Note on Volatility Surface

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Abstract

The note documents methodologies of volatility surface modeling. The contents are mainly extracted from lecture notes of Tankov[4] and Touzi[3] from École Polytechnique and the book of Derman[1]. The credits go to these three authors.

Contents

1 Black-Scholes models and implied volatility

In this section, we review the elements of Black-Scholes model.

1.1 Black-Scholes model

The main hypotheses of Black-Scholes model is

1. No transaction cost.
2. No restrictions on transaction size.
3. The market is arbitrage-free.
4. The underlying stock price follows geometric Brownian motion under the historical measure P

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.1}$$

   where W_t is a standard Brownian motion, μ is the drfit and σ the volatility, μ 和 σ are both constants.

5. Money deposited in the bank is risk free and earns a constant interest rate r.

1.2 Self-financing portfolio

Definition 1.1. *A self-financing strategy is a strategy where where no funds are added to or withdrawn from the portfolio after the initial date.*

Proposition 1.1. *Let V_t be the value of a self-financing portfolio at time t which is consisted of stock and bonds. δ_t denotes the number of stocks in the portfolio then the dynamics of V_t follows*

$$dV_t = (V_t - \delta_t S_t)\, r dt + \delta_t dS_t$$

Proof. B_t denotes the value of bonds in the portfolio at time t, then

$$V_t = \delta_t S_t + B_t$$

At discrete times $\{t_i\}_{i=1}^N$, by condition of self-financing, one can write

$$\begin{aligned} & B_{t_i-} + \delta_{t_i-} S_{t_i} = B_{t_i} + \delta_{t_i} S_{t_i} \\ \Rightarrow \quad & e^{r(t_i - t_{i-1})} B_{t_{i-1}} + \delta_{t_i-} S_{t_i} = B_{t_i} + \delta_{t_i} S_{t_i} \end{aligned}$$

Therefore

$$\begin{aligned} V_{t_i} - V_{t_{i-1}} &= (e^{r(t_i - t_{i-1})} - 1) B_{t_{i-1}} + \delta_{t_{i-1}} (S_{t_i} - S_{t_{i-1}}) \\ &= (e^{r(t_i - t_{i-1})} - 1)(V_{t_{i-1}} - \delta_{t_{i-1}} S_{t_{i-1}}) + \delta_{t_{i-1}} (S_{t_i} - S_{t_{i-1}}) \end{aligned}$$

Pass $\delta t \to 0$, finally it is obtained

$$dV_t = (V_t - \delta_t S_t)\, r dt + \delta_t dS_t$$

□

Suppose $V_t = C(t, S_t)$ where $C(t, S_t)$ is a regulized function. Then $dV_t = dC(t, S_t)$ and by Itô lemma

$$\begin{aligned} dC(t, S_t) &= \frac{\partial C}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{\partial^2 V}{\partial S^2} < dS_t, dS_t > \\ &= \sigma \frac{\partial C}{\partial S} dW_t + (\frac{\partial C}{\partial t} + S_t \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}) dt \end{aligned}$$

Because of $dV_t = dC(t, S_t)$ we necessarily have

$$\begin{aligned}\delta_t &= \frac{\partial C(t, S_t)}{\partial S} \\ rC(t, S) &= \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \qquad (1.2)\end{aligned}$$

From the above, we can conclude that if function $C(t, S_t)$ satisfies equation(1.2), the portfolio with $\delta_t = \frac{\partial C(t,S_t)}{\partial S}$ and $B_t = C(t, S_t) - \delta_t S_t$ is self-financing. By no-arbitrage principle, this implis that the price of an option with payoff $h(S_T)$ is the solution of equation(1.2) with the boundary condition $C(T, S) = h(S_T)$.

1.3 Risk-neutral pricing

By assumption of Black-Scholes model, under the historical measure P,

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Let Q be an equivalent probability measure of P where

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\frac{\mu - r}{\sigma}W_t - \frac{(\mu - r)^2}{2\sigma^2}t\right)$$

By Girsanov theorem $\widehat{W}_t := W_t + \frac{\mu - r}{\sigma}t$ is a Brownian motion under probability measure Q. and

$$dS_t = \mu S_t dt + \sigma S_t d\widehat{W}_t$$

Let $\tilde{C}(t, S_t) = e^{-r(T-t)}\mathbb{E}^Q[h(S_T)|S_t]$, then for $s < t$

$$\begin{aligned}\mathbb{E}^Q[e^{-rt}\tilde{C}(t, S_t)|\mathcal{F}_s] &= \mathbb{E}^Q[e^{-rT}\mathbb{E}^Q[h(S_T)|S_t]|\mathcal{F}_s] \\ &= e^{-rT}\mathbb{E}^Q[\mathbb{E}^Q[h(S_T)|S_t]|\mathcal{F}_s] \\ &= e^{-rT}\mathbb{E}^Q[h(S_T)|\mathcal{F}_s] \qquad \text{(by tower property)} \\ &= e^{-rs}e^{-r(T-s)}\mathbb{E}^Q[h(S_T)|\mathcal{F}_s] \\ &= e^{-rs}\tilde{C}(s, S_s)\end{aligned}$$

therefore $e^{-rt}\tilde{C}(t, S_t)$ is a Q-martingale.

Another non-rigorous proof can be done by showing

$$e^{-rt}\tilde{C}(t, S_t) = e^{-rT}\mathbb{E}^Q[h(S_T)|S_t] = \mathbb{E}^Q[\frac{h(S_T)}{e^{rT}}|S_t]$$

because the numéraire under probability measure Q is money market account(i.e. $e^r t$), therefore $e^{-rt}\tilde{C}(t, S_t)$ is a Q-martingale.

Besides by Itô we have

$$d(e^{-rt}\tilde{C}(t, S_t)) = e^{-rt}\left(-r\tilde{C} + \frac{\partial \tilde{C}}{\partial t} + rS\frac{\partial \tilde{C}}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \tilde{C}}{\partial S^2}\right)dt + e^{-rt}\sigma S\frac{\partial \tilde{C}}{\partial S}d\widehat{W}_t$$

So

$$e^{rt}d\left(e^{-rt}\tilde{C}(t, S_t)\right) = \left(-r\tilde{C} + \frac{\partial \tilde{C}}{\partial t} + rS\frac{\partial \tilde{C}}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \tilde{C}}{\partial S^2}\right)dt + \sigma S\frac{\partial \tilde{C}}{\partial S}d\widehat{W}_t$$

Because $e^{-rt}\tilde{C}(t, S_t)$ is a Q-martingale then its drift term is zero, which further shows that $\tilde{C}(t, S_t)$ is a solution of equation(1.2). Because of the uniqueness of the solution we can deduce that the price of the option with payoff $h(S_T)$, which is a solution of equation(1.2), in the Black-Scholes model is

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^Q(h(S_t e^{r(T-t)-\frac{1}{2}\sigma^2(T-t)+\sigma\widehat{W}_{T-t}})|\mathcal{F}_t)$$

The above is a special case of fundamental theorem of asset pricing, which says that a market is arbitrage-free if and only if there exists a probability measure Q(i.e. risk neutral measure), equivalent to historical measure P where the instrument price is the discounted expected value of future payoff.

1.4 Implied volatility

1.4.1 Option delta

Use I to denote the implied volatility of option $C(t, S_t)$, by definition we can write

$$C(t, S_t) = C^{BS}(t, S_t, I)$$

If I does not depend on S_t but only on K(Derman calls this *sticky strike*), then we have

$$\frac{\partial C(t, S_t)}{\partial S} = \frac{\partial C^{BS}(t, S_t)}{\partial S}$$

The option delta equals to delta from the Black-Scholes model. However in reality this rarely happens and we in fact must write

$$\frac{\partial C(t, S_t)}{\partial S} = \frac{\partial C^{BS}(t, S_t, I)}{\partial S} + \frac{\partial C^{BS}(t, S_t, I)}{\partial I}\frac{\partial I}{\partial S}$$

To handle the second term in above equation, practitioners usually propose models of smile evolutions under certain assumptions. For example, Derman proposed regimes of *sticky delta* where I depends on K/S but not on them separately. In that case set $I = I(K/S)$ we have

$$\frac{\partial C(t, S_t)}{\partial S} = \frac{\partial C^{BS}(t, S_t, I)}{\partial S} + \frac{\partial C^{BS}(t, S_t, I)}{\partial I}\frac{K}{S^2}\frac{\partial I^{'}}{\partial S}$$

For details please go to see Derman's book[1].

1.4.2 Compared with historical volatility

In the Black-Scholes model, the implied volatility of all options on the same underlying must be the same and equal to the historical volatility (standard deviation of annualized returns) of the underlying. However, when it is computed from market-quoted option prices, one observes that

- The implied volatility is always greater than the historical volatility of the underlying.
- The implied volatility of different options on the same underlying depend on their strikes and maturity dates.

2 Delta hedging strategies

Realized volatility is the amount of noise in the stock price, it is the coefficient of the Wiener process in the stock returns model, it is the amount of randomness that actually transpires. Implied volatility is how the market is pricing the option currently. Since the market does not have perfect knowledge about the future these two numbers can and will be different. [1]

2.1 Hedged option position

We set up the problem in the following way

1. Assume the option is hedged at discrete points in time, $t_0, t_1, ...t_n$, such that $t_i - t_{i-1} = \delta t$ and with tn representing the expiration time of the option.
2. Use $C_i = C(t_i, S_i)$ to denote the market price of the option at time t_i when the stock price is S_i; use $\Delta_i = \Delta(t_i, S_i)$ to denote the number of shares of the stock S that we short at the start of each period i.
3. Any cash received is invested at the riskless rate r, and any cash borrowed is funded at the same rate.

We begin by set up a portfolio V_0 by holding the option worth C_0. At first time step we hedge it with Δ_0 of the stock. The composition of the portfolio at $t = 0$ is shown in table(2.1).

Component	Value
Option	C_0
Stock	$-\Delta_0 S_0$
Cash	$\Delta_0 S_0$

Table 2.1: Portfolio composition and values at $t = t_0$

At the second period the composition updates, and one needs to rebalance the delta hedging position, as shown in table(2.2)

Component	Value before rebalance	Value after rebalance
Option	C_1	C_1
Stock	$-\Delta_0 S_1$	$-\Delta_1 S_1$
Cash	$\Delta_0 S_0 e^{r\delta t}$	$\Delta_0 S_0 e^{r\delta t} + (\Delta_1 - \Delta_0) S_1$

Table 2.2: Portfolio composition and values at $t = t_1$

At maturity the total value of the portfolio becomes

$$V_n = C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\delta t} + (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\delta t} + (\Delta_2 - \Delta_1) S_2 e^{(n-2)r\delta t} + \cdots + (\Delta_n - \Delta_{n-1}) S_n$$

Let $n \to \infty$ with $n\delta t = t_n - t_0 \equiv T$, we have

$$V_T = C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S_t d\Delta_t$$

In the idealized Black-Scholes world, the option is perfectly hedged at every instant, and therefore the final P&L is independent of the stock price path. Because the instantaneously hedged option is riskless, the hedging strategy replicates a riskless bond and therefore, by the law of one price, must have the same final value. Therefore

$$V_T = C_0 e^{rT} = C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S_t d\Delta_t$$

[1] This section is mainly extracted from Derman [1] and Willmot [2]

2.2 Hedging with realized volatility

We now use $C(t, S_t; \sigma)$ to denote the Black-Scholes price with volatility σ.

Theorem 2.1. *By delta hedging with realized volatility, the present value of P&L of a hedged option position is*

$$C(S_0, t_0; \sigma^r) - C(S_0, t_0; \sigma^i)$$

where σ^r is the realized volatility and σ^i the implied volatility.

Proof. For simplicity, we use $C^r = C(S_0, t_0; \sigma^r), C^i = C(S_0, t_0; \sigma^i)$

Initially we buy the option at its implied volatility and hedged at known realized volatility, the hedged portfolio at any time t is given by

$$\pi = C^i - \Delta^r S$$

After δt the mark to market profit becomes

$$d\pi = dC^i - \Delta^r dS - (C^i - \Delta^r S)rdt \tag{2.1}$$

where the first term is the increase in the value of the long position in the option, the second is the decrease in the value of the short position in the stock and the last term, $(C^i - \Delta^r S)rdt$, represents the interest on the cost of borrowing an amount $(C^i - \Delta^r S)$ used to set up the initial hedge portfolio.

Besides if we value an option at σ^i and hedged at σ^i, the hedging strategy becomes riskless one that leads to BSM equation, With the riskless hedging strategy, the increase in value of the hedge portfolio should be no different from the interest earned on the position at the riskless rate, meaning

$$0 = dC^r - \Delta^r dS - (C^r - \Delta^r S)rdt \tag{2.2}$$

By rearranging the teams in equation (2.2) we obtain

$$\Delta^r Srdt - \Delta^r dS = -dC^r + C^r rdt \tag{2.3}$$

Substituting equation (2.3) in equation (2.1) we arrive at

$$d\pi = dC^i - dC^r - (C^i - C^r)rdt = e^{rt} d\left(e^{-rt}(C^i - C^r)\right)$$

The present value of this profit can be obtained by discounting

$$e^{-r(t-t_0)} e^{rt} d\left(e^{-rt}(C^i - C^r)\right) = e^{rt_0} d\left(e^{-rt}(C^i - C^r)\right)$$

The entire profit of the hedging strategy is

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}(C^i - C^r)\right) = e^{rt_0}\left(e^{-rt}(C^i - C^r)\right)\Big|_{t_0}^{T}$$

At expiration, when t = T, the value of the option is simply its intrinsic value, where $C^i_T = C^r_T$, therefore the profit is then

$$e^{rt_0}\left(e^{-rt_0}(C^i_{t_0} - C^r_{t_0})\right) = C^r_{t_0} - C^i_{t_0}$$

□

Provided we know the future realized volatility and provided that we can hedge continuously, the final P&L at the expiration of the option is known and deterministic and is equal to the difference between the value of the option based on realized volatility and the value of the option based on implied volatility.

2.3 Hedging with implied volatility

By hedging with implied volatility we are balancing the random fluctuations in the mark-to-market option value with the fluctuations in the stock price. The evolution of the portfolio value is then 'deterministic' as we shall see.

Theorem 2.2. *By delta hedging with implied volatility, the present value of P&L of a hedged option position is*

$$\frac{1}{2}\left((\sigma^i)^2 - (\sigma^r)^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S_t^2 \Gamma^i dt$$

Proof. By Itô lemma, we have

$$dC^i = \Theta^i dt + \Delta^i dS_t + \frac{1}{2}(\sigma^r)^2 S_t^2 \Gamma^i dt$$

then the instantaneous profit of the hedged position is

$$\begin{aligned} d\pi_t &= dC^i - \Delta^i dS - (C^i - \Delta^i S) r dt \\ &= \Theta^i dt + \Delta^i dS_t + \frac{1}{2}(\sigma^r)^2 S_t^2 \Gamma^i dt - \Delta^i dS - (C^i - \Delta^i S) r dt \\ &= \left(\Theta^i + \Delta^i r S + \frac{1}{2}(\sigma^r)^2 S_t^2 \Gamma^i - rC^i\right) dt \end{aligned} \tag{2.4}$$

Because of Black-Scholes PDE

$$\Theta^i + \Delta^i r S_t + \frac{1}{2}(\sigma^i)^2 S_t^2 \Gamma^i - rC^i = 0$$

equation(2.4) becomes

$$\begin{aligned} d\pi_t &= \left(\Theta^i + \Delta^i r S + \frac{1}{2}(\sigma^r)^2 S_t^2 \Gamma^i - rC^i\right) dt \\ &= \frac{1}{2}\left((\sigma^i)^2 - (\sigma^r)^2\right) S_t^2 \Gamma^i dt \end{aligned}$$

Integrate the present value of all of these profits over the life of the option to get a total profit of

$$\frac{1}{2}\left((\sigma^i)^2 - (\sigma^r)^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S_t^2 \Gamma^i dt$$

□

2.4 Hedging with arbitrary constant volatility

Theorem 2.3. *By delta hedging with constant volatility σ^h, the present value of P&L of a hedged option position is*

$$C(S_0, t_0; \sigma^h) - C(S_0, t_0; \sigma^i) + \int_{t_0}^{T} e^{-r(t-t_0)}((\sigma^r)^2 - (\sigma^h)^2)\frac{S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t; \sigma^h) dt$$

Proof. Similar with previous proof, the instantaneous profit of the hedged position is

$$\begin{aligned} d\pi_t &= dC^i - \Delta^h dS - (C^i - \Delta^h S) r dt \\ &= \left(dC^i - dC^h - r(C^i - C^h) dt\right) + \left(dC^h - \Delta^h dS - r(C^h - \Delta^h dS) dt\right) \end{aligned} \tag{2.5}$$

From proofs in Theorem(2.1) , we can see that the present values of first term in equation(2.5) equals

$$C_{t_0}^h - C_{t_0}^i$$

From proofs in Theorem(2.2) , the present value of second item in equation(2.5) arrives

$$\frac{1}{2}\left((\sigma^i)^2 - (\sigma^r)^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S_t^2 \Gamma^i dt$$

Therefore the present value of total P&L is

$$C(S_0, t_0; \sigma^h) - C(S_0, t_0; \sigma^i) + \int_{t_0}^{T} e^{-r(t-t_0)}((\sigma^r)^2 - (\sigma^h)^2)\frac{S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(S_t, t; \sigma^h)dt$$

$\square$

3 Local volatility models

In section(1) we see that the Black-Scholes model with constant volatility cannot reproduce all the option prices observed in the market for a given underlying because their implied volatility varies with strike and maturity.

To take into account the market implied volatility smile while staying within a Markovian and complete model (one risk factor), a natural solution is to model the volatility as a deterministic function of time and the value of the underlying

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t \tag{3.1}$$

where r is the interest rate, assumed to be constant, and W_t is the Brownian motion under the risk-neutral measure Q. The equation(3.1) defines a *local volatility model*.

By the same procedure of building self-financing portfolio, shown in section(1), we see that the price of an option with payoff $h(S_T)$ at date date T is given by

$$C(t, S_t) = \mathbb{E}^Q[e^{-r(T-t)}h(S_T)|S_t]$$

and is characterized by the partial differential equation

$$rC(t, S) = \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma(t, S)^2 S^2 \frac{\partial^2 C}{\partial S^2}, \quad C(T, S) = h(S)$$

The pricing equation has the same form as in the Black-Scholes model, but one can no longer deduce an explicit pricing formula, because the volatility is now a function of the underlying.

Practicioners have adopted a *model calibration* approach which allows to use all observed prices of quoted options as an input for their pricing and hedging activities. The parameters obtained by calibration are of course different from those which would be obtained by historical estimation. But this does not imply any problem related to the presence of arbitrage opportunities.

The model calibration approach is adopted in view of the fact that financial markets do not obey to any fundamental law except the simplest no-dominance or the slightly stronger no-arbitrage. There is no universally accurate model in finance, and any proposed model is wrong. Therefore, practitioners primarily base their strategies on comparison between assets, this is exactly what calibration does.

3.1 Dupire's formula

Theorem 3.1. *If S_t follows*

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t, S_{t_0} = S_0$$

and suppose that

1. S_t is square integrable, i.e. $\mathbb{E}[\int_{t_0}^T S_t^2 dt] < \infty, \forall T$.
2. For $\forall t > t_0$, S_t has a density $p(t, x)$ which is continue on $(t_0, \infty) \times (0, \infty)$.
3. Diffusion coeffient $\sigma(t, x)$ is continue on $(t_0, \infty) \times (0, \infty)$.

Then the value of the call option $C(T, K) = e^{-r(T-t)}\mathbb{E}[(S_T - K)^+]$ satisfies Dupire's equation

$$\frac{\partial C}{\partial T} = \frac{\sigma(T, K)^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K}, \quad (T, K) \in [t_0, \infty) \times [0, \infty)$$

with the initial condition $C(t_0, K) = (S_0 - K)^+$

Proof. Please see section (10.3) in Touzi [3]. □

3.1.1 Calibration of local volatility models

Modelling local volatility by Dupire's formula is criticized for:

- Market prices are not known for all strikes and all maturities. They must be interpolated and the final result is very sensitive to the interpolation method used.
- The only risk factor is underlying price and therefore is impossible to incorporate volatility risk.
- Because of the need to calculate the second derivative of the option price function $C(T, K)$, small data errors lead to very large errors in the solution (ill-posed problem).

Due to these problems, in practice, Dupire's formula is not used directly on the market prices. To avoid solving the ill-posed problem, practitioners typically use one of two approaches to calibrate the model:

1. Start by a preliminary calibration of a parametric functional form to the implied volatility surface (for example, a function quadratic in strike and exponential in time may be used). With this smooth parametric function, recalculate option prices for all strikes, which are then used to calculate the local volatility by Dupire's formula.
2. Reformulate Dupire's equation as an optimization problem by introducing a penalty term to limit the oscillations of the volatility surface.

4 Stochastic volatility models

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dW_t \\ d\sigma_t &= a_t dt + b_t dW_t^{'}, \quad d < W, W^{'} >= \rho dt \end{aligned} \tag{4.1}$$

where $a_t = a(t, \sigma_t, S_t), b_t = b(t, \sigma_t, S_t)$, $\rho \in (-1, 1)$. Because there are two source of risks, two underlyings are desired for hedging the risks.

Assume there exists a liquid underlying with the price

$$C_t^0 = C^0(t, \sigma_t, S_t)$$

where the deterministic function $C^0(t, \sigma_t, S_t)$ is known and $\frac{\partial C^0(t,\sigma,S)}{\partial \sigma} > 0$ for all t, σ, S. Besides some assumptions of assuring the existence and regularity of the solution has to be made but not listed in details here.

Use V_t to denote the value of a self-financing portfolio with δ_t stocks and ω_t quantity of C^0. By self-financing condition and I'tô lemma, we have

$$\begin{aligned} dV_t \quad &= \left(V_t - \delta_t S_t - \omega_t C_t^0\right) rdt + \delta_t S_t + \omega_t C_t^0 \\ &= \left(V_t - \delta_t S_t - \omega_t C_t^0\right) rdt + \left(\delta_t + \omega_t \frac{\partial C^0}{\partial S}\right) dS_t + \omega_t \frac{\partial C^0}{\partial \sigma} d\sigma_t + \omega_t \mathcal{L}_t C^0 dt \end{aligned} \tag{4.2}$$

where

$$\mathcal{L}_t = \frac{\partial}{\partial t} + \frac{1}{2} S_t^2 \sigma_t^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2} b_t^2 \sigma_t^2 \frac{\partial^2}{\partial \sigma^2} + S_t \sigma_t b_t \rho \frac{\partial^2}{\partial S \partial \sigma}$$

Suppose this self-financing portfolio replicates a deterministic payoff function $C : V_t = C(t, \sigma_t, S_t)$. By I'tô lemma, it is obtained that

$$dV_t = \mathcal{L}_t C dt = \mathcal{L}_t C dt + \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial \sigma} d\sigma_t$$

To equalize the above dynamics and equation(4.2), we arrive

$$\begin{aligned} \omega_t &= \frac{\partial C / \partial \sigma}{\partial C^0 / \partial \sigma} \\ \sigma_t &= \frac{\partial C}{\partial S} - \omega_t \frac{\partial C^0}{\partial S} \\ \mathcal{L}_t C - rC + rS_t \frac{\partial C}{\partial S} &= \frac{\partial C}{\partial \sigma} \frac{\mathcal{L}_t C^0 - rC^0 + rS_t \frac{\partial C^0}{\partial S}}{\partial C^0 / \partial \sigma} \end{aligned}$$

Set $\lambda = -\frac{\mathcal{L}_t C^0 - rC^0 + rS_t \frac{\partial C^0}{\partial S}}{\partial C^0 / \partial \sigma}$, note that λ does not depend on values of option that we are going to hedge, but only on values of the stock that has been chosen initially. By replication we arrive at the equation that the price of the option with payoff $h(S_T)$ satisfies

$$\mathcal{L}_t C - rC + rS_t \frac{\partial C}{\partial S} + \lambda(t, \sigma, S) \frac{\partial C}{\partial \sigma} = 0, \quad C(T, \sigma, S) = h(S)$$

where

$$\mathcal{L}_t = \frac{\partial}{\partial t} + \frac{1}{2} S_t^2 \sigma_t^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2} b_t^2 \sigma_t^2 \frac{\partial^2}{\partial \sigma^2} + S_t \sigma_t b_t \rho \frac{\partial^2}{\partial S \partial \sigma}$$

This is the generalization of the Black-Scholes equation with stochastic volatility. The hedging strategy with two types of underlyings whose hedging ratio satisfying $\omega_t = \frac{\partial C / \partial \sigma}{\partial C^0 / \partial \sigma}$ and $\sigma_t = \frac{\partial C}{\partial S} - \omega_t \frac{\partial C^0}{\partial S}$ is called delta-vega hedging(risk of vega is risk of volatility).

4.1 Symmetric principle of implied volatility

As in the Black-Scholes model

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

$e^{-rt}S_t$ is a martingale under risk neutral measure Q. We introduce a new measure defined by

$$\left.\frac{dQ^*}{dQ}\right|_{\mathcal{F}_t} = \frac{S_t}{e^{rt}S_0} = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

then by Girsanov's theorem $W_t^* = W_t - \sigma t$ is a Brownian Motion under measure Q^*. By symmetry property $\tilde{W} = -W^*$ is an equivalent Brownian Motion under measure Q^*. So we can find that

$$\begin{aligned} e^{2rt}\frac{S_0^2}{S_t} \quad &= e^{2rt}S_0 e^{-\sigma W_t - (r - \frac{\sigma^2}{2})t} \\ &= S_0 e^{-\sigma(W_t^* + \sigma t) + (r + \frac{\sigma^2}{2})t} \\ &= S_0 e^{-\sigma W_t^* + (r - \frac{\sigma^2}{2})t} \\ &= S_0 e^{\sigma \tilde{W}_t + (r - \frac{\sigma^2}{2})t} \end{aligned}$$

which implies that the distribution of $e^{2rt}\frac{S_0^2}{S_t}$ under measure Q^* is the same with that of S_t under measure Q.

Theorem 4.1. *Use $C^{BS}(K,T,\sigma), P^{BS}(K,T,\sigma)$ denote the Black-Scholes call and put price with strike K, maturity T and volatility σ. Then*

$$C^{BS}(K,T,\sigma) = \frac{K}{s_0 e^{rT}} P^{BS}\left(\frac{(r^{rT}S_0)^2}{K}, T, \sigma\right)$$

Proof. By using the result derived before the theorem, we can write

$$\begin{aligned} C^{BS}(K,T,\sigma) \quad &= e^{-rT}\mathbb{E}[(S_T - K)^+] \\ &= e^{-rT}\frac{K}{S_0 e^{rT}}\mathbb{E}\left[\frac{S_T}{e^{rT}S_0}\left(\frac{(e^{rT}S_0)^2}{K} - \frac{(e^{rT}S_0)^2}{S_T}\right)^+\right] \\ &= e^{-rT}\frac{K}{S_0 e^{rT}}\mathbb{E}^*\left[\left(\frac{(e^{rT}S_0)^2}{K} - \frac{(e^{rT}S_0)^2}{S_T}\right)^+\right] \\ &= e^{-rT}\frac{K}{S_0 e^{rT}}\mathbb{E}^*\left[\left(\frac{(e^{rT}S_0)^2}{K} - S_0 e^{\sigma\tilde{W}_t + (r - \frac{\sigma^2}{2})t}\right)^+\right] \\ &= \frac{K}{s_0 e^{rT}} P^{BS}\left(\frac{(r^{rT}S_0)^2}{K}, T, \sigma\right) \end{aligned}$$

□

4.2 The characteristic solution

First introduce the following theorem without proof.

Theorem 4.2. *(Mixing Theorem, Romano and Touzi, 1997) Let $C(S_0, V_0, T)$ be the call option price under the risk-adjusted, stochastic volatility process (4.1). Let $C^{BS}(S_0, V_0, T)$ be the Black-Scholes price given* *today's price S_0 and volaility V_0 of underlying. Define*

$$S_T^e = S_0 \exp\left(-\frac{1}{2}\int_0^T \rho_t^2\sigma_t^2 dt + \int_0^T \rho_t\sigma_t dW_t\right)$$
$$V_T^e = \frac{1}{T}\int_0^T (1-\rho^2)\sigma_t^2 dt$$
$$\left\langle C^{BS}(S_T^e, V_T^e, T)\right\rangle = \lim_{\Delta_t\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...C^{BS}(S_T^e, V_T^e, T)\prod_{t=0}^{T-\Delta t}\exp(-\frac{1}{2}W_t^2)\frac{dW_t}{(2\pi)^{0.5}}$$

In words, the option value under stochastic volatility is a weighted sum or mixture of the Black-Scholes values with an effective stock price and effective volatility. The effective variables depend only upon the volatility process. Hence, the valuation reduces to a pricing expectation over the risk-adjusted volatility process alone.

4.2.1 Zero correlation case

In general $< S_T^e, S_T^e >= S_0$, when $\rho = 0$, then $S_T^e = S_0$. In this case, we introduce

$$C(T,K) = \int_0^{\infty} C^{BS}(T,K,)p(U_t)dU_T = \mathbb{E}[C^{BS}(T,K,V_T)]$$

Suppose V_T is close to $\mathbb{E}[V_T]$, i.e, $VarV_T << E[V_T^2]$, by Taylor expansion, we have

$$C(T,K) \approx C^{BS}(T,K,E[V_T]) + \frac{1}{2}VarV_T\frac{\partial^2}{\partial V^2}C^{BS}(T,K,E[V_T])$$

where

$$\frac{\partial^2}{\partial V^2}C^{BS}(T,K,E[V_T]) = \frac{SN(d_1)}{4V^{3/2}}\left(\frac{k^2}{V} - \frac{V}{4} - 1\right), k = \log\frac{K}{Se^{rT}}$$

Besides, because $\delta\sigma << \sigma$,

$$C^{BS}(T,K,\sigma+\delta\sigma) \approx C^{BS}(T,K,\sigma) + \delta\sigma\frac{\partial C^{BS}(T,K,\sigma)}{\partial\sigma}$$

4.3 Pricing formula of caplet and calibration to caps market

Consider the Libor rate for time period $[T_s, T_e]$, which is fixed at time $t_f \le T_s$. The market quotes give Black volatilities of caplets for various strikes. Suppose a caplet has strike K and the year fraction of $[T_s, T_e]$ is τ. The market price for this caplet is

$$V_0^{mkt} = P(0,T_e)\tau\mathrm{Bl}(K, F, \sigma_B\sqrt{t_f}, 1)$$

where $F = F(0; T_s, T_e)$ is the forward rate for $[T_s, T_e]$ at time 0, σ_B is the market quoted Black vol, and $\mathrm{Bl}(K, F, v, w)$ is given by

$$\mathrm{Bl}(K,F,v,w) = Fw\Phi(wd_1) - Kw\Phi(wd_2),\ d_1 = \frac{\ln(F/K)+v^2/2}{v},\ d_2 = \frac{\ln(F/K)-v^2/2}{v}$$

The theoretical price of the above caplet based on one-factor LGM model is

$$V_0^{model} = P(0,T_e)\tau\mathrm{Bl}\left(K+\frac{1}{\tau}, F+\frac{1}{\tau}, [H(T_e)-H(T_s)]\sqrt{\zeta_{t_f}}, 1\right)$$

where $H(t) = \int_0^t e^{-\kappa s}ds$ and $\zeta_t = \int_0^t e^{2\kappa s}\sigma_s^2 ds$ ($\sigma_\cdot$ is the volatility parameter in the corresponding one-factor Hull-White model).

Therefore, calibration to caplets in order to obtain ζ_{t_f} requires solving the following equation

$$\boxed{\mathrm{Bl}(K, F, \sigma_B\sqrt{t_f}, 1) = \mathrm{Bl}\left(K + \frac{1}{\tau}, F + \frac{1}{\tau}, [H(T_e) - H(T_s)]\sqrt{\zeta_{t_f}}, 1\right)}$$

We note $\mathrm{Bl}(K, F, v, 1)$ is a monotone increasing function of v with a range of $((F-K)^+, F)$. So the above calibration equation always has a solution.

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