Part II. Solution.

1. From the assumption, $X_i \sim \exp(\lambda)$ and $f_{X_i}(x_i, \lambda) = \lambda e^{-\lambda x}$. Then the likelihood function is given by

$$\mathcal{L}(\lambda; x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(\lambda; x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

and the log-likelihood function is given by

$$\ell(\lambda; x_1, \ldots, x_n) = \ln\{\mathcal{L}(\lambda; x_1, \ldots, x_n)\} = \sum_{i=1}^{n} \ln \mathcal{L}(\lambda; x_i).$$

Observe

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\ell(\lambda; x_1, \ldots, x_n) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sum_{i=1}^{n} \ln \mathcal{L}(\lambda; x_i)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sum_{i=1}^{n} \ln\left(\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sum_{i=1}^{n} \ln\left(\lambda^n e^{-\lambda \sum_{i=1}^{n}}\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(n \ln \lambda - \lambda \sum_{i=1}^{n} x_i\right) \\
&= \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \\
&= 0 \iff \lambda = \frac{n}{\sum_{i=1}^{n} x_i}. \qquad (1)
\end{aligned}$$

Noting

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left(\ell(\lambda; x_1, \ldots, x_n) = -\frac{n}{\lambda^2}\right) < 0 \quad \forall \lambda \in \mathbb{R}$$

shows the value for λ in (1) is indeed a maximum and the argmax. Thus, $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$.

2. We wish to approximate the distribution of the MLE $\hat{\lambda}$ using its Fisher score and information. From the results in the previous part, the Fisher score can be given by

$$S_n(\lambda) = \ell_n'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i,$$

and the Fisher information by

$$I_n(\lambda) = -\mathbb{E}_\lambda \ell_n''(\lambda) = -\mathbb{E}\left(-\frac{n}{\lambda^2}\right) = \frac{n}{\lambda^2}$$

Note that $\ell(\lambda)$ is a smooth, thrice-differentiable function. So. Finally, from the asymptotic normality of maximum likelihood estimators theorem, we have

$$\frac{\hat{\lambda}-\lambda}{\sqrt{\mathrm{Var}(\hat{\lambda})}} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$$

where

$$\mathrm{Var}(\hat{\lambda}) \xrightarrow{d} \frac{1}{-\mathbb{E}\left(-\frac{n}{\lambda^2}\right)} = \frac{\lambda^2}{n},$$

and

$$\frac{\hat{\lambda}-\lambda}{\mathrm{se}(\hat{\lambda})} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$$

where $\mathrm{se}(\hat{\lambda})$ denotes the asymptotic standard error of $\hat{\lambda}$ and is given by

$$\mathrm{se}(\hat{\lambda}) = \frac{1}{\sqrt{I_n(\lambda)}} = \frac{\lambda}{\sqrt{n}},$$

so that we can say

$$\hat{\lambda} \overset{\mathrm{appr.}}{\sim} \mathrm{N}(\lambda, \frac{\lambda^2}{n}).$$

3. The hypotheses to be tested are:

$$H_0 : \lambda = 1 \quad \text{(average waiting time matches question proposal)}$$

versus

$$H_1 : \lambda \neq 1 \quad \text{(average waiting time does not match question proposal).}$$

The test statistic we will use is

$$\frac{\hat{\lambda}-\lambda}{S/\sqrt{5}}$$

from the previous part, which has a t_13 distribution if the null hypothesis is true. The observed value is

$$\frac{\hat{\lambda}-\lambda}{S/\sqrt{5}} = \frac{\overline{x}-1}{\frac{0.2}{\sqrt{5}}} = \frac{1.18092-1}{\frac{0.2}{\sqrt{5}}} = 2.0227\ldots$$

Then, using t-distribution tables,

$$\begin{aligned} P\text{-value} &= \mathbb{P}_{\lambda=1}\left(\frac{\hat{\lambda}-1}{S/\sqrt{5}}\right) \\ &= 2\mathbb{P}(T > 2.0227), \quad T \sim t_13 \\ &\approx 0.114 \end{aligned}$$

so there is little or no evidence against H_0.