Math2901: Group Assignment

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Question 1 Solutions

Let $A, B \subseteq \Omega$.

1.

Given that event A is independent of itself, this means that $\mathbb{P}(A|A) = \mathbb{P}(A)$. This implies $\mathbb{P}(A) = \mathbb{P}(A)$ and so $\mathbb{P}(A)$ must equal either 1 or 0.

2.

Given event A such that $\mathbb{P}(A) = 1$ or $\mathbb{P}(A) = 0$, we observe the conditional probability between events A and B.

For the case where $\mathbb{P}(A) = 1$:

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\ &= \mathbb{P}(B \cap A) \\ &= \mathbb{P}(B).\end{aligned}$$

We reach this result because the intersection with a guaranteed event will always be the probability of the other event.

For the case where $\mathbb{P}(A) = 0$:

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(A \cup B)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}(B) - \mathbb{P}(B)}{\mathbb{P}(B)} \\ &= 0 \\ &= \mathbb{P}(A).\end{aligned}$$

Since the union with an impossible event always returns the probability of the other event, we reach our result.

3.

Note, by the Total Law of Probability, $\mathbb{P}(A \cap B) \leq 1$.

Now observe,

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
1 &\geq \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
\mathbb{P}(A \cap B) \ &\geq \mathbb{P}(A) + \mathbb{P}(B) - 1.
\end{aligned}$$

The inequality has been proven, thus we are done.

4.

Using the previous proven inequality,

$$\begin{aligned}
\mathbb{P}(A_1 \cap A_2) \ &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
\mathbb{P}(A_1 \cap A_2 \cap A_3) \ &\geq \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - 1 + \mathbb{P}(A_3) - 1 \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - 2 \\
&\vdots \\
\mathbb{P}(\bigcap_{i=1}^{n} A_i) \ &\geq \sum_{i=1}^{n} A_i - (n-1).
\end{aligned}$$

We have thus proved the inequality.

Question 2 Solutions

Question 3 Solutions

1.

We can see that $\tilde{X_n} = \bar{X_n} + \frac{1}{n}(2X_1) - \frac{1}{n}\left(X_{n-1} + X_n\right)$. Now calculating the bias of $\tilde{X_n}$

$$\begin{aligned}
\text{Bias}(\tilde{X_n}) &= \mathrm{E}(\tilde{X_n}) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i + 2X_1 - (X_{n-1} + X_n)\right) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{1}{n}\mathrm{E}(2X_1) - \frac{1}{n}\mathrm{E}(X_{n-1} + X_n) - \mu \\
&= \frac{1}{n}\mathrm{E}\left(\sum_{i=1}^{n} X_i\right) + \frac{2}{n}\mathrm{E}(X_1) - \frac{1}{n}\mathrm{E}(X_{n-1}) - \frac{1}{n}\mathrm{E}(X_n) - \mu.
\end{aligned}$$

Because the random sample is independent identically distributed, the expected value of each variable in the sample is equal to the mean. This means that our simplified equation becomes

$$\mu + \frac{2}{n}\mu - \frac{1}{n}\mu - \frac{1}{n}\mu - \mu = 0.$$

We have thus shown that $\tilde{X_n}$ is an unbiased estimator of μ.

2.

The mean square error can be written as

$$\begin{aligned}
\text{MSE}(\tilde{X_n}) &= \text{Var}(\tilde{X_n}) + (\text{Bias}(\tilde{X_n}))^2 \\
&= \text{Var}(\tilde{X_n}) \\
&= \text{Var}\left(\frac{3X_1 + \sum_{i=2}^{n-2} X_i}{n}\right) \\
&= \frac{1}{n^2}\left(9\text{Var}(X_1) + \sum_{i=2}^{n-2} \text{Var}(X_i)\right) \\
&= \frac{9\sigma^2 + \sigma^2(n-3)}{n} \\
&= \frac{6\sigma^2 + n\sigma^2}{n^2}.
\end{aligned}$$

Thus, the MSE is $\frac{1}{n^2}(6\sigma^2 + n\sigma^2)$.

3.

$$\begin{aligned}
\lim_{n\to\infty} \text{MSE}(\tilde{X_n}) &= \lim_{n\to\infty} \frac{1}{n^2}(6\sigma^2 + n\sigma^2) \\
&= 0.
\end{aligned}$$

4.

To measure the better estimate of μ, we can observe the variances of each

$$\text{Var}(\bar{X_n}) = \frac{\sigma^2}{n} \leq \frac{1}{n^2}(6\sigma^2 + n\sigma^2) = \text{Var}(\tilde{X_n}).$$

Since the variance of $\bar{X_n}$ is lesser, it is the better estimator.

Question 4 Solutions

Part I:

a) Null Hypothesis: H_0: $\mu = 5$

Alternative Hypothesis: H_1: $\mu \neq 5$

NOTE: For parts b) and c), we will use the rejection region method instead of the asymptotic test due to content restrictions.
b)

The rejection region method can be used here because we reject our null hypothesis, especially after the results show a different mean ($\mu = 6$), supporting the alternative hypothesis. The test statistic is

$$T_\mu(X) := \left|\frac{\bar{X} - \mu}{\frac{\sigma}{n}}\right|,$$

and substituting our values into the statistic gives

$$\left|\frac{6-5}{\frac{1.5}{100}}\right| = 6.67 \text{ (to two decimal places)}.$$

c)

We set the rejection region to be

$$\begin{aligned}
R &= \left\{x; x \in \mathbb{R}^n,\ \left|\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right| > c\right\} \\
&= \{x; x \in \mathbb{R}^n,\ 6.67 > c\}.
\end{aligned}$$

We shall reject H_0 if $x \in R$ or equivalently $c < 6.67$.

Since the null hypothesis states that $\mu = 5 < 6.67$, we conclude that the average wait time is different from the target value of 5 minutes.

Part II:

a)

b)

c)