Math2901: Group Assignment

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Question 1 Solutions

Let $A, B \subseteq \Omega$.

1.

Given that event A is independent of itself, this means that $\mathbb{P}(A|A) = \mathbb{P}(A)$. This implies $\mathbb{P}(A) = \mathbb{P}(A)$ and so $\mathbb{P}(A)$ must equal either 1 or 0.

2.

Given event A such that $\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$, we observe the conditional probability between events A and B.

For the case where $\mathbb{P}(A) = 1$:

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\ &= \mathbb{P}(B \cap A) \\ &= \mathbb{P}(B).\end{aligned}$$

We reach this result because the intersection with a guaranteed event will always be the probability of the other event.

For the case where $\mathbb{P}(A) = 0$:

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(B)}{\mathbb{P}(B)} \\ &= 0 \\ &= \mathbb{P}(A).\end{aligned}$$

Since the union with an impossible event always returns the probability of the other event, we reach our result.

3.

Note, by the Total Law of Probability, $\mathbb{P}(A \cap B) \leq 1$.

Now observe,

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
1 &\geq \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
\mathbb{P}(A \cap B) &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1.
\end{aligned}$$

The inequality has been proven, thus we are done.

4.

Using the previous proven inequality,

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
\mathbb{P}(A_1 \cap A_2 \cap A_3) &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 + \mathbb{P}(A_3) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - 2 \\
&\vdots \\
\mathbb{P}(\bigcap_{i=1}^{n} A_i) &\geq \sum_{i=1}^{n} A_i - (n-1).
\end{aligned}$$

We have thus proved the inequality.

Question 2 Solutions

Question 3 Solutions

1.

We can see that $\tilde{X_n} = \bar{X_n} + \frac{1}{n}(2X_1) - \frac{1}{n}(X_{n-1} + X_n)$. Now calculating the bias of $\tilde{X_n}$

$$\begin{aligned}
\text{Bias}(\tilde{X_n}) &= \mathrm{E}(\tilde{X_n}) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i + 2X_1 - (X_{n-1} + X_n)\right) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{1}{n}\mathrm{E}(2X_1) - \frac{1}{n}\mathrm{E}(X_{n-1} + X_n) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{2}{n}\mathrm{E}(X_1) - \frac{1}{n}\mathrm{E}(X_{n-1}) - \frac{1}{n}\mathrm{E}(X_n) - \mu.
\end{aligned}$$

Because the random sample is independent identically distributed, the expected value of each variable in the sample is equal to the mean. This means that our simplified equation becomes

$$\mu + \frac{2}{n}\mu - \frac{1}{n}\mu - \frac{1}{n}\mu - \mu = 0.$$

We have thus shown that $\tilde{X_n}$ is an unbiased estimator of μ.

2.

The mean square error can be written as

$$\begin{aligned}
\text{MSE}(\tilde{X_n}) &= \text{Var}(\tilde{X_n}) + (\text{Bias}(\tilde{X_n}))^2 \\
&= \text{Var}(\tilde{X_n}) \\
&= \text{Var}\left(\frac{3X_1 + \sum_{i=2}^{n-2} X_i}{n}\right) \\
&= \frac{1}{n^2}\left(9\text{Var}(X_1) + \sum_{i=2}^{n-2} \text{Var}(X_i)\right) \\
&= \frac{9\sigma^2 + \sigma^2(n-3)}{n} \\
&= \frac{6\sigma^2 + n\sigma^2}{n^2}.
\end{aligned}$$

Thus, the MSE is $\frac{1}{n^2}(6\sigma^2 + n\sigma^2)$.

3.

$$\begin{aligned}
\lim_{n\to\infty} \text{MSE}(\tilde{X_n}) &= \lim_{n\to\infty} \frac{1}{n^2}(6\sigma^2 + n\sigma^2) \\
&= 0.
\end{aligned}$$

4.

To measure the better estimate of μ, we can observe the variances of each

$$\text{Var}(\bar{X_n}) = \frac{\sigma^2}{n} \leq \frac{1}{n^2}(6\sigma^2 + n\sigma^2) = \text{Var}(\tilde{X_n}).$$

Since the variance of $\bar{X_n}$ is lesser, it is the better estimator.

Question 4 Solutions

Part I:

a) Null Hypothesis: H_0: $\mu = 5$

Alternative Hypothesis: H_1: $\mu \neq 5$

b)

Part II:

a)