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CS 5084

Assignment 1

## Stable Matching Problem

2. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

*True or false? Consider an instance of the Stable Matching Problem in which there exists a man $m$ and a woman $w$ such that $m$ is ranked first on the preference list of $w$ and $w$ is ranked first on the preference list of $m$. Then in every stable matching $S$ for this instance, the pair $(m, w)$ belongs to $S$.*

True. Since the pair (m, w) rank each other first on their preference list, any other possible matching result would occur as an unstable match since the pair (m, w) each rank the other first over any prospective partners.

3. There are many other settings in which we can ask questions related to some type of "stability" principle. Here's one, involving competition between two enterprises.

Example:

Suppose Network A has two shows (x, y) with ratings (11, 15)

Suppose Network B has two shows (w, z) with ratings (7, 13)

If the resulting pairs are (x, w) and (y, z) then Network A would win both slots, but Network B would want to switch the order of the shows in its schedule so it could win at least one slot rather than win none.

If Network B changed its line up so the new pairs were (x, z) and (y, w) then each network would win one slot, however network A would just change the line up again so that it won both slots.

Thus, the result is no stable pair of schedules because both networks will continue to restructure their schedules to win more slots.

5. The Stable Matching Problem, as discussed in the text, assumes that all men and women have a fully ordered list of preferences. In this problem we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before we have a set $M$ of $n$ men and a set $W$ of $n$ women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that $m_1$ is ranked in first place; second place is a tie between $m_2$ and $m_3$ (she has no preference between them); and $m_4$ is in last place. We will say that $w$ *prefers* $m$ to $m'$ if $m$ is ranked higher than $m'$ on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions for stability. And for each, we can ask about the existence of stable matchings, as follows.

**(a)** A *strong instability* in a perfect matching $S$ consists of a man $m$ and a woman $w$, such that each of $m$ and $w$ prefers the other to their partner in $S$. Does there always exist a perfect matching with no

The answer is yes. A straightforward way to accomplish this would be to break the ties and then run the stable matching algorithm on the resulting preference lists. For example, we could assign the indifferent men or women based on the order they appear in the set.

If m were to appear before m': m = $m_1$ & m' = $m_2$ then m would be assigned the higher rank on the woman's preference list.

If m' were to appear before m: m' = $m_1$ & m = $m_2$ then m' would be assigned the higher rank on the woman's preference list.

This same technique can be done for the women with respect to the men's preference list as well.

**(b)** A *weak instability* in a perfect matching $S$ consists of a man $m$ and a woman $w$, such that their partners in $S$ are $w'$ and $m'$, respectively, and one of the following holds:

- $m$ prefers $w$ to $w'$, and $w$ either prefers $m$ to $m'$ or is indifferent between these two choices; or
- $w$ prefers $m$ to $m'$, and $m$ either prefers $w$ to $w'$ or is indifferent between these two choices.

In other words, the pairing between $m$ and $w$ is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

The answer is no. If a man is indifferent between any of the women and both women prefer the same man over the other, it would result in a weak stability.

For example: The men are ($m_1$, $m_2$) and our women are ($w_1$, $w_2$)

Let $m_1$ be indifferent between $w_1$, and $w_2$

Let both women prefer $m_1$ to $m_2$

The choice that $m_2$ would make has no significance. Thus, there is no matching without weak stability, since no matter which women is matched with $m_1$ it would form a weak instability.

8. For this problem, we will explore the issue of *truthfulness* in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman $w$. Suppose $w$ prefers man $m$ to $m'$, but both $m$ and $m'$ are low on her list of preferences. Can it be the case that by switching the order of $m$ and $m'$ on her list of preferences (i.e., by falsely claiming that she prefers $m'$ to $m$) and running the algorithm with this false preference list, $w$ will end up with a man $m''$ that she truly prefers to both $m$ and $m'$? (We can ask the same question for men, but will focus on the case of women for purposes of this question.)

Resolve this question by doing one of the following two things:

(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

(b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

For B:

| Preference List 1 | | | |
|---|---|---|---|
| $m_1$ | $w_1$ | $w_2$ | $w_3$ |
| $m_2$ | $w_2$ | $w_1$ | $w_3$ |
| $m_3$ | $w_1$ | $w_3$ | $w_2$ |

| Preference List 2 | | | |
|---|---|---|---|
| $w_1$ | $m_2$ | $m_1$ | $m_3$ |
| $w_2$ | $m_1$ | $m_2$ | $m_3$ |
| $w_3$ | $m_1$ | $m_2$ | $m_3$ |

Our resulting pairs = {($m_1$, $w_1$), ($m_2$, $w_2$), ($m_3$, $w_3$)}

This means that $w_1$ was able to get her <u>second</u> choice.

If we change the table to reflect $w_1$ change in preferences

| $W_1$ changes preference | | | |
|---|---|---|---|
| $w_1'$ | $m_2$ | $m_3$ | $m_1$ |
| $w_2$ | $m_1$ | $m_2$ | $m_3$ |
| $w_3$ | $m_1$ | $m_2$ | $m_3$ |

Our resulting pairs = $\{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$

This means that $w_1$ got her first choice. Thus, showing she improved the pair she got by switching her preference.