MTH 215 (ORDINARY DIFFERENTIAL EQUATION)

BY

AKEYEDE, I. AND HABU, P. N.

DEPARTMENT OF MATHEMATICS

FEDERAL UNIVERSITY LAFIA

1.1 BASIC DEFINITIONS AND TERMINOLOGY

Derivative of differential equations from primitive, geometry, physics etc.

In calculus, we learnt that given a function y = f(x), the derivative

$\frac{dy}{dx} = f'(x)$, is itself a function of x and is found by some appropriate rule. For example, if

$$y = e^{x^2} then\ \frac{dy}{dx} = 2xe^{x^2} or\ \frac{dy}{dx} = 2xy$$

Definition: An equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation.

1.2.1 Classification of Differential Equations

Differential Equations are classified according to the following three properties: Classification by type (b) Classification by order and (c) Classification as linear or nonlinear

(a) Classification by type: If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is then said to be an ordinary differential equation. For examples,

(i) $\frac{dy}{dx} - 5y = 1$

(ii) $(x + y)dx - 4ydy = 0$

(iii) $\frac{du}{dx} - \frac{dy}{dx} = x$

(iv) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$

are ordinary differential equations

Definition: An equation involving partial derivatives of one or more dependent variables with respect to two or more independent variables is called a partial differential equation. For examples, (ii) and (iv) are partial differential equations while (i) and (iii) are ordinary

Which of the following is/are partial differential equation

(i) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}$

(ii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$

(iii) $\frac{\partial^2 u}{\partial x^2} = x + y$

(iv) $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2k\frac{\partial u}{\partial t}$

are partial differential equations

(b) Classification by Order: The order of the highest derivatives in a differential equation is called the order of the equation. For example,

$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^2 - 4y = x$ is called a second order ordinary differential equation. The equation

$x^2\frac{dy}{dx} + dy = 0$ is an example of a first order ordinary differential equation. The equation,

$c^2\frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^2 u}{\partial t^2} = 0 \; is\ a\ forth\ order\ partial\ differential\ equation$

Definition: The degree of an ordinary differential equation is the highest exponent (power) of the highest order derivative, after fractions or radicals involving the dependent variable or its derivatives must first be removed from the equation, e.g,

$\frac{(y'')^{3/2}}{y+(y'')^2} = k$ ………………………………………………………………………………. (1)

To know the degree and the order of (1) we write (1) as

$k^2(y'')^4 - (y'')^3 + 2k^2y(y'')^2 + k^2y^2 = 0$, now showing the order of (1) is 2 and the degree is 4

Although the partial differential equations are very important, their study demands a good foundation in the theory of the ordinary differential equations. Therefore in this course, we shall confine our attention to ordinary differential equations

A general nth order, ordinary differential equation is often represented by
$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}\right) = 0$

From now on, in this course, when we say a differential equation, we shall mean an ordinary differential equation.

(c) Classification as Linear or Non-linear: A differential equation is said to be linear if it has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Notice that linear differential equations are characterized by two properties: (a) the dependent variable y and all its derivatives are occur to the first degree, i.e the power of each term involving y is 1(b) Each coefficient of derivatives depends only on the independent variable x.

An equation that is not linear is said to be nonlinear. For example,

(i) xdy + ydx = 0 is a linear first-order ordinary differential equation
(ii) y'' +2y' + y = 0 is a linear second-order ordinary differential equation
(iii) $x^3\frac{d^3y}{dx^3} - x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x$ is a linear third-order ordinary differential equation

On the other hand,

$(i)\ \frac{dy}{dx} = xy^{1/2}$ is a non-linear first-order ordinary differential equation

$(ii)\ yy'' - 2y = x + 1$ is a nonlinear second-order ordinary differential equation

$(iii)\ \frac{d^3y}{dx^3} + y^2 = 0$ is a nonlinear third-order ordinary differential equation

Our goal in this course, is to solve or find solutions of the ordinary differential equations

Definition: Any function f defined on some interval I which when substituted into a differential equation reduces the equation to an identity, is said to be solution of the equation on the interval I. Alternatively, a solution of the general n^{th} order ordinary differential equation, F(x, y, y', y'', . . ., y^n) =0 is a function y = ø(x), that is differentiable a suitable number of times in some interval I, containing the independent variable x, and which has the property that F(x, ø(x), ø(x)', ø(x)'', . . ., ø(x)^n,) = 0, ~~∀~~x ϵ I

In other words, a solution of differential equation

$F\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0\ is\ another\ function\ y = f(x)$
$possesing\ atleast\ n\ derivatives\ such\ that$

F(x, f(x), f'(x), f''(x), . . ., f^n(x)) = 0, for every x in the interval I

Example: Show that the function $y = \frac{x^4}{16}$ is a solution of the nonlinear equation $\frac{dy}{dx} - xy^{1/2} = 0$ … (1)

Solution: A solution of the differential equation $\frac{dy}{dx} - xy^{1/2} = 0\ is\ a\ function\ of\ y = f(x)$ possessing only one derivative $\frac{dy}{dx}$ such that substituting $y = \frac{x^4}{16}$ and result of $\frac{dy}{dx}$ into (1) will satisfy the condition in (1) i.e

$\frac{dy}{dx} = 4\frac{x^3}{16} = \frac{x^3}{16}$

$\therefore \frac{dy}{dx} - xy^{\frac{1}{2}} = \frac{x^3}{16} - x\frac{x^4}{16} = \frac{x^3}{16} - x\frac{x^2}{16} = 0$

$\therefore\ y = \frac{x^4}{16}$ is a solution of the nonlinear equation $\frac{dy}{dx} - xy^{1/2} = 0$

Definition: An explicit solution of a differential equation is of the form f(x), while an implicit solution is of the form f(x, y).

Example: Show that $y = xe^x\ is\ an\ explicit\ solution\ of$

Y'' – 2y'' + y = 0

To see this, we compute $'y' = xe^x + e^x$

And $y'' = xe^x + 2e^x$

From $y = xe^x$

Y'' – 2y'' + y = $= xe^x + 2e^x - 2(xe^x + e^x) + xe^x$

$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$

$\therefore y = xe^x$ =f(x) is an explicit solution of Y'' – 2y'' + y = 0

Example: Show that f(x,y) = $x^2 + y^2 - 4 = 0$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solution:

$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$x^2 + y^2 - 4 = f(x, y) = 0$ is an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$

1.2 Origins of Differential equations

We shall see how specific differential equations arise not only out of consideration of families of geometric curves, but also how differential equations result from an attempt to describe, in mathematical terms, physical problems in the sciences and engineering. We wish to state that differential equation form the backbone of subjects such as physics, fluid mechanics and electrical engineering, and even provide an important working tool in such diverse areas as Biology, Economics, etc

1.2.1 Differential equations of a Family of Curves

Suppose that we now seek to find the differential equation of two-parameter family

$$y = c_1 e^x + c_2$$

The first two derivatives are

$$\frac{d^2y}{dx^2} = c_1 e^x$$

Thus,

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

i.e

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

By taking two derivatives we find that the differential equation of the two –parameter family of straight lines

$Y = c_1x + c_2$

is simply $\frac{d^2y}{dx^2} = 0$

i.e

$\frac{dy}{dx} = c_1$ and $\frac{d^2y}{dx^2} = 0$

Example: Find the differential equation of the family

$y = cx^3$ ………………………………………………………………………………. (1)

Solution

$\frac{dy}{dx} = 3cx^2$ … ... (2)

From $c = \frac{y}{x^3}$ … . (3)

$\therefore \frac{dy}{dx} = 3\left(\frac{y}{x^3}\right)x^2 = 3\frac{y}{x}$ … … … … … … … … … …. ………………………….(4)

Equation 4 can be written as

$\frac{dy}{dx} - 3y = 0$ ……………………………………………………………………….(5) which is a linear first order differential equation.

A first-order differential equation is often given in differential form. In differential form, equation(5) can be written as

$xdy - 3ydx = 0$…………………………………………………………………(6)

where dy and dx are called differentials

Example: Find the differential equation of the two parameter family

$$y = c_1e^{2x} + c_2e^{-2x}$$

Taking the derivatives we have;

$$\frac{dy}{dx} = 2c_1e^{2x} - 2c_2e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4c_1e^{2x} + 4c_2e^{-2x}$$

$$= 4[c_1e^{2x} + c_2e^{-2x}]$$

Since $y = c_1e^{2x} + c_2e^{-2x}$, we can write

$$\frac{d^2y}{dx^2} = 4y \; or \; y'' - 4y = 0$$

2.2.1 Some Physical origin of Differential Equation

It is well known that free-falling objects close to the surface of the earth accelerate at a constant rate g. if we denote distance by s, then velocity, v is given by $\frac{ds}{dt}$

i.e $\frac{ds}{dt} = v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

$\frac{d^2s}{dt^2} = \frac{dv}{dt}$ ……………………………………………….(2)

Thus, if we assume that the upward direction is positive, then equation (2) can be written for free-falling objects close to the surface of the earth as

$\frac{d^2s}{dt^2} = -g \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Which is the differential equation governing the vertical distance that the falling body travels. The minus sign is used since the weight of the body is a force directed opposite to the positive direction. If we suppose further that a stone is tossed off the roof of a building of height so, with an initial upward velocity, say V_o, then $\frac{d^2s}{dt^2} = -g$ $0 < t < t_1$

Subject to the side conditions;

$s(0) = s_o$, $s'(o) = V_o$

here t =0 is taken to be initial time when the stone leaves the roof of the building and t_1 is the time required to hit the ground. Since the stone is thrown upward it would naturally be assumed that $V_o > 0$. This information of the problem ignores other forces such as air resistance acting on the body of the stone

Example: Newton law of cooling states that the time rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If T(t) denotes the temperature of the body at any time t, and T_o is the constant temperature of the outside medium, then time rate = $\frac{dT}{dt}$, difference between the temperature of the body T(t) and T_o constant temperature of the outside medium = T - T_o

$$\therefore \frac{dT}{dt} \alpha (\mathrm{T} - T_0)$$

$$\Rightarrow \frac{dT}{dt} = k(\mathrm{T} - T_0)$$

Example: In the spread of the a contagious disease, for example a flu virus, it is reasonable to assume that the rate, $\frac{dx}{dt}$, at which the disease spreads is proportional not only to the number of people , x(t) who have contacted but also to the number of people, y(t), who have not yet been exposed

Solution

$$\frac{dx}{dt} \alpha\, x(t)y(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$\frac{dx}{dt} = kxy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Where k is the constant of proportionality.

If one infected person is introduced into a fixed position of n people, then, x and y are related by x + y = n + 1 ………………………………………………………………(3)

Using (3) to eliminate y in (2), gives

$\frac{dx}{dt} = kx(n + 1 - x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$

The obvious boundary condition accompany equation (4) is x(0) = 1

METHODS OF SOLVING DIFFERENTIAL EQUATION

1. **Direct substitution**

Example

Find the general solution to the differential equation

$$\frac{dy}{dx} = x\sqrt{x-3}$$

$$\Rightarrow \int dy = \int x\sqrt{x-3}dx$$

$$y =$$

2. **Separating Variables**

Consider the differential equation M(x) +N(y)$\frac{dy}{dx} = 0$, where M(x) is a function of x only and N(y) is a function of x only and they are both continuous functions. Then the differential equation is solved by the method of separation of variables and the equation itself is said to be separable.

The steps involved in solving the problem are:

i. Express the given equation in differential form
M(x)dx + N(y)dy = 0 or M(x)dx = -N(y)dy

ii. Integrate to obtain
$\int \mathrm{M(x)dx} + \int \mathrm{N(y)dy} = \mathrm{c}$ or $\int \mathrm{M(x)dx} = -\int \mathrm{N(y)dy} + \mathrm{c}$

Example 1: find the general solution of

$$(x^2+4)\frac{dy}{dx} = xy$$

$$\Rightarrow \int \frac{dy}{y} = \int \left(\frac{x}{x^2+4}\right) dx + c$$

$$lny = \frac{1}{2}\ln(x^2+4) + lnk$$

$$y = k\sqrt{x^2+4}$$

Example 2: find the particular solution for the initial condition problem

$$xydx + e^{-x^2}(y^2-1)dy, subject\ to\ y(0) = 1$$

$$\frac{x}{e^{-x^2}}dx = -\frac{y^2-1}{y}dy$$

$$\int\left(\frac{1}{y}-y\right)dy = \int xe^{x^2}dx + c$$

$$lny - \frac{1}{2}y^2 = \frac{1}{2}e^{x^2} + c$$

$$2lny - y^2 = e^{x^2} + 2c$$

When x = 0, y = 1, we have c = -1

$$2lny - y^2 - e^{x^2} = -2$$

$$y^2 = x^2(e^{y^2-2})$$

3. Homogeneous (Substitution Method)

A homogeneous differential equation expressed in the form

$$\frac{dy}{dx} = f\left(\frac{x}{y}\right) \dots\dots\dots\dots\dots\dots\dots.(1)$$

To find a solution to (1), we use the substitution y = vx, where v is a function of x only,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

then equation 1 becomes

$$v + x\frac{dv}{dx} = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

$$\frac{dx}{x} = \frac{1}{f(v)-v}dv \dots\dots\dots\dots\dots\dots..(3)$$

(3) has now been reduced to equation seperable.

Example:

Solve the initial boundary problem

$$(2y + x)y' = (2x + y), y(1) = 2.$$

5 Exact Differential Equation

A differential equation of the form

M(x, y)dx, + N(x, y)dy = 0 ……………………………….. (5.1)

With the property that M(x, y) and N(x, y) are related to a differentiable function F(x, y) by the equations M(x, y) = $\frac{\partial F}{\partial x}$, N(x, y) = $\frac{\partial F}{\partial y}$, is said to be exact ……………(5.2)

(6) If the functions M(x, y) and N(x, y) and their derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous and furthermore,

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere, then the differential equation

M(x,y)dx, + N(x,y)dy = 0 is exact.

(7) From equations 5.2, we may obtain the function

F(x, y) by integrating

$\frac{\partial F}{\partial x} = M(x, y)$ in respect to x

i.e F(x,y) = $\int M(x, y)dx + A(y) \dots \dots \dots \dots (7.1)$,

where A(y) is some function of y and acts as a constant as regard partial differential equation with respect to x. Thus, A(y) takes the place of the constant of integration that would occur were M to be only a function of x.

The determination of A(y) then follows immediately by differentiating (7.1) partially with respect to y, and using

$\frac{\partial F}{\partial y} = N(x, y)$ to determine $\frac{\partial A}{\partial y}, from$ which A(y) follows by integration in the form

A(y) = $\int \left\{ N(x, y) - \frac{\partial[\int M(x,y)dx]}{\partial y} \right\} dy + c \dots \dots \dots \dots (7.2)$,

(8) Example 8.1: Let us solve the differential equation

(2x + 3cosy)dx + (2y – 3xsiny)dy = 0 for the function F(x,y)

Solution

The equation is of the form

M(x,y)dx, + N(x,y)dy = 0

M(x,y) = (2x + 3cosy) and N(x,y) = (2y – 3xsiny)

$\frac{\partial M}{\partial y} = -3siny \ and \ \frac{\partial N}{\partial x} = -3siny$

$\therefore \ the \ equation \ is \ exact \ because \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -3siny$

Since the equation is exact, it is also implies that $\exists \ F(x,y)$ such that

$\frac{\partial F}{\partial x} = M(x,y)$ = 2x + 3cosy and $\frac{\partial F}{\partial y} = N(x,y)$

$F = \int (2x + 3cosy)dx + A(y)$

$x^2 + 3xcosy + A(y) \ldots \ldots \ldots \ldots (8.1)$

Differentiating (8.1) partially with respect to y we get

$$\frac{\partial F}{\partial y} = -3xsiny + \frac{dA}{dy}$$

$$since \frac{\partial F}{\partial y} = 2y - 3xsiny$$

$$\Rightarrow -3xsiny + \frac{dA}{dy} = 2y - 3xsiny$$

$$\Rightarrow \frac{dA}{dy} = 2y, hence \ A(y) = y^2 + c'$$

The function F(x,y) is thus

F(x,y) = x^2 +3cosy + y^2 + c'

And so the general solution is

x^2 +3cosy + y^2 = c

If y(0) = $\frac{1}{2}\pi \ then \ c = \frac{\pi^2}{4}$

$\Rightarrow$ x^2 +3cosy + y^2 = $\frac{\pi^2}{4}$

(9) Not all differential equations of the form M(x,y)dy + N(x,y)dx = 0,…………………… (9.1) is not exact, then it can be made exact by multiplying it by some simple factor $\mu(x,y)$, called an integrating factor, which is not identically zero. So, if $\mu(x,y)$ is an integrating factor of (9.1), then

$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (9.2) is\ exact$, and by virtue of (6), the integrating factor must satisfy the equation

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (9.3)$$

(10) Two special forms of differential equation in which an integrating factor may always be found:

(a) M(x,y)dx, + N(x,y)dy = 0 …………………(10.1)

(b) P(x)Q(x)dx + R(x)S(y)dy = 0 ……………. (10.2) [separating variable]

The integrating factor for equation (10.2), in which the variables are separable and $\mu(x,y) = {}^{1}/_{[R(x)Q(x)]}$ and leads to the result

$$\int \frac{S(y)}{Q(y)} dy = -\int \frac{P(x)}{R(x)} dx + c \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10.3)$$

Similarly, the integrating factor for equation (10.2) with homogeneous coefficient is $(x,y) = {}^{1}/_{[R(x)Q(x)]}$, where s = y/x and gives rise to solution

$\int \frac{Q(s)}{P(s)+sQ(s)} ds = ln\left|\frac{c}{x}\right| \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10.4)$, where is an arbitrary constant

(11) Trial and error methods to search for an integrating factor using;

$$\mu(x,y) = x^m y^n$$

Example: Solve the differential equation

$(2xy + y^2)dx + (2x^2 + 3xy)dy = 0$ ……………………………… (11.1)

Solution

Firstly, we noticed that

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, so\ the\ equation\ is\ not\ exact$$

An integrating factor $\mu(x, y) is\ required$. As M and N are simple algebraic functions, the integrating factor $\mu(x, y)$ is believed to be of s simple form in which only certain constants may be determined, and we shall now try an expression of the form;

$\mu(x, y) = x^m y^n$ in which the constants m and n must be determined so that multiplying equation (11.1) by $\mu(x, y) = x^m y^n$, we get,

$x^m y^n(2\text{xy} + y^2)\text{dx} + x^m y^n(2\text{x}^2 + 3\text{xy})\text{dy} \ldots \ldots \ldots \ldots (11.2) so\ that\ (11.2) is\ exact.$

By condition, $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial y}$, this implies that if $\mu(x, y) = x^m y^n$ is in actual fact an integrating factor, then m and n must be chosen so that

$$\frac{\partial[x^m y^n(2\text{xy} + y^2)]}{\partial y} = \frac{\partial[(2\text{x}^2 + 3\text{xy})]}{\partial y} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (11.3)$$

This condition gives rise to the equation

$$nx^m y^{n-1}(2\text{xy} + y^2) + x^m y^n(2\text{x} + 2\text{y}) = mx^{m-1}y^n(2\text{x}^2 + 3xy) + x^m y^n(4\text{x} + 3\text{y})$$

$$x^{m+1}y^n(2n + 2) + x^m y^{n+1}(n + 2) = x^{m+1}y^n(2n + 4) + (3m + 3)x^m y^{n+1}$$

From which we must determine m and n if the chosen form of integrating factor is correct. Since this expression must be an identity, we now equate coefficients of terms of equal degrees in x and y, and if possible, select m and n such that all conditions are satisfied. In this case, only two conditions arise:

(a) Term involving $x^m y^{n+1}, we\ have, n + 2 = 3m + 3$
(b) Terms involving $x^{m+1}y^{n+1}$, we have 2n + 2 = 2m + 4,

These conditions are satisfied if m = 0, n = 1, so an integrating factor of the type assumed exists and in this case $\mu = y$. The exact differential equation is thus

$(2xy^2 + y^3)dx + (2x^2y + 3xy^2)dy = 0$…………………….. (11.4),

Which is easily seen to have the general solution

$X^2y^2 + xy^3 = c$.

When values of m and n cannot be found that will produce an identity from condition $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial y}$, then the integrations factor is not of the form

$\mu(x, y) = x^m y^n$.

(12) Solution of the general linear differential equation,

$\frac{dy}{dx} + P_{(x)y} = Q(x) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (12.1)$

To solve $\frac{dy}{dx} + P_{(x)y} = Q(x) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (12.1)$

Solution

$\mu\left(\frac{dy}{dx}\right) + \mu P_{(x)y} = Q(x)\ a\ derivative, namely\ \frac{d(\mu y)}{dx}.$

Since equation is linear μ must be independent of y, so that we need only to consider μ to be non-linear$\mu = \mu(x)$. i.e $\mu(x, y) or\ \mu(y)$ will make (12.1) to be non-linear so $\mu\ must\ be\ \mu(x)$. Thus, the integrating factor μ is required to be a solution of the equation $\left(\frac{d(\mu y)}{dx}\right) = \mu\left(\frac{dy}{dx}\right) + \mu P_y$. Expanding the left hand side and simplifying, we get

$y\left(\frac{d\mu}{dx}\right) - \mu p_y = 0 \ldots \ldots \ldots \ldots \ldots \ldots (12.2)$. As the solution of (12.1) is not identically zero (i.e y =0), it follows that μ must be

The solution of

$\frac{d\mu}{dx} = P(x)\mu \ldots \ldots \ldots \ldots \ldots \ldots \ldots (12.3)$, the variable x and μ are separable, giving $\frac{d\mu}{\mu} = P(x)dx$, showing that $ln|\mu| = \int P(x)dx + c'$, where c' is an arbitrary constant. Taking exponents, we find that the most general integrating factor is $\mu = e^{c'}e^{\int P(x)dx}$.

However, as the arbitrary constant $e^{c'}$ is always non-zero in ($-\infty \leq c' \leq \infty$), and so may be cancelled when this expression, $e^{c'}$ is used as a multiplier in $\frac{dy}{dx}$ + P(x)y =Q(x), we may take as the integration factor the expression $\mu = e^{\int P(x)dx}$……………………………….(12.4). now multiply

$\frac{dy}{dx}$ + P(x)y = Q(x) by μ and using the properties of an exact equation, then gives $\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$

After a final integration and simplification, we obtain the general solution

Y = $e^{-\int P(x)dx}\{c + \int Q(x)e^{\int P(x)dx}dx$, where c is the arbitrary constant of the final integration and must be retained

(13) Example: Solve $\frac{dy}{dx} + ky = asinmx$, subject to the initial condition y = 1 when x = 0.

Solution

In this case, P(x) = k, so that the integrating factor

$\mu = e^{kx}. hence,$

$\frac{d}{dx}(ye^{kx}) = ae^{kx} sinmx,\ \ giving\ rise\ to$

$y = e^{-kx}[c + a \int e^{kx} sinmxdx]$

Performing the indicated integration gives the general solution,

$y = ce^{-kx} + \frac{a}{k^2+m^2}(ksinmx - mcosmx),$

The first term being the complementary function and the second term the particular integral.

To determine the constant c we now utilize the initial condition y(0) = 1by writing

$1 = c - \frac{am}{k^2+m^2}$

The particular solution is thus,

$y = \left(\frac{k^2+m^2+am}{k^2+m^2}\right) e^{-kx} + \frac{a}{k^2+m^2}(ksinmx - mcosmx).$

(14) To solve the class differential equations:

$\frac{d^2y}{dx^2} + p_{(x)} \frac{dy}{dx} = Q(x) \dots \dots \dots \dots \dots \dots \dots \dots . (14.1)$

(15) Solution of differential equations of the form;

$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_ny = 0 \dots \dots \dots \dots \dots (15.1)$

$p(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ \dots + a_n$ is called the characteristic equation of (15.1)

(16) A differential equation

$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \ \dots + a_ny \dots \dots \dots \dots \dots (16.1)$

Which has n distinct roots λ_i of its characteristic equation P(λ) = 0 has n linearly independent solutions

$y_i = c_ie^{\lambda_i x}$

Its general solution, the complementary function, is of the form

$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + \ \dots + c_ne^{\lambda_n x} \dots\dots\dots\dots (16.2)$

(17) Example: Solve the differential equation:

y'' + 3y' + 2y = 0

Solution

P(λ) = $\lambda^2 + 3\lambda + 2$ and the roots of P(λ) = 0 are $\lambda_1 = -1, \lambda_2 = -2$. The linearly independent solutions are

$y_1 = c_1e^{-x}, y_2 = c_2e^{-2x}$

∴ the general solution or the complementary function is

$y = c_1e^{-x} + c_2e^{-2x}$

When r of the roots of the characteristic polynomial P(λ) coincide and equal λ^*, say, then $\lambda = \lambda^*$ is said to be a root of multiplying r.

(19) When $\lambda = \lambda_1$ is a root of multiplying r of the characteristic equation P(λ) = 0 belonging to

$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \ \dots + a_ny = 0 \dots\dots\dots\dots\dots (19.1)$ then

$e^{\lambda_1 x}, xe^{\lambda_1 x}, x^2e^{\lambda_1 x}, \dots, x^{r-1}e^{\lambda_1 x} \dots\dots\dots\dots (19.2)$ are linearly independent solutions of the differential equation corresponding to the r-fold root $x\ \lambda_1$

(20a) Example: Solve y''' + 4y'' + 5y' + 2y = 0

Solution

P(λ) = $\lambda^3 + 4\lambda^2 + 5\lambda + 2$, using factorization, we have

$\lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda + 1)^2(\lambda + 2)$and the roots of P($\lambda$) = 0 are $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -2$. The root $\lambda = -1$ is a double root and so the linearly independent solutions are

$y_1 = c_1e^{-x},\ y_2 = c_2xe^{-x}, y_3 = c_3e^{-2x}$

∴ the general solution or the complementary function is

$y = c_1e^{-x} + c_2xe^{-x} + c_3e^{-2x} \dots \dots \dots \dots \dots \dots \dots (20.2)$

(20b) Example: To determine the particular solution appropriate to, say, the initial conditions y = 1, y' = 0, y'' =1 when x = 0

Solution

From (20.2), we have

$y = c_1e^{-x} + c_2xe^{-x} + c_3e^{-2x}$

$y' = -c_1e^{-x} + c_2(1-x)e^{-x} - 2c_3e^{-2x}$

$y'' = c_1e^{-x} - c_2(2-x)e^{-x} + 4c_3e^{-2x}$

Substituting the initial conditions give rise to the three simultaneous equations

$1 = c_1 + c_2$, $0 = -c_1 + c_2 - 2c_3$ and $1 = c_1 - 2c_2 + 4c_3$, which have as their solutions $c_1 = -1$, $c_2 = 3$, $c_3 = 2$, hence the required particular solution is

$y = (3x - 1)\, e^{-x} + 2e^{-2x}$

(21) Where λ = u + iv and its complex conjugate λ = u – iv are single root of the characteristic equation P(λ) = 0 of the differential equation;

$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \ \dots + a_ny = 0 \dots \dots \dots \dots \dots (21.1)$ and the remaining roots; $\lambda_3, \lambda_4, \dots, \lambda_n$ are real and distinct then the general solution has the form

$y = e^{\mu x}(c_1 sinvx + c_2 cosvx) + c_3e^{\lambda_3 x} + c_4e^{\lambda_4 x} + \cdots + c_ne^{\lambda_n x}$

$y = e^{(u\,+\,iv)x} + e^{(u-\,iv)x} + c_3e^{\lambda_3 x} + c_4e^{\lambda_4 x} + \cdots + c_ne^{\lambda_n x}$

(22) Example: Solve y'' + 4y' + 13y = 0

Solution

$P(\lambda) = \lambda^2 + 4\lambda + 13 = (\lambda+2+3i)(\lambda+2-3i)$ and the roots of P(λ) = 0 are therefore λ = -2 – 3i and λ = -2 + 3i. The general solution is

$y = e^{-2x}(c_1 sin3x + c_2 cos3x) = e^{(-2+3i)x} + e^{(-2-3i)x}$

(23) Solution of differential equations of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x) \ldots\ldots\ldots\ldots\ldots (23.1)$$

With constant coefficients $a_1, a_2, \ldots, a_n$.

Let the RHS of equation (23.1) be defined as

L(y) $\equiv y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y \ldots\ldots\ldots\ldots\ldots (23.2)$, then

Where $a_1, a_2, \ldots, a_n$ are constants

(24) The general solution of the homogeneous equation (23.1)

L(y) = f(x), is of the form

$y(x) = y_c(x) + y_p(x)$ ……………………………………………….(24.1)

Where $y_c(x)$ is the general solution or complementary function of the reduced equation (homogeneous equation)

L(y) = 0, and $y_p(x)$ is a particular solution of L(y) = f(x)

(25) $The\ equation\ \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x)$ can be solved using any of the following methods

(a) The method of undetermined coefficients

(b) The operator D method

(c) Variation of parameters method

(d) Oscillatory Solutions

(e) Coupled Oscillations and normal modes method and

(f) Laplace Transform method (for boundary value problems)

(26)(a) Using the method of undetermined coefficients

When f(x) is a polynomial in x, in the differential equation

$y'' + y = x^2 - 2$ ………………………. (26.1)

Solution

When f(x) is a polynomial in x, we use the trial function

$y_p = ax^2 + bx + c$ ………………………… (26.2)

To find the particular integral y_p

$y_p'' = 2a$ and substituting into (26.1), we have

$2a + ax^2 + bx + c = x^2 - 2$ ………………… (26.3)

Equating the coefficient of the corresponding powers of x shows that

a = 0, b = 0, so that c = -2

$y_p = x^2 - 2$.

To get y_c from (26.1), we get the homogeneous equation as

$Y'' + y = 0$

The characteristic equation is

$\lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 = -1, \lambda = \pm i \; and \; \lambda_1 = i \; and \; \lambda_2 = -i$

$y = c_1 e^{ix} + c_2 e^{-ix}$

$= c_1(cosx + isinx) + c_2(cosx - isinx)$

$= (c_1 + c_2)cosx + (c_1 - c_2)isinx$

$= (c_1 + c_2)cosx + [i(c_1 - c_2)sinx]$

$= c_1 cosx + c_2 sinx$

$\therefore y = y_c(x) + y_p(x),$

$y = c_1 cosx + c_2 sinx + x^2 - 1$

(b) When f(x) is an exponential function, e.g

Solve $y'' + 3y' + 2y = 3e^{2x}$

Solution

By having

$y'' + 3y' + 2y = 0$, we find that

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_p = ke^{2x}$

$y_p' = 2ke^{2x}$

$y_p'' = 4ke^{2x}$

Substituting $y_p, y_p', and\ y_p''$ into $y'' + 3y' + 2y = 3e^{2x}$, shows that

$4k + 6k + 2k = 3$ or $k = ¼$

$y_p = ¼e^{2x}$

(c) When f(x) is a trigonometric function

$y_p = a\cos mx + b\sin mx$

(d) When f(x) is a product of exponential and trigonometric functions, then

$y_p = e^{kx}[a\cos mx + b\sin mx]$

(A) Using the D operator to solve n $\geq$ 2 order of homogeneous differential equations

(27) The differential operator, D, is such that $D \equiv \frac{d}{dx}, D^2 \equiv \frac{d^2}{dx}\ and\ in\ general\ D^n \equiv \frac{d^n}{dx}, where\ for\ the\ moment\ n\ is\ a\ positive\ integer$

(28) We define the polynomial operator,

$P(D) \equiv D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n$ By

$P(D)f(x) \equiv (D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n)f(x)$, where f(x) is any suitable differentiable function.

(29) The factorization theorem: Suppose that the polynomial

$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ has the factor $\lambda - \lambda_1, \lambda - \lambda_2, \ldots, \lambda - \lambda_n$ then

$D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n \equiv (D - \lambda_1)(D - \lambda_2) \ldots (D - \lambda_n)$

(30) Let us now briefly recapitulate our discussion of the solution of the reduced equation, but this time using the D operator(i.e finding y_c using the D operator)

Example: For distinct roots, consider the general second order differential equation which in factorised form may be written as

$(D-\lambda_1)(D-\lambda_2)y = 0$

Where $\lambda_1 \neq \lambda_2$. Now set $(D-\lambda_2)$ = u……………. (30.2), so that the equation becomes

$(D-\lambda_1)$u = 0 or

$\frac{du}{dx} - \lambda_1 u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(30.3)$

This has the solution

u = $c_1e^{\lambda_1 x}$ so that now we must solve

$(D-\lambda_2)$y = $c_1e^{\lambda_1 x}$ $\ldots\ldots\ldots\ldots\ldots\ldots.(30.4), which$ is simply the familiar first order linear differential equation

$\frac{dy}{dx} - \lambda_2 y = c_1e^{\lambda_1 x}$ *with integrating factor*

$\mu = e^{-\lambda_2 x}. Hence\ \frac{d}{dx}\left(ye^{-\lambda_2 x}\right) = c_1e^{(\lambda_1-\lambda_2)x}$, so that the general solution is

$y_c = \left(\frac{c_1}{\lambda_1-\lambda_2}\right)e^{\lambda_1 x} + c_2e^{\lambda_2 x}$, where C_2 is another arbitrary constant of integration.

(30) (b) for repeated root

Example: This time let us consider a third order equation but assume that two of the roots are equal, so that in factorise form it may be written

$(D-\lambda_1)^{\mathbf{2}}(D-\lambda_2)y = 0$

Changing the order of the factors and setting $(D-\lambda_1)^2 y = u, the\ equation\ becomes$

$(D-\lambda_2)u = 0\ with\ the\ solution$

$u = c_1e^{\lambda_2 x}. hence\ we\ must\ now\ solve$

$(D-\lambda_1)^2 y = c_1e^{\lambda_2 x}\ so\ written\ v = (D-\lambda_1)y$ this equation simplify to

$(D-\lambda_1)v = c_1 e^{\lambda_2 x}$. The integrating factor is $u = e^{-\lambda_1 x}$ and application of the argument of the previous example with redefinition of constants where necessary bring us to the solution

$v = c_1 e^{\lambda_2 x} + c_2 e^{\lambda_1 x}. finally\ we\ must\ solve$

$(D-\lambda_1)y = c_1 e^{\lambda_2 x} + c_2 e^{\lambda_1 x}$

This also has an integrating factor

$u = e^{-\lambda_1 x}$, so that

$\frac{d}{dx}\left(ye^{-\lambda_1 x}\right) = c_1 e^{(\lambda_2-\lambda_1)x} + c_2 and\ hence,$

$y_c = c_1 e^{\lambda_2 x} + (c_2 x + c_3)e^{\lambda_1 x}$

(c) **Example for Complex Conjugate roots:**

(i) If a polynomial P(λ) with real coefficients is such that P(λ) = 0 has a complex root

λ = u + iv, then we know that it must also have a root λ = u – iv. Consequently, as in our previous study, we know that the corresponding term in the particular integral must be $e^{ux}(c_1\cos vx + c_2\sin vx)$

(ii) Also, if the roots have simplicity m, then the corresponding term must be

$e^{ux}(P_{m-1}\cos vx + Q_{m-1}\sin vx)$, where m-1 having arbitrary coefficients. These terms must be added to the other terms that arise from the real roots of P(λ) = 0 to obtain the complementary function y_c.

Consider the equation

$(D^5 - 5D^4 + 12D^3 - 16D^2 + 12D - 4)y = 0$

In factorial form this becomes,

$(D-1)(D-1-i)^2(D-1+i)^2y = 0$, showing that the real factor (D-1) has multiplicity 1and the complex conjugate factors in which u = 1, v = 1, have multiplicity 2. The complementary function y_c is thus

$y_c = e^x[c_1 + (c_2+c_3x)\cos x + (c_4+c_5x)\sin x]$

(B) The real advantage of the D operator method is in the determination of particular integrals y_p, in homogeneous equations of the form $P_{(D)y} = f(x)$, where

$y_p = 1/P(D)\, f(x)$ for the particular integral. Finding

y_p = for $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x)$

(a) Inverse operator, $\frac{1}{D-\lambda}$. We define the effect of the inverse operator $(D-\lambda)^{-1}$ acting on a function f(x) by the expression

$$\frac{1}{D-\lambda} f(x) = e^{\lambda x} \int f(x) e^{-\lambda x} dx = y_p.$$

(b) Example: Determine the particular integral y_p, of

$(D-1)(D-2)y = e^x$

Solution

First using the inverse operator $\frac{1}{D-1}$ and identifying λ with 1 and f(x) with e^x, we have

$(D-2)y = \frac{1}{D-1} e^x$

$= e^x \int e^x e^{-x} dx = xe^x$

i.e $(D-2y)y = xe^x$ ……………………………. (1)

Now using the inverse operator $\frac{1}{D-2}$ with $\lambda = 2$ and $f(x) = xe^x$, we have,

$y_p = \left(\frac{1}{D-2}\right) xe^x$

$= e^{2x} \int xe^x e^{-2x} dx = -(x+1)e^x$

(31) (c) The application of the operator $\frac{1}{D-\lambda}$ to complex factors is equally straight forward. To find y_p for $(D^2 - 2D + 2)y = e^x$. after factorization we have,

$(D - 1 - i)(D - 1 + i)y = e^x$

$(D - 1 + i)y = \frac{1}{(D-1-i)} e^x,\ \lambda = 1 + i$

$= e^{(1+i)x} \int e^x e^{-(1+i)x} dx = ie^{ix}$. Hence,

$y_p = \frac{1}{(D-1+i)} ie^x, \lambda = (1-i)$

$e^{(1-i)x} \int ie^x e^{-(1-i)x} dx = e^x.$

$\therefore$ *the required particular solution is* $y_p = e^x$

(32) Some Simple Rules for y_p: in special cases the inverse operator, $\frac{1}{D-\lambda}$, just defined above, simplifies to give some easy rules for determination of y_p.

Rule 1: When f(x) is an exponential function:

(1)(a) if $P(\alpha) \neq 0$, then,

$$y_p = \frac{1}{p(D)} e^{\alpha x} = e^{\alpha x} \frac{1}{p(\alpha)}, f(x) = e^{\alpha x}$$

$$if\ p(\alpha) = 0\ then\ use\ \frac{1}{D-\lambda} f(x) = e^{\lambda x} \int f(x)\, e^{-\lambda x} dx = y_p$$

(b) Example: Find y_p, for $(D^2 - 2D + 2)y = e^x$

Solution

$P(D) = D^2 - 2D + 2$ and

$y_p = \frac{1}{D2 - 2D + 2} e^x, here\ \alpha = 1\ so\ that\ P(\alpha) = 1 - 2 + 2 = 1$

$$y_p = \frac{1}{p(D)} e^{\alpha x}$$

$$= e^{\alpha x} \frac{1}{p(\alpha)},$$

$$= e^{\alpha x} \frac{1}{1}$$

$$e^x, \alpha = 1,$$

The rule is applicable in $(D - 1)(D - 2)y = e^x$, because in that case $P(\alpha) = 0$, and so $\frac{1}{D-\lambda}(fx) = e^{\lambda x} \int f(x)\, e^{-\lambda x} dx$ must be used

Rule 2: If P(D) only contains even powers of D and so may be written in the form $P(D^2)$ then providing $P(-m^2) \neq 0$, and f(x) is a trigonometric function

$\frac{1}{P(D^2)}\sin(mx) = \frac{\sin(mx)}{P(-m^2)}$ *and* $\frac{1}{P(D^2)}\cos(mx) = \frac{cosmx}{P(-m^2)}$

Example: Solve for y_p from

$(D^4 – 3D^2 + 2)y = Cos\ 2x$

Solution:

Since m = 2, and $P(D^2) = D^4 – 3D^2 + 2$, it follows that

$P(-m^2) = (-4)^2 – 3(-4) + 2 = 30$

$\therefore\ by\ this\ rule, y_p = \frac{cos2x}{30}$

Rule 3: $\frac{1}{D-\lambda} = \frac{-1}{\lambda - D} = -\frac{1}{\lambda}\left(1 + \frac{D}{\lambda} + \frac{D^2}{\lambda^2} + \cdots\right)$ using the binomial theorem gives the same result when applied to x^5 as an application of $\frac{1}{D-\lambda}f(x) = e^{\lambda x}\int f(x)\,e^{-\lambda x}dx$, because $D^{s+1}x^s = 0$, the expansion may therefore be terminated after the term D^s. Applying this operator to a polynopmial of degree m establishes:

$$\frac{1}{D-\lambda}(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m) = -\frac{1}{\lambda}\left(1 + \frac{D}{\lambda} + \frac{D^2}{\lambda^2} + \cdots\right)(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m)$$

(34) Example: Find y_p, for $(D - 4)y = 1+x^2$

Solution

$y_p = \frac{1+x^2}{D-4}$, by rule 3, we need to set m = 2 and λ = 4, so that

$y_p = \frac{1+x^2}{D-4}$,

$= -\frac{1}{4}\left(1 + \frac{D}{4} + \frac{D^2}{4^2}\right)(1 + x^2)$ performing the indicated differentiations, we find that

$y_p = -(9 + 4x + 8x^2)/32$

Alternatively, the particular integral corresponding to the more general expression, $\mathrm{P(D)y} = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$, may be deduced by factorizing the polynomial operator P(D) and making repeated use of rule 3 when f(x) is a polynomial in x

(35) Rule 3 Modified: If P(D) has any quadratic factors corresponding to pairs of complex conjugate root of characteristic polynomial, then Rule 3 is as follows; if

$P(D) \equiv D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_{n-1}D + a_n$, ……………………………………………………(35.1)
and P(D) is expressed in the form

$P(D) \equiv a_n[1 - Q(D)]$……………………………………………………………………….(35.2) so that

$Q(D) \equiv -(\frac{1}{a_n}D^n + \frac{a_1}{a_n}D^{n-1} + \frac{a_2}{a_n}D^{n-2} \ldots + \frac{a_{n-1}}{a_n}D \ldots \ldots \ldots \ldots \ldots$ (35.3) the using (35.2)

$\frac{1}{P(D)}(b_0 + b_1x + b_2x^2 + \cdots b_mx^m = \frac{1}{a_n(1-Q(D))}(b_0 + b_1x + b_2x^2 + \cdots b_mx^m)$

$= \frac{1}{a_n}\left(1 - Q(D)\right)^{-1}(b_0 + b_1x + b_2x^2 + \cdots b_mx^m)$

$= \frac{1}{a_n}[1 + Q(D) + Q^2(D) + Q^m(D) + \cdots + Q^m(D)][b_0 + b_1x + b_2x^2 + \cdots b_mx^m] \ldots (35.4)$

Example: Find y_p, for $(D^2 - 3D + 2)y = x^2 + 1$.

Solution

$P(D) \equiv D^2 - 3D + 2$, and using P(D) in the above notation, $a_1 = -3$, $a_2 = 2$, so that
$Q(D) \equiv -\frac{1}{2}D^2 + \frac{3}{2}D$, using $a_n[1-Q(D)] = D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_{n-1}D + a_n$.

Since the polynomial is of degree 2, we have m = 2 and the modified Rule 3 now gives

$$\frac{1}{D^2 - 3D + 2}(x^2 + 1) = \frac{1}{2}\left[1 + \left(-\frac{1}{2}D^2 + \frac{3}{2}D\right) + \left(-\frac{1}{2}D^2 + \frac{3}{2}D\right)^2\right](1 + x^2).$$

From (35.4) performing the indicated differentiations we finally arrive that

$y_p = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{4}$.

(36) Rule 4: If v(x) is any suitably differentiable function, then

$\frac{1}{P(D)}[e^{\mu x}v(x)] = e^{\mu x}\left(\frac{1}{P(D+\mu)}\right)v(x)$. $f(x) = e^{\mu x}v(x)$

Example: Find y_p, for $(D^2 + 5D + 6)y = xe^{-x}$,

Solution

Here, $P(D) = D^2 + 5D + 6$ and $\mu = -1$,

So that $P(D + \mu) = (D-1)^2 + 5(D-1) + 6$

$$= D^2 + 3D + 2$$

Then by Rule 4, $y_p = e^{-x}(D^2 + 3D + 2)^{-1}x$

Factorizing this result and using

$\frac{1}{D-\lambda} = \frac{-1}{\lambda-D} = \frac{-1}{\lambda}(1 + \frac{D}{\lambda} + \frac{D^2}{\lambda^2} + \cdots)$, we then get

$$y_p = e^{-x}[\frac{1}{(1+D)(2+D)}]x$$

$$= e^{-x}[(1)[1 - D + D^2 - D^3 + D^4]\left(\frac{1}{2}\right)\left[1 - \frac{D}{2} + \frac{D^2}{2} - \cdots\right)]x$$

Expanding the external square bracket as far as terms involving D, because the operator is only acting on x, we find that

$$y_p = e^{-x}\left(1 - D - \frac{D}{2}\right)\frac{1}{2}x,$$

$$y_p = \frac{1}{2}e^{-x}\left(1 - \frac{3D}{2}\right)x$$

$$\frac{1}{2}e^{-x} - \frac{3}{4}e^{-x}.$$

Neglecting higher degrees of D since x of only 1 degree i.e v(x) is of only one degree

Application of Differential Equation

1. Bacteria in a certain culture increase at a rate proportional to the number present. If the number N increases from 1, 000 to 2,000 in 1 hour, how many are present between the end of 1.5 hours.
2. Radium distingrates at a rate proportional to the amount of radium present at any instant. If half-life of radium is 1,600 years, what % of the original amount Q^o will remain after 1,200 years?