Performance of Distillation Protocols in the Presence of Noise using a Quantum Network Simulator

Adelina Avram (6187803), Daglar Coban (4783417), Dragos Ileana (5577659), and Lukas Kingma (6323847)

Faculty of Applied Sciences, Delft University of Technology, The Netherlands

April 14, 2025

Abstract

To enable quantum information tasks such as teleportation and key distribution, high-fidelity entangled pairs are essential. This work investigates the performance of four protocols designed to distill such pairs, BBPSSW, DEJMPS, EPL, and the 3→1 protocol by Chi et al., under realistic noise conditions. Using the NetQASM–NetSquid software stack, each protocol is simulated with noise models that include Werner states for initial entanglement and depolarizing channels for gate operations. By systematically varying these parameters, the fidelity and probability of success of each protocol is analyzed, offering insight into their performance in practical quantum networks. This study highlights the strengths and limitations of each protocol, with a focus on how they handle different levels and types of imperfections in the quantum channel.

1 Introduction

Entanglement is a key resource in many quantum information protocols, enabling tasks such as quantum teleportation, superdense coding, and quantum key distribution. However, practical quantum channels are prone to noise, resulting in degraded or "mixed" entangled states. To mitigate these imperfections and recover high-fidelity entanglement, various distillation protocols have been proposed [1–5]. These protocols take one or more noisy entangled pairs as input and, with some probability of success, produce one higher-fidelity pair.

In this work, we focus on four bipartite distillation schemes: BBPSSW [4], DEJMPS [1], Extreme Photon Loss (EPL) [2,3], and the 3→1 protocol by Chi et al. [5]. The first three are 2→1 protocols, each consuming two entangled pairs to produce one purified pair, while the 3→1 protocol uses three pairs at once. Historically, BBPSSW was among the earliest proposed purification methods, while DEJMPS refined the procedure by improving its tolerance to noise. EPL addresses scenarios characterized by large photonic losses, and the 3→1 protocol can achieve higher-fidelity outputs by leveraging additional resources. For a more detailed description of the four protocols, see Appendix A.

To investigate these protocols under realistic conditions, we rely on NetQASM [6] and NetSquid [7], a software stack that allows us to simulate quantum networks at the discrete-event level while incorporating various noise models. More specifically, we consider two main sources of errors: noise in the shared entangled pairs (assumed to be in Werner states) and gate noise, represented by depolarizing channels applied after local operations. By systematically varying these noise parameters, we characterize each protocol's final entangled state fidelity to the target Bell state $|\phi_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and probability of success. We compare the results obtained from NetQASM simulations to the theoretical formulas, but also to outcomes obtained from additional simulations done using only NumPy.

2 Research Questions

Quantum entanglement distillation protocols are crucial for purifying noisy entangled states in practical quantum communication applications. However, the performance of such protocols can vary significantly depending on the noise characteristics of both the initial entangled states and the quantum operations (gates) used during distillation. In this work, we aim to understand how different entanglement distillation protocols behave under various noise conditions, with a particular focus on their ability to improve fidelity and operate reliably under realistic assumptions.

We pose three key research questions to guide this investigation:

1. **How does each protocol's final entangled fidelity scale with the EPR channel fidelity when gate fidelity is perfect (1.0)?**
   This question examines the fundamental capability of each protocol to enhance fidelity as a function of the input state's quality. By varying the initial EPR channel fidelity while keeping the gates perfect, we aim to identify the regime where each protocol becomes effective—i.e., whether it improves fidelity beyond the input and how the probability of success varies with channel quality. This provides insight into the theoretical limits of each protocol under ideal gate conditions.

2. **How does each protocol's performance degrade as gate fidelity decreases when the EPR channel fidelity is perfect (1.0)?**
   This question investigates the impact of gate imperfections on protocol performance when starting with perfect entangled pairs. By analyzing both the final fidelity and success probability as functions of decreasing gate fidelity, we can determine each protocol's sensitivity to operational errors. This helps identify which protocols remain viable under realistic gate noise conditions even when high-quality entanglement is available.

3. **Which protocol offers the best balance between final fidelity and success probability across different noise regimes?**
By comparing the results from our first two questions, we can evaluate the overall effectiveness of each protocol. This involves analyzing the trade-offs between achieving high final fidelity and maintaining acceptable success probability under various combinations of channel and gate noise. Such analysis is essential for practical implementations where both the quality of the output state and the resource efficiency of the protocol matter.

Together, these questions aim to map out the operating regimes of different distillation protocols, providing insights into their practicality and efficiency under varying physical constraints. The answers may also help identify critical thresholds and guide the design of distillation strategies for emerging quantum networks.

3 Methodology

As mentioned in the Introduction, we consider the following four distillation protocols: BBPSSW, DEJMPS, EPL, and 3→1. The performance of these distillation protocols is measured based on their probability of success and the fidelity of the output entangled state to the pure Bell state $|\phi_{00}\rangle$ conditioned on that the protocol succeeds. Combining these aspects, we also analyzed the trade-off between the fidelity of the output state and the probability of success. These metrics not only provide insight into the likelihood of a protocol to succeed and the EPR pair fidelity when the protocol succeeds, but also allows for an analysis of the overall trade-off between the success probability and the gain in fidelity of a protocol (some protocols may yield larger fidelity conditioned on a success that is less likely to happen!).

Depending on the protocol, the "success" of a protocol is established by the communicating parties (Alice and Bob) either when they obtain equal Z-basis measurements (as is the case with BBPSSW and DEJMPS), or when they both measure the eigenvalue 1 in the Z-basis (EPL), or when they measure the same Bell state in Bell basis measurements (3→1). All protocols are assumed to start with the Werner state as their initial state, which is modelled as a depolarizing channel with parameter p applied to the pure Bell state $|\phi_{00}\rangle$:

$$\rho_{\text{Werner}}(p) = p\,|\phi_{00}\rangle\langle\phi_{00}| + \frac{1-p}{4}\mathbb{I}_4, \tag{1}$$

Throughout this report, we consider the following formula for the fidelity between a mixed state ρ and a pure state $|\psi\rangle$:

$$\text{Fidelity}(\rho, |\psi\rangle) = \langle\psi|\,\rho\,|\psi\rangle\,. \tag{2}$$

To analyze the performance of each protocol, we performed simulations using NetQASM, which explored a subspace of the parameter grid space considered relevant for the research questions discussed in this report. The two parameters involved are (1) the EPR channel fidelity $p \in [0,1]$ defining the initial Werner state, and (2) the gate fidelity $g \in [0,1]$, which characterizes noise in unitary operations involved in distillation. Noisy unitaries are modelled as depolarizing channels applied after the unitaries, with g as the channel parameter:

$$U_{\text{noisy}}\rho U_{\text{noisy}}^{\dagger} = gU\rho U^{\dagger} + \frac{1-g}{4}\mathbb{I}_4 \tag{3}$$

We make a distinction between the *EPR channel fidelity* p, which characterizes the parameter of the depolarizing channel applied to the pure state $|\phi_{00}\rangle$ to obtain the Werner state $p\,|\phi_{00}\rangle\langle\phi_{00}| + \frac{1-p}{4}\mathbb{I}_4$, and the *EPR pair fidelity* which is given as the fidelity F of the Werner state to the pure state $|\phi_{00}\rangle$, where $F = p + \frac{1-p}{4} = \frac{1+3p}{4}$.

In the NetQASM simulations for each protocol, the first parameter sweep considered a fixed perfect gate fidelity $g = 1$ and varied the EPR channel fidelity p over the range $[0,1]$ in steps of 0.2. Conversely, the second parameter sweep considered a fixed perfect EPR channel fidelity $p = 1$ and varied the gate fidelity g over the range $[0,1]$ in steps of 0.2. For each parameter configuration (p, g), we performed a number of 500 samples, where a *sample* represents a single run of the NetQASM simulation program for a distillation protocol. We consider this amount to be large enough to result in statistically relevant measurements of the protocol performance. This amounts to a total of 5500 samples for each protocol. After collecting the outcomes from NetQASM simulations, we computed the success probability and the fidelity given success and compared these results to the theoretical formulas presented in the original publications of these protocols for the expected value of these metrics.

Additionally, we also compared the NetQASM simulation outcomes to results obtained from NumPy simulations of the protocols, which directly observes the evolution of the system state during distillation, without the need for sampling. For this latter experiment, we used our own implementations of the protocols using only the NumPy library (i.e., without NetQASM). This allows us to confirm the NetQASM simulations' outcomes, while also providing an approximate method of analysing the success probability and output state fidelity on the whole parameter grid, without the performance overhead involved in generating many samples.

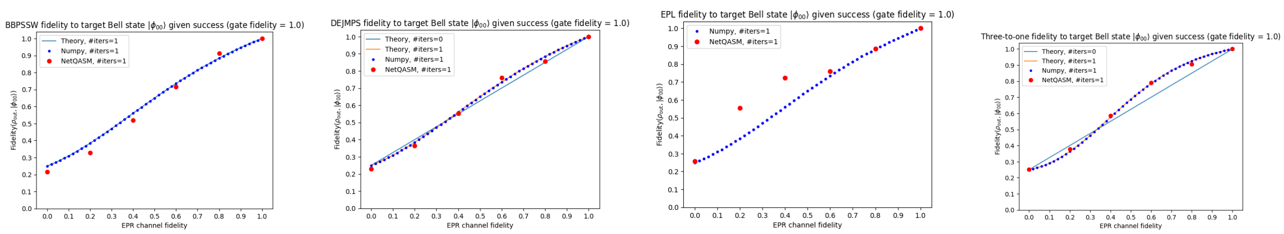
4 Results

In this section, we present the simulation results for each of the three research questions introduced earlier. For each question, we describe the experimental setup, the key performance metrics considered (such as final fidelity and success probability), and the trends observed across the four distillation protocols: BBPSSW, DEJMPS, EPL, and the 3→1 protocol. The results are summarized in graphical formats where appropriate, with plots illustrating protocol performance as a function of relevant noise parameters.

4.1 RQ1: Fidelity Scaling with EPR Channel Fidelity under Perfect Gates

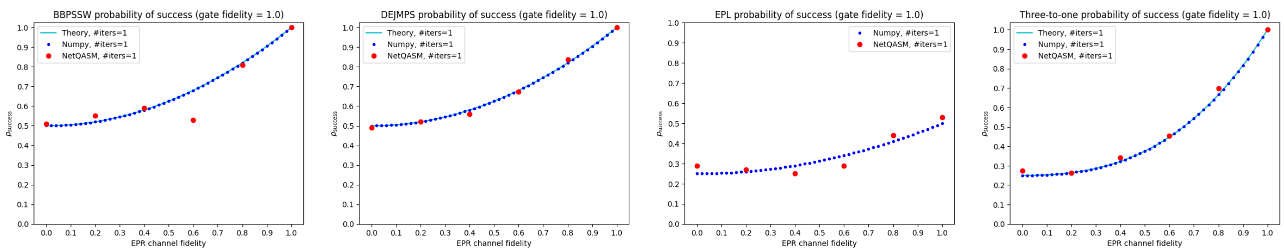
This section investigates how the final fidelity achieved by each protocol scales with the EPR channel fidelity. We focus on the case of perfect gates (gate fidelity = 1.0) to isolate the impact of initial state quality on protocol performance. The aim is to identify fidelity thresholds where each protocol starts offering improvement and to compare their performance in both low- and high-fidelity regimes.

Results Overview



BBPSSW DEJMPS EPL 3→1

(a) Fidelity to target Bell state given success (gate fidelity = 1.0)



BBPSSW DEJMPS EPL 3→1

(b) Probability of success (gate fidelity = 1.0)

Figure 1: *Protocol performance as a function of EPR channel fidelity with perfect gates. (a) shows the final fidelity of the output state given successful distillation. (b) shows the probability of successful distillation for each protocol.*

Discussion of fidelity results: The plots in Figure 1(a) reveal distinct performance characteristics across the four protocols. DEJMPS and the 3→1 protocol demonstrate superior fidelity enhancement, particularly in the mid-range of channel fidelities (0.6-0.9). Both protocols begin to outperform the input fidelity at lower thresholds compared to BBPSSW and EPL. The 3→1 protocol achieves the highest output fidelities overall, especially as the channel fidelity increases beyond 0.7, leveraging its additional resource requirement to produce higher-quality outputs. BBPSSW shows modest improvement but requires higher initial fidelities to be effective, while EPL demonstrates a more gradual improvement curve, only outperforming the input fidelity at very high initial fidelities, with some simulation outliers in the NetQASM results for EPL at lower EPR channel fidelities.

Discussion of success probability: Figure 1(b) illustrates an important trade-off: protocols that achieve higher fidelity improvements often do so at the cost of lower success probabilities. The 3→1 protocol, despite its superior fidelity performance, exhibits the lowest success probability across most of the channel fidelity range. DEJMPS maintains a reasonable balance between fidelity improvement and success probability, with success rates increasing as channel fidelity improves. BBPSSW shows a similar trend to DEJMPS but with slightly higher success probabilities. EPL shows the lowerst success probabilities across the channel fidelities.

Summary of Findings

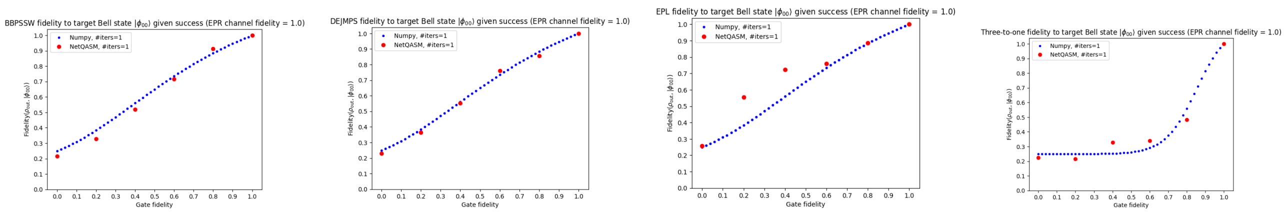
- With perfect gates, DEJMPS and the 3→1 protocol begin improving fidelity from channel fidelities as low as 0.6-0.65, while BBPSSW requires higher initial fidelities (around 0.75) to show improvement. EPL only improves fidelity at very high initial channel fidelities (above 0.9).
- The 3→1 protocol achieves the highest output fidelities but at the cost of the lowest success probabilities, demonstrating a clear resource-quality trade-off.
- DEJMPS offers a balanced performance profile with good fidelity improvement and moderate success probabilities, making it potentially more practical for many applications.
- BBPSSW performs similarly to DEJMPS in terms of fidelity improvement, but with slightly higher success probabilities.

These findings provide a foundational understanding of how each protocol behaves across different input fidelities under ideal gate conditions. The clear trade-offs between fidelity improvement and success probability highlight the importance of choosing the appropriate protocol based on specific application requirements. In the next sections, we will explore complementary scenarios where gate noise plays a more prominent role and evaluate overall robustness and practicality of each protocol.

4.2 RQ2: Protocol Performance under Perfect EPR Channel Fidelity and Varying Gate Fidelity

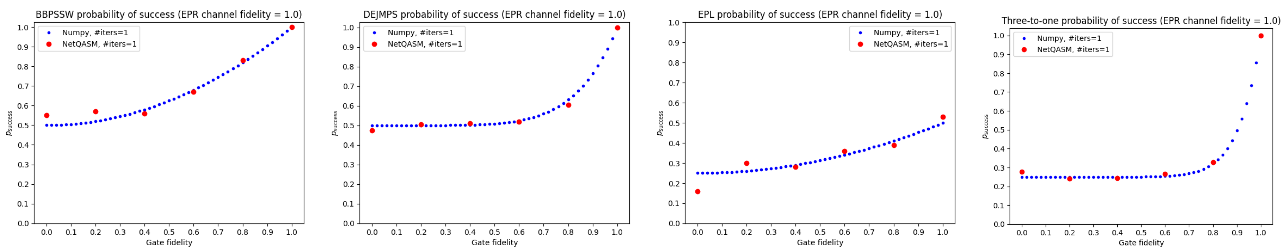
This section investigates how each protocol performs when starting from a perfect EPR channel fidelity (1.0) but using gates with varying fidelity. This scenario allows us to isolate the impact of gate imperfections on protocol performance when the initial entanglement is ideal. Our goal is to determine which protocol strikes the best balance between maintaining high output fidelity and achieving reasonable success probability as gate quality decreases.

Results Overview



BBPSSW DEJMPS EPL 3→1

(a) Fidelity to target Bell state given success (EPR Channel fidelity = 1.0)



BBPSSW DEJMPS EPL 3→1

(b) Probability of success (EPR Channel fidelity = 1.0)

Figure 2: *Protocol performance with perfect EPR channel fidelity (1.0) and varying gate fidelity. (a) shows final output fidelity given successful distillation; (b) shows probability of successful distillation.*

Discussion of fidelity results: Figure 2(a) reveals how gate fidelity directly impacts the quality of the distilled state. The 3→1 protocol demonstrates superior fidelity maintenance at high gate fidelities, but experiences the steepest degradation as gate fidelity decreases. DEJMPS shows resilience, maintaining relatively high output fidelity even as gate quality declines, outperforming other protocols in the mid-range of gate fidelities (0.8-0.95). BBPSSW exhibits a more gradual decline but achieves lower overall fidelities compared to DEJMPS. EPL shows moderate performance, with output fidelity declining at a rate between that of DEJMPS and BBPSSW. Notably, all protocols converge toward similar low fidelities when gate quality becomes very poor, suggesting a fundamental limit to distillation under severe gate noise.

Discussion of success probability: Figure 2(b) illustrates significant variations in success probability across protocols. EPL maintains the highest success probability across most of the gate fidelity range, making it particularly resource-efficient when throughput is prioritized. DEJMPS and BBPSSW show comparable success rates, with moderate decline as gate fidelity decreases. The 3→1 protocol exhibits the lowest success probability throughout, which is expected given its more complex structure requiring three initial pairs. Interestingly, the success probability curves flatten at very low gate fidelities, suggesting that beyond a certain noise threshold, additional gate errors have diminishing impact on protocol success rates.

Summary of Findings

- With perfect gates, all protocols achieve high output fidelity, with the 3→1 protocol reaching the highest values, followed closely by DEJMPS.
- As gate fidelity decreases, DEJMPS demonstrates the best resilience among the 2→1 protocols, maintaining higher output fidelity compared to BBPSSW and EPL.
- The 3→1 protocol is most sensitive to gate imperfections, showing the steepest decline in output fidelity as gate quality decreases.
- EPL offers the lowest success probability across most gate fidelity values, making it less suitable for applications with low gate fidelity.
- The trade-off between fidelity and success probability is protocol-dependent: DEJMPS offers a balanced compromise with good fidelity resilience and moderate success rates, while EPL prioritizes success probability at some cost to output fidelity.

These findings highlight the importance of considering both output fidelity and success probability when selecting a distillation protocol for practical implementations. While the 3→1 protocol can achieve superior fidelity with near-perfect gates, DEJMPS may be more suitable for realistic scenarios with moderate gate noise due to its balanced performance profile.

4.3 RQ3: Protocol Comparison Across Different Noise Regimes

This section provides a comprehensive comparison of all four protocols under different combinations of EPR channel fidelity and gate fidelity. By examining these two key noise sources, we can identify optimal protocol choices for different operating regimes and understand the fundamental trade-offs involved in entanglement distillation.

Results Overview

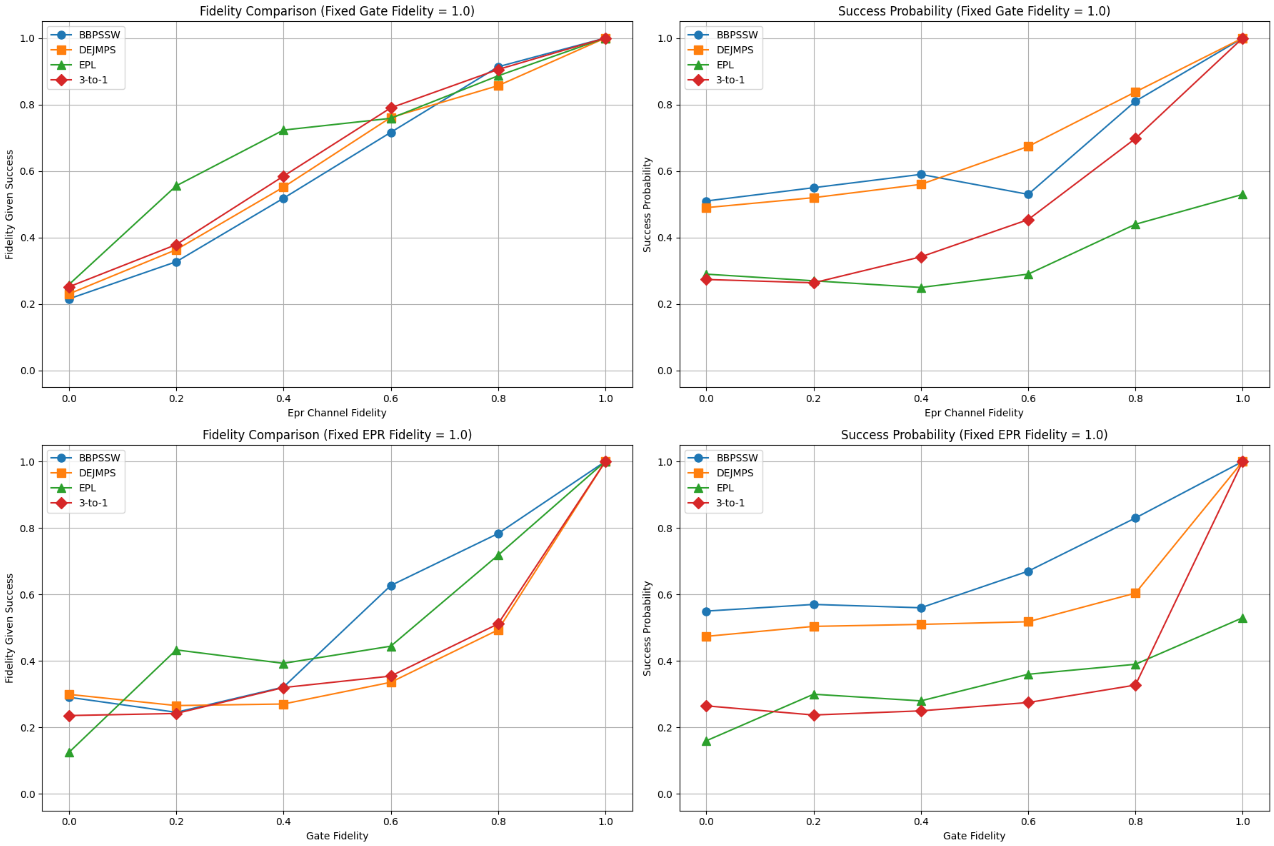


Figure 3: *Performance comparison of the four protocols. Top-left: Final fidelity of the output state as a function of EPR channel fidelity with perfect gates. Bottom-left: Final fidelity as a function of gate fidelity with perfect EPR channel. Top-right: Success probability as a function of EPR channel fidelity with perfect gates. Bottom-right: Success probability as a function of gate fidelity with EPR channel fidelity set to 1.*

Discussion of fidelity results: The top-left plot from Figure 3 shows how each protocol's output fidelity scales with EPR channel fidelity when gates are perfect. The 3→1 protocol achieves the highest output fidelities across most of the EPR channel fidelity range, followed closely by DEJMPS. BBPSSW shows modest improvement but requires higher initial fidelities to be effective, while EPL demonstrates a more gradual improvement curve, with some simulation outliers given very low EPR channel fidelities.

The bottom-left plot from Figure 3 reveals how gate fidelity impacts distillation performance when the initial EPR channel is perfect. DEJMPS demonstrates superior resilience to gate noise, maintaining relatively high output fidelity even as gate quality declines. The 3→1 protocol performs well at high gate fidelities but experiences the steepest degradation as gate fidelity decreases. BBPSSW exhibits a more gradual decline but achieves lower overall fidelities compared to DEJMPS.

Discussion of success probability: The success probability plots in Figure 3 highlight important trade-offs. EPL has the lowest success probabilities across most parameter combinations, especially when the gate fidelity is 1.0. DEJMPS and BBPSSW show comparable success rates, with moderate decline as noise increases. The 3→1 protocol exhibits the lowest success probability throughout, which is expected given its more complex structure requiring three initial pairs, only performing slighlty better with perfect gate fidelity, which is expected as it has one of the most amount of gates. This suggests that in resource-constrained environments, EPL may be preferable despite its more modest fidelity improvements.

Summary of Findings

- **Protocol specialization:** Each protocol demonstrates distinct strengths in different noise regimes. The 3→1 protocol excels at high gate fidelities, DEJMPS offers the best resilience to gate noise.
- **Fidelity-success trade-off:** Protocols that achieve higher output fidelities typically do so at the cost of lower success probabilities. This fundamental trade-off must be considered when selecting a protocol for

specific applications.

- **Noise sensitivity:** Gate noise generally has a more detrimental effect on protocol performance than EPR channel noise, suggesting that improving gate operations should be prioritized over initial state preparation in many scenarios.
- **Practical recommendation:** For near-term quantum networks with moderate noise levels, DEJMPS offers the most balanced performance profile, with good fidelity improvement, reasonable success rates, and strong resilience to gate noise.

These findings provide a comprehensive understanding of protocol behavior across different noise regimes, enabling informed protocol selection based on specific operating conditions and application requirements. The clear performance trade-offs identified here highlight the importance of matching distillation strategies to the particular noise characteristics of a given quantum network implementation.

5 Conclusions

In this work, we evaluated four prominent entanglement distillation protocols—BBPSSW, DEJMPS, EPL, and the 3→1 protocol—under realistic noise conditions using NetSquid and NetQASM simulations. Our analysis was structured around three core research questions, each targeting a distinct aspect of protocol performance: sensitivity to input fidelity, performance under gate noise, and robustness across noisy settings.

In response to **RQ1**, we found that all protocols improve fidelity when both the initial Werner-state fidelity and gate fidelity are sufficiently high. DEJMPS and the 3→1 protocol stand out by achieving improved output fidelity even at relatively modest initial fidelities, provided gate operations are nearly perfect. The threshold plots revealed that protocols vary significantly in the minimum input fidelity required for successful distillation, especially as gate noise increases.

Addressing **RQ2**, we observed that under perfect input EPR channel fidelity and moderately noisy gates ($p_U \simeq 0.8$), protocol performance diverges more clearly. DEJMPS showed a strong balance between maintaining high final fidelity and retaining a relatively high success probability. In contrast, protocols like 3→1, while capable of yielding very high output state fidelity, suffered from significantly lower success rates. These trade-offs are especially relevant in practical applications where throughput and reliability are critical.

For **RQ3**, we compared the performance of all the four protocols. As expected, increasing gate noise reduced performance across all protocols, but to different extents. DEJMPS and 3→1 proved particularly sensitive to gate imperfections, while BBPSSW and EPL retained acceptable performance down to $p_U \approx 0.60$. The 3→1 protocol, while powerful in idealized settings, required higher-quality gates to outperform its 2→1 counterparts, limiting its utility in noisy intermediate-scale quantum (NISQ) systems.

One limitation in our research is represented by the fact that we considered a fixed total of 500 samples for each parameter configuration of the gate fidelity and the EPR channel fidelity in the NetQASM simulations. More specifically, for lower values of these parameters, it is expected that the success probability will also be lower, resulting in less samples used in approximating the output state fidelity given conditioned on success. Nevertheless, we are still countering this effect to a certain extent, since the success probability generally decreases to $\frac{1}{5}$ (BBPSSW and DEJMPS) or $\frac{1}{2}$ (EPL and 3→1) for lower parameter values, meaning that at least ~ 100 samples are used for approximating the output state fidelity given success, which we still consider as a large enough amount.

Overall, our findings emphasize that the choice of distillation protocol should be carefully matched to the noise landscape of a given system. While high-resource protocols may offer stronger performance under ideal conditions, simpler protocols like DEJMPS and EPL may offer greater robustness and efficiency in near-term devices. Understanding these trade-offs is essential for designing scalable, noise-resilient quantum networks.

A Protocols

Here the four protocols whose properties are analyzed in this work are presented in more detail.

A.1 BBPSSW

The BBPSSW protocol uses two noisy Bell pairs and applies CNOT gates followed by measurement on one pair to improve the fidelity of the other (see figure 4).

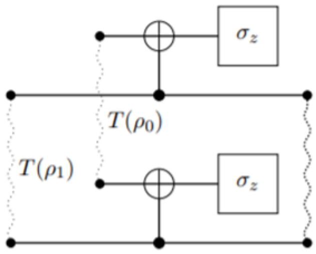


Figure 4: *Schematic circuit of the BBPSSW protocol. After a measurement in Z basis is performed on the* $|\Psi_{0,A}\rangle$ *and* $|\Psi_{0,B}\rangle$ *pair, the entanglement between the qubits forming* $T(\rho_0)$ *is broken, while the entanglement between those making up* $T(\rho_1)$ *remains intact.*

The theoretical fidelity of the state obtained after applying BBPSSW is:

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}, \tag{4}$$

where F is the fidelity of the initial state.

The probability of success is:

$$p_{\text{succ}} = F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2 \tag{5}$$

A.2 DEJMPS

An improvement over BBPSSW, DEJMPS also uses two entangled pairs but applies local basis rotations (unitaries) before the CNOT gates (see figure 5). This reduces phase-flip errors more effectively and is particularly useful in scenarios where noise is not symmetric.

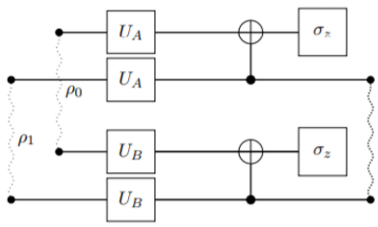


Figure 5: *Schematic circuit of the DEJMPS protocol. Bilateral local operations are applied on all four qubits:* $|\Psi_{0,A}\rangle$, $|\Psi_{1,A}\rangle$ *and* $|\Psi_{0,B}\rangle$, $|\Psi_{1,B}\rangle$ *After a measurement in Z basis is performed on the* $|\Psi_{0,A}\rangle$ *and* $|\Psi_{0,B}\rangle$ *pair, the entanglement between the qubits forming* ρ_0 *is broken, while the entanglement between those making up* ρ_1 *remains intact.*

The unitaries that the protocol makes use of are the following:

$$\begin{aligned} U_A|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \\ U_A|1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - i|0\rangle) \\ U_B|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\ U_B|1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + i|0\rangle) \end{aligned} \tag{6}$$

The fidelity of the output state, as well as the probability of success have the same expressions as for BBPSSW.

A.3 EPL

The Extreme Photon Loss Protocol (EPL) is an entanglement purification method designed for quantum channels where photons are frequently lost. It uses a single photon detection system, which means it is successful when exactly one photon is detected. Currently, there are two versions of this protocol: the original one, introduced in [2], and a simplified version described in [3], which is more practical, as it uses fewer resources and simpler operations. While the former method uses CPHASE gates and involves performing measurements in the X basis, the latter uses CNOTs and measures in Z basis, being similar with BBPSSW in this respect(see figure 6). However, unlike BBPSSW, which succeeds when the measurement outcomes are equal (either 00 or 11), EPL only succeeds when both outcomes are 1.

The optimal type of input state when implementing this protocol is:

$$\rho_\phi = p|\Phi_\phi\rangle\langle\Phi_\phi| + (1-p)|11\rangle\langle 11|, \text{where } |\Phi_\phi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + e^{i\phi}|10\rangle \tag{7}$$

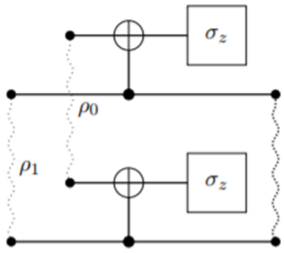


Figure 6: *Schematic circuit of the EPL protocol. After a measurement in Z basis is performed on the $|\Psi_{0,A}\rangle$ and $|\Psi_{0,B}\rangle$ pair, the entanglement between the qubits forming ρ_0 is broken, while the entanglement between those making up ρ_1 remains intact.*

The probability of success with these conditions is:

$$p_{\text{succ}} = \frac{p^2}{2} \tag{8}$$

In this work, we used a Werner input state, for which the literature does not provide theoretical values for the fidelity and success probability. This is why the corresponding theoretical curves are absent from the plots.

A.4 3-to-1 Protocol

The 3-to-1 protocol uses three low-fidelity entangled pairs to probabilistically distill one higher-fidelity pair. It involves more rounds of gates and measurements (see figure 7) compared to 2-to-1 schemes, but can achieve better error tolerance and fidelity. It is useful in contexts where pair generation is cheap or errors are high.

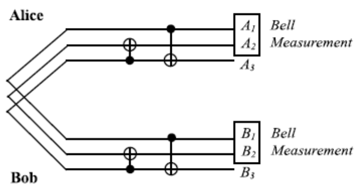


Figure 7: *Schematic circuit of the 3-to-1 protocol. Bell measurements are applied on the two-qubit states $|\Psi_{1,A}\rangle \otimes |\Psi_{2,A}\rangle$ and $|\Psi_{1,B}\rangle \otimes |\Psi_{2,B}\rangle$. The entanglement between these pairs is broken, while the entanglement between qubits $|\Psi_{A,3}\rangle$ and $|\Psi_{B,3}\rangle$ remains intact.*

The theoretical fidelity of the circuit's output state is:

$$F' = \frac{2 - 7F + 14F^2}{7 - 14F + 16F^2}, \tag{9}$$

whereas the probability of success is:

$$p_{\text{succ}} = \frac{(1+2F)(7-14F+16F^2)}{27} \tag{10}$$

References

[1] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, *Quantum privacy amplification and the security of quantum cryptography over noisy channels*, Physical Review Letters, vol. 77, no. 13, p. 2818, 1996.

[2] E. T. Campbell and S. C. Benjamin, *Measurement-based entanglement under conditions of extreme photon loss*, Physical Review Letters, vol. 101, no. 13, p. 130502, 2008.

[3] N. H. Nickerson, J. F. Fitzsimons, and S. C. Benjamin, *Freely scalable quantum technologies using cells of 5-to-50 qubits with very lossy and noisy photonic links*, Physical Review X, vol. 4, no. 4, p. 041041, 2014.

[4] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, *Purification of noisy entanglement and faithful teleportation via noisy channels*, Physical Review Letters, vol. 76, no. 5, p. 722, 1996.

[5] D. P. Chi, T. Kim, and S. Lee, *Efficient three-to-one entanglement purification protocol*, Physics Letters A, vol. 376, no. 3, pp. 143–146, 2012.

[6] A. Dahlberg, B. van der Vecht, C. Delle Donne, M. Skrzypczyk, I. te Raa, W. Kozlowski, and S. Wehner, *NetQASM – A low-level instruction set architecture for hybrid quantum-classical programs in a quantum internet*, Quantum Science and Technology, 2022.

[7] T. Coopmans, R. Knegjens, A. Dahlberg, D. Maier, L. Nijsten, J. de Oliveira Filho, M. Papendrecht, J. Rabbie, F. Rozpedek, M. Skrzypczyk, et al., *NetSquid, a network simulator for quantum networks*, 2020. [Online]. Available: https://www.netsquid.org