Swing-up control of inverted pendulum using pseudo-state feedback

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Swing-up control, that is the transfer of a pendulum from a pendant state to the inverted one, is a good laboratory experiment of optimal control theory for non-linear control systems. The optimal control can be determined by the maximum principle and obtained as a function of time. Since the control is, however, determined in a feedforward fashion, the control is not robust to disturbances and uncertainties of the system, and the transfer of the state of the pendulum is not assured. In the paper, a robust swing-up control using a subspace projected from the whole state space is proposed. Based on the projected state space or pseudo-state, the control input is determined depending on the partitioning of the state as a bang-bang type control. The control algorithm is applied for a new type of pendulum (TITech pendulum), and the effectiveness and robustness of the proposed control are examined by experiments.

1 INTRODUCTION

Inverted-pendulum control has been widely used in control laboratories to demonstrate the effectiveness of control systems (**1–3**) in an analogy to the control of launching a rocket. The stabilization of not only a single but also a double inverted pendulum and also multiple inverted pendulums using hinge control have also been studied (**4–7**). Control such as stabilization using feedback is one aspect of control, but there exists the other important one to transfer from one state to the other. In the late 1950s and the early 1960s, optimal theory considered this kind of problem. The maximum principle is known to give a solution of this kind of problem. Examples, however, could be hardly found in the literature.

Transferring a pendulum from the pendant state to the inverted state may be a typical example of this kind of problem (**1**). In this case, the stabilizing control may be used in the neighbourhood of the upright position. Thus a pendulum can be used as a benchmark to test the algorithm for the problem. The control input for swing-up of the pendulum based on optimal theory has been given as a time function and has been injected in the feedforward fashion. This feedforward control requires an accurate model and is not robust to uncertainties.

This paper proposes a robust type of swing-up control based on the subspace projected from the state space. The system is of fourth order, and the control law is difficult to determine by the maximum principle using state space.

In the proposed control, the partitioned state composed of the angle and its velocity, which are selected as psuedo-state, are assigned different accelerations of the cart. Based on the vector fields in the partitions of the subspace, the trajectories connecting the pendant state to the upright state in the pseudo-state space is determined. Each vector field corresponds to a control law giving the prescribed acceleration. In the trajectory determination, the angular velocity of the motor is neglected, and it is assumed that the acceleration is of the bang-bang type.

The control is applied to a pendulum system which has been also improved by having the cart fixed to the rotating arm connected to a direct drive motor, called the TITech pendulum. This kind of rotating type pendulum excludes the limitation of the movement of the cart, which was the serious problem in the linear moving cart on a rail. Compared with the conventional inverted pendulum, the new type requires less space and has fewer unmodelled dynamics owing to a power transmission mechanism, since the shaft around which the pendulum is rotated is directly attached to the motor shaft. The proposed algorithm is verified by the pendulum, and the assumptions made in the synthesis are shown to be appropriate; and experimental results demonstrate the effectiveness and the robustness of the proposed control algorithm.

2 MODELLING OF CONTROLLED OBJECT

In swing-up control, we use a non-linear model to determine a control to swing up the pendulum into the neighbourhood of the upright position, and its linearized model is used for stabilization in the neighbourhood. The non-linear model is derived by the Euler–Lagrange method, and derivation of the linearized model is based on the non-linear model. The parameters of the models are identified as physical quantities, which are estimated by individual identification procedures.

2.1 Calculation of energy and input torque

For the definition of notations, see Fig. 1 and Table 1. Subscripts 0 and 1 stand for variables for the arm and the pendulum, respectively. The kinetic energy T_i, the potential energy V_i, and dissipation energy D_i ($i = 0, 1$) are obtained as follows. Three kinds of total energy T, V, D are obtained as sums of those of the arm and pen-

The MS was received on 22 September 1992 and was accepted for publication on 17 December 1992.

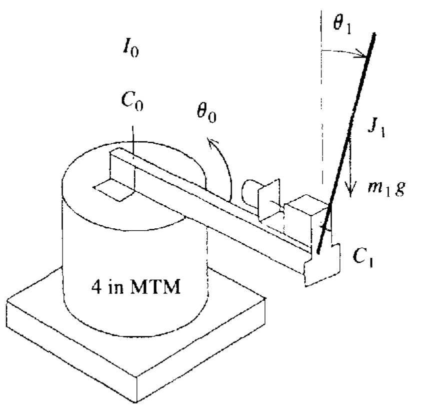

Fig. 1 Illustration of variables (TIT pendulum)

dulum. Since input torque is applied only to the arm, these are

1. *The arm*

$$T_0 = \tfrac{1}{2} I_0 \dot{\theta}_0^2, \quad V_0 = 0, \quad D_0 = \tfrac{1}{2} C_0 \dot{\theta}_0^2 \tag{1}$$

2. *The pendulum*

$$\begin{aligned} T_1 &= \frac{1}{2} J_1 \dot{\theta}_1^2 \\ &+ \frac{1}{2} m_1 \left[\left\{ \frac{d}{dt} (L_0 \sin \theta_0 + l_1 \sin \theta_1 \cos \theta_0) \right\}^2 \right. \\ &+ \left\{ \frac{d}{dt} (L_0 \cos \theta_0 - l_1 \sin \theta_1 \sin \theta_0) \right\}^2 \\ &+ \left. \left\{ \frac{d}{dt} (l_1 \cos \theta_1) \right\}^2 \right] \end{aligned} \tag{2}$$

$$V_1 = m_1 l_1 g \cos \theta_1, \quad D_1 = \frac{1}{2} C_1 \dot{\theta}_1^2 \tag{3}$$

3. *Total energy T, V, D*

$$T = \sum_{i=0}^{1} T_i, \quad V = \sum_{i=0}^{1} V_i, \quad D = \sum_{i=0}^{1} D_i \tag{4}$$

4. *Input torque F*

$$F_0 = \tau, \quad F_1 = 0 \tag{5}$$

where it is assumed that the pendulum has a moment inertia J_1 along the rotating axis.

Table 1 List of measured and estimated parameter values

Physical quantity	Symbol	Units
Total length	L_0	0.215 m
Inertia of arm	I_0	1.75×10^{-2} kg m^2
Friction coefficient	C_0	0.118 Nms
Mass	m_1	5.38×10^{-2} kg
Distance to centre of gravity	l_1	0.113 m
Inertia around centre of gravity	J_1	1.98×10^{-4} kg m^2
Coefficient of friction	C_1	8.3×10^{-5} Nms

2.2 Euler–Lagrange dynamic equations

Non-linear model

According to the Euler–Lagrange method,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{\theta}_i} - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} + \frac{\partial D}{\partial \dot{\theta}_i} = F_i \qquad (i = 0, 1) \tag{6}$$

then the following equations are obtained:

$$\begin{aligned} &\begin{bmatrix} I_0 + m_1(L_0^2 + l_1^2 \sin^2 \theta_1) & m_1 l_1 L_0 \cos \theta_1 \\ m_1 l_1 L_0 \cos \theta_1 & J_1 + m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \end{bmatrix} \\ &+ \begin{bmatrix} C_0 + \frac{1}{2} m_1 l_1^2 \sin 2\theta_1 \dot{\theta}_1 & - m_1 l_1 L_0 \sin \theta_1 + \frac{1}{2} m_1 l_1^2 \sin 2\theta_1 \dot{\theta}_0 \\ -\frac{1}{2} m_1 l_1^2 \sin 2\theta_1 \dot{\theta}_0 & C_1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -m_1 l_1 g \sin \theta_1 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \end{aligned} \tag{7}$$

Linear model

From dynamic equation (7), each coefficient matrix of $\ddot{\theta}_i$, $\dot{\theta}_i$, θ_i $(i = 0, 1)$ involves θ_1 in terms of $\sin \theta_1$ and $\cos \theta_1$, so this equation is a non-linear ordinary equation. By linearizing this around the vertical inverted state $(\theta_1 = 0, \ \dot{\theta}_1 = 0)$ the following equation is obtained:

$$\begin{aligned} &\begin{bmatrix} I_0 + m_1 L_0^2 & m_1 l_1 L_0 \\ m_1 l_1 L_0 & J_1 + m_1 l_1^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} C_0 & 0 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & -m_1 l_1 g \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \end{aligned} \tag{8}$$

3 CONSTRUCTION OF SWING-UP CONTROL SYSTEM

In this section, two swing-up methods are discussed, and stabilizing control in the neighbourhood of the inverted state is then described.

3.1 Feedforward control method

In this section, using the non-linear model of equation (7), we construct a feedforward control law to transfer the pendulum from the pendant to the inverted state via the steepest-descent method. Particularly, the resulting control law is derived in the form of a bang-bang control sequence (see Fig. 2), where the set of switching instances is

$$T^{(n)} = (T_1, T_2, \ldots, T_n) \tag{9}$$

They minimize the criterion function

$$J = \theta_{\text{end}}^T Q \theta_{\text{end}} \tag{10}$$

where

$$\theta_{\text{end}} = [\theta_0(T_n), \theta_1(T_n), \dot{\theta}_0(T_n), \dot{\theta}_1(T_n)]^T \tag{11}$$

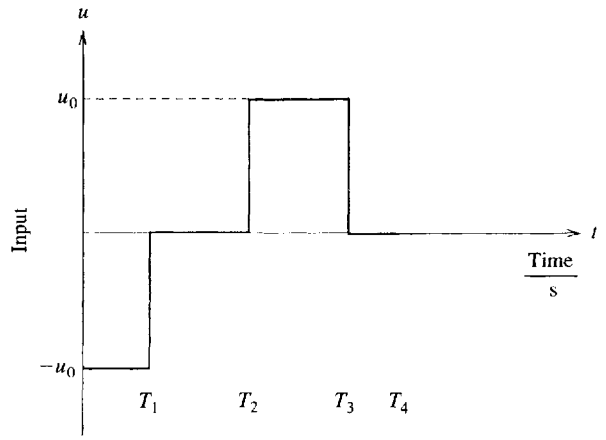

Fig. 2 Profile of control input to swing up the pendulum

with the following restrictions:

$$Gx(k) \leqslant 0 \tag{12}$$

where

$$Gx(k) := [-T_1(k), -\{T_2(k) - T_1(k)\}, -\{T_3(k) - T_2(k)\}, -\{T_4(k) - T_3(k)\}]^T \leqslant 0$$

To minimize equation (10) under the constraint of equation (12), let us consider a Lagrangian given by

$$L(x) := J(x)^2 + \lambda^T G \tag{13}$$

For the Lagrangian, a modified steepest-descent method is employed to obtain its minimum value. The algorithm is given as follows:

$$\lambda = [\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4]^T \geqslant 0 \tag{14}$$

$$L(x + \varepsilon z) \simeq \left(J(x) + \varepsilon \frac{\partial J}{\partial x} z\right)^2 + \lambda^T \left(\frac{\partial G}{\partial x} \varepsilon z + G\right) \tag{15}$$

$$L(x + \varepsilon z) - L(x) = \left(2J(x) \frac{\partial J}{\partial x} z + \lambda^T \frac{\partial G}{\partial x} z\right)\varepsilon + \varepsilon^2 \left(\frac{\partial J}{\partial x} z\right)^2 \tag{16}$$

Then

$$z = \left(2J(x) \frac{\partial J}{\partial x} + \lambda^T \frac{\partial G}{\partial x}\right)^T \tag{17}$$

$$\varepsilon = -\frac{z^T z}{2\left(\frac{\partial J}{\partial x} z\right)^2} \tag{18}$$

so that the criterion function will decrease in the steepest-descent way. The algorithm is as follows:

Step 1 Given $x(0)$
Step 2 Find $J\{x(k)\}$
Step 3 Compute z and ε
Step 4 If $\| z \| < 10^{-8}$ then stop
Step 5 Set $x(k + 1) = x(k) + \varepsilon z$
Step 6 Set $\lambda(k + 1) = \lambda(k) + \Lambda G(\Lambda < 0)$
Step 7 Increase k by 1 ($k = k + 1$)
Step 8 Reset if $\lambda(k + 1) < 0$
Step 9 Reset if $x(k + 1) < 0$
Step 10 Go to step 2.

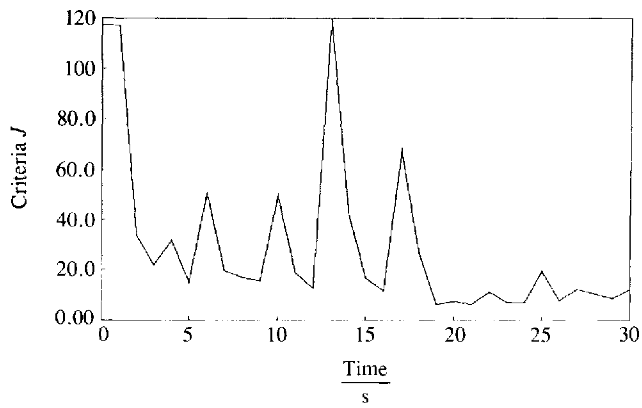

Fig. 3 Convergence of criteria by steepest-descent method

On repeating the above algorithm 30 times using the identified parameters, the cost function J is obtained as shown in Fig. 3, and the minimum J was found to be 6.205. The optimal switching times were $t_1 = 60$ ms, $t_2 = 245$ ms, $t_3 = 355$ ms and $t_4 = 710$ ms.

3.2 Bang-bang psuedo-state feedback control method

The feedforward control given in the preceding section requires accurate knowledge of the plant parameters. In order to overcome this drawback, a feedback control is considered. Let us consider the non-linear model of equation (7). It is observed that

$$(J_1 + m_1 l_1^2)\ddot{\theta}_1 + C_1\dot{\theta}_1 - m_1 l_1 g \sin\theta_1 = -m_1 l_1 L_0(\cos\theta_1)\ddot{\theta}_0 + \tfrac{1}{2} m_1 l_1^2(\sin 2\theta_1)\dot{\theta}_0^2 \tag{19}$$

Now suppose that the last term of the right-hand side of equation (19) can be neglected, that is

$$\frac{1}{2} m_1 l_1^2(\sin 2\theta_1)\dot{\theta}_0^2 \simeq 0 \tag{20}$$

(This assumption is valid provided that the pendulum is light enough to be swung up with small $\dot{\theta}_0$.) Consequently, equation (19) is reduced to

$$(J_1 + m_1 l_1^2)\ddot{\theta}_1 + C_1\dot{\theta}_1 - m_1 l_1 g \sin\theta_1 = -m_1 l_1 L_0 (\cos\theta_1)\ddot{\theta}_0 \tag{21}$$

Next, equation (21) is rewritten in the state-space form as

$$\frac{d}{dt}\begin{bmatrix}\theta_1 \\ \dot{\theta}_1\end{bmatrix} = \begin{bmatrix}\dot{\theta}_1 \\ (J_1 + m_1 l_1^2)^{-1}\{-C_1\dot{\theta}_1 + m_1 l_1 g \sin\theta_1 - m_1 l_1 L_0 (\cos\theta_1)\ddot{\theta}_0\}\end{bmatrix} \tag{22}$$

When $\ddot{\theta}_0$ is considered as an input of the system of equation (22), projected vector fields onto $(\theta_1, \dot{\theta}_1)$ phase plane of equation (22) for $\ddot{\theta}_0 = 0$ rad/s^2, $\ddot{\theta}_0 = +100$ rad/s^2 and $\ddot{\theta}_0 = -100$ rad/s^2 can be obtained as shown in Figs 4a to c respectively. The next step combines these projected vector fields to construct a bang-bang state feedback. For simplicity, it is supposed here that there is no modelling error in the motor model so that the relation between the control τ and the acceleration of the motor shaft angle $\ddot{\theta}_0$ can be exactly derived. Consequently, the required control law can be obtained to

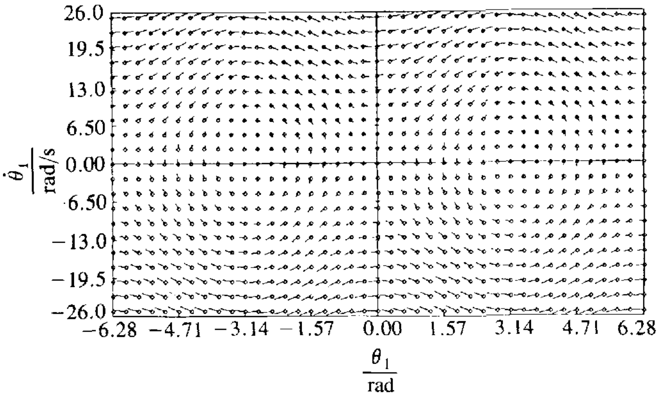

Fig. 4a Vector field of $(\theta_1, \dot{\theta}_1)$ for $\ddot{\theta}_0 = 0.0$ rad/s^2

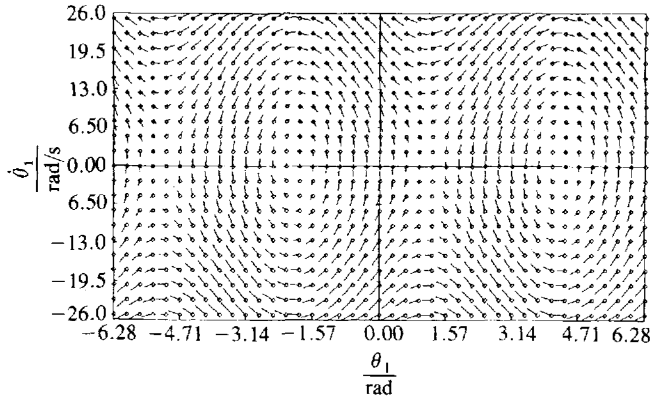

Fig. 4b Vector field of $(\theta_1, \dot{\theta}_1)$ for $\ddot{\theta}_0 = 100.0$ rad/s^2

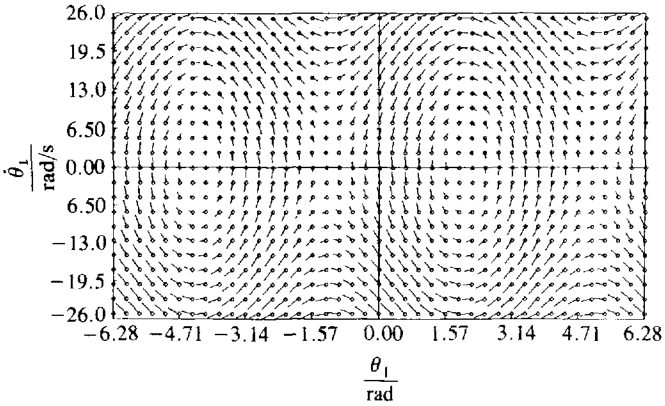

Fig. 4c Vector field of $(\theta_1, \dot{\theta}_1)$ for $\ddot{\theta}_0 = -100.0$ rad/s^2

determine a switching law of $\ddot{\theta}_0$ which makes the pendulum swing up to the upright position.

It can be observed that there are many switching patterns of $\ddot{\theta}_0$ which satisfy the aforementioned objective. One pattern used here is such that the corresponding $\dot{\theta}_0$ is small and thus expression (20) is valid. A proposed pattern is shown in Fig. 5, where $u = -u_0$ in area A, $u = 0$ in areas B and D, and $u = u_0$ in area C. The input u is switched according to where $(\theta_1, \dot{\theta}_1)$ are in the phase plane.

3.3 Control of upright state

A linear state feedback is used around the upright position when the pendulum is transferred to that region by the psuedo-state feedback control mentioned in the preceding section. Therefore, this section describes the feedback controller based on LQ control which is given by the following three steps:

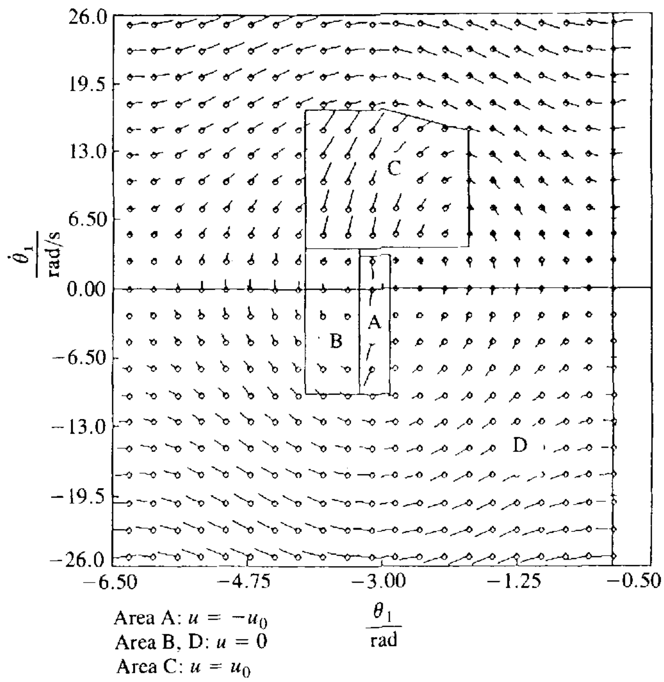

Fig. 5 Vector field of $(\theta_1, \dot{\theta}_1)$ by bang-bang control

1. *Design of the optimal regulator*
Consider the quadratic criterion function J

$$J = \int_0^\infty (x^T Q x + r u^2)\, dt \tag{23}$$

and the optimal feedback gain f for the criterion for the dynamic system of equation (8) where x and u stand for $[\theta_0\ \theta_1\ \dot{\theta}_0\ \dot{\theta}_1]^T$ and τ, respectively. If the design parameters Q and r are specified, then f can be obtained by a standard method.

2. *Design of observer*
Since only output can be obtained from potentiometer and resolver which measure θ_0 and θ_1 in principle, a minimum-order observer is used to estimate $\dot{\theta}_0$ and $\dot{\theta}_1$ for the state feedback. And observer poles are selected as design parameters, and P, Q, R, S_1 and S_2 are obtained by Gopinath's method in the following equations (7):

$$\dot{\hat{z}} = P\hat{z} + Qy + Ru \tag{24}$$

$$\hat{x} = S_1 \hat{z} + S_2 y \tag{25}$$

3. *Design of closed-loop system*
Using the optimal regulator and the minimum-order observer obtained in 1 and 2 above, a closed-loop system is constructed with

$$u = -f^T \hat{x} \tag{26}$$

The linear state feedback is used around the upright position.

4 SYSTEM CONFIGURATION

The experimental set-up consists of the rotational type inverted pendulum shown in Fig. 1 and a personal computer. The proposed controller is realized on a personal computer based on 80286 (+80287). The details of software and hardware are as follows.

4.1 Hardware

The details of the parts are as follows:

1. *Controlled object: pendulum on rotating arm*
The apparatus requires little space compared with the conventional type. In fact, the apparatus needs only about 50 × 50 cm area.

2. *Sensors: potentiometer and resolver*
Two kinds of variable are observed, and each one is measured with the following detectors:

(1) The rotational angle of arm θ_0 is measured by a resolver with a precision of 102400 pulses/360°.
(2) The rotational angle of the 1st pendulum θ_1 is obtained by a potentiometer with a gain of $0 \sim 10$V/340°.

3. *Actuator*: Direct-drive (DD) motor (4-inch Megatorque Motor: NSK) is used, and a signal within $-10 \sim 10$V is injected as a torque command.

4. *Signal converter*: *I/O board*
There are interface boards on which the signal inside and outside the personal computer is communicated.

(a) A/D board: converting θ_1 to 12-bit data.
(b) D/A board: converting 12-bit data to voltage.
(c) Counter board: converting pulse signal to 24-bit data.

5. Controller: Personal computer with the mathematic co-processor 80287.

4.2 Software

Original real-time software (MRCOS Version 2.01) which was developed in the authors' laboratory was used. The program consists of

(a) off-line part and on-line control calculation module written in C language;
(b) on-line input-output part and interrupt module written in a macro assembler.

The control law can be easily modified by changing the control calculation part written in C.

5 EXPERIMENTAL RESULTS

The control algorithm was tested using the rotating inverted pendulum and compared the feedforward control with state-feedback control using the phase-plane information.

The comparison was done for two cases; first where all the parameters were correct (as in Table 1) and where the parameter uncertainty was introduced by attaching a 15 g weight to an end point of the pendulum. The experimental conditions were as follows:

Linear control law was used when the pendulum is around the upright state ($\theta_1 = 0.0 \pm 0.5$ rad).

The control range for swing-up control was from $-\pi$ to 0 rad of θ_1.

The weight of regulator $Q = \text{diag}\ [30\ 2000\ 0.1\ 10]$, $r = 1$.

The design parameters of the observer were multiple poles at -50.

The control period was 5 ms.

5.1 Swing-up control using feedforward

The experiment results are shown in Figs. 6a to c when the parameters were correct, that is without a weight, and when a weight is attached at a tip of the pendulum, respectively.

5.2 Swing-up control using psuedo-state feedback

The experiment results by the psuedo state feedback control are shown in Figs 7a to d for each case.

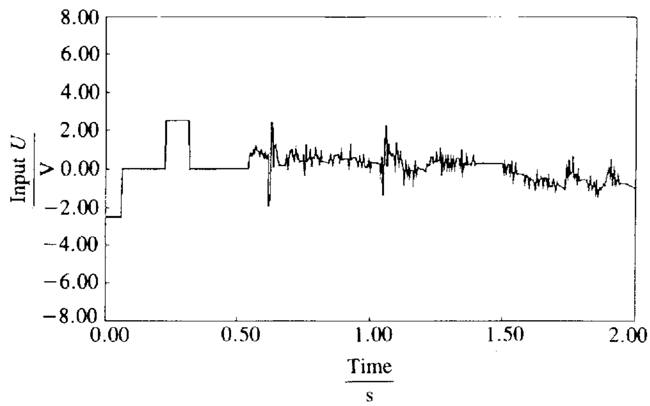

Fig. 6a Feedforward control (without load)

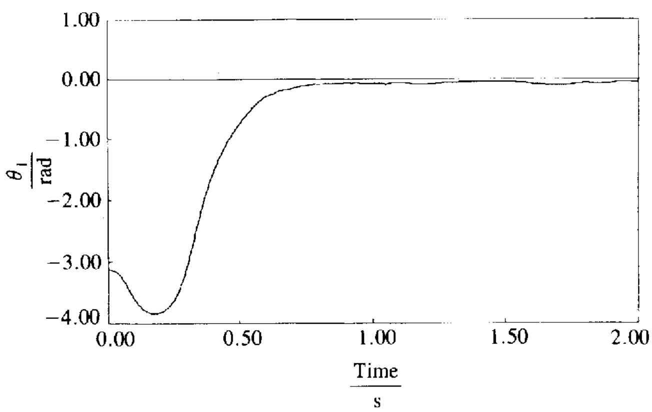

Fig. 6b Response of pendulum to feedforward control without load)

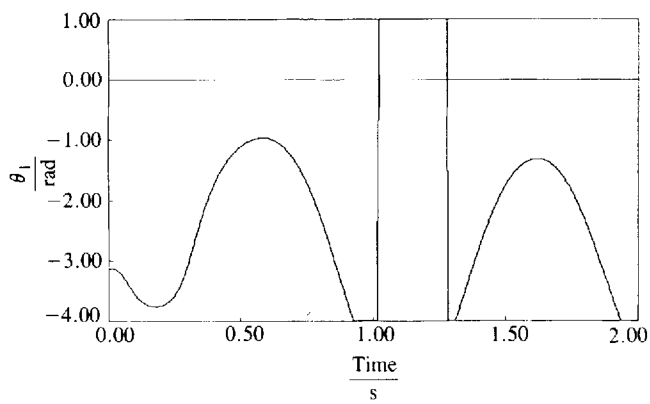

Fig. 6c Response of pendulum to feedforward control (with load)

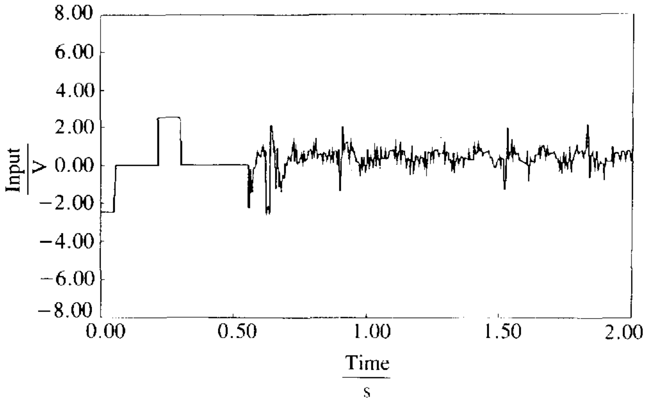

Fig. 7a Feedback control (without load)

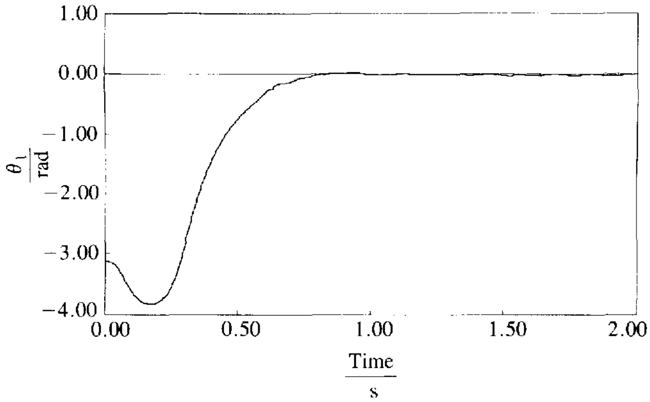

Fig. 7b Response of pendulum to feedback control (without load)

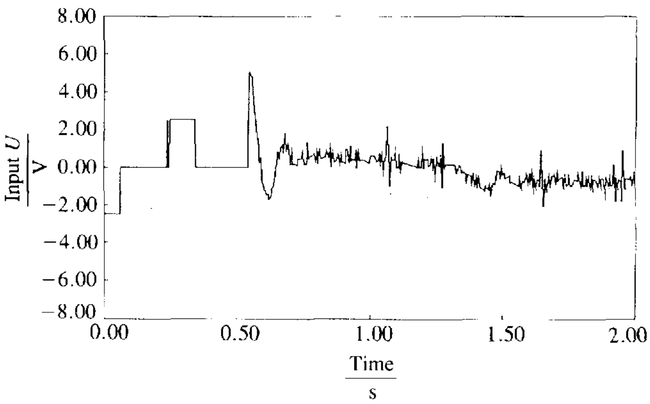

Fig. 7c Feedback control (with load)

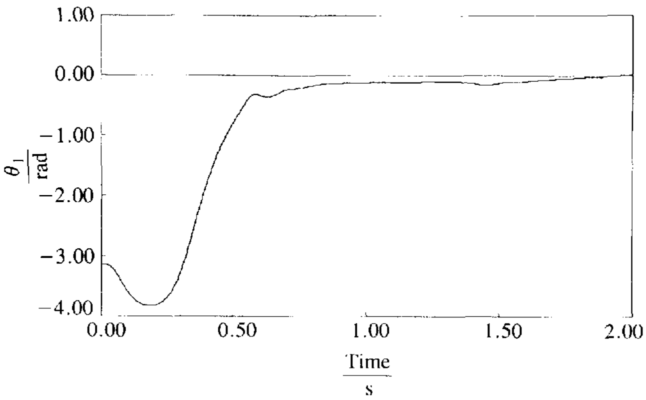

Fig. 7d Response of pendulum to the feedback control (with load)

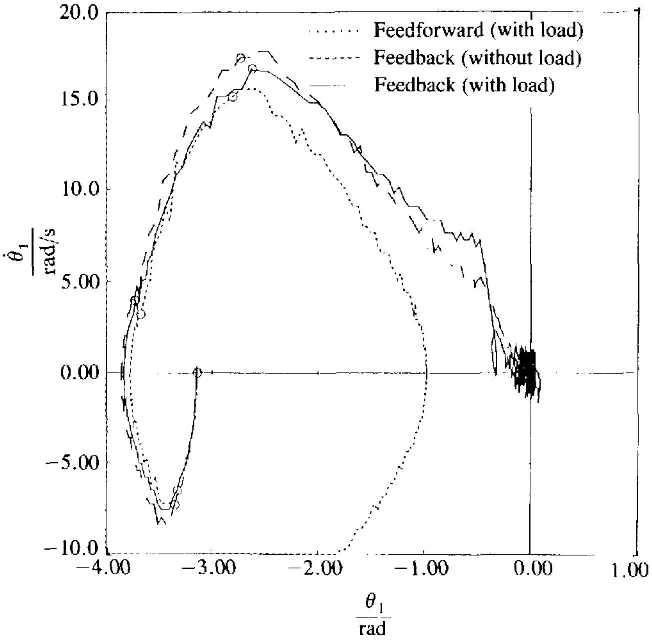

Fig. 8 Trajectory in (θ_1, $\dot{\theta}_1$) plane to feedforward and feedback control

As shown in the figures, when the parameters were correct, both feedforward and state-feedback results agreed with the simulation results, and the swing-up control was successful.

When the parameter uncertainty was introduced, it can be seen that the feedforward control failed to swing up the pendulum, however, the state-feedback control realizes the transfer successfully. The trajectory of the successful transfer in the phase plane is shown in Fig. 8. In Fig. 9 two functions, $-m_1 l_1 L_0$ (cos θ_1) $\ddot{\theta}_0$ and $\frac{1}{2} m_1 l_1^2$ (sin $2\theta_1)\dot{\theta}_0^2$, in the case of no load are plotted to check the assumption of equation (20). From Fig. 9 it can be observed that the amplitude of $\frac{1}{2} m_1 l_1^2$ (sin $2\theta_1)\dot{\theta}_0^2$ is sufficiently smaller than that of $-m_1 l_1 L_0$(cos $\theta_1)\ddot{\theta}_0$, which justifies the proposed algorithm. Finally, the sequential movements of the swing-up control are shown in Fig. 10.

6 CONCLUDING REMARKS

In this paper, a new bang-bang-type state-feedback control algorithm which can swing up a pendulum from a pendant position to an upright position has been pro-

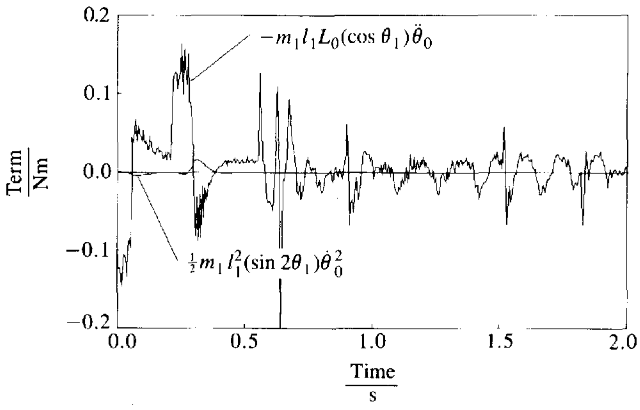

Fig. 9 Two functions in right-hand side of equation (19)

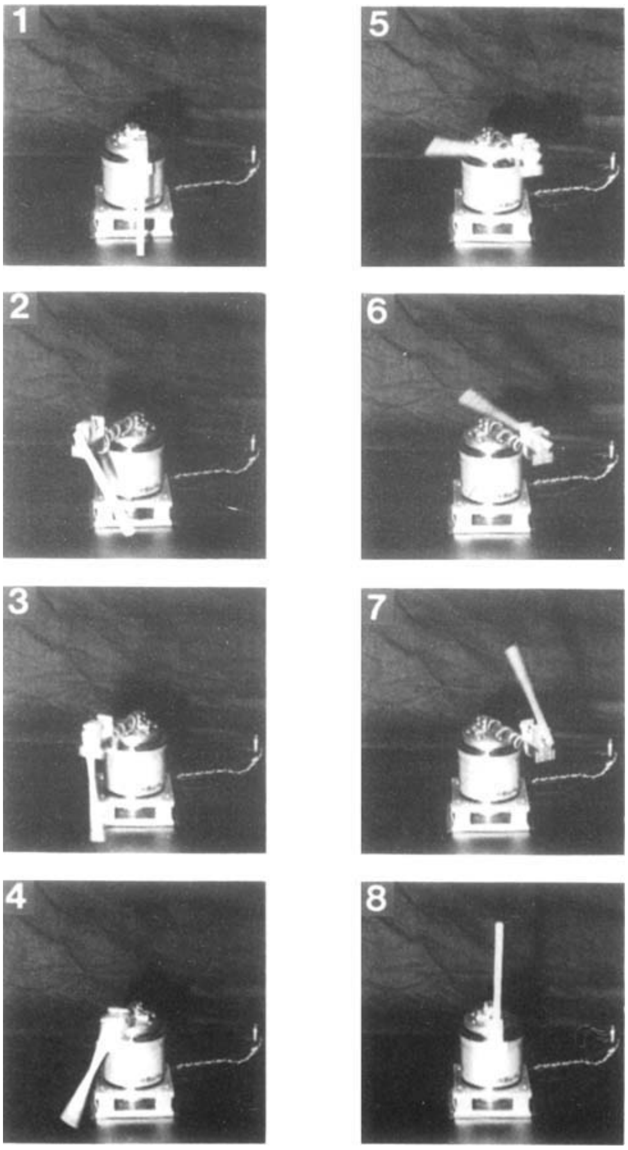

Fig. 10 Movements of swing-up control

posed. In the control system, a conventional LQ (**5**) control is also employed in the neighbourhood of the upright position for stabilization after the pendulum has been swung up in the neighbourhood of the position by the proposed bang-bang control. Experimental results have shown that the proposed method is robust for parameter uncertainties of the controlled system compared with a feedforward control (**1**).

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