

# A deep mathematical understanding of DNNs

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## Abstract

Frameworks such as TensorFlow or PyTorch make deep learning developments easy. They have made this field wide spread for every enthusiast. Implementations only needs an instinctive understanding of deep learning. The proper math aspect is little by little forgotten.

Normalized vector spaces theories are supposed known. The objective is to do a collection of the important propositions explaining dense neural network (DNN) theories. These propositions will be mathematically proven. The subject used as reference is a multi-class classification problem with – dense layers, *ReLU* and *SoftMax* activation layers, Categorical cross-entropy loss and Stochastic gradient descent optimizer. But all the elements below can be easily re-used or re-defined to cover regressions.

## 1 – Prerequisites and Notations

### 1.1 – Matrices

**Definition** – Let  $a_{i,j} \in \mathbb{R}$  for  $i \in \llbracket 1, n \rrbracket$  and  $j \in \llbracket 1, m \rrbracket$ . Then the ordered rectangular array

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}$$

is a real matrix of dimension  $n \times m$ .

The following notations are considered

$$\forall i \in \llbracket 1, n \rrbracket, \forall j \in \llbracket 1, m \rrbracket, A_{i,j} = a_{i,j}$$

$$\forall j \in \llbracket 1, m \rrbracket, A_{:,j} = \begin{bmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{bmatrix}$$

$$\forall i \in \llbracket 1, n \rrbracket, A_{i,:} = [a_{i,1} \ a_{i,2} \ \cdots \ a_{i,n}]$$

The notation  $M_{n,m}$  means the matrix set of dimension  $n \times m$  with coefficients in  $\mathfrak{R}$ .

The notation  $M_{n,m}(E)$  means the matrix set of dimension  $n \times m$  with coefficients in  $E \subset \mathfrak{R}$ .

**Convention** – A vector is a matrix with only one row. Thus, the real vector set  $\mathfrak{R}^m$  is equivalent to  $M_{1,m}$ .

**Notation** – The matrix transpose operation will be noted as  $A^T$ .

**Definition** – Let  $A \in M_{n,m}$ , and  $B \in M_{m,p}$ , and let the product noted  $A \times B$  or  $AB$  be

$$C = A \times B = AB$$

where  $C$  is a  $n \times p$  matrix with

$$\forall i \in \llbracket 1, n \rrbracket, \forall j \in \llbracket 1, p \rrbracket, C_{i,j} = \sum_{k=1}^m A_{i,k} \times B_{k,j}$$

**Definition** – Let  $a \in \mathfrak{R}$  and  $B \in M_{n,m}$ . Let the scalar wise product noted as  $a \times B$  be

$$C = a \times B = B \times a$$

where  $C$  is in  $M_{n,m}$  with

$$\forall i \in \llbracket 1, n \rrbracket, \forall j \in \llbracket 1, m \rrbracket, C_{i,j} = a \times B_{i,j}$$

**Notation** – Let  $a \in \mathfrak{R}^n$  and  $b \in \mathfrak{R}^n$ . Let the scalar product on  $\mathfrak{R}^n$  between two vectors noted as  ${}_n(a|b)$  be

$${}_n(a|b) = a \times b^T = b \times a^T = {}_n(b|a)$$

Let  $c \in \mathfrak{R}^n$ . Let the Euclidean norm on  $\mathfrak{R}^n$  noted as  ${}_n\|c\|$  be

$${}_n\|c\| = \sqrt{(c|c)} = \sqrt{c \times c^T}$$

**Theorem** – Let  $a \in \mathfrak{R}^n$  and  $b \in \mathfrak{R}^n$ . Then

$${}_n(a|b) \leq {}_n\|a\| \times {}_n\|b\|$$

This theorem is named Cauchy-Schwartz inequality.

Proof: TO DO.

## 1.2 – Function regularity

**Definition** – Let  $\Omega \neq \emptyset$  subset of  $\mathfrak{R}^m$ ,  ${}_m\|\cdot\|$  a norm on  $\mathfrak{R}^m$ ,  ${}_m\|\cdot\|$  a norm on  $\mathfrak{R}^{m'}$ , and  $f: \Omega \rightarrow \mathfrak{R}^{m'}$ . Then  $f$  continuous function is equivalent to

$$\forall a \in \Omega,$$

$$\forall \epsilon > 0, \exists \eta > 0, \forall x \in \Omega, {}_m\|x - a\| < \eta \Rightarrow {}_{m'}\|f(x) - f(a)\| < \epsilon$$

The notation  $\zeta(\Omega, \mathfrak{R}^{m'})$  means the set of continuous functions from  $\Omega$  to  $\mathfrak{R}^{m'}$ .

The notation  $\zeta(\mathfrak{R}^m)$  means the set of continuous functions from  $\mathfrak{R}^m$  to  $\mathfrak{R}^{m'}$ .

**Definition** – Let  $\Omega \neq \emptyset$  subset of  $\mathfrak{R}^m$ ,  ${}_m\|\cdot\|$  a norm on  $\mathfrak{R}^m$ ,  ${}_m\|\cdot\|$  a norm on  $\mathfrak{R}^{m'}$ , and  $f: \Omega \rightarrow \mathfrak{R}^{m'}$ . Then  $f$  derivable is equivalent to

$$\forall x \in \Omega, \forall a \in \Omega, \exists \frac{\partial f}{\partial x}(a) \in \mathfrak{R}^{m'},$$

$$\forall \epsilon > 0, \exists \eta > 0, \forall \tau \in \mathfrak{R}, |\tau| < \eta \Rightarrow \left\| \frac{f(\tau \times x + a) - f(a)}{\tau} - \frac{\partial f}{\partial x}(a) \right\| < \epsilon$$

In the notation  $\frac{\partial f}{\partial x}(a)$ ,  $x$  and  $a$  mean the direction and point of derivation respectively.

The notation  $\frac{\partial f}{\partial x}(a)$  can also be wrote as  $f'(a)$  when  $m=1$ .

The notation  $D(\Omega, \mathfrak{R}^{m'})$  means the set of derivable functions from  $\Omega$  to  $\mathfrak{R}^{m'}$ .

The notation  $D(\mathfrak{R}^m)$  means the set of derivable functions from  $\mathfrak{R}^m$  to  $\mathfrak{R}^{m'}$ .

**Definition** – Let  $\Omega \neq \emptyset$  subset of  $\mathfrak{R}^m$  and  $f \in D(\Omega, \mathfrak{R}^{m'})$ . Then its Jacobian matrix  $J_f$  is defined as the application

$$J_f: \left\{ \begin{array}{l} \Omega \rightarrow M_{m',m} \\ x \mapsto \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{m'}}{\partial x_1}(x) & \cdots & \frac{\partial f_{m'}}{\partial x_m}(x) \end{bmatrix} \end{array} \right.$$

The notation  $f_i$  means the application  $f_i: \left\{ \begin{array}{l} \Omega \rightarrow \mathbb{R} \\ x \mapsto f(x)_i \end{array} \right.$  and  $x_i$  means the matrices  $x_i = [0 \ \cdots \ 0 \ \underset{\text{at index } i}{1} \ 0 \ \cdots \ 0] \in \mathbb{R}^m$ .

The Jacobian matrix is also named Gradient matrix when  $m'=1$ .

**Notation** – Let  $f: \Omega \rightarrow \Omega'$  and  $g: \Omega' \rightarrow \Omega''$ . Then the notation  $g \circ f$  means the application  $g \circ f: \left\{ \begin{array}{l} \Omega \rightarrow \Omega'' \\ x \mapsto g(f(x)) \end{array} \right.$ .

Let  $(f_i)_{i \in [1, n]}$  with  $f_i: \Omega_i \rightarrow \Omega_{i+1}$  for  $i \in [1, n]$ . Then the notation  $\bigcirc_{i=1}^n f_i$  means the application  $\bigcirc_{i=1}^n f_i: \left\{ \begin{array}{l} \Omega_1 \rightarrow \Omega_{n+1} \\ x \mapsto f_n(f_{n-1}(\cdots f_2(f_1(x)))) \end{array} \right.$ .

**Proposition** – Let  $U \neq \emptyset$  and  $V \neq \emptyset$  subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ . Let  $f: \left\{ \begin{array}{l} U \rightarrow V \\ x \mapsto f(x) \end{array} \right.$  and  $g: \left\{ \begin{array}{l} V \rightarrow \mathbb{R}^p \\ y \mapsto g(y) \end{array} \right.$  such as  $f \in \zeta(U, V)$  and  $g \in \zeta(U, V)$ . Then  $g \circ f: \left\{ \begin{array}{l} U \rightarrow \mathbb{R}^p \\ x \mapsto g(f(x)) \end{array} \right.$  is in  $\zeta(U, \mathbb{R}^p)$ .

Proof: TO DO.

**Theorem** – Let  $U \neq \emptyset$  and  $V \neq \emptyset$  subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ . Let  $f: \left\{ \begin{array}{l} U \rightarrow V \\ x \mapsto f(x) \end{array} \right.$  and  $g: \left\{ \begin{array}{l} V \rightarrow \mathbb{R}^p \\ y \mapsto g(y) \end{array} \right.$  such as  $f \in \zeta(U, V) \cap D(U, V)$  and  $g \in \zeta(U, V) \cap D(V, \mathbb{R}^p)$ . Then

$g \circ f: \begin{cases} U \rightarrow \mathbb{R}^p \\ x \mapsto g(f(x)) \end{cases}$  is in  $D(U, \mathbb{R}^p)$  and its Jacobian is

$$J_{g \circ f}: \begin{cases} U \rightarrow M_{p,n} \\ x \mapsto J_g(f(x)) \times J_f(x) \end{cases}$$

Proof: TO DO.

### 1.3 – Function convexity and smoothness

**Definition** – Let  $\Omega \neq \emptyset$  convex of  $\mathbb{R}^m$  and  $f: x \mapsto f(x)$  such as  $f \in \mathcal{C}(\Omega, \mathbb{R})$ . Then  $f$  convex on  $\Omega$  is equivalent to

$$\begin{aligned} & \forall (y, z) \in \Omega^2, \quad \forall t \in [0, 1], \\ & f(t \times y + (1-t) \times z) \leq t \times f(y) + (1-t) \times f(z) \end{aligned}$$

**Proposition** – Let  $\Omega \neq \emptyset$  convex of  $\mathbb{R}^m$  and  $f \in D(\Omega, \mathbb{R})$  convex on  $\Omega$ . Then  $f$  convex on  $\Omega$  is equivalent to

$$\begin{aligned} & \forall (y, z) \in \Omega^2, \\ & f(y) + \frac{df}{dx}(y) \times (z - y)^T \leq f(z) \end{aligned}$$

Proof: TO DO.

**Proposition** – Let  $\Omega \neq \emptyset$  convex of  $\mathbb{R}^m$  and  $f: x \mapsto f(x)$  such as  $f \in D(\Omega, \mathbb{R})$  and convex. Then

$$\exists X^* \subset \Omega \setminus \{\emptyset\}, \quad \forall x^* \in X^*, \quad f: x \mapsto f(x^*) \leq f(x) \quad \text{<TODO existing and that's all>}$$

**Definition** – Let  $\Omega \neq \emptyset$  subset of  $\mathbb{R}^m$ ,  $\|\cdot\|_m$  a norm on  $\mathbb{R}^m$ , and  $f: x \mapsto f(x)$  such as  $f \in D(\Omega, \mathbb{R})$ . Let  $L > 0$ . Then  $f$   $L$ -smooth on  $\Omega$  is equivalent to

$$\forall (y, z) \in \Omega^2, \quad \left\| \frac{df}{dx}(y) - \frac{df}{dx}(z) \right\|_m \leq L \times \|y - z\|_m$$

**Proposition** – Let  $\Omega \neq \emptyset$  convex of  $\mathfrak{R}^m$  and  $f: x \mapsto f(x)$  such as  $f \in D(\Omega, \mathfrak{R})$ . Then  $f$  first order integral form Taylor expansion is

$$\forall (y, z) \in \Omega^2, \\ f(y) = f(z) + \int_0^1 \frac{df}{dx}(z + \tau(y-z))(y-z)^T d\tau$$

Proof: TO DO.

**Proposition** – Let  $\Omega \neq \emptyset$  convex of  $\mathfrak{R}^m$ ,  $L > 0$ , and  $f: x \mapsto f(x)$  such as  $f \in D(\Omega, \mathfrak{R})$  plus  $L$ -smooth on  $\Omega$ . Then

$$\forall (y, z) \in \Omega^2, \\ f(z) \leq f(y) + \frac{df}{dx}(y) \times (z-y)^T + \frac{L}{2} \times_m \|z-y\|^2$$

$$\forall x \in \Omega, \\ f(x - \frac{1}{L} \times \frac{df}{dx}(x)) - f(x) \leq -\frac{1}{2L} \times_m \left\| \frac{df}{dx}(x) \right\|^2$$

Proof: TO DO.

**Proposition** – Let  $\Omega \neq \emptyset$  convex of  $\mathfrak{R}^m$ ,  $\|\cdot\|_m$  a norm on  $\mathfrak{R}^m$ , and  $f: x \mapsto f(x)$  such as  $f \in D(\Omega, \mathfrak{R})$ , convex and  $L$ -smooth. Then  $f$  is co-coercive

$$\forall (y, z) \in \Omega^2, \\ \frac{1}{L} \times_m \left\| \frac{df}{dx}(y) - \frac{df}{dx}(z) \right\|^2 \leq \left( \frac{df}{dx}(y) - \frac{df}{dx}(z) \right) \times (x-y)^T$$

Proof: TO DO.

### 1.3 – Others

**Notation** – Let  $f: \begin{cases} \Omega_1 \times \dots \times \Omega_n \rightarrow \Omega'_1 \times \dots \times \Omega'_m \\ (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n) \end{cases}$  an application with  $n$  parameters and  $m$  outputs. Then for  $k \in \llbracket 1, n \rrbracket$  the notation  $f(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_n)$  means the application  $f(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_n): \begin{cases} \Omega_k \rightarrow \Omega'_1 \times \dots \times \Omega'_m \\ x_k \mapsto f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) \end{cases}$ .

**Notation** – Let  $\Omega \in \mathfrak{R}^m$ . The notation  $1_\Omega$  means the  $\Omega$  indicator function on  $\mathfrak{R}^m$ .

$$1_\Omega: \begin{cases} \Omega \rightarrow \mathfrak{R}^m \\ x \mapsto \begin{cases} 1 & x \in \Omega \\ 0 & x \notin \Omega \end{cases} \end{cases}$$

## 2 – Activation functions

**Definition** – Let  $F_{act} \in D(\mathfrak{R}^m)$ . Then the vector wise application

$$F_{act}: \begin{cases} \mathfrak{R}^m \rightarrow \mathfrak{R}^m \\ z \mapsto f(z) \end{cases}$$

is an activation function.

**Definition** –  $ReLU$  is the following vector wise application

$$ReLU: \begin{cases} \mathfrak{R}^m \rightarrow \mathfrak{R}^m \\ z \mapsto \max(0, z) \end{cases}$$

with  $\max$  the element-wise maximum operation between two vectors.

**Hypothesis** – The notation  $ReLU_j$  means the application corresponding to the coefficient  $j$  of the function  $ReLU$ . Let  $z \in \mathfrak{R}^m$  then

$$\forall j \in \llbracket 1, m \rrbracket, \quad ReLU_j(z_j) = \max(0, z_j) = ReLU(z)_j$$

$ReLU$  is supposed derivable on every coefficients at 0

$$\forall j \in \llbracket 1, m \rrbracket, \text{ReLU}_j'(0) = 0$$

**Proposition** –  $\text{ReLU}$  is an activation function. Its Jacobian matrix is

$$\frac{d \text{ReLU}}{dz} : \left\{ \begin{array}{l} \mathfrak{R}^m \rightarrow M_{m,m} \\ z \mapsto \begin{bmatrix} 1_{\mathfrak{R}_{\setminus \{0\}}^+(z_1)} & 0 & \cdots & 0 \\ 0 & 1_{\mathfrak{R}_{\setminus \{0\}}^+(z_2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1_{\mathfrak{R}_{\setminus \{0\}}^+(z_m)} \end{bmatrix} \end{array} \right.$$

Proof: TO DO.

**Proposition** – The following vector wise application is an activation function

$$\text{SoftMax} : \left\{ \begin{array}{l} \mathfrak{R}^m \rightarrow ]0, 1[^m \\ z \mapsto \frac{e^{z_j}}{\sum_{j'=1}^m e^{z_{j'}}} \end{array} \right.$$

with  $e$  the element-wise exponential operation.

The  $\text{SoftMax}$  function will be denoted as  $S$  for simplicity.

Its Jacobian matrix is

$$\frac{dS}{dz} : \left\{ \begin{array}{l} \mathfrak{R}^m \rightarrow M_{m,m} \\ z \mapsto \frac{dS}{dz}(z) \end{array} \right.$$

where  $\forall z \in \mathfrak{R}^m, \forall (j, j') \in \{1, 2, \dots, m\}^2$ ,

$$\frac{dS}{dz}(z)_{j,j'} = S(z)_j \times (\delta_{j,j'} - S(z)_{j'})$$

with  $\delta_{j,j'}$  the Kronecker delta.

Proof: TO DO.



### 3 – Loss

**Definition** – Let  $\hat{\Omega} \in M_{n,m}$  and  $\Omega \subseteq M_{n,m}$  non empty subsets. Let  $\hat{y} \in \hat{\Omega}$  and  $F_{loss}^{\hat{y}} \in D(\Omega, \mathfrak{R})$ . Then  $F_{loss}^{\hat{y}}$  is a loss function is equivalent to the application

$$F_{loss}^{\hat{y}} \circ g: \begin{cases} E \rightarrow \mathfrak{R} \\ \epsilon \mapsto (F_{loss}^{\hat{y}} \circ g)(\epsilon) = F_{loss}^{\hat{y}}(\hat{y} + \epsilon) \end{cases}$$

is an increasing function according each coefficient with  $E \subseteq M_{n,m}$  such as  $F_{loss}^{\hat{y}} \circ g$  is always defined.

The  $\hat{y}$  matrix is named the ground truth.

**Proposition** – Let  $\hat{y} \in \{0,1\}^m$  a ground truth matrix. Then the application

$$\xi^{\hat{y}}: \begin{cases} ]0,1[^m \rightarrow \mathfrak{R} \\ y \mapsto -\sum_{j=1}^m \hat{y}_j \log(y_j) \end{cases}$$

is a loss function. The application is named Categorical cross-entropy loss.

Its Gradient matrix is

$$\frac{d\xi^{\hat{y}}}{dy}: \begin{cases} ]0,1[^m \rightarrow \mathfrak{R}^m \\ y \mapsto -\begin{bmatrix} \frac{\hat{y}_1}{y_1} & \dots & \frac{\hat{y}_m}{y_m} \end{bmatrix} \end{cases}$$

Proof: TO DO.

**Proposition** – Let  $\hat{y} \in \{0,1\}^m$  a ground truth matrix. Let  $S: \mathfrak{R}^m \rightarrow ]0,1[^m$  and  $\xi^{\hat{y}}: ]0,1[^m \rightarrow \mathfrak{R}$  the *SoftMax* activation and Categorical cross-entropy loss functions. Then  $\xi^{\hat{y}} \circ S: \mathfrak{R}^m \rightarrow \mathfrak{R}$  is derivable on  $\mathfrak{R}^m$  and its Gradient matrix is

$$\frac{d(\xi^{\hat{y}} \circ S)}{dz}: \begin{cases} \mathfrak{R}^m \rightarrow \mathfrak{R}^m \\ z \mapsto S(z) - \hat{y} \end{cases}$$

Proof: TO DO.

## 4 – Dense layers

**Definition** – The application

$$L: \begin{cases} \mathbb{R}^m \times M_{m',m} \times \mathbb{R}^{m'} \rightarrow \mathbb{R}^{m'} \\ (y, W, b) \mapsto y \times W^T + b \end{cases}$$

defines a dense layer with  $y$  named the input vector,  $W$  named the weight matrix and  $b$  named the bias matrix.

The notation  $L_j$  means the application  $L_j: \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$  corresponding to the row  $j$  of the second matrix component of the dense layer  $L$ . Let  $y \in \mathbb{R}^m$  an input vector,  $W \in M_{m',m}$  a weight matrix and  $b \in \mathbb{R}^{m'}$  a bias matrix then

$$\forall j \in \llbracket 1, m' \rrbracket, L_j(y, W_{j,:}, b_j) = y \times (W_{j,:})^T + b_j = L(y, W, b)_j$$

**Proposition** – Let  $L: \mathbb{R}^m \times M_{m',m} \times \mathbb{R}^{m'} \rightarrow \mathbb{R}^{m'}$  a dense layer function. Then  $L$  is derivable according the first and third variables on  $\mathbb{R}^m$  and  $\mathbb{R}^{m'}$  respectively.

Let  $y \in \mathbb{R}^m$  an input vector and  $b \in \mathbb{R}^{m'}$  a bias matrix. Then  $L_{j,:}: \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$  is also derivable according the second variable for all  $j \in \llbracket 1, m' \rrbracket$ .

Its Gradient or Jacobian matrices are

$$\begin{aligned} \frac{\partial L}{\partial y} &: \begin{cases} \mathbb{R}^m \rightarrow M_{m',m} \\ y \mapsto W \end{cases} \\ \forall j \in \llbracket 1, m' \rrbracket, \frac{\partial L_j}{\partial w} &: \begin{cases} \mathbb{R}^m \rightarrow \mathbb{R}^m \\ w \mapsto y \end{cases} \\ \frac{\partial L}{\partial b} &: \begin{cases} \mathbb{R}^{m'} \rightarrow M_{m',m'} \\ b \mapsto I_{m'} \end{cases} \end{aligned}$$

with  $I_{m'}$  the identity matrix of size  $m' \times m'$ .

Proof: TO DO.

**Proposition** – Let  $L: \mathfrak{R}^m \times M_{m',m} \times \mathfrak{R}^{m'} \rightarrow \mathfrak{R}^{m'}$  and  $ReLU: \mathfrak{R}^{m'} \rightarrow \mathfrak{R}^{m'}$  the dense layer and

$ReLU$  activation functions. Let  $F^{upstream}: \begin{cases} \mathfrak{R}^{m'} \rightarrow \mathfrak{R} \\ y' \mapsto F^{upstream}(y') \end{cases}$  such as  $F^{upstream} \in D(\mathfrak{R}^{m'}, \mathfrak{R})$ .

Then  $F^{upstream} \circ ReLU \circ L(\cdot, W, b): \mathfrak{R}^m \times M_{m',m} \times \mathfrak{R}^{m'} \rightarrow \mathfrak{R}$  is derivable according the first and third variables on  $\mathfrak{R}^m$  and  $\mathfrak{R}^{m'}$  respectively.

The notation  $F_{j'}^{upstream}$  means the application corresponding to the coefficient  $j'$  of  $F^{upstream}$ .  
Let  $y' \in \mathfrak{R}^{m'}$  then

$$\forall j' \in \llbracket 1, m' \rrbracket, F_{j'}^{upstream}(y'_j) = F^{upstream}(y')_{j'}$$

Let  $y \in \mathfrak{R}^m$  an input vector and  $b \in \mathfrak{R}^{m'}$  a bias matrix. Then

$F_{j'}^{upstream} \circ ReLU_{j'} \circ L_{j'}(\cdot, w, b_{j'}) : \mathfrak{R} \rightarrow \mathfrak{R}$  is also derivable for all  $j' \in \llbracket 1, m' \rrbracket$ .

Its Gradient matrices are

$$\frac{\partial (F^{upstream} \circ ReLU \circ L(\cdot, W, b))}{\partial y} : \begin{cases} \mathfrak{R}^m \rightarrow \mathfrak{R}^m \\ y \mapsto \frac{\partial (F^{upstream} \circ ReLU \circ L(\cdot, W, b))}{\partial y}(y) \end{cases}$$

where  $\forall y \in \mathfrak{R}^m, \forall j \in \llbracket 1, m \rrbracket$ ,

$$\frac{\partial (F^{upstream} \circ ReLU \circ L(\cdot, W, b))}{\partial y}(y)_j = \sum_{j'=1}^{m'} \frac{d F^{upstream}}{d y'}(ReLU(L(y, W, b)))_{j'} \times 1_{\mathfrak{R}_{\setminus \{0\}}^+}(L(y, W, b)_{j'}) \times W_{j', j}$$

with  $W \in M_{m',m}$  a weight matrix,  $b \in \mathfrak{R}^{m'}$  a bias matrix.

$$\forall j' \in \llbracket 1, m' \rrbracket, \frac{\partial (F_{j'}^{upstream} \circ ReLU_{j'} \circ L_{j'}(\cdot, w, b_{j'}))}{\partial w} : \begin{cases} \mathfrak{R}^m \rightarrow \mathfrak{R}^m \\ w \mapsto \frac{\partial (F_{j'}^{upstream} \circ ReLU_{j'} \circ L_{j'}(\cdot, w, b_{j'}))}{\partial w}(w) \end{cases}$$

where  $\forall w \in \mathfrak{R}^m, \forall j \in \llbracket 1, m \rrbracket$ ,

$$\frac{d (F_{j'}^{upstream} \circ ReLU_{j'} \circ L_{j'}(\cdot, w, b_{j'}))}{d w}(w)_j = F_{j'}^{upstream'}(ReLU_{j'}(L_{j'}(y, w, b_{j'}))) \times 1_{\mathfrak{R}_{\setminus \{0\}}^+}(L_{j'}(y, w, b_{j'})) \times y_j$$

with  $y \in \mathfrak{R}^m$  an input matrix,  $b \in \mathfrak{R}^{m'}$  a bias matrix.

$$\frac{\partial (F^{upstream} \circ \text{ReLU} \circ L(\cdot, W, b))}{\partial b} : \begin{cases} \mathfrak{R}^{m'} \rightarrow \mathfrak{R}^{m'} \\ b \mapsto \frac{\partial (F^{upstream} \circ \text{ReLU} \circ L(\cdot, W, b))}{\partial b}(b) \end{cases}$$

where  $\forall b \in \mathfrak{R}^{m'}$ ,  $\forall j' \in \llbracket 1, m' \rrbracket$ ,

$$\frac{\partial (F^{upstream} \circ \text{ReLU} \circ L(\cdot, W, b))}{\partial b}(b)_{j'} = \frac{d F^{upstream}}{d y'}(\text{ReLU}(L(y, W, b)))_{j'} \times 1_{\mathfrak{R}_{\setminus \{0\}}^+}(L(y, W, b)_{j'})$$

with  $y \in \mathfrak{R}^m$  a weight matrix and  $W \in M_{m', m}$  a weight matrix.

Proof: TO DO.

## 5 – Neural Networks

**Definition** – A training data set is defined as couples of vectors  $(X^i, \hat{Y}^i) \in \mathfrak{R}^m \times \mathfrak{R}^l$  for  $i \in \llbracket 1, n \rrbracket$ . The  $X^i$  are named input or feature matrices and the  $\hat{Y}^i$  target or label matrices.

**Definition** – Let  $p$  dense layers with activation functions  $F_{act}^k \circ L^k(\cdot, W^k, b^k) : \mathfrak{R}^{m_k} \rightarrow \mathfrak{R}^{m_{k+1}}$  for  $k \in \llbracket 1, p \rrbracket$  with  $W^k \in M_{m_{k+1}, m_k}$  and  $b^k \in \mathfrak{R}^{m_{k+1}}$  the  $L^k$  weight and bias matrices. Let a training data set  $(X^i, \hat{Y}^i) \in \mathfrak{R}^{m_1} \times \mathfrak{R}^{m_{p+1}}$  for  $i \in \llbracket 1, n \rrbracket$ . Let  $F_{loss}^{\hat{Y}^i} : \mathfrak{R}^{m_{p+1}} \rightarrow \mathfrak{R}$  loss functions for  $i \in \llbracket 1, n \rrbracket$  with  $(\hat{Y}^i)_{i \in \llbracket 1, n \rrbracket}$  as ground truth matrices respectively.

Then a neural network is defined as the application  $N : \begin{cases} \mathfrak{R}^{m_1} \rightarrow \mathfrak{R}^{m_{p+1}} \\ y \mapsto \bigcirc_{k=1}^n (F_{act}^k \circ L^k(\cdot, W^k, b^k))(y) \end{cases}$ .

The optimization problem is  $\min_{(W_{1,:}^k, \dots, W_{m_{k+1},:}^k, b^k)_{k \in \llbracket 1, p \rrbracket}} \sum_{i=1}^n F_{loss}^{\hat{Y}^i}(N(X^i))$  and

$F_{loss}^{global} : ((W_{1,:}^k, \dots, W_{m_{k+1},:}^k, b^k)_{k \in \llbracket 1, p \rrbracket}) \mapsto \sum_{i=1}^n F_{loss}^{\hat{Y}^i}(N(X^i))$  is named the objective function or global

loss.

**Theorem** – Let  $p$  dense layers with activation functions –  $ReLU^k \circ L^k(\cdot, W^k, b^k): \mathfrak{R}^{m_k} \rightarrow \mathfrak{R}^{m_{k+1}}$  for  $k \in \llbracket 1, p-1 \rrbracket$  and  $S \circ L^p(\cdot, W^p, b^p): \mathfrak{R}^{m_p} \rightarrow \mathfrak{R}^{m_{p+1}}$ .  $W^k \in M_{m_{k+1}, m_k}$  and  $b^k \in \mathfrak{R}^{m_{k+1}}$  are defined as the  $L^k$  weight and bias matrices. Let a training data set  $(X^i, \hat{Y}^i) \in \mathfrak{R}^{m_1} \times \mathfrak{R}^{m_{p+1}}$  for  $i \in \llbracket 1, n \rrbracket$ . Let  $\xi^{\hat{Y}^i}: \mathfrak{R}^{m_{p+1}} \rightarrow \mathfrak{R}$  Categorical cross-entropy losses for  $i \in \llbracket 1, n \rrbracket$  with  $(\hat{Y}^i)_{i \in \llbracket 1, n \rrbracket}$  as ground truth matrices respectively.

Then the following application  $N: \mathfrak{R}^{m_1} \rightarrow \mathfrak{R}^{m_{p+1}}$  with

$$N = S \circ L^p(\cdot, W^p, b^p) \circ \bigcirc_{k=1}^{p-1} (ReLU^k \circ L^k(\cdot, W^k, b^k))$$

is a neural network and its objective function is

$$\xi_{loss}^{global}: ((W_{1,:}^k, \dots, W_{m_{k+1},:}^k, b^k)_{k \in \llbracket 1, p \rrbracket}) \mapsto \sum_{i=1}^n \xi_{loss}^{\hat{Y}^i}(N(X^i))$$

Let  $k \in \llbracket 1, p \rrbracket$ . For all  $i \in \llbracket 1, n \rrbracket$ , let

$$y^{downstream(k), X^i} = \begin{cases} \bigcirc_{l=1}^{k-1} (ReLU^l \circ L^l(\cdot, W^l, b^l))(X^i) & k \geq 2 \\ X^i & k = 1 \end{cases}$$

$$F^{upstream(k), \hat{Y}^i}: \begin{cases} \mathfrak{R}^{m_{k+1}} \rightarrow \mathfrak{R} \\ y \mapsto \begin{cases} \xi^{\hat{Y}^i} \circ S(y) & k = p \\ \xi^{\hat{Y}^i} \circ S \circ L^p(\cdot, W^p, b^p)(y) & k = p-1 \\ \xi^{\hat{Y}^i} \circ S \circ L^p(\cdot, W^p, b^p) \circ \bigcirc_{l=k+1}^{p-1} (ReLU^l \circ L^l(\cdot, W^l, b^l))(y) & k \leq p-2 \end{cases} \end{cases}$$

$$\text{then } \frac{d F^{upstream(k), \hat{Y}^i}}{d y}: \begin{cases} \mathfrak{R}^{m_{k+1}} \rightarrow \mathfrak{R} \\ y \mapsto \begin{cases} S(y) - \hat{Y}^i & k = p \\ (S(y) - \hat{Y}^i) \times W^p & k = p-1 \text{ where} \\ \frac{\partial (F^{upstream(k+1), \hat{Y}^i} \circ ReLU^{k+1} \circ L^{k+1}(\cdot, W, b))}{\partial y}(y) & k \leq p-2 \end{cases} \end{cases}$$

$$\forall k \in \llbracket 1, p-2 \rrbracket, \forall y \in \mathfrak{R}^{m_{k+1}}, \forall j \in \llbracket 1, m \rrbracket,$$

$$\frac{\partial (F^{upstream(k+1), \hat{Y}^i} \circ ReLU^{k+1} \circ L^{k+1}(\cdot, W^{k+1}, b^{k+1}))}{\partial y}(y)_j$$

$$= \sum_{j'=1}^{m'} \frac{d F^{upstream(k+1), \hat{Y}^i}}{d y'} (ReLU^{k+1}(L^{k+1}(y, W^{k+1}, b^{k+1})))_{j'} \times 1_{\mathfrak{R}_{\setminus \{0\}}^+}(L^{k+1}(y, W^{k+1}, b^{k+1}))_{j'} \times W_{j', j}^{k+1}$$

with  $W^{k+1} \in M_{m', m}$  a weight matrix,  $b^{k+1} \in \mathfrak{R}^{m'}$  a bias matrix.

Let  $k = p$ . Then  $\xi_{loss}^{global}$  Gradient matrices are

$$\forall j' \in \llbracket 1, m_{k+1} \rrbracket, \quad \frac{\partial \xi_{loss}^{global}}{\partial W_{j', :}^k} : \left\{ \begin{array}{c} \mathfrak{R}^{m_k} \rightarrow \mathfrak{R}^{m_k} \\ w \mapsto \sum_{i=1}^n \frac{\partial (F_{j'}^{upstream(k), \hat{Y}^i} \circ L_{j'}^k(\cdot, w, b_{j'}^k))}{\partial w}(w) \end{array} \right.$$

where  $\forall i \in \llbracket 1, n \rrbracket$ ,  $\forall w \in \mathfrak{R}^{m_k}$ ,  $\forall j \in \llbracket 1, m_k \rrbracket$ ,

$$\frac{\partial (F_{j'}^{upstream(k), \hat{Y}^i} \circ L_{j'}^k(\cdot, w, b_{j'}^k))}{\partial w}(w)_j$$

$$= F_{j'}^{upstream(k), \hat{Y}^i} (L_{j'}^k(y^{downstream(k), X^i}, w, b_{j'}^k)) \times y_j^{downstream(k), X^i} \quad \text{with } b^k \in \mathfrak{R}^{m_{k+1}} \text{ a bias matrix.}$$

$$\frac{\partial \xi_{loss}^{global}}{\partial b^k} : \left\{ \begin{array}{c} \mathfrak{R}^{m_{k+1}} \rightarrow \mathfrak{R}^{m_{k+1}} \\ b^k \mapsto \sum_{i=1}^n \frac{\partial (F^{upstream(k), \hat{Y}^i} \circ L^k(\cdot, W^k, b^k))}{\partial b^k}(b^k) \end{array} \right.$$

where  $\forall i \in \llbracket 1, n \rrbracket$ ,  $\forall b^k \in \mathfrak{R}^{m_k}$ ,  $\forall j' \in \llbracket 1, m_{k+1} \rrbracket$ ,

$$\frac{\partial (F^{upstream(k), \hat{Y}^i} \circ L^k(\cdot, W^k, b^k))}{\partial b^k}(b^k)_j,$$

$$= \frac{d F^{upstream(k), \hat{Y}^i}}{d y} (L^k(y^{downstream(k), X^i}, W^k, b^k))_j \times 1_{\mathfrak{R}_{\setminus \{0\}}^+}(L^k(y^{downstream(k), X^i}, W^k, b^k))_j,$$

with  $W^k \in M_{m_{k+1}, m_k}$  a weight matrix.

Let  $k \in \llbracket 1, p-1 \rrbracket$ . Then  $\xi_{loss}^{global}$  Gradient matrices are

$$\forall j' \in \llbracket 1, m_{k+1} \rrbracket, \quad \frac{\partial \xi_{loss}^{global}}{\partial W_{j',:}^k} : \left( w \mapsto \sum_{i=1}^n \frac{\partial (F_{j'}^{upstream(k), \hat{Y}^i} \circ ReLU_{j'}^k \circ L_{j'}^k(\cdot, w, b_{j'}^k))}{\partial w} \right) (w)$$

where  $\forall i \in \llbracket 1, n \rrbracket, \forall w \in \mathfrak{R}^{m_k}, \forall j \in \llbracket 1, m_k \rrbracket,$

$$\frac{\partial (F_{j'}^{upstream(k), \hat{Y}^i} \circ ReLU_{j'}^k \circ L_{j'}^k(\cdot, w, b_{j'}^k))}{\partial w} (w)_j$$

$$= F_{j'}^{upstream(k), \hat{Y}^i} (ReLU_{j'}^k (L_{j'}^k (y^{downstream(k), X^i}, w, b_{j'}^k))) \times 1_{\mathfrak{R}_{\setminus \{0\}}^+} (L_{j'}^k (y^{downstream(k), X^i}, w, b_{j'}^k)) \times y_j^{downstream(k), X^i}$$

with  $b^k \in \mathfrak{R}^{m_{k+1}}$  a bias matrix.

$$\frac{\partial \xi_{loss}^{global}}{\partial b^k} : \left( b^k \mapsto \sum_{i=1}^n \frac{\partial (F^{upstream(k), \hat{Y}^i} \circ ReLU^k \circ L^k(\cdot, W^k, b^k))}{\partial b^k} \right) (b^k)$$

where  $\forall i \in \llbracket 1, n \rrbracket, \forall b^k \in \mathfrak{R}^{m_k}, \forall j' \in \llbracket 1, m_{k+1} \rrbracket,$

$$\frac{\partial (F^{upstream(k), \hat{Y}^i} \circ ReLU^k \circ L^k(\cdot, W^k, b^k))}{\partial b^k} (b^k)_{j'}$$

$$= \frac{d F^{upstream(k), \hat{Y}^i}}{d y} (ReLU^k (L^k (y^{downstream(k), X^i}, W^k, b^k)))_{j'} \times 1_{\mathfrak{R}_{\setminus \{0\}}^+} (L^k (y^{downstream(k), X^i}, W^k, b^k)_{j'})$$

with  $W^k \in M_{m_{k+1}, m_k}$  a weight matrix.

Proof: TO DO.

## 6 – Optimizations

**Definition** – Let  $f \in D(\Omega, \mathfrak{R})$  with  $\Omega \in \mathfrak{R}^m$ . <TODO>

## 7 – References