Math2901: Group Assignment

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Question 1 Solutions

Let $A, B \subseteq \Omega$.

1.

Given that event A is independent of itself, this means that $\mathbb{P}(A|A) = \mathbb{P}(A)$. This implies $\mathbb{P}(A) = \mathbb{P}(A)$ and so $\mathbb{P}(A)$ must equal either 1 or 0.

2.

Given event A such that $\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$, we observe the conditional probability between events A and B.

For the case where $\mathbb{P}(A) = 1$:

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\ &= \mathbb{P}(B \cap A) \\ &= \mathbb{P}(B).\end{aligned}$$

We reach this result because the intersection with a guaranteed event will always be the probability of the other event.

For the case where $\mathbb{P}(A) = 0$:

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(B)}{\mathbb{P}(B)} \\ &= 0 \\ &= \mathbb{P}(A).\end{aligned}$$

Since the union with an impossible event always returns the probability of the other event, we reach our result.

3.

Note, by the Total Law of Probability, $\mathbb{P}(A \cap B) \leq 0$.

Now observe,

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
1 &\geq \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
\mathbb{P}(A \cap B) \ &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1.
\end{aligned}$$

The inequality has been proven, thus we are done.

4.

Using the previous proven inequality,

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) \ &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
\mathbb{P}(A_1 \cap A_2 \cap A_3) \ &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 + \mathbb{P}(A_3) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - 2 \\
&\vdots \\
\mathbb{P}(\bigcap_{i=1}^{n} A_i) \ &\geq \sum_{i=1}^{n} A_i - (n-1).
\end{aligned}$$

We have thus proved the inequality.

Question 2 Solutions

Question 3 Solutions

1.

We can see that $\tilde{X_n} = \bar{X_n} + \frac{1}{n}(2X_1) - \frac{1}{n}(X_{n-1} + X_n)$. Now calculating the bias of $\tilde{X_n}$

$$\begin{aligned}
\text{Bias}(\tilde{X_n}) &= \mathrm{E}(\tilde{X_n}) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i + 2X_1 - (X_{n-1} + X_n)\right) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{1}{n}\mathrm{E}(2X_1) - \frac{1}{n}\mathrm{E}(X_{n-1} + X_n)\mu.
\end{aligned}$$

Because the random sample is independent identiclaly distributed, the expected value of each variable in the sample is equal to the mean. This means that our simplified equation becomes

$$\begin{aligned}
\frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{2}{n}\mathrm{E}(X_1) - \frac{1}{n}\mathrm{E}(X_{n-1}) - \frac{1}{n}\mathrm{E}(X_n)\mu &= \mu + \frac{2}{n}\mu - \frac{1}{n}\mu - \frac{1}{n}\mu - \mu \\
&= 0.
\end{aligned}$$

We have thus shown that $\tilde{X_n}$ is an unbiased estimator of μ.

2.

The mean square error can be written as

$$
\begin{aligned}
\mathrm{MSE}(\tilde{X_n}) &= \mathrm{Var}(\tilde{X_n}) + (\mathrm{Bias}(\tilde{X_n}))^2 \\
&= \mathrm{Var}(\tilde{X_n}) \\
&= \mathrm{E}(\tilde{X_n}^2) - \mathrm{E}(\tilde{X_n})^2 \\
&= \mathrm{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i + 2X_1 - X_{n-1} - X_n\right)^2\right] - \mu^2 \\
&= \frac{1}{n^2}\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2 + 2\left(\sum_{i=1}^{n}\right)(2X_1 - X_{n-1} - X_n) + (2X_1 - X_{n-1} - X_n)^2\right] - \mu^2 \\
&= \frac{1}{n^2}\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] + \frac{2}{n^2}\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)(2X_1 - X_{n-1} - X_n)\right] + \frac{1}{n^2}\mathrm{E}\left[(2X_1 - X_{n-1} - X_n)^2\right] - \mu^2 \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) \times \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{2}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) \times \frac{1}{n}\mathrm{E}(2X_1 - X_{n-1} - X_n) + \frac{1}{n}\mathrm{E}(2X_1 - X_{n-1} - X_n) \times \frac{1}{n}\mathrm{E}(2X \\
&= \mu^2 + 2\mu \times \frac{1}{n} + \left[\frac{1}{n}\mathrm{E}(2X_1 - X_{n-1} - X_n)\right]^2 - \mu^2 \\
&= 2\mu \times \frac{1}{n}(2\mu - \mu - \mu) + \frac{1}{n^2}(2\mu - \mu - \mu)^2 \\
&= 0.
\end{aligned}
$$

Thus, the MSE is 0.

3.

$$
\begin{aligned}
\lim_{n\to\infty} \mathrm{MSE}(\tilde{X_n}) &= \lim_{n\to\infty} 0 \\
&= 0.
\end{aligned}
$$

4.

Measure consistency of both estimators.

Question 4 Solutions