Question 2. Let $U_1 \sim U_{[x_1,y_1]}$ and $U_2 \sim U_{[x_2,y_2]}$ be two independent random variables and, for simplicity, we suppose that $x_1 \le y_1 \le x_2 \le y_2$.

1. Compute the probability density of $Y := U_1 + U_2$ and sketch it.

2. For what values of the parameters $x_i, y_i \in \mathbb{R}_+$ for $i = 1,2$ does Y follow the triangular distribution, i.e. the probability function of Y is of the form given below in (1).

A random variable X is said to follow a triangular distribution with parameters (a, b, c) if

$$f_X(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & a < x \le c \\ \frac{2(b-x)}{(b-a)(b-c)} & c < x \le b \end{cases} \tag{1}$$

To answer the following questions, let us take $a = 0$ and $c = \frac{a+b}{2} = \frac{b}{2}$.

3. For a fixed n, find the MLE $\hat{b}$ of b.

4. Find the method of moment estimate of b.

Solution.

1. For clarity, let $Z = U_1 + U_2$ instead. First note that U_1 and U_2 are independent uniformly distributed continuous random variables. Then from the convolution formula,

$$f_Z(z) = \int_{x_2}^{y_2} f_{U_1}(z-u) f_{U_2}(u)\, \mathrm{d}u. \tag{1}$$

Note that

$$f_{U_1}(u) = I_{\{x_1<u<y_1\}\times\frac{1}{y_1-x_1}} \implies f_{U_1}(z-u) = I_{\{x_1<z-u<y_1\}\times\frac{1}{y_1-x_1}}$$
$$= I_{\{z-y_1<u<z-x_1\}\times\frac{1}{y_1-x_1}}$$

and

$$f_{U_2} = I_{\{x_2<u<y_2\}} \times \frac{1}{y_2-x_2}.$$

Then substituting f_{U_1} and f_{U_2} into $(*)$ gives

$$\begin{aligned} f_Z(z) &= \int_{x_2}^{y_2} I_{\{z-y_1<u<z-x_1\}} \left(\frac{1}{y_1-x_1} \times \frac{1}{y_2-x_2}\right) \mathrm{d}u \\ &= \int_{\max(x_2,z-y_1)}^{\min(y_2,z-x_1)} \left(\frac{1}{y_1-x_1} \times \frac{1}{y_2-x_2}\right) \mathrm{d}u. \end{aligned}$$

But the integrand above is a constant. Hence we have

$$f_Z(z) = (\min(y_2, z - x_1) - \max(x_2, z - y_1)) \left(\frac{1}{y_1 - x_1} \times \frac{1}{y_2 - x_2} \right), \tag{2}$$

for $z \in (x_2, y_2) \cap (z - y_1, z - x_1)$.

We wish to simplify the interval. This will depend on the value of z. Consider the following cases.

(a) First take $z \le x_1 + x_2$. Then $z - y_1 \le z - x_1 \le x_2$ so $(x_2, y_2) \cap (z - y_1, z - x_1) = \emptyset$ and

$$f_Z(z) = 0.$$

(b) Next take $x_1 + x_2 \le z < \min(x_1 + y_2, x_2 + y_1)$. If $\min(x_1 + y_2, x_2 + y_1) = x_1 + y_2$ then

$$z \le x_1 + y_2 < x_2 + y_1 \implies z - x_1 \le y_2 < x_2 + y_1 - x_1, \quad z - y_1 \le x_1 + y_2 - y_1 < x_2$$
$$\therefore (2) \implies f_Z(z) = (z - x_1 - x_2) \left(\frac{1}{y_1 - x_1} \times \frac{1}{y_2 - x_2} \right),$$

and, similarly, if $\min(x_1 + y_2, x_2 + y_1) = x_2 + y_1$ then

$$f_Z(z) = (z - x_1 - x_2) \left(\frac{1}{y_1 - x_1} \times \frac{1}{y_2 - x_2} \right).$$

(c) Next consider $\min(x_1+y_2, x_2+y_1) < z < \max(x_1+y_2, x_2+y_1)$. If $\min(x_1+y_2, x_2+y_1) = x_1+y_2$ so $\min(x_2 + y_1, x_1 + y_2) = x_1 + y_2$ then

$$x_1 + y_2 < z < x_2 + y_1 \implies y_2 < z - x_1 < x_2 + y_1 - x_1, \quad x_1 + y_2 - y_1 < z - y_1 < x_2$$
$$\therefore (2) \implies f_Z(z) = (y_2 - x_2) \left(\frac{1}{y_1 - x_1} \times \frac{1}{y_2 - x_2} \right),$$

and, similarly, if $\min(x_1 + y_2, x_2 + y_1) = x_2 + y_1$ then

$$f_Z(z) = (y_1 - x_1) \left(\frac{1}{y_1 - x_1} \times \frac{1}{y_2 - x_2} \right).$$

But $x_1 + y_2 < x_2 + y_1 \implies y_2 - x_2 < y_1 - x_1$ and $x_2 + y_1 < x_1 + y_2 \implies y_1 - x_1 < y_2 - x_2$. Hence we can summarise the result of this third case as

$$f_Z(z) = \frac{\min(y_1 - x_1, y_2 - x_2)}{(y_1 - x_1)(y_2 - x_2)}.$$

(d) Now take $\max(x_1+y_2, x_2+y_1) \le z < y_1+y_2$. First, if $\max(x_1+y_2, x_2+y_1) = x_1+y_2$ then

$$x_2+y_1 \le x_1+y_2 \le z \implies x_2+y_1-x_1 \le y_2 \le z-x_1,\ x_2 \le x_1+y_2 \le z-y_1$$
$$\therefore (2) \implies f_Z(z) = (y_2 - z + y_1)\left(\frac{1}{y_1-x_1} \times \frac{1}{y_2-x_2}\right),$$

and second, if $\min(x_1+y_2, x_2+y_1) = x_2+y_1$ then, similarly,

$$f_Z(z) = (y_2 - z - y_1)\left(\frac{1}{y_1-x_1} \times \frac{1}{y_2-x_2}\right).$$

(e) Finally, take $z \ge y_1+y_2$. Then $z-x_1 \ge z-y_1 \ge y_2$ so $(x_2,y_2)\cap(z-y_1,z-x_1) = \emptyset$ and

$$f_Z(z) = 0.$$

From these cases, we have that the probability density of $Z = U_1+U_2$ in full is given by

$$f_Z(z) = \begin{cases} \dfrac{z-(x_1+x_2)}{(y_1-x_1)(y_2-x_2)} & x_1+x_2 < z \le \min(x_1+y_2, x_2+y_1) \\ \dfrac{\min(y_1-x_1, y_2-x_2)}{(y_1-x_1)(y_2-x_2)} & \min(x_1+y_2, x_2+y_1) < z < \max(x_1+y_2, x_2+y_1) \\ \dfrac{-z+(y_2-y_1)}{(y_1-x_1)(y_2-x_2)} & \max(x_1+y_2, x_2+y_1) \le z < y_1+y_2 \\ 0 & \text{otherwise} \end{cases}$$

It is sketched below.

2. First note that the shape of the probability density function of $f_Z(z)$ is a trapezium. Then, clearly, we achieve a triangular distribution the interval

$$|\min(x_1+y_2, x_2+y_1) < z < \max(x_1+y_2, x_2+y_1)| = 0, \tag{3}$$

and so the top edge of trapezium vanishes.

But

$$(3) \iff \min(x_1+y_2, x_2+y_1) = \max(x_1+y_2, x_2+y_1),$$

and this is only possible when $x_1 + y_2 = x_2 + y_1$.

3. From the question, the probability density function is

$$\begin{aligned} f_X(x) &= \begin{cases} \frac{4x}{b^2} & 0 < x \leq \frac{b}{2} \\ \frac{4(b-x)}{b^2} & \frac{b}{2} < x \leq b \end{cases} \\ &= \frac{4x}{b^2} I_{\{0<x\leq\frac{b}{2}\}} + \frac{4(b-x)}{b^2} I_{\{\frac{b}{2}<x\leq b\}} \end{aligned}$$

Then

$$\mathcal{L}(x; b_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x) = \prod_{i=1}^{n} \left(\left(\frac{4x}{b^2}\right)^i I_{\{0<x\leq\frac{b}{2}\}} + \left(\frac{4(b-x)}{b^2}\right)^i I_{\{\frac{b}{2}<x\leq b\}} \right).$$

Fixing $n = 1$ we have

$$\mathcal{L}(x; b_1) = \frac{4x}{b^2} I_{\{0<x\leq\frac{b}{2}\}} + \frac{4(b-x)}{b^2} I_{\{\frac{b}{2}<x\leq b\}}.$$

Note that $\mathcal{L}(x; b_1)$ is strictly increasing for $0 < x \leq \frac{b}{2}$ and strictly decreasing for $\frac{b}{2} < x \leq b$. Thus $b = 2x$ maximises $f_X(x; b)$ and so $\hat{b} = 2x$.

4. For $k = 1$ the method of moment equations are

$$\begin{aligned} \mathbb{E}(X) &= \int_a^b x f_X(x)\, \mathrm{d}x \\ &= \int_0^b \frac{4x^2}{b^2} I_{\{0<x\leq\frac{b}{2}\}}\, \mathrm{d}x + \int_{\frac{b}{2}}^b \frac{4x(b-x)}{b^2} I_{\{\frac{b}{2}<x\leq b\}}\, \mathrm{d}x \\ &= \frac{1}{2} - \frac{b}{3}. \end{aligned}$$

But

$$\mathbb{E}(X) = \frac{b}{2},$$

which leads to the system of equations

$$\frac{1}{2} - \frac{b}{3} = \frac{b}{2}.$$

Thus, $b = 0.6$.