ES 656 Human-Robot Interaction

Activity 5

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> **Problem 1: Formulation: Derive expressions for the following: Jacobian Matrix J, Structure Matrix A, and Task Space Stiffness Matrix Kx**

Solution 1.

Assumptions

- The limb is a planar two-link system with revolute joints.
- The link lengths are l_1 and l_2.
- The joint angles are q_1 and q_2.
- The end-effector position in Cartesian coordinates is (x, y).
- The system has $m = 3$ actuators (muscles).
- Zero muscle tension $T = 0$ and no external force $F = 0$.

1. Jacobian Matrix J

The end-effector position is:

$$x = l_1 \cos(q_1) + l_2 \cos(q_1 + q_2)$$

$$y = l_1 \sin(q_1) + l_2 \sin(q_1 + q_2)$$

The Jacobian Matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix}$$

Computing the partial derivatives:

$$\frac{\partial x}{\partial q_1} = -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2), \quad \frac{\partial x}{\partial q_2} = -l_2 \sin(q_1 + q_2)$$

$$\frac{\partial y}{\partial q_1} = l_1 \cos(q_1) + l_2 \cos(q_1 + q_2), \quad \frac{\partial y}{\partial q_2} = l_2 \cos(q_1 + q_2)$$

Thus,

$$J = \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

2. Structure Matrix A

Let $f = [f_1, f_2, f_3]^T$ be the muscle forces, and $\tau = [\tau_1, \tau_2]^T$ be the joint torques. The relationship is given by:

$$\tau = Af$$

The structure matrix A is a 2×3 matrix of muscle moment arms:

$$A = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}$$

Here, r_{ij} is the moment arm of the j-th muscle about the i-th joint. These depend on the muscle geometry and configuration.

3. Task Space Stiffness Matrix K_X

Let the muscle stiffness matrix be:

$$K_m = \text{diag}(k_1, k_2, k_3)$$

The joint space stiffness matrix K_q is:

$$K_q = AK_mA^T$$

The task space stiffness matrix is then:

$$K_X = (J^T)^{-1}K_qJ^{-1} = (J^T)^{-1}AK_mA^TJ^{-1}$$

This is valid under the assumption that J is invertible (i.e., the configuration is not singular).

> **Problem 2: Configuration Analysis: Choose five different limb configurations (i.e., five sets of q1,q2) based on real-world postures. Examples: Reaching for an object on a table, Blocking or throwing a punch (boxing), Playing a carrom game, etc. Use realistic numerical values for link lengths, cable attachment locations, spring stiffness, and other required parameters.**

Solution 2

Task 2: Configuration Analysis

We approximate the human upper and lower arm with the following lengths:

- Upper arm length $l_1 = 0.3\,\text{m}$
- Forearm length $l_2 = 0.25\,\text{m}$

Joint Angle Configurations

Five real-world limb postures were selected for analysis. The joint angles q_1 and q_2 are expressed in radians, where:

- q_1: angle of link 1 (upper arm) with respect to the horizontal axis
- q_2: angle of link 2 (forearm) with respect to link 1

No.	**Configuration**	q_1 (rad)	q_2 (rad)
1	Reaching Forward	$\frac{\pi}{6}$ (30°)	$-\frac{\pi}{6}$ (30°)
2	Reaching Up	$\frac{\pi}{2}$ (90°)	0 (0°)
3	Blocking Punch	$\frac{2\pi}{3}$ (120°)	$-\frac{\pi}{2}$ (90°)
4	Throwing Punch	$\frac{\pi}{9}$ (20°)	$-\frac{\pi}{9}$ (20°)
5	Carrom Strike	$-\frac{\pi}{4}$ (45°)	$\frac{\pi}{2}$ (90°)

Structure Matrix A

The Structure Matrix $A \in \mathbb{R}^{2\times 3}$ relates actuator forces to joint torques. It is defined using simplified constant effective moment arms using the formula from first question:

$$A = \begin{bmatrix} 0.05 & 0.01 & 0.03 \\ 0.01 & 0.04 & 0.02 \end{bmatrix}$$

This matrix indicates:

- Actuator 1 primarily influences joint 1.
- Actuator 2 primarily influences joint 2.
- Actuator 3 influences both joints.

Muscle Stiffness Matrix K_m

We model the muscles as linear springs. The stiffness matrix $K_m \in \mathbb{R}^{3\times 3}$ is diagonal and defined as $k_1 = 100\,\text{N/m}, k_2 = 80\,\text{N/m}, k_3 = 120\,\text{N/m}$. The moment arm matrix A is approximated by:

$$K_m = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 120 \end{bmatrix} \quad \text{(in N/m)}$$

These values represent passive elastic stiffnesses of the three actuators/muscles.

Problem 3: Visualization: For each of the five configurations, plot velocity manipulability ellipses, force manipulability ellipses, and task space stiffness ellipses. Assumptions for stiffness calculations: no external force applied (F = 0), muscle tension (T) assumed to be zero (i.e., ignoring muscle contribution) due to lack of OpenSim-based data.

Solution 3

Task 3: Visualization (Calculations)

In this task, we compute the matrices required to generate task space ellipses for each of the five configurations defined in Task 2. These ellipses help visualize different capabilities of the limb in task space:

- **Velocity Manipulability Ellipse**
- **Force Manipulability Ellipse**
- **Task Space Stiffness (Compliance) Ellipse**

Each ellipse is described by a quadratic form of the general type:

$$\delta\mathbf{x}^T M\,\delta\mathbf{x} = 1$$

where $\delta\mathbf{x} = \begin{bmatrix}\delta x \\ \delta y\end{bmatrix} \in \mathbb{R}^2$ represents a small displacement in task space, and $M \in \mathbb{R}^{2\times 2}$ is a symmetric positive-definite matrix that characterizes the geometry of the ellipse.

1. Velocity Manipulability Ellipse

This ellipse represents the set of possible end-effector velocities under the constraint of unit norm joint velocities $\dot{\mathbf{q}}^T\dot{\mathbf{q}} = 1$. It is defined by:

$$\delta\mathbf{x}^T(JJ^T)^{-1}\delta\mathbf{x} = 1$$

Here, J is the Jacobian matrix, and JJ^T maps joint velocities to task space velocities.

2. Force Manipulability Ellipse

This ellipse indicates the set of forces $\mathbf{F} \in \mathbb{R}^2$ that can be applied at the end-effector resulting from unit norm joint torques $\boldsymbol{\tau}^T\boldsymbol{\tau} = 1$. Using the relationship $\mathbf{F} = (J^T)^{-1}\boldsymbol{\tau}$, we derive the ellipse as:

$$\mathbf{F}^T(JJ^T)\mathbf{F} = 1$$

This shows that the inverse of the velocity manipulability matrix also defines the force manipulability ellipse.

3. Task Space Stiffness (Compliance) Ellipse

The task space stiffness matrix K_X relates small displacements in task space to restoring forces via:

$$\delta\mathbf{F} = K_X \delta\mathbf{x}$$

The potential energy stored in the system for a small displacement $\delta\mathbf{x}$ is:

$$E = \frac{1}{2}\delta\mathbf{x}^T K_X \delta\mathbf{x}$$

To visualize the ease of displacement (compliance), we consider the compliance matrix K_X^{-1}. The compliance ellipse is then given by:

$$\delta\mathbf{x}^T K_X \delta\mathbf{x} = 1$$

This ellipse represents the set of displacements $\delta\mathbf{x}$ that require the same amount of restoring force or correspond to equal potential energy. A larger ellipse indicates lower stiffness and higher compliance.

Computational Steps for Each Configuration

For each configuration (q_1, q_2), we perform the following steps:

1. Compute the Jacobian matrix J
2. Compute the velocity manipulability matrix JJ^T
3. Compute the force manipulability matrix $(JJ^T)^{-1}$
4. Compute the joint space stiffness matrix:

   $$K_q = AK_mA^T$$

   where A is the Structure Matrix and K_m is the Muscle Stiffness Matrix.
5. Compute the task space stiffness matrix:

   $$K_X = J^{\dagger^T} K_q J^\dagger$$

   where $J^\dagger$ denotes the Moore-Penrose pseudoinverse of the Jacobian J.
6. Compute the compliance matrix K_X^{-1}

These matrices are then used to generate the respective ellipses using the equations above.

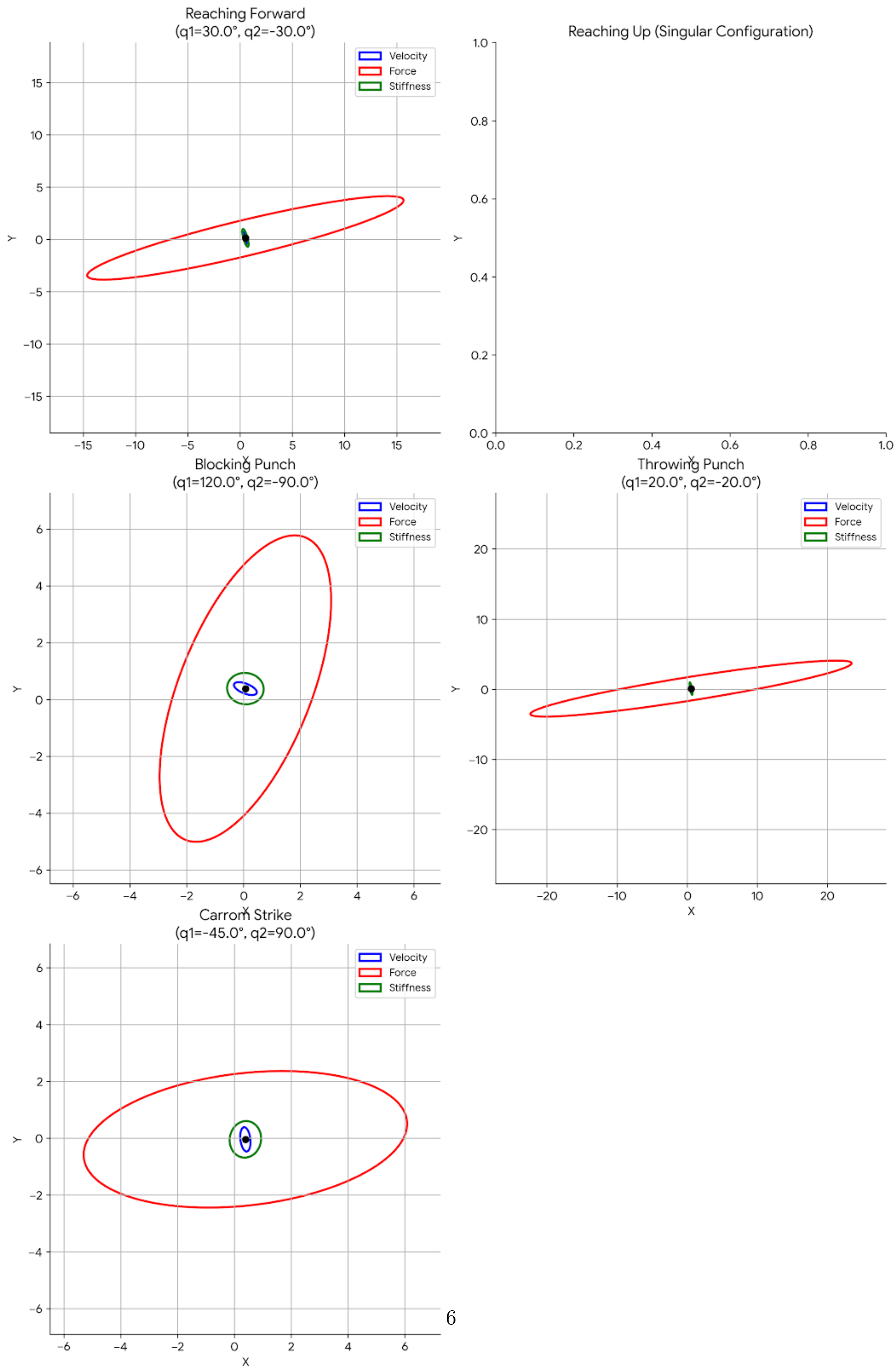

Figure 1: Velocity & Force Manipulability Ellipse, Task Space Stiffness Ellipse

Problem 4: Discussion: Interpret the physical meaning and implications of the ellipse plots regarding limb motion capability, force generation ability, and resistance to deflection or perturbation. Compare the computational results qualitatively with your subjective impression of limb behavior in the five configurations.

Solution 4

Task 4: Discussion – Qualitative Comparison with Subjective Impression

Let us interpret the results from the ellipse plots for each configuration and compare them with how we intuitively expect a human arm to behave in those postures. The ellipses provide insight into limb capabilities such as velocity generation, force production, and stiffness in task space.

1. Reaching Forward

Configuration: Arm mostly extended horizontally.
Joint Angles: $q_1 = \frac{\pi}{6}$, $q_2 = -\frac{\pi}{6}$

- **Velocity:** The velocity ellipse is elongated perpendicular to the arm's direction, consistent with the ease of making small sweeping movements side-to-side.
- **Force:** The arm feels stronger when pulling or pushing along its length than sideways. The force ellipse confirms this by elongating along the arm.
- **Stiffness:** The compliance ellipse is smaller perpendicular to the reach direction, showing greater resistance to sideways perturbations.

2. Reaching Up (Singular Configuration)

Configuration: Arm extended vertically overhead.
Joint Angles: $q_1 = \frac{\pi}{2}$, $q_2 = 0$

- **Velocity:** The Jacobian becomes near-singular. Movements along the arm's direction become constrained and less dexterous.
- **Force:** Force generation in certain directions is poor due to the singular nature of the configuration.
- **Stiffness:** The arm feels "locked" in place, with reduced flexibility for sideways motion.

3. Blocking Punch

Configuration: Arm bent with forearm raised.
Joint Angles: $q_1 = \frac{2\pi}{3}$, $q_2 = -\frac{\pi}{2}$

- **Velocity:** A bent arm permits quick rotational adjustments, which is reflected in a moderately sized velocity ellipse.
- **Force:** This posture is meant to resist impact. The force ellipse indicates capability to generate significant opposing force.
- **Stiffness:** The compliance ellipse is small, showing the posture is stiff and resists displacement — ideal for blocking.

4. Throwing Punch

Configuration: Arm extending forward slightly downward.
Joint Angles: $q_1 = \frac{\pi}{9}$, $q_2 = -\frac{\pi}{9}$

- **Velocity:** The velocity ellipse is elongated in the forward direction, consistent with punching dynamics.
- **Force:** The force ellipse also aligns forward, suggesting good capacity to deliver force in the punch direction.
- **Stiffness:** The compliance ellipse shows lower stiffness in the direction of movement and higher stiffness orthogonal to it.

5. Carrom Strike

Configuration: Arm bent low for a precise strike.
Joint Angles: $q_1 = -\frac{\pi}{4}$, $q_2 = \frac{\pi}{2}$

- **Velocity:** Enables small, controlled movements needed for precision. The velocity ellipse reflects this precision control.
- **Force:** The force ellipse reveals effective force generation in specific directions useful for a carrom strike.
- **Stiffness:** The compliance ellipse is relatively small in critical directions, confirming the need for stability during striking.

General Observations from Ellipses

- The **shape and orientation** of ellipses vary significantly with joint configuration.
- **Velocity** and **force manipulability** ellipses are inversely related and share principal axes.
- The **stiffness (compliance) ellipse** indicates how the limb responds to external forces. A smaller ellipse implies higher resistance.
- The **blocking** configuration shows a small compliance ellipse, indicating maximum resistance — aligning with its function.
- **Punching** and **reaching** configurations show elongation in task-aligned directions — suggesting readiness for movement or force production in those directions.

Limitations and Model Assumptions

This analysis is based on a simplified planar two-link model with linear spring-like muscles. Real human arms have:

- More degrees of freedom (e.g., wrist, scapula movement)
- Complex muscle paths and attachments

Singularities and Redundancy

The singular configuration (e.g., arm vertically extended) demonstrates limitations in manipulability. While the overall system includes actuation redundancy ($m > n$), the calculation of task space stiffness assumes a non-singular Jacobian. This highlights how posture affects control and capability, even in redundant systems.