# Basics of Set Theory

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## 1 Sets and Set Operations

In high school, you already learned various ways to describe a set: listing all its elements, or describing the property that defines the set. You also learned about the classification of sets. For example, you learned what is an empty set, or a finite set, or an infinite set. You must have frequently used the natural number set, the integer set, the real number set, and many more. Of course, you also learned about subsets and proper subsets, and also set operations like union, intersection, complement. We assume you are familiar with formulas like $A = A \cup A$, and you understand how to count the number of subsets.

In this course, we introduce some new stuff that might surprise you. First of all, we ask you to rethink: What is a set? Your high school teacher might have told you that a set is nothing but a collection of objects. While you might have been quite comfortable with this explanation, unforunately we have to remind you that you would run into great trouble if you understood sets in this way.

Let's define a set $S = \{x | x \notin x\}$. Now ask yourself: Is it true that $S \in S$ ?

- You might want to say yes. But if $S \in S$, then by definition of $S$ we must have $S \notin S$. Contradiction!
- You might want to say no. But if $S \notin S$, then by definition of $S$ we must have $S \in S$. Contradiction again!

This is called *Russell's Paradox*, proposed by British mathematician, logician and philosopher Bertrand Russell. Since the answer to Russell's question above can be neither yes nor no, the only thing we can say is that people's intuitive understanding of sets is plainly wrong.

There are many similar paradoxes.

- Liar Paradox: We write down a statement—"This statement is wrong." Is the above statement right or wrong? Unfortunately, we find that if the statement is wrong, then it must be right. On the other hand, if the statement is right, then it is wrong. Consequently, the statement can be neither right nor wrong.
- The Barber Paradox: In a village, a strange barber shaves, and only shaves, all villagers who do not shave themselves. Using the notations of sets, we write

$$\{p | \text{Barber shaves } p\} = \{p | p \text{ does not shave } p\}.$$

But then the strange barber is confused by his own rule: Should he shave himself? If the barber shaves himself (*i.e.*, belongs to the above set), then he should not shave himself (*i.e.*, he does not belong to the above set). If the barber does not shave himself (*i.e.*, does not belong to the above set), then he should shave himself (*i.e.*, he belongs to the above set).

- Unexpected Hanging Paradox: A prisoner is sentenced to death and his execution is scheduled in the next week. The precise date of the execution is such that it should not be known before the day of execution itself. Using the notations of sets, we write

  $$D = \{d | \text{ if the execution is on day } d, \text{ then it is not known before day } d\}.$$

  But what days belong to $D$? Saturday, the last day of the week, clearly does not belong to $D$. Given that we can exclude Saturday from consideration, Friday becomes the last possible day, and thus it does not belong to $D$ either ... Keeping going this way, we can exclude all days of the week! Does that mean we can always know what day is the execution day in advance?

With respect to the semantics, the main problem of the above paradoxes is that they kind of refer to themselves. So you might want to restrict the use of self references if you want no trouble. One possibility is that you classify mathematical objects into various levels, and require that any object can only refer to objects at lower levels. In this way, self references could be avoided, and paradoxes could also be avoided. Nevertheless, classifying objects into levels without hurting the power of expressions is not an easy task. You might unexpectedly illegalize many interesting sets when you attempt to avoid paradoxes associated with sets.

If we look at the paradoxes from a different angle, we might find that the main problem comes from the so-called "Comprehension Principle:" **You can pick an arbitrary property $p$, and define a set $S$ of all the objects having property $p$, *i.e.*, $S = \{x|p(x)\}$.** This was one of the fundamental approaches to define a set when we first learned about sets. Unfortunately, now we have to make it clear that the approach is problematic. You can easily see that, if we disallow the use of Comprehension Principle, then the paradoxes listed above no longer exist.

Wait a minute... Are we saying we are going to disallow defining sets like $\{x|x > 1, x \in \mathbb{R}\}$? That would be rediculous. In fact, we won't set up a new rule saying that anything defined using the Comprehension Principle is not a set. In stead, we are going to call a mathmatical object defined this way by a different name: a *class*. Some classes, like $\{x|x > 1, x \in \mathbb{R}\}$, which we are familiar with and always use, remain to be sets. But other classes, like $\{p|p \text{ does not shave } p\}$, or $\{x|x \notin x\}$, which could bring us trouble, are not sets. Abrupt speaking, we allow a class to be a set if that would not produce a paradox. If a paradox could be produced, then we call the class a *proper class* (*i.e.*, a class that is not a set).

In summary, now we have classes and sets. Sets are definitely classes. However, classes may be either sets or proper classes (but definitely not both). What classes are sets, precisely? We need a number of axioms, in order to answer this question. If you have strong interests in the axioms, you can refer to a standard textbook of axiomatic set theory, *e.g.*, [2], for details. In this course, we do not spend time discussing these axioms, because we are more interested in ideas than in details.

There is one exception, though. We are highly interested in one particular axiom:

**Axiom 1.1** *(Axiom of Regularity) Every non-empty set contains an element that is disjoint from itself. Formally,*

$$\forall \textit{ set } A \neq \phi, \exists x \in A \textit{ s.t. } x \cap A = \phi.$$

For example, the set $\{x|x > 1, x \in \mathbb{R}\}$ contains an element 2 such that $2 \cap \{x|x > 1, x \in \mathbb{R}\} = \phi$. (Actually, in the above statement, you can replace 2 with any other element[1] from $\{x|x > 1, x \in \mathbb{R}\}$. The statement will remain true.)

The Axiom of Regularity tells us that many classes that look like sets are not really sets. For example, $\{\{\{\ldots\}\}\}$ is not a set, because its only element is itself, which means that it does not have any element disjoint from itself. Below, we give two examples that are a little more sophisticated:

**Example 1.1** *Show that there are no sets $A, B$ such that $A \in B$ and $B \in A$.*

**Solution:** By contradiction. If there are sets $A, B$ such that $A \in B$ and $B \in A$, we construct $S = \{A, B\}$. Note that $S$ must be a set because we have listed all its elements, which are sets $A$ and $B$. Therefore, by the Axiom of Regularity, there exists $x \in S$ such that $x \cap S = \phi$. We distinguish two cases:

Case 1: $x = A$. Since $B \in A$ and $B \in S$, $A \cap S \neq \phi$. Contradiction.

Case 2: $x = B$. Similar to Case 1. □

**Example 1.2** *[7] Prove that there is no infinite sequence of sets $(S_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $S_{n+1} \in S_n$.*

**Solution:** (Adapted from [7]) By contradiction. Consider $S = \{S_n|n \in \mathbb{N}\}$. $S$ is a set since we have enumerated all its elements, which are all sets. Since $S$ is a set, there exists $S_x \in S$ such that $S_x \cap S = \phi$. Since $S_{x+1} \in S$, $S_{x+1} \notin S_x$.

The above contradicts the definition of sequence $(S_n)_{n\in\mathbb{N}}$. □

Given the above discussions of what is a set, hopefully you have got a better picture of sets. Next, we study operations on sets. We learned a good number of set operations in high school, *e.g.*, set complement, set union, set intersection, . . . So we'll skip them and focus on a few operations that might be new to you: set difference, power set, Cartesian product, and power of set.

When we remove from a set $A$ those elements from another set $B$, we are calculating a *set difference*:

$$A \setminus B \stackrel{def}{=} \{x|x \in A \text{ and } x \notin B\}$$

Below are a few simple examples.

**Example 1.3**

$$\{1, 2, 3, 4, 5\} \setminus \{2, 4, 6, 8, 10\} = \{1, 3, 5\}; \quad \mathbb{R} \setminus \mathbb{Q} = \mathbb{I};$$
$$\mathbb{Z} \setminus \{2x|x \in \mathbb{Z}\} = \{2x + 1|x \in \mathbb{Z}\}; \quad \mathbb{Z} \setminus \mathbb{R}^+ = \mathbb{Z}^- \cup \{0\};$$

[1] Those familiar with the construction of natural numbers might argue that 3 (and bigger integers) can't be used for this purpose, because $3 = \{0, 1, 2\}$ and thus 3 is not disjoint from $\{x|x > 1, x \in \mathbb{R}\}$. We must point out that $3 = \{0, 1, 2\}$ is only one way to define the natural number 3, although it might be the current standard way. There are other ways to define 3, so that 3 is still disjoint from $\{x|x > 1, x \in \mathbb{R}\}$.

*where $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ is the set rational numbers, $\mathbb{I}$ is the set of irrational numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}^+$ is the set of positive real numbers, and $\mathbb{Z}^-$ is the set of negative integers.*

You might find that the operation of set difference is like subtraction of numbers in many ways. For instance, just like subtraction of numbers, set difference is neither commutative, nor associative. We believe you can easily find counter examples for both properties. Our next example asks you to find a counter example for a more sophisticated formula.

**Example 1.4** *Is set difference really like subtraction of numbers? For all sets $A, B, C$, do we have $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$ ? If you don't think this formula is valid, please find a counter example.*

If $A \setminus (B \cup C)$ is not equal to $(A \setminus B) \cup (A \setminus C)$, what does it equal?

**Proposition 1.1** *(De Morgan's Law[2]) For all sets $A, B, C$,*

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C); \quad A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

*In particular, we have*

$$\overline{B \cup C} = \overline{B} \cap \overline{C}; \quad \overline{B \cap C} = \overline{B} \cup \overline{C}.$$

There is absolutely no difficulty in proving De Morgan's Law. However, in order to demonstrate how to write a rigorous mathematical proof, we present a proof of the first half below. The second half can be shown similarly.

**Proof:** For all $x \in A \setminus (B \cup C)$, we must have $x \in A$ but $x \notin B \cup C$. Hence, $x \notin B$ and $x \notin C$. Therefore, $x \in A \setminus B$ and $x \in A \setminus C$. Equivalently, $x \in (A \setminus B) \cap (A \setminus C)$. Consequently, we get that

$$A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C).$$

For all $x \in (A \setminus B) \cap (A \setminus C)$, we must have that $x \in A \setminus B$ and $x \in A \setminus C$. So, $x \in A$ but $x \notin B$ and $x \notin C$. Therefore, $x \notin B \cup C$, which implies $x \in A \setminus (B \cup C)$. Hence, we get that

$$(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C).$$

Combining the above results, we obtain that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

$\square$

The *power set* of a given set $S$ is simply the set of all subsets of $S$. We use the notation $2^S$ to represent it. Formally,

$$2^S \stackrel{def}{=} \{T | T \subseteq S\}.$$

[2] Augustus De Morgan (1806-1871) was a British mathematician born in India, when his father worked with East India Company. He received his B.A. from Trinity College, Cambridge University, but could not enter their graduate program, because he refused to take a mandatory theological test. At the age of twenty-one, he because a professor of mathematics at the newly founded London University (called UCL today). His last years were miserable, because his son, a mathematical genius, and later also his daughter, died before him.

**Example 1.5** *Let $S = \{1, 2, 3\}$. The power set of $S$ is $2^S = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.*
*The power set of the real number set $\mathbb{R}$ is the set of real number sets: $2^{\mathbb{R}} = \{T | T \text{ is a set of some real numbers}\}$.*

The *Cartesian product* of two sets $X$ and $Y$ is simply the set of ordered pairs where the first element of each pair comes from $X$, and the second element from $Y$.

$$X \times Y \stackrel{def}{=} \{(x, y) | x \in X, y \in Y\}.$$

In general, the *Cartesian product* of $n$ sets $X_1, X_2, \ldots, X_n$ is the set of ordered $n$-tuples where the first element of each tuple comes from $X_1$, the second from $X_2$, ..., the $n$th from $X_n$.

$$X_1 \times X_2 \times \ldots \times X_n \stackrel{def}{=} \{(x_1, x_2, \ldots, x_n) | x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n\}.$$

When $X_1 = X_2 = \ldots = X_n$, this Cartesian product becomes the *$n$th power of* $X_1$, written as $X_1^n$. Please be cautious here, because we are talking about *the power of a set*, not the power set discussed earlier.

**Example 1.6**

$$\{1, 2\} \times \{2, 3, 4\} = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}.$$
$$\mathbb{Z} \times \mathbb{R} = \{(x, y) | x \in \mathbb{Z}, y \in \mathbb{R}\}.$$
$$\{1, 2\} \times \{2, 3, 4\} \times \{5\} = \{(1, 2, 5), (1, 3, 5), (1, 4, 5), (2, 2, 5), (2, 3, 5), (2, 4, 5)\}.$$
$$\mathbb{R} \times \mathbb{Z} \times \mathbb{Q} = \{(x, y, z) | x \in \mathbb{R}, y \in \mathbb{Z}, z \in \mathbb{Q}\}.$$
$$\{1, 2\}^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}.$$

*The plane of Euclidean coordinates is the Cartesian product of two number axes, or the square of a number axis. That's why we often use $\mathbb{R}^2$ to represent an Euclidean plane. Notice that this is a typical example of a power of a set, very different from a power set. In comparison, the power set of $\mathbb{R}$ is discussed in Example 1.5.*

By the way, we have been talking about ordered pairs since middle school, but what is an *ordered pair*? What is the precise definition? Centuries ago, people used to say "an ordered pair is an ordered pair—no formal definition needed." In modern mathematics, however, we can actually define it formally. One possible definition is

$$(x, y) = \{\{x, y\}, x\}.$$

Intuitively, we first describe the two elements involved in the ordered pair, and the specify which of them goes first. It is not hard to see that such a definition works. Nevertheless, we should notice two issues:

- This is not the only possible definition. There are other definitions that can also work. (Can you propose one?)

- Having a definition like this mainly serves the purpose of establishing a (hopefully) complete axiomatic system based on a few very simple axioms. Using such a definition, we no longer need the informal concept of a pair being "ordered." Everything becomes formal and elegant. But it is not meant to help us solve any practical problems.

Modern mathematics can be viewed as a huge axiomatic system, the foundation of which is (axiomatic) set theory. In this sense, almost every mathematical object is actually a set. Why did mathematicians choose such an approach? It was in response to the foundational crisis at the end of 19th century and the beginning of 20th century. Interested readers can search for "foundational crisis of mathematics" and read the story.

## References

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