возникновение автоколебаний




$x_1(t) = k e^{-\frac{t}{T}}$

$x_2(t) = A \sin(wt+\varphi)$

Возникают вибрации



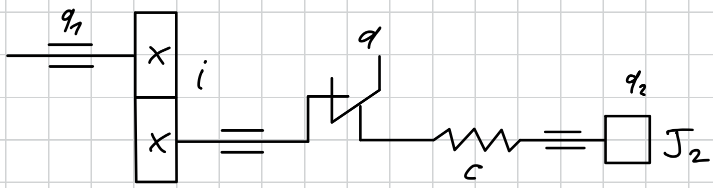

а) $x(t)$
б) при $\Delta x > 0$ —
в) при $\Delta x < 0$ — } устойчивое состояние
г) $\Delta x > 0$ —
д) $\Delta x < 0$ — } неуст.

## Устойчивость

Устойчивость нелинейных систем зависит от начальных условий

1. Устойчивость неасимптотическая $|x| < \delta_{доп}$
2. Устойчивость асимптотическая $|x| \to 0$
3. Устойчивость на конечном интервале времени
4. Устойчивость в целом (2 при любых н.у.)
5. Абсолютная устойчивость (4 + заданный класс нелинейности)

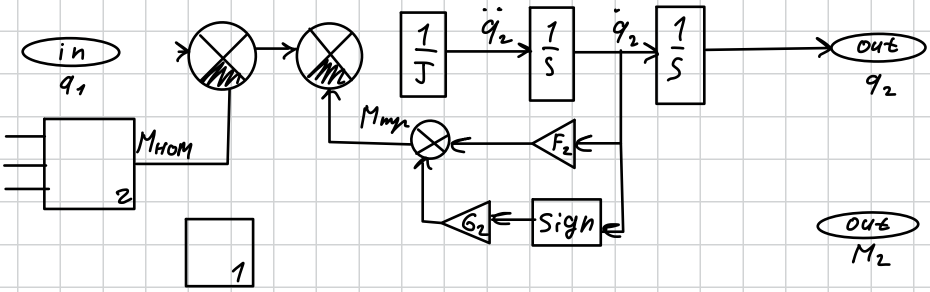
## Структурная схема редуктора




$$\Theta = \frac{q_1}{i - q_2}$$

$$M_{упр} = f(\Theta)\cdot c + (F_1\cdot\Theta' + G_1)\cdot f_1(\Theta)$$
$$M_{тр} = F_2\cdot q_2' + G_2\cdot \mathrm{sign}(\dot q_2)$$
$$M_{нат} = G\cdot|M_{упр} - M_{тр}|\cdot \mathrm{sign}(\dot q_2)$$
$$M_{упр} - M_{тр} - M_{нат} = J_2\cdot\ddot q_2$$

$$f(\Theta) = \begin{cases} \Theta - a, & \Theta > a \\ \Theta + a, & \Theta < -a \\ 0, & |\Theta| < a \end{cases}; \quad f_1(\Theta) = \dot f(\Theta) = \begin{cases} 1, & |\Theta| \geq a \\ 0, & |\Theta| < a \end{cases}$$

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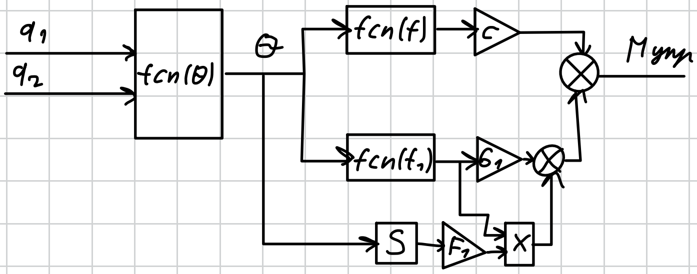
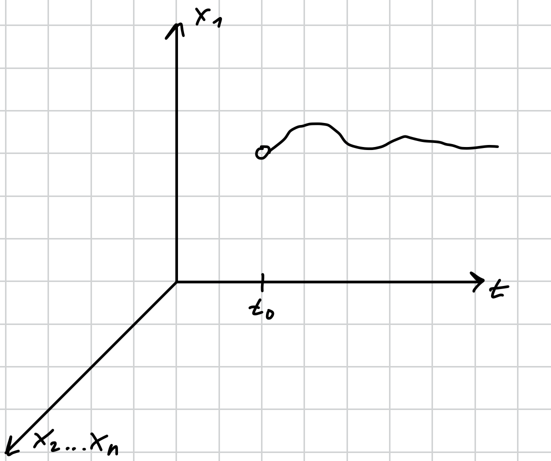



## Фазовые пространства и фазовые портреты САУ

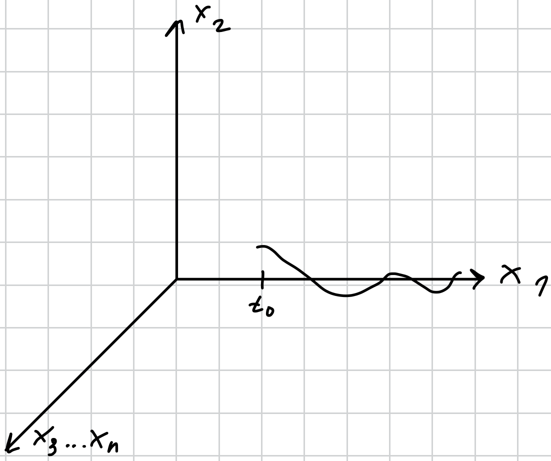
## Фазовые пространства линейных систем

$$X = \Phi(X, t)$$

$$X = \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}$$

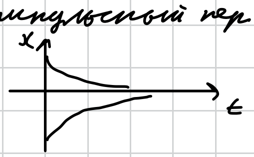
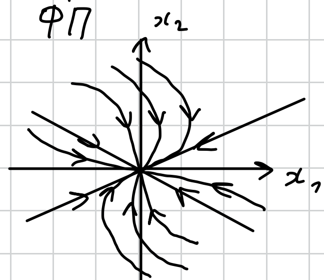
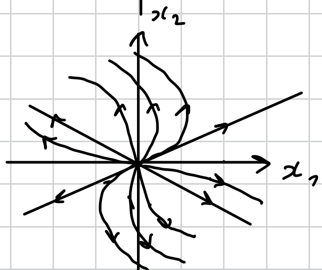
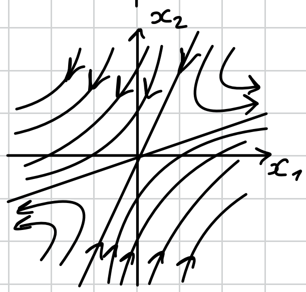
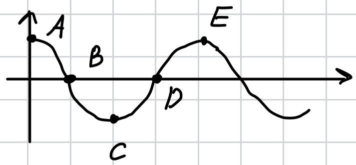
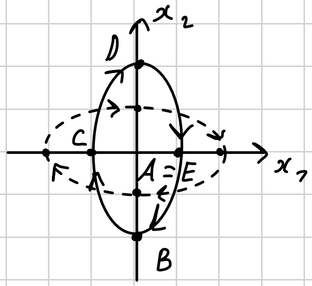
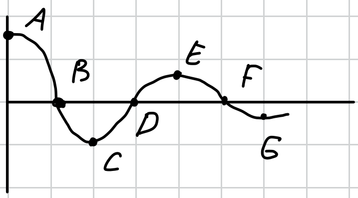
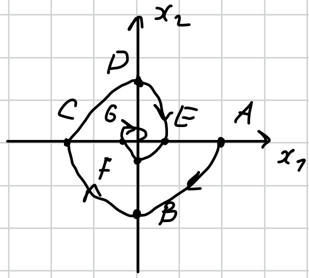
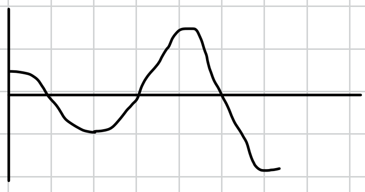
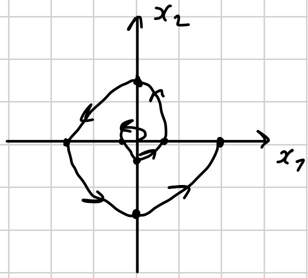



$$\dot{X} = \Phi(x)$$




$$a_0 \ddot{x} + a_1 \dot{x} + a_2 x = 0$$

$\lambda_1=\alpha_1+j\beta$, $\lambda_2=\alpha_2-j\beta$

| Корни | Импульсный пер. процесс | фазовый портрет ↑ ФП | |
|---|---|---|---|
| 1) $\beta=0$, $\alpha_1<0$, $\alpha_2<0$ |  |  | Устойчивый узел |
| 2) $\beta=0$, $\alpha_1>0$, $\alpha_2>0$ |  |  | Неустойчивый узел |
| 3) $\beta=0$, $\alpha_1>0$, $\alpha_2<0$; $x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$ при $C_1=0$ | |  | Седло |
| 4) $\alpha_1=\alpha_2=0$, $\beta\neq0$ |  |  | Особая точка типа центр |
| 5) $\beta\neq0$, $\alpha_1<0$, $\alpha_2<0$ |  |  | Устойчивый фокус |
| 6) $\beta\neq0$, $\alpha_1>0$ |  |  | Неустойчивый фокус |

## Методика построения фазовых траекторий для динамических нелин. систем

1. Ищем точки равновесного состояния, решая систему

$$\begin{cases} \dfrac{dx_1}{dt}=\Phi_1(x_1,x_2)=0 \\ \dfrac{dx_2}{dt}=\Phi_2(x_1,x_2)=0 \end{cases}$$

2. В окр-ти каждой особой точки проводим линеаризацию и получаем систему

3. Определяем тип особой точки и строим приближ. ФП

4. Уточняем вид ФП с помощью изоклин и разделяем фазовую плоскость с помощью сепаратрис

Сепаратриса – это линия, которая разделяет области влияния каждой особой точки

Метод изоклин – позвол. постр. изокл – точки на плоскости с равным наклоном касательной $\left(\frac{dx_2}{dx_1} = c = const\right)$

Пример

$$\frac{dx_1}{dt} = -x_1(1+x_1^2) - 2x_2 = 0$$

$$\frac{dx_2}{dt} = x_1 + x_2 = 0$$

$$\begin{cases} x_1 = -x_2 \\ x_2(1+x_2^2) - 2x_2 = 0 \end{cases}$$

$$x_2 + x_2^3 - 2x_2 = 0$$

$$x_2^3 - x_2 = 0$$
$$x_2(x_2^2 - 1) = 0$$
$$x_2^{(1)} = 0, \quad x_2^{(2,3)} = \pm 1$$

$A(0;0)$, $B(1,-1)$, $C(-1;1)$

При $A(0;0)$: $$\frac{d(0+\Delta x_1)}{dt} = -(0+\Delta x_1)(1+\cancel{(0+\Delta x_1)^2}^{\,0}) - 2\cdot(0+\Delta x_2)$$

$$\frac{d(\Delta x_1)}{dt} = -\Delta x_1 - 2\Delta x_2$$

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \qquad A - \lambda E = \begin{pmatrix} -1-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$$

$$\det = -1 + \lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 = -1$$

$\lambda_{1,2} = \pm i$ – центр

При $B(1;-1)$: $$\frac{d(\Delta x_1)}{dt} = -(1+\Delta x_1)(1+(1+\Delta x_1)^2) - 2(-1+\Delta x_2)$$

$$\frac{d(\Delta x_1)}{dt} = -(1+\Delta x_1)(2+2\Delta x_1) + 2 - 2\Delta x_2$$

$$\frac{d(\Delta x_1)}{dt} = -2 - 4\Delta x_1 - \cancel{2\Delta x_1^2}^{\,0} + 2 - 2\Delta x_2$$

$$\frac{d(\Delta x_1)}{dt} = -4\Delta x_1 - 2\Delta x_2$$

$$\frac{d(\Delta x_2)}{dt} = 1 + \Delta x_1 - 1 + \Delta x_2 = \Delta x_1 + \Delta x_2$$

$$A = \begin{pmatrix} -4 & -2 \\ 1 & 1 \end{pmatrix} \qquad A - \lambda E = \begin{pmatrix} -4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$$

$$\det = -4 + 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 3\lambda - 2 = 0$$
$$D = 9 + 8 = 17$$

$$\lambda_{1,2} = -3 \pm \sqrt{17}$$ — седло

При $C(-1;1)$:

$$\frac{d(\Delta x_1)}{dt} = -(-1+\Delta x_1)(1 + (-1+\Delta x_1)^2) - 2(1 + \Delta x_2)$$

$$\frac{d(\Delta x_1)}{dt} = 2 + 4\Delta x_1 - 2\cancel{\Delta x_1^2}^{\,0} - 2 - 2\Delta x_2$$

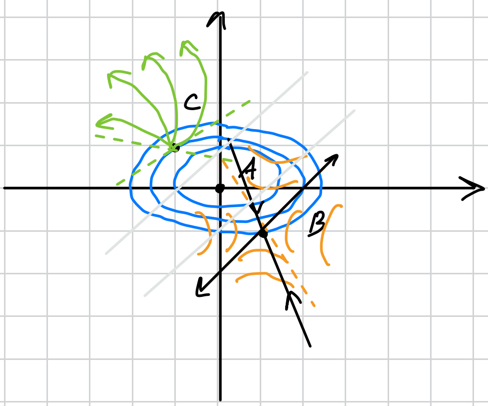
$$\frac{d(\Delta x_1)}{dt} = 4\Delta x_1 - 2\Delta x_2$$

$$\frac{d(\Delta x_2)}{dt} = -1 + \Delta x_1 + 1 + \Delta x_2 = \Delta x_1 + \Delta x_2$$

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \qquad A - \lambda E = \begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$$

$$\det = 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \qquad \lambda_1 = \frac{5-1}{2} = 2$$
$$D = 1 \qquad\qquad \lambda_2 = \frac{5+1}{2} = 3$$ $\rightarrow$ неуст. узел



$x_1 = c_2 e^{t}$

$\dot{x}_1 = c_2 e^{t} = x_1$

$x_2 = c_1 e^{-7t}$

$\dot{x}_2 = -7c_1 e^{-7t} = -7x_2$

$$\frac{dx_1}{dt} = -4x_1 - 2x_2$$

$$\frac{dx_2}{dt} = x_1 + x_2$$

$$\frac{dx_2}{dx_1} = \frac{x_1 + x_2}{-4x_1 - 2x_2}$$

## Лекция №2

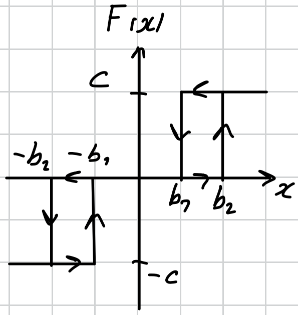
## Построение фазовых характеристик нелинейных систем, содержащих статическую нелинейность

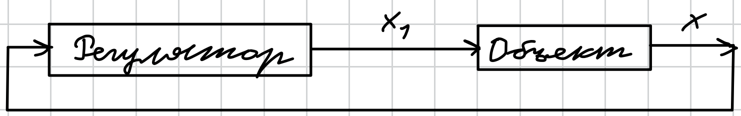
### Методика

1. Разбиение фазовой плоскости на части, где система ведёт себя как линейная

2. В каждой области построить фазовый портрет

3. Корректно соединить фазовые траектории на линии переключения



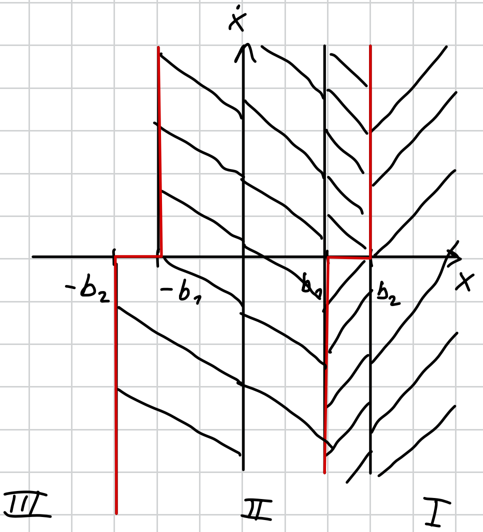




$(T_1 p + 1)x = -k_1 X_1$

Уравнение регулятора

$pX_1 = F(x)$

$$F(x) = \begin{cases} c, & x > b_2;\ x \in [b_1; b_2],\ \dot{x} < 0 \\ -c, & x < -b_2;\ x \in [-b_2; -b_1],\ \dot{x} > 0 \\ 0, & x \in [-b_2; -b_1],\ \dot{x} < 0;\ x \in [-b_1; b_1];\ x \in [b_1; b_2],\ \dot{x} > 0 \end{cases}$$




$T_1 \dot{x} + x = -K_1 x_1$

$T_1 \ddot{x} + \dot{x} = -K_1 \dot{x}_1$

$T_1 \ddot{x} + \dot{x} = -k_1 F(x)$

$F(x) = 0 \Rightarrow T_1 \ddot{x} + \dot{x} = 0 \Rightarrow \ddot{x} = -\frac{\dot{x} + k_1 F(x)}{T}$

$x_1 = x,\ x_2 = \dot{x}$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{x_2 + k_1 F(x)}{T} = -\frac{x_2}{T}$$

$$\frac{dx_1}{dx_2} = \frac{x_2}{-\frac{x_2}{T}} = -T$$

$$dx_1 = -T\,dx_2$$

$$x_1 = -T x_2 + C$$

$$x_2 = -\frac{x_1}{T} + C$$

При $F(x) = c$ : $T_1 \ddot{x} + \dot{x} = -Kc$

$x_1 = x$, $x_2 = \dot{x}$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dT} = -\frac{x_2 + Kc}{T}$$

$$\frac{dx_2}{dx_1} = \frac{-x_2 - Kc}{T x_2}$$

$$\frac{dx_2\, T x_2}{-x_2 - Kc} = dx_1$$

$$\left(\frac{x_2}{-x_2 - Kc} = -1 + \frac{Kc}{x_2 + Kc}\right)$$

$$-Tx_2 + TKc\ln|x_2 + Kc| + C = x_1$$

При $F(x) = -c$ : $T_1 \ddot{x} + \dot{x} = Kc$

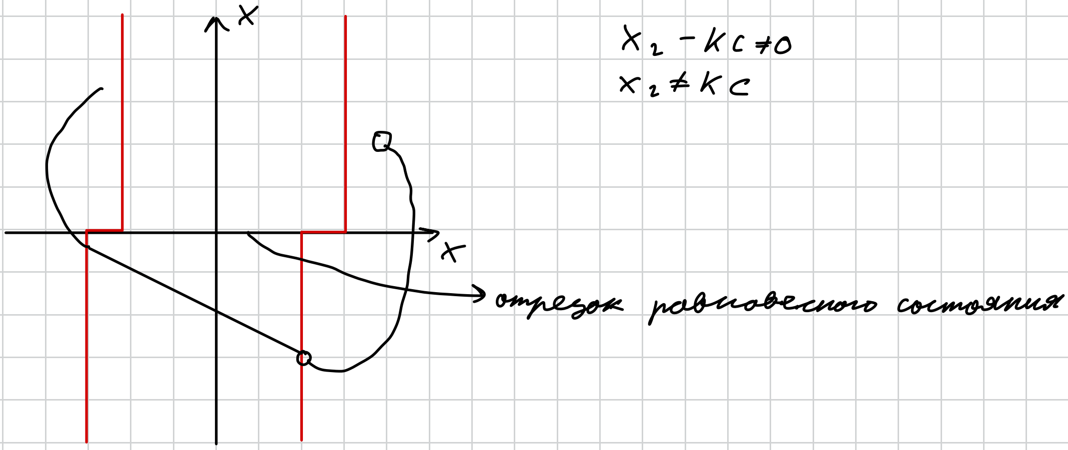
$x_1 = x$, $x_2 = \dot{x}$

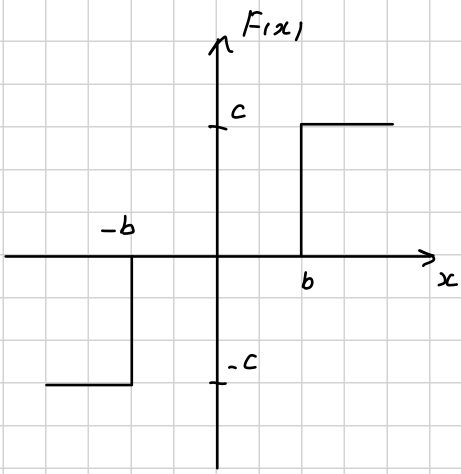
$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\frac{x_2 - Kc}{T}, \qquad \frac{dx_2}{dx_1} = \frac{-x_2 + Kc}{T x_2}$$

$$\frac{T x_2\, dx_2}{-x_2 + Kc} = dx_1 \Rightarrow \quad C - Tx_2 - TKc\ln|x_2 - Kc| = x_1$$

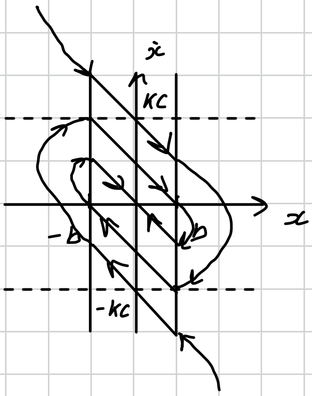
$x_2 - Kc \neq 0$

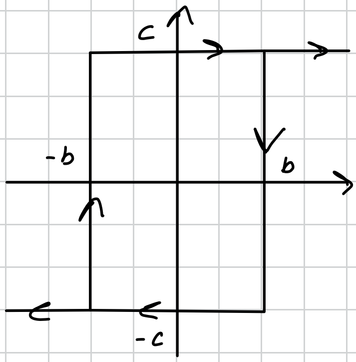
$x_2 \neq Kc$



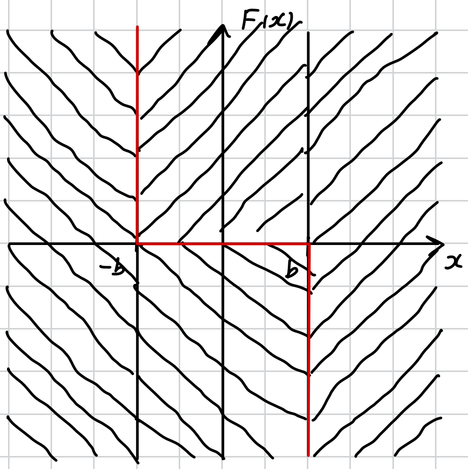



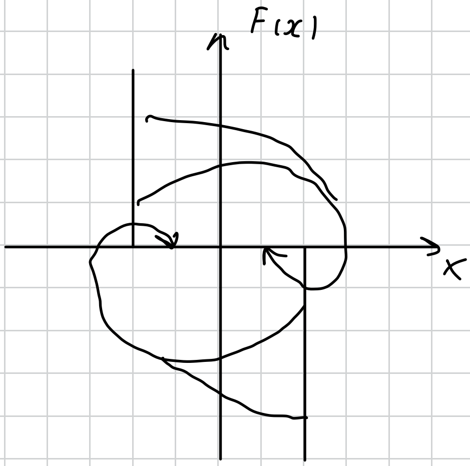

$$F(x) = \begin{cases} c, & x > b \\ 0, & x \in [-b; b] \\ -c, & x < -b \end{cases}$$

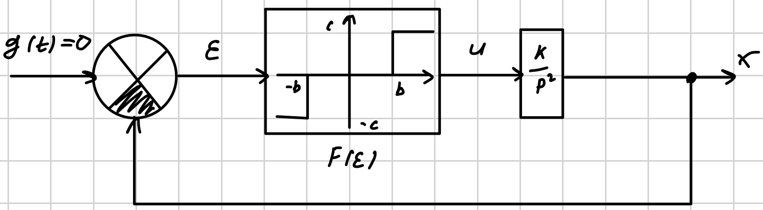



$$F(x)=\begin{cases} c, & x>b;\ x\in[-b;b],\ \dot{x}>0 \\ -c, & x<-b;\ x\in[-b;b],\ \dot{x}<0 \end{cases}$$



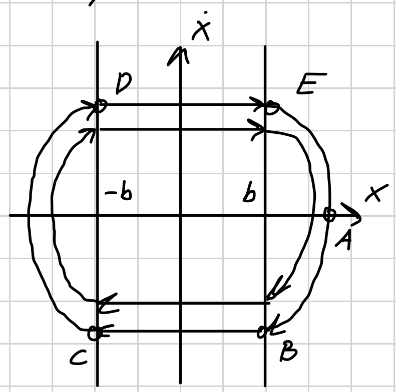

$\varepsilon = \cancel{g(t)}^{\,0} - x = -x$

$u = F(\varepsilon)$

$$x = u \cdot \frac{k}{p^2} \Rightarrow \ddot{x} = ky = k \cdot F(\varepsilon) = k \cdot (-F(x))$$

$$F(\varepsilon) = \begin{cases} c, & \varepsilon > b \\ 0, & \varepsilon \in [-b; b] \\ -c, & \varepsilon < -b \end{cases}$$

$$F(x) = \begin{cases} c, & x < -b \\ 0, & x \in [-b; b] \\ -c, & x > b \end{cases}$$




$x_1 = x, \quad x_2 = \dot{x}$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = k(-F(x))$$

При $f(x) = 0$: $\dfrac{dx_1}{dt} = x_2$, $\dfrac{dx_2}{dt} = 0 \Rightarrow x_2 = const = c$

$X_1 = ct + c_1$

При $f(x) = c$: $\dfrac{dx_1}{dt} = x_2$, $\dfrac{dx_2}{dt} = -kc$

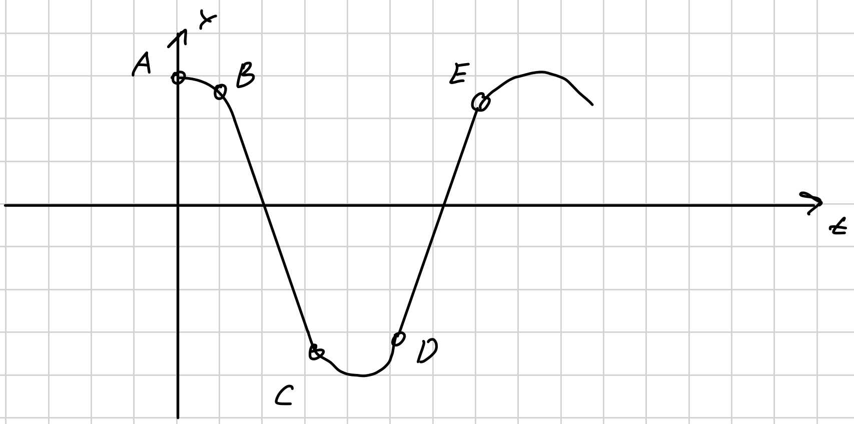
$x_2 = -kct + c_2 \quad \Rightarrow t = \dfrac{-x_2}{kc} - \dfrac{c_2}{c} \Rightarrow x_1 = -kc \dfrac{\left(-\frac{x_2}{kc} - \frac{c_2}{c}\right)^2}{2} + c\left(-\frac{x_2}{kc} - \frac{c_2}{c}\right) + c_3$

$x_1 = -kc\dfrac{t^2}{2} + c_2 t + c_3$

При $f(x) = -c$:

$x_2\, dx_2 = kc\, dx_1$

$$\frac{x_2^2}{2} = kcx_1 + c_4$$




## Лекция №3

# Метод припасовывания

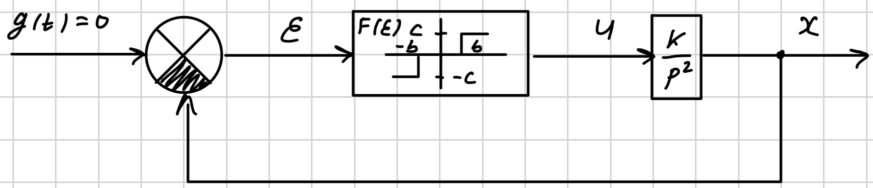
Заключается в том, что на каждом участке движения мы решаем ДУ движения в общем случ. Конечное условие предыдущ. участка являются начальными для следующего.

$x(0)$, $\dot{x}(0)$ - известно

$x(t) = F_1(t)$

$\dot{x}(t) = F_2(t)$

## Продолжение примера с прошлой лекции




$\varepsilon = -x$, $u = F(\varepsilon)$, $x = u\dfrac{k}{p^2}$ $\qquad F(x) = 0 \qquad F(x) = -c$

$\ddot{x} = k\,F(\varepsilon) = -k\,F(x)$ $\qquad x_2 = C_1 \qquad x_2 = -kC_3 t + C_4$

$\qquad x_1 = C_1 t + C_2 \qquad x_1 = -\dfrac{kC_3 t^2}{2} + C_4 t + C_5$

$x_1 = x$, $x_2 = \dot{x}$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -k\,F(x)$$

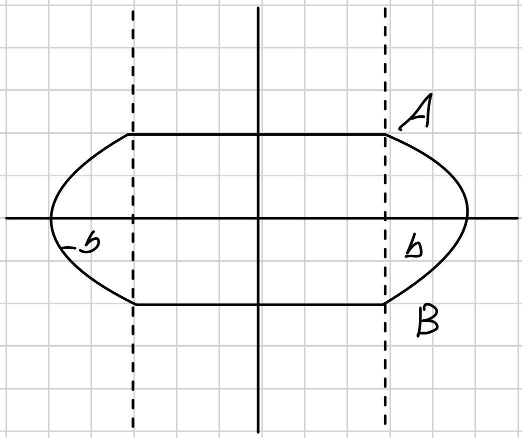
1. $F(\varepsilon) = c$

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -kc$$

$$\frac{dx_2}{dx_1} = -\frac{kc}{x_2}$$

$$x_2\,dx_2 = -kc\,dx_1$$

$$\frac{x_2^2}{2} = -kc\,x_1 + C_1$$




$t = 0 \quad A(b, \dot{x}_0)$

$t = t_k \quad B(b, \dot{x}_k)$

$x_1(0) = C_5 = b$

$x_2(0) = C_4 = \dot{x}_0$

$$\cancel{b} = -\frac{kC_3 t_k^2}{2} + \dot{x}_0 t_k + \cancel{b}$$

$$\frac{kC_3 t_k^2}{2} - \dot{x}_0 t_k = 0 \qquad t_k = 0, \quad t_k = \frac{2\dot{x}_0}{kC_3}$$

$$x_{2B} = -kc\cdot\frac{2\dot{x}_0}{kc} + \dot{x}_0 = -2\dot{x}_0 + \dot{x}_0 = -x_0$$

2-ой участок:

$t = 0 \quad B(b; -\dot{x}_0)$

$x_2 = C_1 = -\dot{x}_0$

$x_1 = C_2 = b$

$C(-b; -\dot{x}_0)$

$-b = -\dot{x}_0 t_K + b$

$t_K = \frac{2b}{\dot{x}_0}$

$x_1 = -\dot{x}_0 \cdot \frac{2b}{\dot{x}_0} + b = -b$

$C(-b; -\dot{x}_0)$

$x_2 = KCt + C_6$
$x_1 = \frac{KCt^2}{2} + C_6 t + C_7$

$x_2(0) = C_6 = -\dot{x}_0$
$x_1(0) = C_7 = -b$

$x_1 = \frac{KCt_K^2}{2} - \dot{x}_0 t_K \cancel{-b} = \cancel{-b}$

$t_K \left(\frac{KCt_K}{2} - \dot{x}_0\right) = 0$

$t_K = \frac{2\dot{x}_0}{KC}$

$x_2(t_K) = KC \cdot \frac{2\dot{x}_0}{KC} - \dot{x}_0 = \dot{x}_0$

---

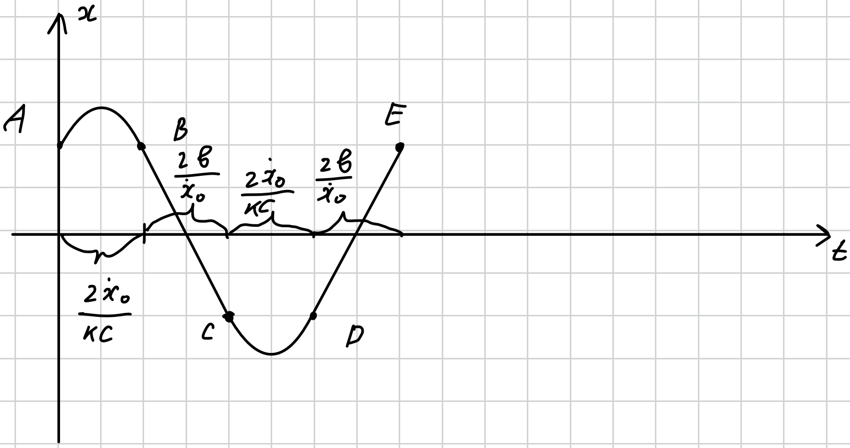
$t = 0 \quad x_1(0) = -b, \; x_2(0) = \dot{x}_0$

$t = t_K \quad x_1 = b, \; x_2 = -?$

$C_1 = \dot{x}_0, \; C_2 = -b$

$b = \dot{x}_0 t_K - b, \; t_K = \frac{2b}{\dot{x}_0}$

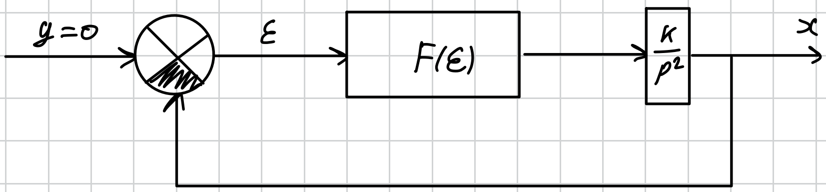
$x_2(t_K) = \dot{x}_0$

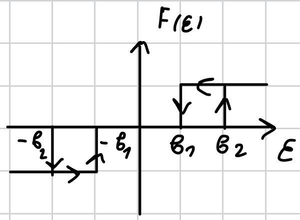


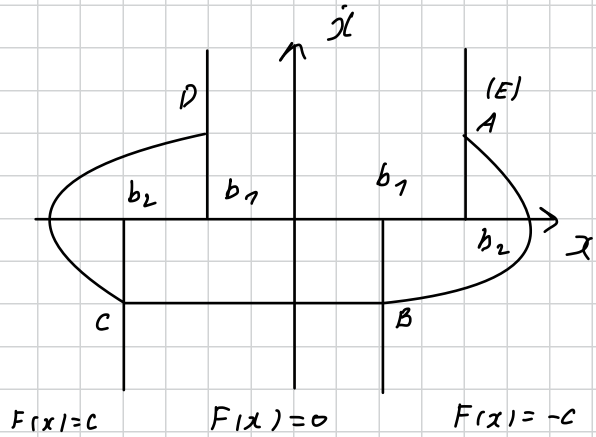

$$T = \frac{4\dot{x}_0}{kc} + \frac{4b}{\dot{x}_0}$$

$$x_1 = -\frac{kc_3 t_k^2}{2} + \dot{x}_0 t_k + b = -\frac{kc_3}{2}\cdot\frac{4\dot{x}_0^2}{k^2c^2} + \dot{x}_0\cdot\frac{2\dot{x}_0}{kc} + b = \frac{2\dot{x}_0^2}{kc} + \frac{2\dot{x}_0^2}{kc} + b = \frac{4\dot{x}_0^2}{kc} + b$$

---




$F(x)=0$ $\qquad$ $F(x)=c$ $\qquad$ $F(x)=-c$

$$\begin{cases} x_1 = C_1 t + C_2 \\ x_2 = C_1 \end{cases} \qquad \begin{cases} x_1 = ck\frac{t^2}{2} + C_3 t + C_4 \\ x_2 = ckt + C_3 \end{cases} \qquad \begin{cases} x_1 = -ck\frac{t^2}{2} + C_5 t + C_6 \\ x_2 = -ckt + C_5 \end{cases}$$

I участок: $A(b_2, \dot{x}_0) \rightarrow x_1 = C_6 = b_2$

$x_2 = C_5 = \dot{x}_0$

$$x_1(t_k) = -ck\frac{t_k^2}{2} + \dot{x}_0 t_k + b_2 = b_1$$

$$D = \dot{x}_0^2 + 4\cdot\frac{ck}{2}\cdot(b_2 - b_1) = \dot{x}_0^2 + 2ck(b_2 - b_1)$$

$$t_k = \frac{-\dot{x}_0 + \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}}{-ck}$$

$$x_2(t_k) = -ckt_k + \dot{x}_0 = \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}$$

## II участок

$B(b, \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)})$

$x_1(0) = c_2 = b_1$

$x_2(0) = c_1 = \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}$

$x_1(t_k) = \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}\, t_k + b_1 = -b_2$

$$t_k = -\frac{b_1 + b_2}{\sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}}$$

$x_2 = c_1 \Rightarrow C(-b_2; -\sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)})$

## III участок

$C(-b_2; -\sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)})$

$x_1(0) = c_4 = -b_2$

$x_2(0) = c_3 = -\sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}$

$x_1(t_k) = \frac{ck}{2} t_k^2 - \sqrt{\dot{x}_0^2 + 2ck(b_2 - b_1)}\, t_k - b_2 = -b_1$

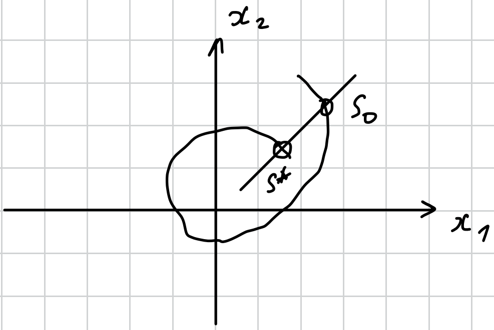
$D = \dot{x}_0^2 + 2ck(b_2 - b_1) - 2ck(b_1 - b_2) = x_0^2 + 4ck(b_2 - b_1)$

$$t_k = \frac{\sqrt{-//-} + \sqrt{x_0^2 + 4ck(b_2 - b_1)}}{ck}$$

$$x_2(t_k) = ck \cdot \frac{\sqrt{-//-} + \sqrt{x_0^2 + 4ck(b_2 - b_1)}}{ck}$$

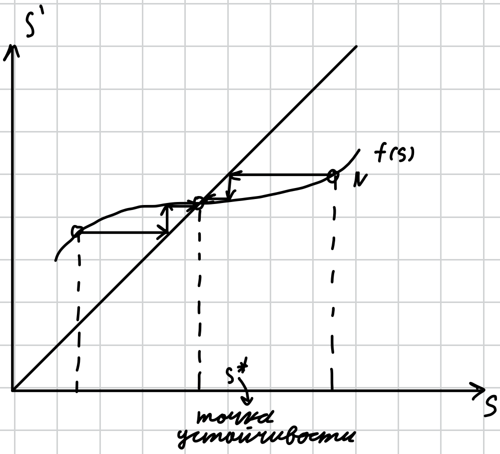
Лекция №4

# Метод точечного преобразования

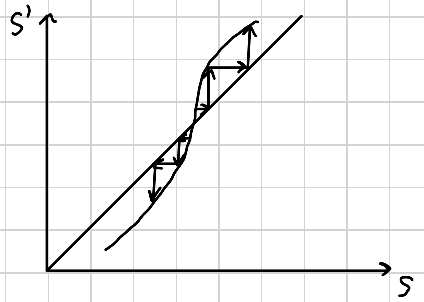


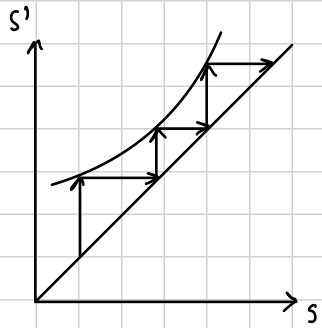

1. $S_0 \to S^*$
2. $S^* = S_{02}$, $S_{02} \to S^*_2$
3. $S^*_2 = S_{03}$, $S_{03} \to S^*_3$

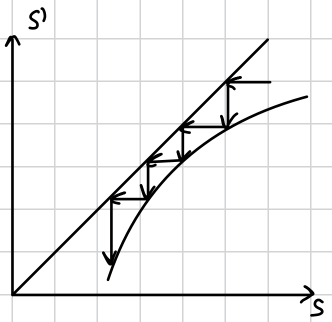
$f(S)$

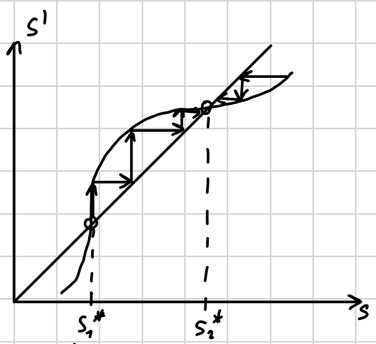


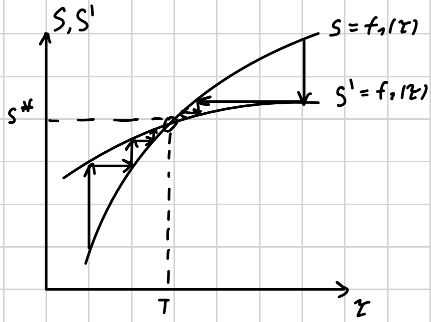
$$\left(\frac{ds'}{ds}\right)_{s=s^*} < 1$$

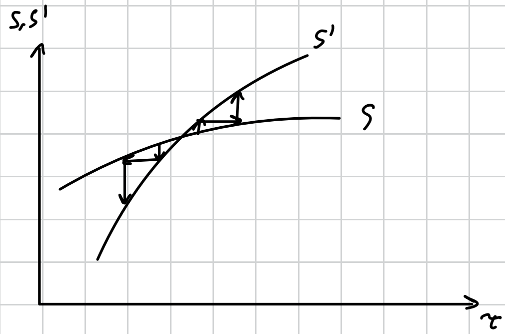


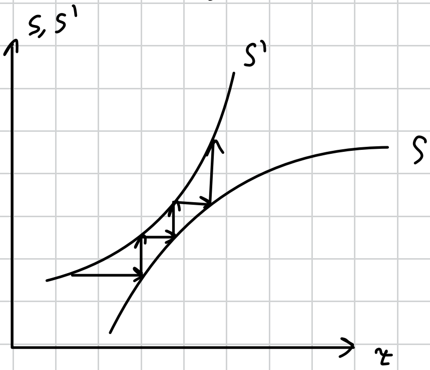



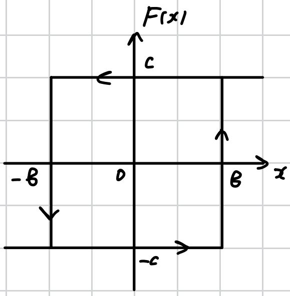



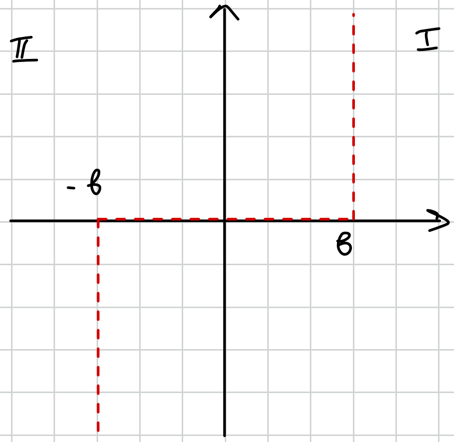




## Пример




$$F(x) = \begin{cases} c, \ x > b; \ x \in [-b; b], \ \dot{x} < 0 & \mathrm{I} \\ -c, \ x < -b; \ x \in [-b; b], \ \dot{x} > 0 & \mathrm{II} \end{cases}$$




$$(T_1 p + 1)x = -k_1 x_1$$

$$p x_1 = F(x)$$

$$x_1 = x, \quad x_2 = \dot{x}$$

$$(T_1 p^2 + p)x = -k_1 F(x)$$

$$\ddot{x} T_1 + \dot{x} = -k_1 F(x)$$

$$\ddot{x} = \frac{-k_1 F(x)}{T_1} - \frac{\dot{x}}{T_1}$$

$$x_1 = -T_1 (\ln|k_1 c + x_2| - k_1 c\, x_2) + C_1$$

$T_1\ddot{x}_1 + \dot{x}_1 = -k_1 c$

Хар. ур-е:

$T_1\lambda^2+\lambda=0$

$\lambda_1=0$

$\lambda_2=-\frac{1}{T_1}$

$x_1 = C_2\cdot 1 + C_3 e^{-\frac{1}{T_1}t}$

$x_{12} = tC_4$

$\dot{x}_{12}=C_4$

$\ddot{x}_{12}=0$

$C_4=-k_1c \Rightarrow x_{12}=-k_1ct$

$x_1 = C_2+C_3e^{-\frac{1}{T_1}t}-k_1ct$

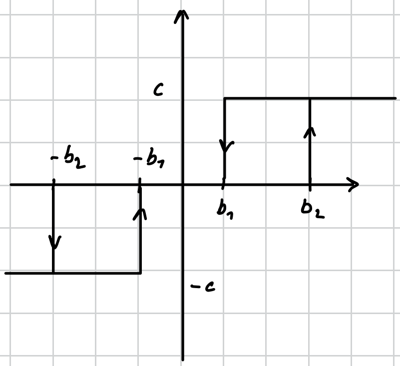
$t=0\to x_1=b \Rightarrow b=C_2+C_3=b$

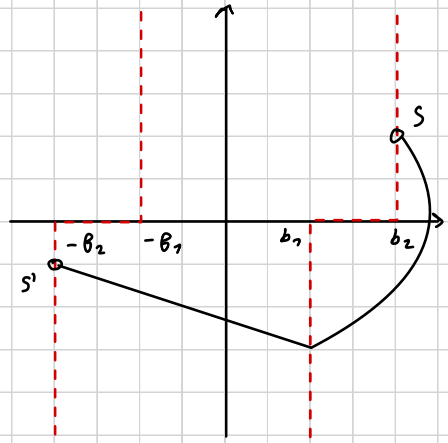
$C_2=b-C_3$

$s=x_2(0)$

$x(t_k)=-b:\ C_2+C_3e^{-\frac{1}{T_1}t_k}-k_1ct_k=-b$

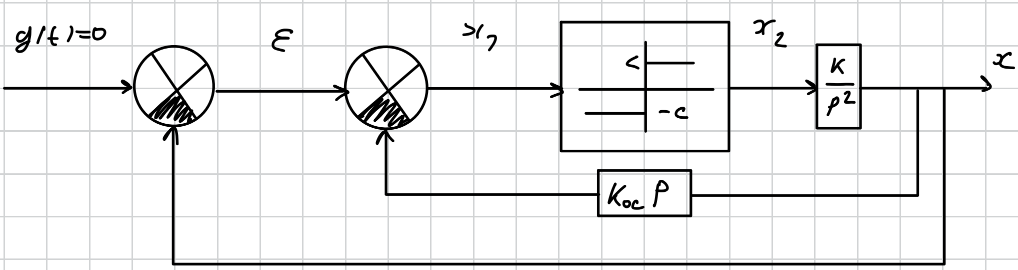
$x_2=-\frac{1}{T_1}C_3e^{-\frac{1}{T_1}t}-k_1c$ , $x_2(0)=-\frac{1}{T_1}C_3-k_1c=s \Rightarrow C_3=-(s+k_1c)T$




## Лекция № 5

## Скользящий процесс

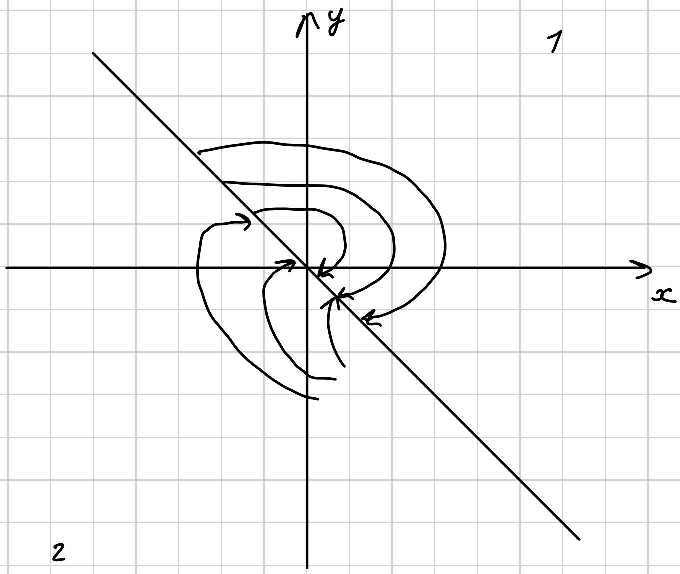


$\varepsilon = g - x = -x$

$x_1 = \varepsilon - k_{oc} p x = \varepsilon - k_{oc}\dot{x}$

$x_2 = F(x_1) = c \cdot \mathrm{sign}(x_1)$

$\ddot{x} = k x_2$




$y = \dot{x}$

$x_1 = -x - k_{oc} y$

$x_2 = c\,\sin(-x - k_{oc} y)$

$\dot{y} = k c\, \mathrm{sign}(-x - k_{oc} y)$

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = k c\, \mathrm{sign}(-x - k_{oc} y)$$

$-x - k_{oc} y = 0$

$x + k_{oc} y = 0$

$y = -\frac{x}{k_{oc}}$

1. $x + k_{oc} y > 0$: $\quad \frac{dy}{dx} = \frac{-kc}{y}$

$y\,dy = -kc\,dx$

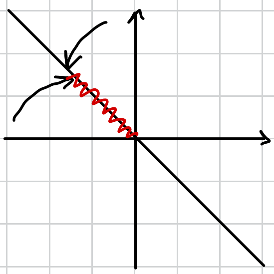
$\frac{y^2}{2} = -kcx + C_1$

2. $x + k_{oc} y < 0$: $\quad \frac{dy}{dx} = \frac{kc}{y}$

$y\,dy = kc\,dx$

$\frac{y^2}{2} = kcx + C_2$

Скользящий процесс – это движение по прямой с бесконечно большой частотой и бесконечно малой амплитудой (теоретический)

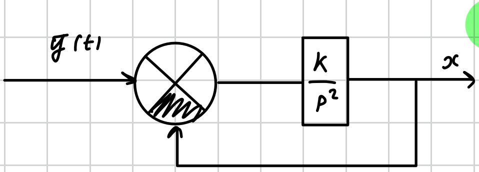


Плюсы: процесс сходится, причем по прямой

Минусы: постоянные переключения

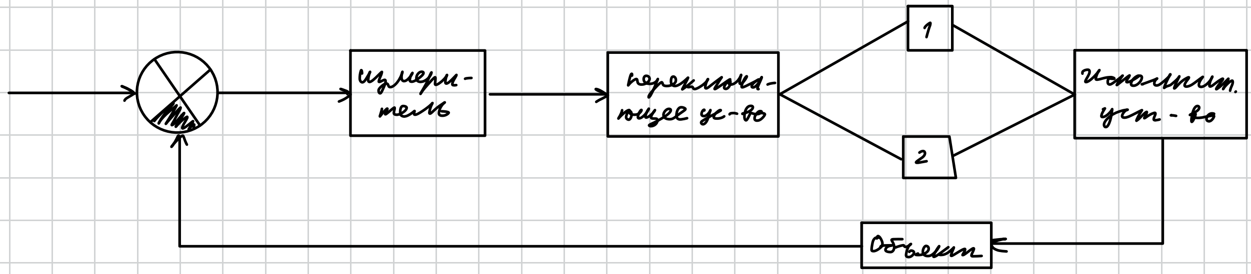
$k_{oc}\dot{x} + x = 0, \quad x = C \cdot e^{-\frac{1}{k_{oc}} t}$ → заменяется на диф. ур. 1-го порядка

**Логическое управление**

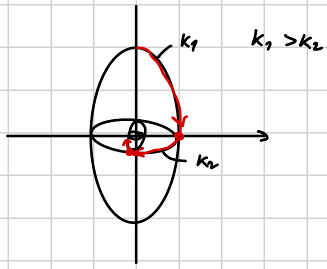



=> на границе устойчивости

$\Phi_x(S) = \frac{W}{1+W} = \frac{K}{S^2+K}$, $D(S) = S^2 + K$ → чисто мнимые корни



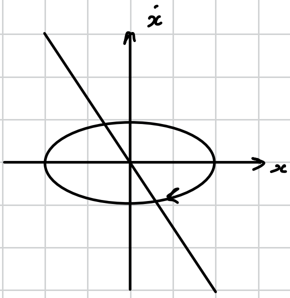

$\frac{K}{p^2}$



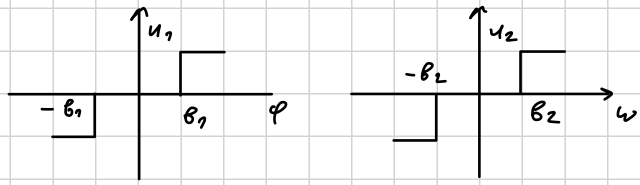

$\dot x > 0,\ x > 0 : k_1$
$\dot x < 0,\ x > 0 : k_2$
$\dot x < 0,\ x < 0 : k_1$
$\dot x > 0,\ x < 0 : k_2$

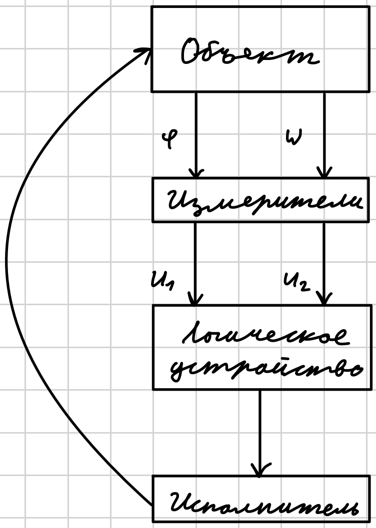
$$\Rightarrow \boxed{\begin{array}{l} \dot x \cdot x > 0 : k_1 \\ \dot x \cdot x < 0 : k_2 \end{array}}$$

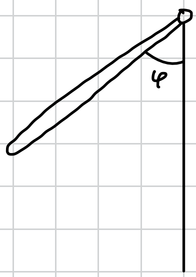
$\dot x = -\frac{1}{k_{контр}} x$




=> нужно изменить знак (-k) и по линии седла вернуться



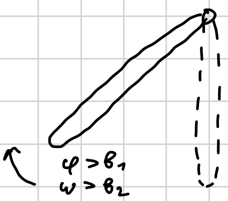



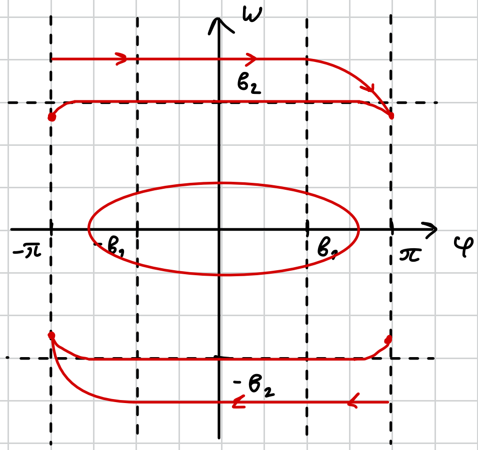




$M = J\ddot\varphi \Rightarrow M = 0, \pm 1$

| $\varphi$ \ $\omega$ | $\omega > в_2$ | $-в_2 < \omega < в_2$ | $\omega < -в_2$ |
|---|---|---|---|
| $\varphi > в_1$ | $-1$ | $-1$ | $0$ |
| $-в_1 < \varphi < в_1$ | $0$ | $0$ | $0$ |
| $\varphi < -в_1$ | $0$ | $+1$ | $+1$ |

$M = 1$ → против часовой стрелки
$M = -1$ → по часовой стрелке




Получилось неверно

$$\frac{d\varphi}{dt} = w$$

$$\frac{dw}{dt} = \frac{M}{J}$$

$$\frac{dw}{dt} = 0 \Rightarrow w = c_1$$

$$\frac{d\varphi}{dt} = C_1$$

$$\varphi = c_1 t + c_2$$

$$\frac{dw}{dt} = \frac{1}{J} \qquad \frac{dw}{d\varphi} = \frac{1}{Jw}$$

$$\frac{d\varphi}{dt} = w \qquad Jw\,dw = d\varphi$$

$$\varphi = \frac{Jw^2}{2} + C_3$$

$$\frac{dw}{dt} = -\frac{1}{J} \qquad \frac{dw}{d\varphi} = \frac{-1}{Jw}$$

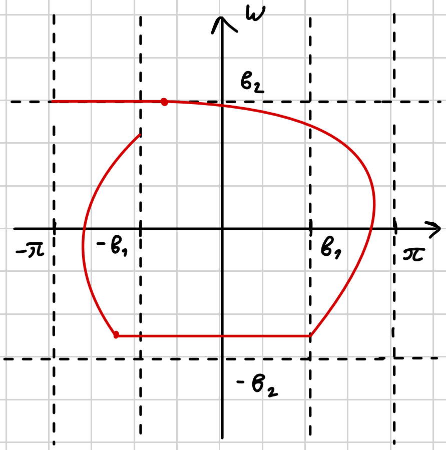
$$\frac{d\varphi}{dt} = w \qquad \varphi = -\frac{Jw^2}{2} + C_4$$

Попробуем составить таблицу снова

| $\varphi$ \ $w$ | $w > b_2$ | $-b_2 < w < b_2$ | $w < -b_2$ |
|---|---|---|---|
| $\varphi > b_1$ | $-1$ | $-1$ | $0$ |
| $-b_1 < \varphi < b_1$ | $-1$ | $0$ | $+1$ |
| $\varphi < -b_1$ | $0$ | $+1$ | $+1$ |

$\Rightarrow$ тоже не получилось

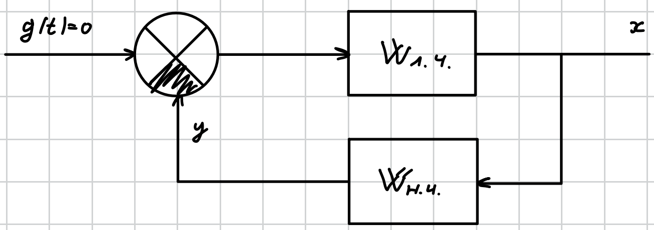
Учтем инерциальность



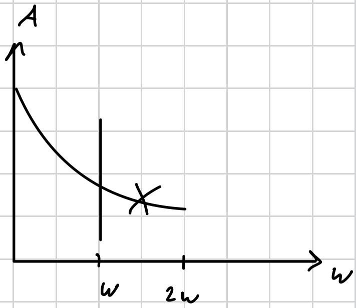

$\Rightarrow$ теперь сходится

## Метод гармонической линеаризации

1. Разбиваем систему на линейную и нелинейную часть



$$y(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty} A_n \sin n\omega t + B_n \cos n\omega t$$




а) $x(t)=a\sin\omega t$

б) $x(t)=x_0+a\sin\omega t$

↓ внеш. сигнал ↓ несимметр. нелин.

## Условия провед. гармонич. линеариз.

1. Лин. часть вып. ф-ю фильтра низких частот
2. К нелин. части применимо преоб. Фурье (условие Дирихле)
3. В системе либо 1 нелин. хар., либо сводима к 1

$x(t)=a\sin\omega t$, $p\,x(t)=a\omega\cos\omega t$

$y(t)=A_1\sin\omega t+B_1\cos\omega t$

$q(a)=\frac{A_1}{a}$, $y(t)=q(a)\,x(t)+\frac{p\,x(t)}{\omega}q'(a)$, $q'(a)=\frac{B_1}{a}$

$y(t)=x(t)\left(q(a)+\frac{p\,q'(a)}{\omega}\right)$

$W_{н.ч.}=q(a)+\frac{p\,q'(a)}{\omega}$

$W_{н.ч.}=q(a)+\gamma\, q'(a)$

$$A_0=\frac{\omega}{\pi}\int_0^{2\pi/\omega}F(a\sin\omega t)\,dt$$

$$A_1=\frac{\omega}{\pi}\int_0^{2\pi/\omega}F(a\sin\omega t)\sin\omega t\,dt$$

$$B_1=\frac{\omega}{\pi}\int_0^{2\pi/\omega}F(a\sin\omega t)\cos\omega t\,dt$$

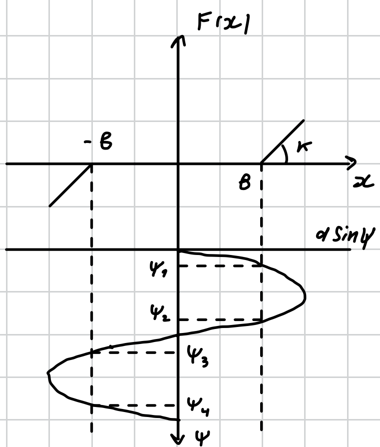
$\psi=\omega t$

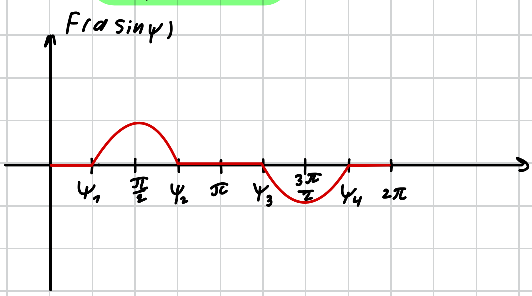
$$q(a)=\frac{A_1}{a}=\frac{1}{\pi a}\int_0^{2\pi}F(a\sin\psi)\sin\psi\,d\psi$$

$q'(a)=\frac{B_1}{a}=\frac{1}{\pi a}\int_0^{2\pi}F(a\sin\psi)\cos\psi\,d\psi=0$ из-за однозначной симметричной нелинейности

$$q(a)=\frac{4}{\pi a}\int_0^{\pi/2}F(a\sin\psi)\sin\psi\,d\psi$$

### Пример




При $a<b$ $q(a)=0$

При $a>b$

$q'(a)=0$

$$q(a)=\frac{4}{\pi a}\int_0^{\frac{\pi}{2}}F(a\sin\psi)\sin\psi\,d\psi=\frac{4}{\pi a}\int_{\psi_1}^{\frac{\pi}{2}}F(a\sin\psi)\sin\psi\,d\psi=\frac{4}{\pi a}\int_{\psi_1}^{\frac{\pi}{2}}K\cdot a\sin^2\psi\,d\psi=\frac{4K}{\pi}\int_{\psi_1}^{\frac{\pi}{2}}\frac{1-\cos 2\psi}{2}\,d\psi=\frac{2K}{\pi}\left(\frac{\pi}{2}-\psi_1\right)-\frac{K}{\pi}(0-\sin 2\psi)$$

$a\sin\psi_1=b$

$\sin\psi_1=\frac{b}{a}$

$\cos\psi_1=\sqrt{1-\left(\frac{b}{a}\right)^2}$

$$q(a)=\frac{2k}{\pi}\left(\frac{\pi}{2}-\arcsin\frac{b}{a}\right)+\frac{2k}{\pi}\cdot\frac{b}{a}\sqrt{1-\left(\frac{b}{a}\right)^2}$$

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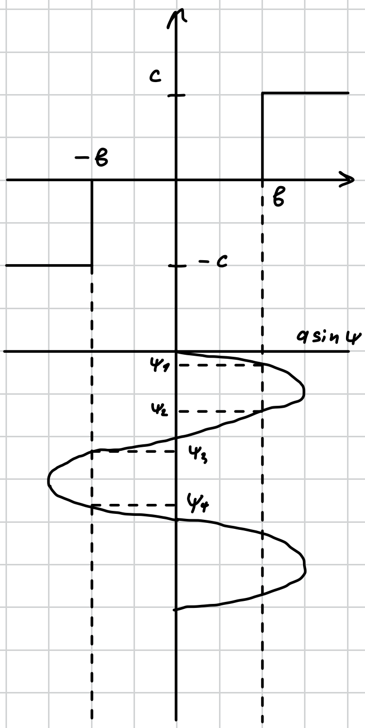
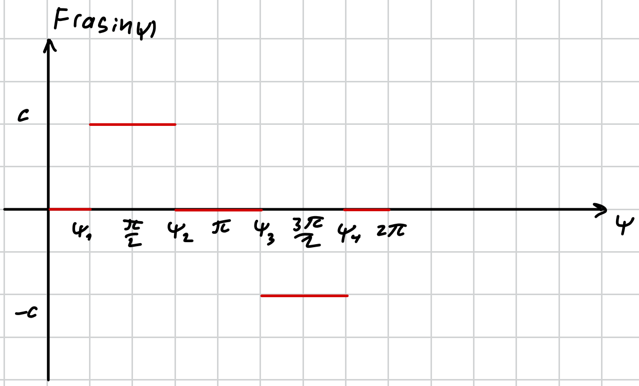



$q'=0$

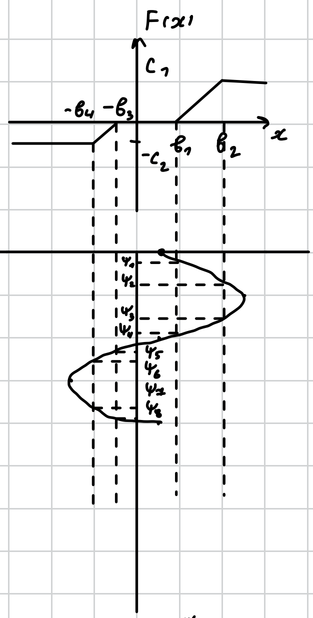
$$q=\frac{4}{\pi a}\int_0^{\frac{\pi}{2}}F(a\sin\psi)\sin\psi\,d\psi=\frac{4}{\pi a}\int_{\psi_1}^{\frac{\pi}{2}}c\sin\psi\,d\psi=-\frac{4c}{\pi a}\cdot\cos\psi\Big|_{\psi_1}^{\frac{\pi}{2}}=\frac{4c}{\pi a}\cos\psi_1=\frac{4c}{\pi a}\sqrt{1-\frac{b^2}{a^2}}$$

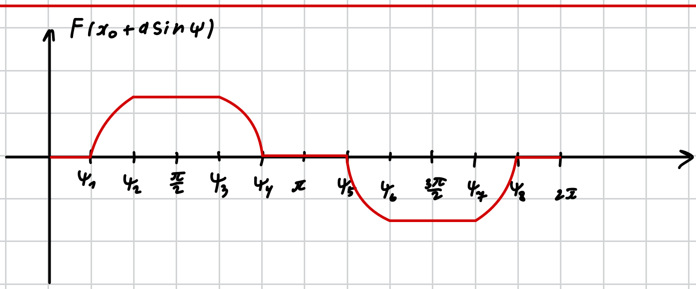
---

$0<x_0<b_1$

$x_0+a>b_2$

$x_0-a<b_4$




Всего 57 случаев

$$x_0=\frac{1}{2\pi}\int_0^{2\pi}F(a\sin\psi)\,d\psi$$

$$F_0(x_0,a)=\frac{1}{2\pi}\left(\int_{\psi_1}^{\psi_2}k_1(x_0+a\sin\psi)\,d\psi+\int_{\psi_2}^{\psi_3}c_1\,d\psi+\int_{\psi_3}^{\psi_4}k(x_0 a\sin\psi)\,d\psi+\int_{\psi_5}^{\psi_6}k_2(x_0+a\sin\psi)\,d\psi-\int_{\psi_6}^{\psi_7}c_2\,d\psi+\int_{\psi_7}^{\psi_8}k_2(x_0+a\sin\psi)\,d\psi\right.$$

$x_0 + a\sin\psi_1 = b_1$ $\quad\quad$ $x_0 + a\sin\psi_5 = b_3$

$x_0 + a\sin\psi_2 = b_2$ $\quad\quad$ $x_0 + a\sin\psi_6 = b_4$

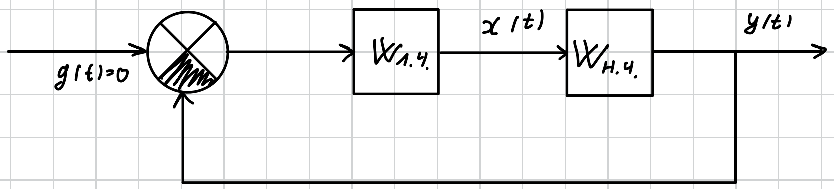
$\psi_2 = \pi - \psi_3$ $\quad\quad$ $\psi_5 = \pi - \psi_7$

$\psi_1 = \pi - \psi_4$ $\quad\quad$ $\psi_6 = \pi - \psi_8$

$$q'(a, x_0) = \frac{1}{2\pi a}\left(\int_{\psi_1}^{\psi_2} k_1(x_0 + a\sin\psi)\cancel{\sin\psi}^{\cos\psi}\,d\psi + \int_{\psi_2}^{\psi_3} c_1 \cancel{\sin\psi}^{\cos\psi}\,d\psi + \int_{\psi_3}^{\psi_4} k_1(x_0 + a\sin\psi)\cancel{\sin\psi}^{\cos\psi}\,d\psi + \int_{\psi_5}^{\psi_6} k_2(x_0 + a\sin\psi)\cancel{\sin\psi}^{\cos\psi}\,d\psi - \int_{\psi_6}^{\psi_7} c_2 \cancel{\sin\psi}^{\cos\psi}\,d\psi + \int_{\psi_7}^{\psi_8} k_2(x_0 + a\sin\psi)\cancel{\sin\psi}^{\cos\psi}\,d\psi\right)$$

## Лекция №6

# Определение параметров автоколебаний (алгебраический способ)




$$W_{л.ч.} = \frac{A(s)}{B(s)}$$

$$y = F(x) = \left(q(a) + \frac{q'(a)}{\omega}p\right)x(t)$$

$$x(t) = -y(t)\frac{A(p)}{B(p)}$$

$$x(t)B(p) + A(p)y(t) = 0$$

$$x(t)B(p) + A(p)F(x) = 0$$

$$x(t)\underbrace{\left(B(p) + A(p)\left[q(a) + \frac{q'(a)p}{\omega}\right]\right)}_{\text{характеристическое уравнение}} = 0$$

$$\lambda = \cancel{\alpha}^{0} \pm j\beta = \pm j\omega$$

Замена: $p = j\omega$

$$B(j\omega) + A(j\omega)\left[q(a) + \frac{q'(a)j\omega}{\omega}\right] = 0$$

$$X(a,\omega) + j\cdot Y(a,\omega) = 0$$

$$\begin{cases} X(a,\omega) = 0 \\ Y(a,\omega) = 0 \end{cases}$$

$x = x^* + \Delta x$, где $x^* = a^*\sin\omega^* t$

$$(x^* + \Delta x)B(p) + A(p)F(x^* + \Delta x) = 0$$

$$B(p)\Delta x + A(p)\frac{\partial F}{\partial x}\bigg|_{x = x^*} = 0$$

$$x = a e^{j\omega t \xi}$$

$$x + \Delta x = (a + \Delta a) e^{j(\omega + \Delta\omega) t \xi}$$

$$X(a, \omega, \xi) + \Delta X(a, \omega, \xi) = X(a + \Delta a, \omega + \Delta\omega, \xi)$$
$$Y(a, \omega, \xi) + \Delta Y(a, \omega, \xi) = Y(a + \Delta a, \omega + \Delta\omega, \xi)$$

$$\left.\frac{\partial X}{\partial a}\right|_{a=a^*} \cdot \left.\frac{\partial X}{\partial \omega}\right|_{\omega=\omega^*} + \left.\frac{\partial Y}{\partial a}\right|_{a=a^*} \cdot \left.\frac{\partial Y}{\partial \omega}\right|_{\omega=\omega^*}$$

$$\frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}(\Delta\omega + \xi j) + j\left[\frac{\partial Y}{\partial a}\Delta a \frac{\partial Y}{\partial \omega}(\Delta\omega + j\xi)\right] = 0$$

$$\frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}\Delta\omega + j\frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}\xi + j\left[\frac{\partial Y}{\partial a}\Delta a \frac{\partial Y}{\partial \omega}\Delta\omega + \frac{\partial Y}{\partial a}\Delta a \cdot \frac{\partial Y}{\partial \omega} j\xi\right]$$

$$\begin{cases} \frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}\Delta\omega - \frac{\partial Y}{\partial a}\Delta a \frac{\partial Y}{\partial \omega}\xi = 0 \\ \frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}\xi + \frac{\partial Y}{\partial a}\Delta a \frac{\partial Y}{\partial \omega}\Delta\omega = 0 \end{cases}$$

$$\Delta\omega = \frac{\frac{\partial Y}{\partial a}\cancel{\Delta a}\frac{\partial Y}{\partial \omega}\xi}{\frac{\partial X}{\partial a}\frac{\partial X}{\partial \omega}\cancel{\Delta a}}$$

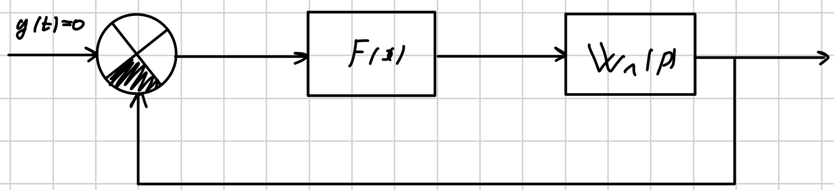
$$\frac{\partial X}{\partial a}\Delta a \frac{\partial X}{\partial \omega}\xi + \frac{\partial Y}{\partial a}\Delta a \frac{\partial Y}{\partial \omega} \cdot$$

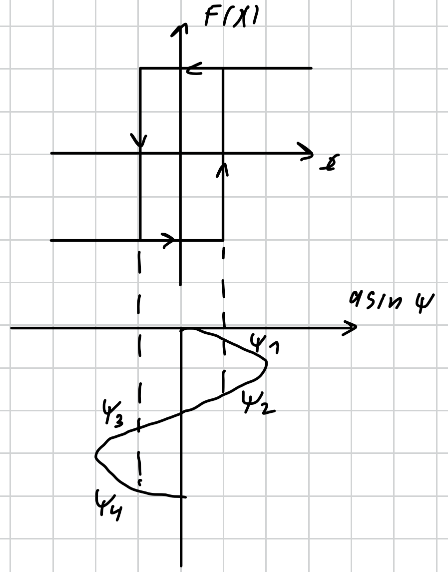
$$\frac{\partial X}{\partial a} \cdot \frac{\partial X}{\partial \omega}\Delta a \xi + \left(\frac{\partial Y}{\partial a}\frac{\partial Y}{\partial \omega}\right)^2 \Delta a \xi \cdot \left(\frac{\partial X}{\partial a}\frac{\partial X}{\partial \omega}\right)^{-1} = 0$$

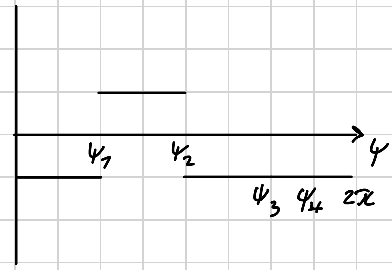
$$\frac{\left(\frac{\partial X}{\partial a} \cdot \frac{\partial X}{\partial \omega}\right)^2 \Delta a \xi + \left(\frac{\partial Y}{\partial a}\frac{\partial Y}{\partial \omega}\right)^2 \Delta a \xi}{\frac{\partial X}{\partial a}\frac{\partial X}{\partial \omega}} = 0$$

$$\frac{\partial X}{\partial a}\frac{\partial Y}{\partial \omega} - \frac{\partial Y}{\partial a}\frac{\partial X}{\partial \omega} > 0$$

$$\xi = \frac{-\left(\frac{\partial X}{\partial a}\frac{\partial Y}{\partial \omega} - \frac{\partial Y}{\partial a}\frac{\partial X}{\partial \omega}\right)\Delta a}{\left(\frac{\partial X}{\partial a}\frac{\partial X}{\partial \omega}\right)^2 + \left(\frac{\partial Y}{\partial a}\frac{\partial Y}{\partial \omega}\right)^2}$$






$$q(a) = \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\psi)\sin\psi\, d\psi$$

$$q(a) = \frac{1}{\pi a}\int_0^{\psi_1}(-c)\sin\psi\, d\psi + \int_{\psi_1}^{\psi_3} c\sin\psi\, d\psi + \int_{\psi_3}^{2\pi}(-c)\sin\psi\, d\psi$$

$$q(a) = \frac{c}{\pi a}(\cos\psi_1 - \cancel{\cos 0} - \cos\psi_3 + \cos\psi_1 + \cancel{\cos 2\pi} - \cos\psi_3)$$

$$q(a) = \frac{c}{\pi a}(2\cos\psi_1 - 2\cos\psi_3) = \frac{2c}{\pi a}(\cos\psi_1 - \cos\psi_3)$$

$$\sin\psi_1 = \frac{b}{c} \Rightarrow \cos\psi_1 = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

$$\cos\psi_3 = -\sqrt{1-\left(\frac{b}{a}\right)^2}$$

$$q(a) = \frac{4c}{\pi a}\sqrt{1-\frac{b^2}{a^2}} = \frac{4c}{\pi a^2}\sqrt{a^2-b^2}$$

$q'(a) = \int$ тоже, но с косинусами $= \frac{2c}{\pi a}(\sin\psi_3 - \sin\psi_1) = -\frac{4c}{\pi a}\frac{b}{a} = -\frac{4cb}{\pi a^2}$

$$W_1 = \frac{7}{p(0{,}1p+1)(0{,}05p+1)}$$

$B(p) + A(p)F(x) = 0$

$p = j\omega$

$$j\omega(0{,}1j\omega+1)(0{,}05j\omega+1) + 7q(a) + 7\frac{q'(a)}{\omega}j\omega = 0$$

$$(-0{,}1\omega^2 + j\omega)(0{,}05j\omega+1) + 7q(a) + 7q'(a)j$$

$$-0{,}005\omega^3 j - 0{,}1\omega^2 - 0{,}05\omega^2 + j\omega + 7q(a) + j7q'(a) = 0$$

$c = 12,\ b = 1{,}5$

$$\begin{cases} X(a,\omega) = -0{,}15\omega^2 + \frac{336}{\pi a}\sqrt{1-\frac{2{,}25}{a^2}} = 0 \\ Y(a,\omega) = -0{,}005\omega^3 + \omega - \frac{336}{\pi a}\cdot\frac{1{,}5}{a} = 0 \end{cases}$$

$$\frac{\partial Y}{\partial a} = -\frac{336\cdot 1{,}5}{\pi}(-2)\cdot\frac{1}{a^3} = \frac{321}{a^3}$$

$$\frac{\partial Y}{\partial \omega} = -0{,}015\omega^2 + 1$$

$$\frac{\partial x}{\partial w} = -0{,}3\,w$$

$$\frac{\partial x}{\partial a} = \left((-1) \cdot \frac{336}{\pi} \cdot \frac{1}{a^2}\right) \sqrt{1 - \frac{2{,}25}{a^2}} + \frac{336}{\pi a} \cdot \frac{1}{2\sqrt{1 - \frac{2{,}25}{a^2}}} \cdot (-2) \cdot \frac{2{,}25}{a^3} = -\frac{107}{a^2} \sqrt{1 - \frac{2{,}25}{a^2}} - \frac{241}{a^4} \frac{1}{\sqrt{1 - \frac{2{,}25}{a^2}}}$$

$a = 6$

$w = 10{,}7$

$$\frac{\partial y}{\partial a} = \frac{321}{6^3} = 1{,}5$$

$$\frac{\partial y}{\partial w} = -0{,}015 \cdot 10{,}7^2 + 1 = 0{,}7$$

$$\frac{\partial x}{\partial w} = -0{,}3 \cdot 10{,}7 = -3{,}21$$

$$\frac{\partial x}{\partial a} = -\frac{107}{6^2} \sqrt{1 - \frac{2{,}25}{6^2}} - \frac{241}{6^4} \cdot \frac{1}{\sqrt{1 - \frac{2{,}25}{6^2}}} = -3$$

$$\frac{\partial x}{\partial a} \frac{\partial y}{\partial w} - \frac{\partial x}{\partial w} \frac{\partial y}{\partial a} = -3 \cdot (-0{,}7) - (-3{,}21) \cdot 1{,}5 > 0$$