Note on LGM Model

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Version 1.1, 2018-12-07.

Abstract

The note documents methodologies of Linear Gaussian Markov model and its applications in fixed income product modeling.

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1 HJM framework

After Ho and Lee (1986) introduced a discrete setting to model the evolution of the entire yield curve, Heath, Jarrow and Morton (HJM) (1992) developed a continuous framework for the modeling of interest rate dynamics(see for instances in [2]). Precisely they choose the instantaneous forward rates, that can be locked in at one time for borrowing at a later time, as the fundamental quantities to model. HJM model is an arbitrage free framework for the stochastic evolution of the entire yield curve, where the forward rates dynamics are fully specified through their instantaneous volatility structures.

1.1 Forward rates

Denote $B(t,T)$ as the price at time t of a *zero coupon* bond maturing at time T and have face value 1. We assume this bond bears no risk of default and for every t and T, the bond price $B(t,T)$ is defined.

One can set up a zero cost portfolio, by taking a short position of size 1 in a T maturity bonds and a long position of size $\frac{B(t,T)}{B(t,T+\delta)}$ in $T+\delta$ maturity bonds. At time T one is required to pay 1 to cover the short position and at $T+\delta$ one receives $\frac{B(t,T)}{B(t,T+\delta)}$, which is strictly larger than 1.

It can be seen that the yield for the time $[T,T+\delta]$ is determined, or "locked in", at earlier time t. By this meaning, one defines the instantaneous *forward rate* at time t for investing at time T to be

$$f(t,T) = -\lim_{\delta\to 0}\frac{\log B(t,T+\delta)-\log B(t,T)}{\delta}$$

More precisely the forward rate is given by

$$f(t,T) = -\frac{\partial}{\partial T}\log B(t,T) \tag{1.1}$$

If $f(t,T)$ is known for all values of $t<T$, one can recover $B(0,T)$ by equation (1.1) that

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \tag{1.2}$$

Theoretically it does not matter whether to build a model for forward rates or for bond prices because of equations (1.1) and (1.2). In practice forward rates are not easy to directly determine from market data as it is sensitive to small changes in bond prices.

At last if letting $T=t$ the interest rate at time t is given by

$$r_t = f(t,t) \tag{1.3}$$

This rate is the instantaneous rate we can lock in at time t for borrowing at time t. It is called spot rate or short rate in practise.

1.2 Dynamics of forward rates and bond prices

In the HJM model, forward rates is modeled under the form

$$f(t,T) = f(0,T) + \int_0^t \gamma(s,T)ds + \int_0^t \sigma^f(s,T)d\hat{W}_s \tag{1.4}$$

where $\hat{W}_s$ is the Brownian motion under the historical probability measure P.

Indeed HJM framework only admits no arbitrage under certain condition where is addressed in the following theorem.

Theorem 1.1. **(HJM non-arbitrage condition)** *A HJM model admits no arbitrage if there exists an adapted process θ_t which satisfies*

$$\gamma(t,T) = \sigma^f(t,T)\left(\theta_t - \sigma^B(t,T)\right)$$

where $\gamma(t,T)$, $\sigma^f(t,T)$ *are the drift and volatility of the forward rate under the historical measure and*

$$\sigma^B(t,T) = -\int_t^T \sigma^f(t,u)\ du$$

By applying theorem **1.1** and equation (1.4) the dynamics under the risk neutral measure is given by

$$df(t,T) = -\sigma^f(t,T)\sigma^B(t,T)dt + \sigma^f(t,T)dW_t$$

where W_t is the Brownian motion under the risk neutral measure Q.

Indeed zero coupon is a tradable asset and under risk neutral measure its drift must be r_t. It can be shown by using Ito lemma and equation (1.2) that

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma^B(t,T)dW_t$$

The following theorem summarizes the discussion.

Theorem 1.2. *In a term-structure model satisfying the HJM no-arbitrage condition stated in theorem(1.1), the forward rates evolve according to*

$$df(t,T) = -\sigma^f(t,T)\sigma^B(t,T)dt + \sigma^f(t,T)dW_t \tag{1.5}$$

and the zero coupon bond prices evolves as

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma^B(t,T)dW_t \tag{1.6}$$

where W_t *is a Brownian motion under a risk-neutral measure* Q。 *Here* $\sigma^B(t,T) = -\int_t^T \sigma^f(t,u)du$ *and* $r_t = f(t,t)$ *is the interest rate. The discounted bond prices satisfy*

$$d\left(D(0,t)B(t,T)\right) = D(0,t)\sigma^B(t,T)B(t,T)dW_t$$

where $D(0,t) = e^{-\int_0^t r_u du}$ *is the discount process. The solution to equation(1.6) is*

$$B(t,T) = B(0,T)\exp\left(\int_0^t r_u du + \int_0^t \sigma^B(u,T)dW_u - \frac{1}{2}\int_0^t (\sigma^B(u,T))^2 du\right)$$

1.3 Markovian property

From equation (1.3) and equation (1.5) it can be verified that under risk neutral measure Q the spot rate is

$$r_t = f(t,t) = f(0,t) + \int_0^t \sigma^f(u,t)\sigma^B(u,t)du + \int_0^t \sigma^f(u,t)dW_u$$

If one focuses on the path dependent term by defining

$$G(t) = \int_0^t \sigma^f(u,t)dW_u$$

Obviously it is generally not Markovian because

$$\begin{aligned}\mathbb{E}^Q(G(T)|G(t)) &= \mathbb{E}^Q\left(G(t) + \int_t^T \sigma^f(u,T)dW_u + (\int_0^t \sigma^f(u,T)dW_u - \int_0^t \sigma^f(u,t)dW_u)|G(t)\right) \\ &\neq E_t^Q(G(T))\end{aligned}$$

unless $\int_0^t \sigma^f(u,T)dW_u - \int_0^t \sigma^f(u,t)dW_u$ is non random or a deterministic function of $G(t)$.

An important area of investigation concerns the conditions under which r_t is Markov or less strictly, can be written as

$$r_t = h(t, x_t)$$

for a deterministic function h and a finite-dimensional Markovian vector of state variables x_t. Following section presents an extended multi factor Hull-White model that satisfies Markovian property under certain conditions and is easy to implement in practice.

2 LGM model

2.1 Model setup

A Linear Gaussian Markov(LGM) model assumes that the volatility $\sigma^f(t,T)$ of the forward rate $f(t,T)$ is deterministic with the form

$$\sigma^f(t,T) = \sigma_t e^{-\int_t^T \lambda_s ds} \tag{2.1}$$

Based on the relation between $\sigma^B(t,T)$ and $\sigma^f(t,T)$ under HJM model, one can deduce that

$$\sigma^B(t,T) = -\int_t^T \sigma^f(t,s)ds = -\sigma_t \Lambda(t,T) \tag{2.2}$$

with

$$\Lambda(t,T) = \int_t^T e^{-\int_t^s \lambda_u du} ds$$

λ_t can be seen the mean reversion rate of the short rate and is also assumed to be deterministic. It is the dampening factor for the volatility of the forward rate. If λ is constant, then $\Lambda_i(t,T) = \frac{1-e^{-\lambda(T-t)}}{\lambda}$.

In practice usually a multi-factor LGM model is used. In following sections by default we are in a three factor LGM world, that is, $f(t,T)$ is driven by a correlated three dimensional Brownian motion W.

By introducing the forward measure Q^T under which the forward rate is a martingale, one writes the dynamics of the forward rate as

$$f(t,T) = \sigma^f(t,T) \cdot dW_t^T$$

By definition in equation (1.3) the spot rate under 3-factor model is expressed as

$$r_t = f(t,t) = f(0,t) + \sum_{i=1}^{3} x_t^i \tag{2.3}$$

$$x_t^i = \int_0^t \sigma_i^f(s,t) dW_s^{t,i} = -\int_0^t \sum_{j=1}^{3} \rho_{ij} \sigma_j^B(s,t) \sigma_i^f(s,t) ds + \int_0^t \sigma_i^f(s,t) dW_s^{*,i} \tag{2.4}$$

where $W_s^{*,i}$ is the Brownian motion under the risk neutral measure Q.

It is noted that the deterministic property of σ_t^i and λ_t functions assures that the spot rate is Markovian. Indeed from equation and (2.3) and (2.4), r_t is a function of t and x_t, where x_t is a sum of a deterministic function and a Gaussian internal. Gaussian process are Markovian so is r_t.

Proposition 2.1. *The factor x_t^i evolves as*

$$dx_t^i = \left(\sum_j \Phi_t^{ij} - \lambda_t^i x_t^i \right) dt + \sigma_t^i dW_t^i \tag{2.5}$$

where $\Phi_t^{ij} = \mathrm{Cov}(x_t^i, x_t^j) = \int_0^t \rho_{ij} \sigma_j^f(s,t) \sigma_i^f(s,t) ds$ is the covariance matrix of the factors.

Proof. By equation (2.4) x_t has the dynamics

$$\begin{aligned} dx_t^i &= -\int_0^t \sum_j \rho_{ij} d(\sigma_j^B(s,t)\sigma_i^f(s,t))ds - \sum_j \rho_{ij} \sigma_j^B(t,t) \sigma_i^f(t,t) dt + \sigma_i^f(t,t) dW_t^{*,i} \\ &+ \int_0^t d\sigma_i^f(s,t) dW_s^{*,i} \end{aligned}$$

From equation (2.1) and (2.2), it can be verified that

$$\begin{aligned} d(\sigma_j^B(s,t)\sigma_i^f(s,t)) &= \sigma_j^B(s,t)d\sigma_i^f(s,t) + \sigma_i^f(s,t)d\sigma_j^B(s,t) \\ &= -\lambda_t^i\sigma_i^f(s,t)\sigma_j^B(s,t)dt - \sigma_i^f(s,t)\sigma_j^f(s,t)dt \end{aligned}$$

As a result, the first term of dx_t^i arrives at

$$\begin{aligned} \int_0^t \sum_j \rho_{ij} d(\sigma_j^B(s,t)\sigma_i^f(s,t))ds &= -\int_0^t \sum_j \rho_{ij}\lambda_t^i\sigma_j^B(s,t)\sigma_i^f(s,t)dtds - \int_0^t \sum_j \rho_{ij}\sigma_i^f(s,t)\sigma_j^f(s,t)dtds \\ &= -\left(\int_0^t \sum_j \rho_{ij}\lambda_t^i\sigma_j^B(s,t)\sigma_i^f(s,t)ds\right)dt - \left(\int_0^t \sum_j \rho_{ij}\sigma_i^f(s,t)\sigma_j^f(s,t)ds\right)dt \end{aligned}$$

By definition of $\sigma^B(t,T)$ in equation (2.2), one obtains

$$\sigma_j^B(t,t) = 0$$

which implies the second term of dx_t^i is zero.

The third term obviously equals to $\sigma_t^i dW_t^{*,i}$.

To address the fourth term of dx_t^i, by applying Fubini's theorem to the stochastic integral, one can evaluate the following integral as

$$\int_0^t d\sigma_i^f(s,t)dW_s^{*,i} = -\int_0^t \lambda_t^i\sigma_i^f(s,t)dtdW_s^{*,i} = -\int_0^t \lambda_t^i\sigma_i^f(s,t)dW_s^{*,i}dt$$

Therefore the dynamics of x_t is equivalent to

$$\begin{aligned} dx_t^i &= \left(\int_0^t \sum_j \rho_{ij}\lambda_t^i\sigma_j^B(s,t)\sigma_i^f(s,t)ds\right)dt + \left(\int_0^t \sum_j \rho_{ij}\sigma_i^f(s,t)\sigma_j^f(s,t)ds\right)dt \\ &+ \sigma_t^i dW_t^{*,i} - \lambda_t^i \int_0^t \sigma_i^f(s,t)dW_s^{*,i}dt \\ &= \left(\int_0^t \sum_j \rho_{ij}\sigma_i^f(s,t)\sigma_j^f(s,t)ds - \lambda_t^i x_t^i\right)dt + \sigma_t^i dW_t^i \\ &= (\sum_j \Phi_t^{ij} - \lambda_t^i x_t^i)dt + \sigma_t^i dW_t^i \end{aligned}$$

where

$$\Phi_t^{ij} = cov(x_t^i, x_t^j) = \int_0^t \rho_{ij}\sigma_i^f(s,t)\sigma_j^f(s,t)ds$$

□

Proposition 2.2. *The covariance matrix Φ_t^{ij} evolves as*

$$d\Phi_t^{ij} = (\rho_{ij}\sigma_t^i\sigma_t^j - (\lambda_t^i + \lambda_t^j)\Phi_t^{ij})dt \tag{2.6}$$

Proof.

$$\begin{aligned} d\Phi_t^{ij} &= \rho_{ij}\sigma_t^i\sigma_t^j dt + \int_0^t \rho_{ij} d(\sigma_i^f(s,t)\sigma_j^f(s,t))ds \\ &= \rho_{ij}\sigma_t^i\sigma_t^j dt + \int_0^t \rho_{ij}(-\lambda_t^i - \lambda_t^j)\sigma_i^f(s,t)\sigma_j^f(s,t)dtds \\ &= (\rho_{ij}\sigma_t^i\sigma_t^j - (\lambda_t^i + \lambda_t^j)\Phi_t^{ij})dt \end{aligned}$$

□

Φ_t^{ij} can be seen as the convexity adjustment due the fact that the short rate, or spot rate is not a martingale under the risk neutral measure.

2.2 Pricing formula

Theorem 2.1. *Under the above assumptions, the relationship between the factors and zero coupon is given by*

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(-\frac{1}{2}\Lambda(t,T)^2\Phi(t) - \Lambda(t,T)(r_t - f(0,t))\right) \tag{2.7}$$

Proof. By defintion of $B(t,T)$ and $B(t,t)$, $\frac{B(0,T)}{B(0,t)}$ is a martingale under forward measure Q^t, therefore it can be shown that

$$\begin{aligned}\frac{B(t,T)}{B(t,t)} =& \frac{B(0,T)}{B(0,t)} \\ &\exp\left(\int_0^t (\sigma^B(s,T) - \sigma^B(s,t))dW_s^t - \int_0^t \frac{1}{2}((\sigma^B(s,T))^2 - (\sigma^B(s,t))^2)ds\right)\end{aligned} \tag{2.8}$$

By definition of $\sigma^B(s,T)$ and $\sigma^B(s,t)$, using Fubini's theorem as well, the stochastic integral is then equivalent to

$$\begin{aligned}\int_0^t (\sigma^B(s,T) - \sigma^B(s,t))dW_s^t &= \int_0^t \sigma_s(\Lambda(s,T) - \Lambda(s,t))dW_s^t \\ &= -\int_0^t \sigma_s(\int_s^T e^{-\int_s^r \lambda_u du}dr - \int_s^t e^{-\int_s^r \lambda_u du}dr)dW_s^t \\ &= -\int_0^t \sigma_s \int_t^T e^{-\int_s^r \lambda_u du}dr dW_s^t \\ &= -\int_0^t \sigma_s e^{-\int_s^t \lambda_u^i du}dW_s^t \int_t^T e^{-\int_t^r \lambda_u^i du}ds \\ &= -x_t\Lambda(t,T)\end{aligned}$$

Its variance indeed is

$$\mathrm{Var}(x_t\Lambda(t,T)) = \Phi_t\Lambda(t,T)\cdot\Lambda(t,T) = \int_0^t \frac{1}{2}(\sigma^B(s,T) - \sigma^B(s,t))^2 ds$$

Therefore equation (2.8) is equivalent to,

$$\frac{B(t,T)}{B(t,t)} = \frac{B(0,T)}{B(0,t)} \exp(-\frac{1}{2}\Phi_t\Lambda(t,T)\cdot\Lambda(t,T) - x_t\Lambda(t,T))$$

□

References

[1] Steven Shreve, Stochastic Calculus for Finance II, *Springer Finance*, Springer, 2004.

[2] Damiano Brigo and Fabio Mercurio. Interest rate models: theory and practice. *Springer Finance, 2nd edition*, Springer, 2006. 2

[3] Harvey Stein. Valuation of exotic interest rate derivatives. *research note*, Bloomberg LP, December, 2007.

[4] Yan Zeng. Elements of LGM model. *work note, version 1.0*, July, 2012.

[5] Changwei Xiong. Introduction to interest rate models, *work note*, Nomura Singapore, June, 2018.