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A Normal Form for Energy Shaping: Application to the Furuta Pendulum

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Abstract

In this paper we derive a nonlinear control law for stabilization of the Furuta pendulum system with the pendulum in the upright position and the rotating rigid link at rest at the origin. The control law is derived by first applying feedback that makes the Furuta pendulum look like a planar pendulum on a cart plus a gyroscopic force. The planar pendulum on a cart is an example of a class of mechanical systems which can be stabilized in full state space using the method of controlled Lagrangians. We consider this class of systems as our normal form and for the case of the Furuta pendulum, we add to the first transforming feedback law, the energy-shaping control law for the planar pendulum system. The resulting system looks like a mechanical system plus feedback-controlled dissipation and an external force that is quadratic in velocity. Using energy as the Lyapunov function we prove local exponential stability and demonstrate a large region of attraction.

1 Introduction

The method of controlled Lagrangians and the equivalent IDA-PBC method use energy shaping for stabilization of underactuated mechanical systems (see [2, 6, 10, 9] and references therein). The method of controlled Lagrangians provides a control law for underactuated mechanical systems such that the closed-loop dynamics derive from a Lagrangian. The objective of the method is to choose the control law to shape the controlled energy for stability. It is also possible to use the control to modify structure, e.g., to introduce or change gyroscopic terms in the Lagrangian setting (see [6]).

In general, computing the control law and controlled energy requires solving a set of PDEs. These PDEs, referred to as "matching conditions", ensure that the proposed energy and structure shaping is realizable with the limited number of actuation directions. Bloch, Leonard and Marsden constructed, for certain classes of systems, a structured family of controlled Lagrangians for which the matching conditions reduce to simple tests and the control design becomes algorithmic.

One such class of systems was described in [4, 2]. In these works, a control law was presented and asymptotic stability proved for any mechanical system satisfying the *simplified matching* conditions. In [2], both the kinetic energy and the potential energy were modified in the closed-loop system so that stabilization in the full state space could be achieved. The examples described in that paper, representative of the class of mechanical systems, were the classic inverted planar pendulum on a cart as well as the inverted spherical pendulum on a 2-D cart.

In this paper we consider the Furuta pendulum as a model system which does not satisfy the simplified matching conditions and for which satisfaction of a somewhat more generalized set of matching conditions (based also on a structured choice of controlled Lagrangian) fails to give a stabilizing control law in the full state space. The Furuta pendulum is a pendulum with fulcrum attached to a rigid link that rotates in the horizontal plane. The pendulum rotates in the plane perpendicular to the rigid link, so the system looks like an inverted pendulum on a cart that is constrained to move in a circle rather than along a line.

The "swing-up" problem for the Furuta pendulum was studied using energy methods by Åström and Furuta [1]. Their controller uses a bang-bang pseudo-state feedback control method. Olfati-Saber [8] proposed a semi-global stabilizing scheme for the Furuta pendulum in the inverted position using fixed point controllers. Taking into account passivity of the system, Fantoni and Lozano [7] proposed an energy-based approach to swing up the Furuta pendulum to a homoclinic orbit. By guaranteeing convergence to the homoclinic orbit, it is guaranteed that the trajectory will enter the basin of attraction of any (local) balancing controller. More details on the Furuta pendulum can be found in [7] and references therein.

In [3], it was demonstrated that the Furuta pendu-

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lum satisfies a set of matching conditions for a structured shaping of kinetic energy. In this case, symmetry in the kinetic energy was retained, that is, the angle of the rotating link with respect to a reference vertical plane remains a cyclic variable for the closed-loop system. Stabilization was proved for the reduced system dynamics meaning that the control law stabilized the pendulum in the inverted position and brought the cart to rest, but the final position of the cart was left arbitrary. Potential shaping, in addition to kinetic shaping, for the Furuta pendulum within the structured choice of controlled Lagrangian is possible; however, this choice does not yield full state-space stabilization.

Chang showed in [5] that a stabilizing control law could be derived for the Furuta pendulum using the most general framework for the method of controlled Lagrangians which allows for external gyroscopic forcing [6].

Here, we consider systems that satisfy the simplified matching conditions as systems in normal form. These systems and the associated stabilizing control law are described in §2. In §3 we use feedback to transform the Furuta pendulum dynamics into normal form, i.e., into the dynamics of a planar pendulum on a cart plus a gyroscopic force. We then use the stabilization law for the planar pendulum and prove stability of the controlled Furuta pendulum system in §4. In §5 we present simulation and experimental results that suggest a large region of attraction. Final remarks are given in §6.

2 Simplified Matching Systems as Normal Form

In [2], using the method of controlled Lagrangians, a control law to asymptotically stabilize a class of systems satisfying simplified matching conditions was derived. Such systems lack gyroscopic forces and the planar inverted pendulum is one such system. Here, q_1^α denotes the coordinates for the unactuated directions with index α going from 1 to m. q_2^a denotes the coordinates for the actuated directions with index a going from 1 to n. The Lagrangian for the given system is given by

$$\begin{aligned} &L(q_1^\alpha, q_2^a, \dot{q}_1^\beta, \dot{q}_2^b) \\ &= \frac{1}{2} g_{\alpha\beta}\dot{q}_1^\alpha \dot{q}_1^\beta + g_{\alpha a}\dot{q}_1^\alpha \dot{q}_2^a + \frac{1}{2} g_{ab}\dot{q}_2^a \dot{q}_2^b - V(q_1^\alpha, q_2^b) \end{aligned}$$

where summation over indices is implied, g is the kinetic energy metric and V is the potential energy. It is assumed that the actuated directions are symmetry directions for the kinetic energy, that is, we assume $g_{\alpha\beta}$, $g_{\alpha a}$, g_{ab} are all independent of q_2^a.

The condition for such a system to satisfy the simplified matching conditions are

- $g_{ab} =$ constant
- $\dfrac{\partial g_{\alpha a}}{\partial q_1^\delta} = \dfrac{\partial g_{\delta a}}{\partial q_1^\alpha}$
- $\frac{\partial^2 V}{\partial q_1^\alpha \partial q_2^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial q_1^\beta \partial q_2^a} g^{ad} g_{\alpha d}$.

Satisfaction of these matching conditions allows for a structured feedback shaping of kinetic *and* potential energy. In particular, a control law is given in [2] such that the closed-loop system is a Lagrangian system with the (parametrized) controlled Lagrangian given by

$$\begin{aligned} &L_c(q_1^\alpha, q_2^a, \dot{q}_1^\beta, \dot{q}_2^b) = \\ &\frac{1}{2}\left(g_{\alpha\beta} + \rho(\kappa+1)(\kappa + \frac{\rho-1}{\rho}) g_{\alpha a} g^{ab} g_{b\beta}\right)\dot{q}_1^\alpha \dot{q}_1^\beta \\ &+ \rho(\kappa+1) g_{\alpha a}\dot{q}_1^\alpha \dot{q}_2^a + \frac{1}{2}\rho g_{ab}\dot{q}_2^a \dot{q}_2^b - V(q_1^\alpha, q_2^b) - V_\epsilon(q_1^\alpha, q_2^b) \end{aligned}$$

where ρ and κ are constant parameters and V_ϵ satisfies

$$-\left(\frac{\partial V}{\partial q_2^a} + \frac{\partial V_\epsilon}{\partial q_2^a}\right)(\kappa + \frac{\rho - 1}{\rho}) g^{ad} g_{\alpha d} + \frac{\partial V_\epsilon}{\partial q_1^\alpha} = 0.$$

The results in [2] then given condition on ρ, κ and V_ϵ that ensure stability of the equilibrium in the full state space. For example, in the case that the equilibrium of interest is a maximum of the original potential energy (as in the case of the inverted pendulum examples), $\kappa > 0$ and $\rho < 0$ and the potential V_ϵ are chosen such that the energy function for the controlled Lagrangian E_c has a maximum at the origin of full state space. Asymptotic stability is obtained by adding a dissipative term to the control law which drives the controlled system to its maximum energy (E_c) state.

Consider the planar inverted pendulum as shown in Figure 2.1.

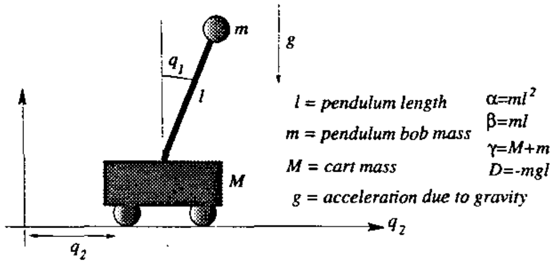

Figure 2.1: The planar pendulum on a cart.

The Lagrangian for this system is

$$L = \frac{1}{2}\alpha \dot{q}_1^2 + \beta \cos q_1 \dot{q}_1 \dot{q}_2 + \frac{1}{2}\gamma \dot{q}_2^2 + D\cos q_1$$

and the equations of motion are

$$\begin{bmatrix} \alpha & \beta\cos q_1 \\ \beta\cos q_1 & \gamma \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} D\sin q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\beta \sin q_1 \dot{q}_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}. \quad (2.1)$$

It can easily be verified that this system satisfies the simplified matching conditions since $g_{ab} = \gamma$ is a constant and the other two conditions are trivially satisfied.

Following [2], the control law which asymptotically stabilizes the system at the origin in full state space is given by

$$u = \frac{\kappa\beta \sin q_1 \left(\alpha \dot{q}_1^2 + D\cos q_1\right) - \dfrac{B\epsilon D\gamma^2 q_2}{\beta^2} + Bu^{\mathrm{diss}}}{\alpha - \dfrac{\beta^2(\kappa+1)\cos^2 q_1}{\gamma}} \tag{2.2}$$

with $u^{\mathrm{diss}} = c\gamma(\dot{q}_2 + p_1 \cos q_1 \dot{q}_1)$, $c > 0$ a constant and

$$B = \frac{1}{\rho}\left(\alpha - \frac{\beta^2\cos^2 q_1}{\gamma}\right), \quad p_1 = (\kappa + \frac{\rho-1}{\rho})\frac{\beta}{\gamma} > 0.$$

In §4, we will prove local exponential stability for the Furuta pendulum using a control law that augments (2.2). To do this we will consider gyroscopic forces. Define

$$EL_\alpha := \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1^\alpha}\right) - \frac{\partial L}{\partial q_1^\alpha}, \ \ EL_a := \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2^a}\right) - \frac{\partial L}{\partial q_2^a}.$$

Then, consider a system with equations of motion given by

$$\begin{pmatrix} EL_\alpha \\ EL_a \end{pmatrix} = \begin{pmatrix} F_\alpha \\ F_a \end{pmatrix} = S(q,\dot{q})\dot{q}$$

where $S(q,\dot{q})$ is gyroscopic i.e., a skew-symmetric matrix. Then, if E_c is the energy for the system, $\dot{E}_c = \dot{q}^T S(q,\dot{q})\dot{q} = 0$. Since the gyroscopic force does not affect the rate of change of energy, it is still natural, even in the presence of a gyroscopic force, to use energy E_c as the Lyapunov function to prove stability.

With this as inspiration, we use a feedback control law in §3 to transform the equations of motion for the Furuta pendulum into the equations of motion for the planar pendulum system plus a gyroscopic force. The idea then is to use the feedback law for the planar pendulum described above to stabilize the transformed Furuta pendulum system. As will be seen this is not as straightforward as it might seem, since a pure energy-shaping control law does not necessarily preserve gyroscopic forces. The general matching conditions for mechanical control systems with gyroscopic forces are given in [6].

3 Furuta Pendulum and Transformation to Normal Form

The Furuta pendulum consists of a pendulum on a cart where the cart is restricted to move on a circle in the horizontal plane as shown in Figure 3.1. The Lagrangian for this system is given by

$$L = \frac{1}{2}\alpha\dot{q}_1^2 + \beta'\cos q_1 \dot{q}_1\dot{q}_2 + \frac{1}{2}\left(\gamma' + \alpha\sin^2 q_1\right)\dot{q}_2^2 + D\cos q_1 \tag{3.1}$$

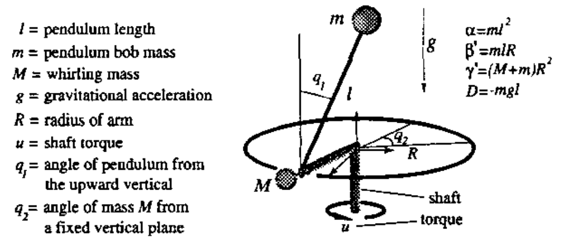

Figure 3.1: The Furuta pendulum.

This system does not satisfy the simplified matching conditions since $g_{ab} = \gamma' + \alpha\sin^2 q_1$ is not constant. Using a more general form of kinetic shaping, relative stability can be obtained i.e., the system can be brought to the state $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (0, 0, q_f, 0)$, where q_f need not be zero [3]. As mentioned in the introduction, combining both the kinetic shaping and potential shaping within the structured choice of controlled Lagrangians does not yield full state space stability. Instead, we look for a stabilizing solution in full state space by first recasting the Furuta pendulum system as a planar pendulum plus a gyroscopic force term.

The equations of motion for the Furuta pendulum with input $u = u_1 + u_2'$ are

$$\begin{bmatrix} \alpha & \beta'\cos q_1 \\ \beta'\cos q_1 & \gamma' + \alpha\sin^2 q_1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} D\sin q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\alpha\sin q_1\cos q_1 \dot{q}_2^2 \\ -\beta'\sin q_1 \dot{q}_1^2 + 2\alpha\sin q_1\cos q_1 \dot{q}_1\dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_1 + u_2' \end{bmatrix} \tag{3.2}$$

Choose u_1 to cancel terms so that the above set of equations transform to

$$\begin{bmatrix} \alpha & \beta'\cos q_1 \\ \beta'\cos q_1 & \gamma' \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} D\sin q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\beta'\sin q_1\dot{q}_1^2 \end{bmatrix} = \begin{bmatrix} F_\alpha \\ F_a \end{bmatrix} + \begin{bmatrix} 0 \\ u_2' \end{bmatrix} \tag{3.3}$$

where

$$F_\alpha = \alpha\sin q_1\cos q_1\dot{q}_2^2, \quad F_a = -\alpha\sin q_1\cos q_1\dot{q}_1\dot{q}_2.$$

The LHS of equation (3.3) is the same as the LHS of the equations of motion for the planar pendulum (2.1), with γ replaced by γ' and β replaced by β'. Observe that $F_\alpha\dot{q}_1 + F_a\dot{q}_2 = 0$. Therefore, the RHS is a gyroscope force term added to the input term $(0, u_2')^T$. We now split u_2' into a sum of u_2 and u^d. The idea is to choose u_2 so that the controlled system derives from a Lagrangian L_2 where

$$\begin{pmatrix} EL_{2\alpha} \\ EL_{2a} \end{pmatrix} = \begin{pmatrix} F_{2\alpha} \\ F_{2a} \end{pmatrix}$$

and the RHS is a gyroscope force term. This can be done for the Furuta pendulum using the control law for the planar pendulum; however, the parameters ρ and

κ must be constrained in such a way that equilibrium cannot be made stable. Since this is unacceptable, we relax the requirement that the RHS be perfectly gyroscopic and use this control law without constraining the choices of ρ and κ at the matching stage. We prove local exponential stability for the Furuta pendulum in the next section and show in §5 that the region of attraction appears to be considerable.

4 Stabilization of the Furuta Pendulum

Denote the LHS of equation (3.3) by $(ELpp_\alpha, ELpp_a)^T$ where pp signifies "planar pendulum". Let $(ELppc_\alpha, ELppc_a)^T$ be the Euler-Lagrange equations for the controlled planar pendulum when the force terms on the RHS of equation (3.3) are ignored. Since we cannot, in this context, obtain gyroscopic forces for the controlled Lagrangian and guarantee stability, we choose to cancel F_a in equation (3.3) using u_1 so that

$$\begin{pmatrix} ELpp_\alpha \\ ELpp_a \end{pmatrix} = \begin{pmatrix} F_\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \end{pmatrix} \tag{4.1}$$

These equations become

$$\begin{pmatrix} ELppc_\alpha \\ ELppc_a \end{pmatrix} = \begin{pmatrix} F_\alpha \\ 0 \end{pmatrix} \tag{4.2}$$

for

$$u_2 = \frac{\kappa\beta' \sin q_1 \left(\alpha \dot{q}_1^2 + D\cos q_1\right) - \dfrac{B\epsilon D\gamma'^2 q_2}{\beta'^2}}{\alpha - \dfrac{\beta'^2(\kappa+1)\cos^2 q_1}{\gamma'}} - \frac{\kappa\beta' F_\alpha \cos q_1}{\alpha - \dfrac{\beta'^2(\kappa+1)\cos^2 q_1}{\gamma'}} \tag{4.3}$$

and B defined as for equation (2.2). When $F_\alpha = 0$ and $u^{\text{diss}} = 0$, (4.3) is the same as (2.2). The calculation of u_2 is done exactly as in [2].

Adding a dissipation term u^d to the input u_2 modifies equation (4.2) to

$$\begin{pmatrix} ELppc_\alpha \\ ELppc_a \end{pmatrix} = \begin{pmatrix} F_\alpha + p_1 \cos q_1 u^d \\ u^d \end{pmatrix} \tag{4.4}$$

where $p_1 = \left(\kappa + \dfrac{\rho - 1}{\rho}\right)\dfrac{\beta'}{\gamma'} > 0$. Let E_c be the controlled energy for the system given by equation (4.2). $\kappa > 0$, $\rho < 0$ and $\epsilon > 0$ are chosen such that E_c has a *maximum* at $(0,0,0,0)$ as in [2]. Then, for the system (4.4), the expression for $\dot{E}_c$ is

$$\dot{E}_c = F_\alpha \dot{q}_1 + u^d(\dot{q}_2 + p_1 \cos q_1 \dot{q}_1). \tag{4.5}$$

If we choose

$$u^d = c\gamma'(\dot{q}_2 + p_1 \cos q_1 \dot{q}_1) \tag{4.6}$$

with $c > 0$, we get

$$\begin{aligned} \dot{E}_c &= F_\alpha \dot{q}_1 + c\gamma'(\dot{q}_2 + p_1 \cos q_1 \dot{q}_1)^2 \\ &= \alpha \sin q_1 \cos q_1 \dot{q}_2^2 \dot{q}_1 + c\gamma'(\dot{q}_2 + p_1 \cos q_1 \dot{q}_1)^2. \end{aligned}$$

Note that $\dot{E}_c$ is *not* positive semi-definite as desired since there exist points arbitrarily close to the origin where $\dot{E}_c < 0$. However, when $\dot{E}_c < 0$, it is necessary that q_1 and $\dot{q}_1$ have opposite signs, i.e., the pendulum must be moving towards the upright position when the energy is decaying away from the maximum value. See Figure 4.1 for plots of the $\dot{E}_c = 0$ surface.

We show next that the closed-loop system is locally exponentially stable by showing asymptotic stability of the linearization of (4.4) about the origin. To do so we use Lyapunov's direct method with Lyapunov function defined by the energy $\tilde{E}_c$ associated with the linearized (Lagrangian) system. Note that the energy-based approach is much easier than an eigenvalue check.

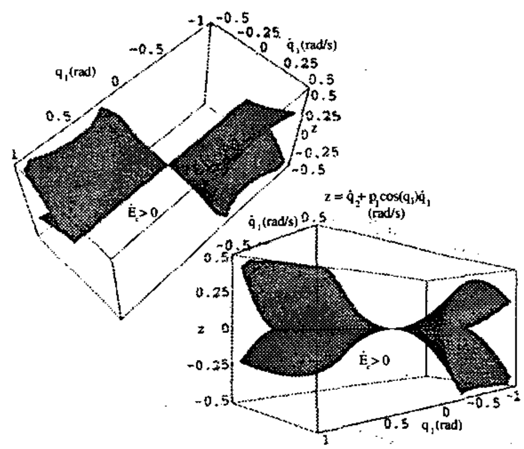

Figure 4.1: The surface $\dot{E}_c = 0$ in $(q_1, \dot{q}_1, \dot{q}_2 + p_1 \cos q_1 \dot{q}_1)$ coordinates.

Proposition 4.1 *The system given by (4.4) is locally exponentially stabilized by the choice of u^d given by (4.6).*

Proof. The linearized controlled Lagrangian for the planar pendulum is

$$\begin{aligned} \tilde{L}_{ppc} &= \frac{1}{2}a_{11}\dot{q}_1^2 + a_{12}\dot{q}_1\dot{q}_2 + \frac{1}{2}a_{22}\dot{q}_2^2 - \frac{1}{2}Dq_1^2 \\ &- \frac{1}{2}p_2\left(q_2 + p_1 q_1\right)^2 \end{aligned}$$

where

$$a_{11} = \alpha + \rho\frac{\beta'^2}{\gamma'}(\kappa+1)\left(\kappa + 1 - \frac{1}{\rho}\right),$$

$$a_{12} = \rho\beta'(\kappa+1), \quad a_{22} = \rho\gamma', \quad p_2 = \frac{\epsilon D\gamma'^2}{\beta'^2} < 0.$$

Let the corresponding energy be $\tilde{E}_c$. Then, we compute

$$\begin{aligned}\dot{\tilde{E}}_c &= u^d\left(\dot{q}_2 + p_2\dot{q}_1\right)\\ &= c\gamma'\left(\dot{q}_2 + p_2\dot{q}_1\right)^2\\ &\geq 0\end{aligned}$$

Let $S = \{(q_1, \dot{q}_1, q_2, \dot{q}_2) \mid \dot{\tilde{E}}_c \geq 0\}$ and $M = \{(q_1, \dot{q}_1, q_2, \dot{q}_2) \mid \dot{\tilde{E}}_c = 0\} \subset S$. The set M is an invariant set. Consider a solution belonging to M. Note that $u^d = \dot{q}_2 + p_1\dot{q}_1 = 0$ on M. Therefore, $q_2 + p_1 q_1 = d_1$, a constant. We have the following three equations for motion on M:

$$a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + Dq_1 + p_1 p_2 d_1 = 0 \tag{4.7}$$

$$a_{12}\ddot{q}_1 + a_{22}\ddot{q}_2 + p_2 d_1 = 0 \tag{4.8}$$

$$\ddot{q}_2 + p_1\ddot{q}_1 = 0. \tag{4.9}$$

Using (4.9) to eliminate $\ddot{q}_2$ from (4.7) and (4.8) gives

$$\alpha\ddot{q}_1 = -Dq_1 - p_1 p_2 d_1 \tag{4.10}$$

$$\beta'\ddot{q}_1 = -p_2 d_1 \tag{4.11}$$

since $\alpha = a_{11} - a_{12}p_1$ and $\beta' = a_{12} - a_{22}p_1$. Equating these two expressions for $\ddot{q}_1$ and using $d_1 = q_2 + p_1 q_1$ gives

$$q_1 = -\frac{a}{1 + ap_1} q_2 \ , \ \text{where}$$

$$a = \left(\frac{p_1}{\alpha} + \frac{1}{\beta'}\right)\frac{\alpha}{D} p_2$$

Thus,

$$d_1 = q_2 + p_1 q_1 = \frac{1}{1 + ap_1} q_2 = \text{ constant.}$$

This implies that q_2 and therefore q_1 is constant. Thus, $\ddot{q}_1 = \ddot{q}_2 = 0$. By (4.11), this implies that $d_1 = 0$ which in turn implies that $q_1 = 0$ by (4.10). It follows that $q_2 = 0$.

Therefore, the only dynamical solution in the set M is the trivial solution and so by the LaSalle Invariance principle the linearized system is asymptotically (and therefore exponentially) stable. Accordingly, the nonlinear system is locally exponentially stable. ∎

Thus we have proved that the Furuta pendulum is locally exponentially stabilized with the control law given by $u = u_1 + u_2'$ where u_2' is the sum of u_2 and u^d. u_1 is the control law that cancels terms. We note that some of these cancelled terms are acceleration terms. We can derive the control law in terms of the state variables by substituting for the expressions for accelerations since we know the closed-loop Euler-Lagrange equations. The complete control law is given as follows.

$$\begin{aligned} u = {} & 2\alpha\sin(q_1)\cos(q_1)\dot{q}_1\dot{q}_2 + \frac{1}{\beta'\cos(q_1)}\{\\ & (\gamma' + \alpha\sin^2(q_1))(F_\alpha - D\sin(q_1))\\ & - \frac{|M_w|}{|M_p|}\left(\gamma'(F_\alpha - D\sin(q_1)) - \beta'(u_2 + u^d + \right.\\ & \left.\beta'\sin(q_1)\dot{q}_1^2)\right)\} - \beta'\sin(q_1)\dot{q}_1^2 \end{aligned} \tag{4.12}$$

where $|M_w|$ is the determinant of the mass matrix of the whirling pendulum (3.2), $|M_p|$ is the determinant of the mass matrix of the corresponding planar pendulum (3.3), u_2 is given by (4.3) and u^d is given by (4.6).

5 Simulation and Experiment.

In §4, we proved local exponential stability of the origin using the controlled energy $\tilde{E}_c$ of the linearized system as a Lyapunov function. The region of attraction for the control law given by (4.12) seems to be large. Figure 5.1 shows a MATLAB simulation for an initial condition as large as $(q_1, \dot{q}_1, q_2, \dot{q}_2) = (1, 2, 1, 2)$. The values chosen for the simulation are $m = 0.14kg$, $M = 0.44kg$, $l = 0.215m$, $R = 1m$, $\kappa = 25$, $\rho = -0.02$, $c = 0.015$ and $\epsilon = 0.00001$. Figure 5.2 is a plot of $\tilde{E}_c$ as a function of time for the same initial condition. It takes a couple of dips initially and then tends to zero staying positive always. The same control law was implemented on an experimental apparatus [11]. The output is shown in Figure 5.3. The initial condition was $q_1 = -0.81$ radian and $q_2 = 0.37$ radian. Initial velocities were zero. As can be seen, the angle q_1 is brought to zero but q_2 keeps oscillating about a point less than zero. This may be because we haven't account for other forces like friction or because of hardware limitations. Another interesting observation is that since we are transforming the whirling pendulum into a planar pendulum problem with some benign forces, the control system evolves on $\mathcal{R} \times \mathcal{S}$ rather than on $\mathcal{S} \times \mathcal{S}$. By this we mean the pendulum doesn't "know" that $q_2 = 0$ and $q_2 = 2\pi$ are in fact the same points.

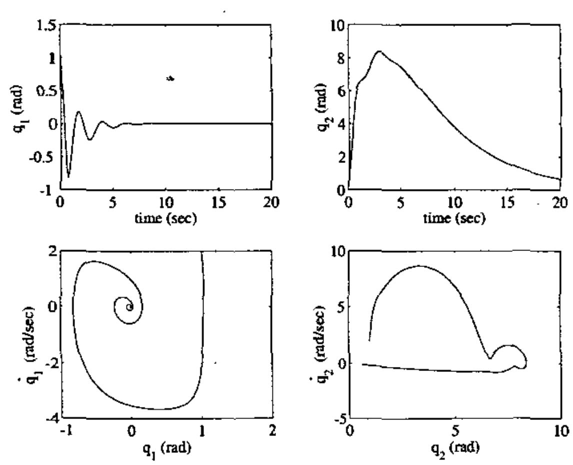

Figure 5.1: Simulation of Furuta pendulum for initial condition $(1, 2, 1, 2)$.

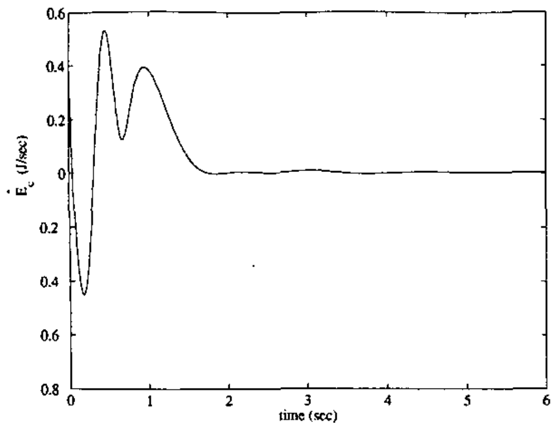

Figure 5.2: Plot of $\dot{E}_c$ versus time for the initial condition $(1, 2, 1, 2)$.

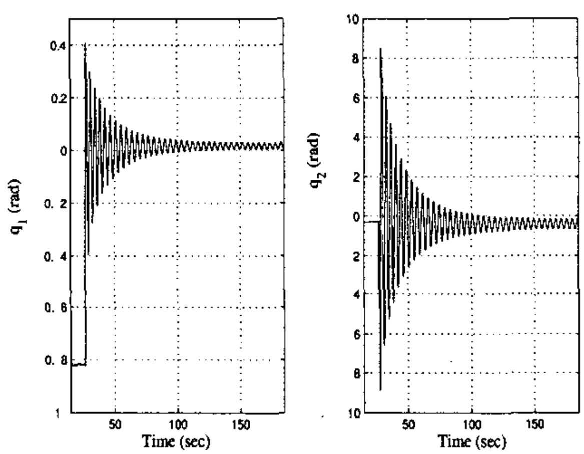

Figure 5.3: Experiment output.

6 Final Remarks

In this paper, a control law for the Furuta pendulum is derived using the fact that the Furuta pendulum can be expressed after a feedback transformation as a planar pendulum plus a gyroscopic force. The planar pendulum belongs to a class of systems which can be asymptotically stabilized in full state space using the method of controlled Lagrangians. It is of interest to explore the possibility of extending this approach to a class of systems that can be brought into normal form, i.e., which can, after a feedback transformation, be made equivalent to a system that satisfies the simplified matching conditions plus a gyroscopic force (or possibly other forces).

Another direction of interest concerns the choice of Lyapunov function to prove stability. As was observed in this application to the Furuta pendulum, our choice of energy function as the Lyapunov function was not so useful with respect to proving a large region of attraction. This suggests combining our technique with other nonlinear control techniques to construct better control Lyapunov functions.

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