به نام خدا

Performance Evaluation of Computer Systems

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Performance Modeling and Design of Computer Systems

5- SAMPLE PATHS, CONVERGENCE, AND AVERAGES

1- Convergence

Definition 5.1 A sequence $\{a_n : n = 1, 2, \ldots\}$ converges to b as $n \to \infty$, written

$$a_n \longrightarrow b, \text{ as } n \to \infty$$

or equivalently,

$$\lim_{n \to \infty} a_n = b$$

if $\forall \epsilon > 0$, $\exists n_0(\epsilon)$, such that $\forall n > n_0(\epsilon)$, we have $|a_n - b| < \epsilon$.

1- Convergence

Definition 5.2 The sequence of random variables $\{Y_n : n = 1, 2, \ldots\}$ ***converges almost surely*** to μ, written

$$Y_n \xrightarrow{a.s.} \mu, \text{ as } n \to \infty$$

or equivalently, the sequence ***converges with probability 1***, written

$$Y_n \longrightarrow \mu, \text{ as } n \to \infty \text{ w.p. } 1$$

if

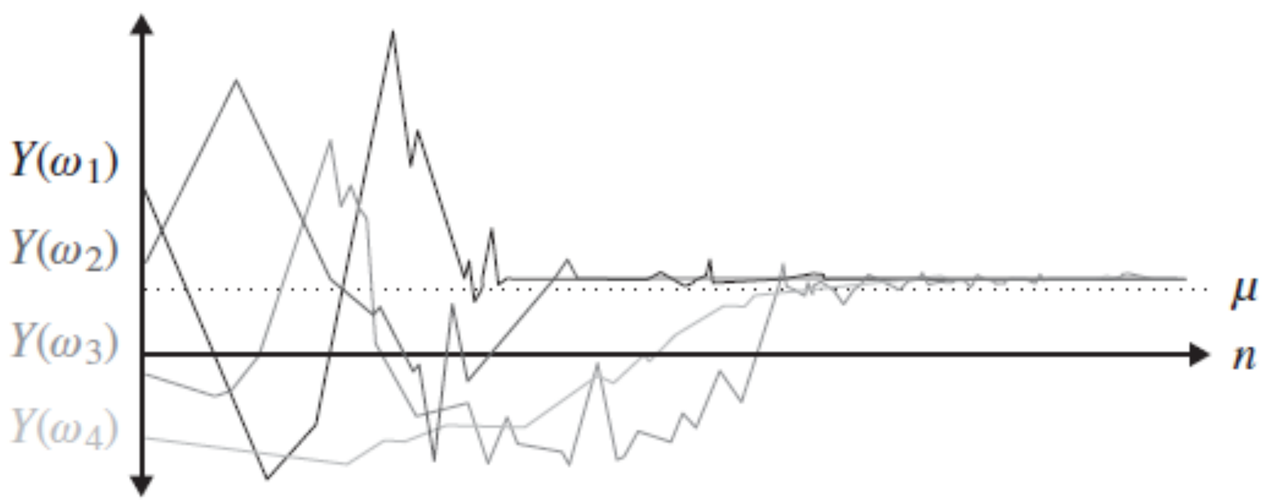
$$\forall k > 0, \mathbf{P}\left\{\lim_{n\to\infty} |Y_n - \mu| > k\right\} = 0.$$

1- Convergence

- Example
  - Sequence of $\{Y_n : n = 1,2, \ldots\}$
  - Y_n denotes the average of the first n coin flips
  - ω , sample path of coin flips
    - $\omega = 0110100101011 \ldots$
    - $\{Y_n(\omega): n = 1,2, \ldots\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{4}, \ldots\right\}$

1- Convergence

- The mass probability of bad sample paths is zero
- For each such bad sample path, ω, the limit is not μ or does not exist
  - $\omega = 1111 \dots$



1- Convergence

Definition 5.3 The sequence of random variables $\{Y_n : n = 1, 2, \ldots\}$ ***converges in probability*** to μ, written

$$Y_n \xrightarrow{P} \mu, \text{ as } n \to \infty$$

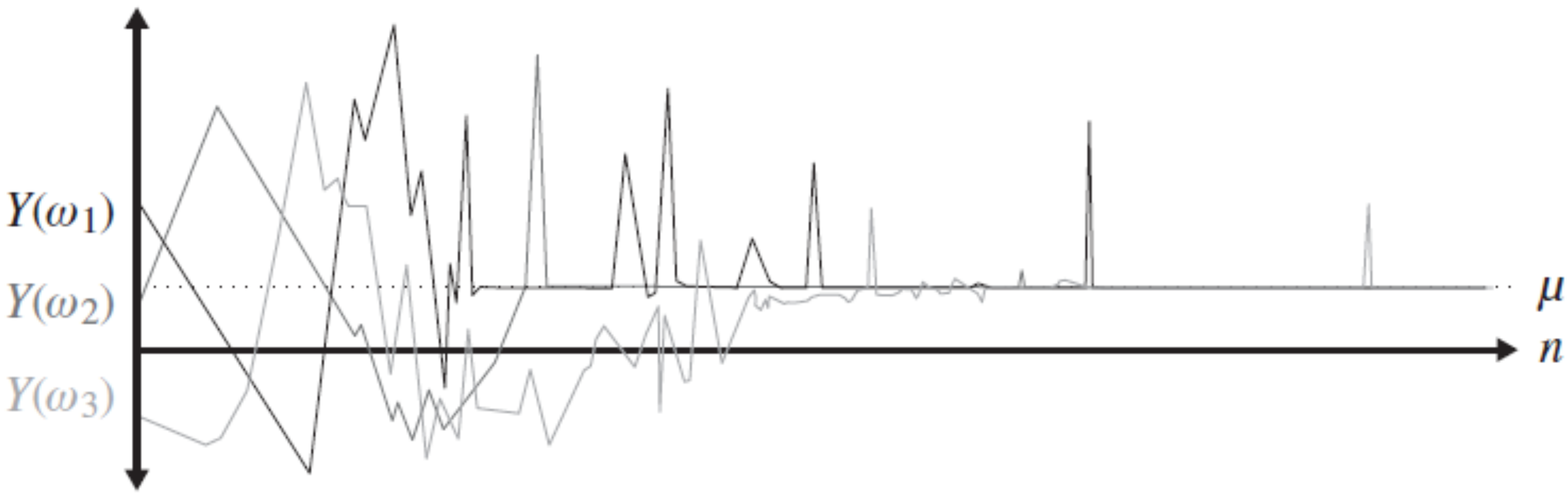
if

$$\forall k > 0, \lim_{n\to\infty} \mathbf{P}\left\{|Y_n - \mu| > k\right\} = 0.$$

1- Convergence

- Almost sure convergence implies convergence in probability
- $\{Y_n\}$ might converge in probability but not almost surely
  - For example, each sample path may have occasional spikes
    - Thus, for no sample path does $\{Y_n(\omega): n = 1,2, \ldots\}$ converge
  - Spikes get further and further apart for large n
    - For any fixed n the fraction of sample paths ω under which $Y_n(\omega)$is far from μ is small and gets smaller as we increase n

1- Convergence



2- Strong and weak laws of large numbers

Theorem 5.4 (Weak Law of Large Numbers) *Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with mean $\mathbf{E}[X]$. Let*

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad Y_n = \frac{S_n}{n}.$$

Then

$$Y_n \xrightarrow{P} \mathbf{E}[X], \text{ as } n \to \infty.$$

This is read as "Y_n converges in probability to $\mathbf{E}[X]$," which is shorthand for the following:

$$\forall k > 0, \lim_{n \to \infty} \mathbf{P}\{|Y_n - \mathbf{E}[X]| > k\} = 0.$$

2- Strong and weak laws of large numbers

Theorem 5.5 (Strong Law of Large Numbers) *Let* $X_1, X_2, X_3, \ldots$ *be i.i.d. random variables with mean* $\mathbf{E}[X]$*. Let*

$$S_n = \sum_{i=1}^{n} X_i \quad \textit{and} \quad Y_n = \frac{S_n}{n}.$$

Then

$$Y_n \xrightarrow{a.s.} \mathbf{E}[X], \textit{ as } n \to \infty.$$

This is read as "Y_n *converges almost surely to* $\mathbf{E}[X]$*" or "*Y_n *converges to* $\mathbf{E}[X]$ *with probability 1," which is shorthand for the following:*

$$\forall k > 0, \mathbf{P}\left\{\lim_{n\to\infty} |Y_n - \mathbf{E}[X]| \geq k\right\} = 0.$$

3- Time average versus ensemble average

- Example
  - FCFS Queue
    - Job is added to queue in every second with probability p
    - Job in service is completed at every second with probability q
    - q > p
    - N(t), number of jobs in the system at time t

3- Time average versus ensemble average

Definition 5.6

$$\overline{N}^{\text{Time Avg}} = \lim_{t \to \infty} \frac{\int_0^t N(v)dv}{t}.$$

– Observing a single sample path over a long period of time

3- Time average versus ensemble average

Definition 5.7

$$\overline{N}^{\text{Ensemble}} = \lim_{t\to\infty} \mathbf{E}\left[N(t)\right] = \sum_{i=0}^{\infty} i p_i$$

where

$$\begin{aligned} p_i &= \lim_{t\to\infty} \mathbf{P}\left\{N(t) = i\right\} \\ &= \text{mass of sample paths with value } i \text{ at time } t. \end{aligned}$$

– Observing all possible paths over a long period of time

3- Time average versus ensemble average

- Another example
  - Average time a job spends in system

$$\overline{T}^{\text{Time Avg}} = \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} T_i}{A(t)}$$

$$\overline{T}^{\text{Ensemble}} = \lim_{i\to\infty} \mathbf{E}\left[T_i\right]$$

3- Time average versus ensemble average

Theorem 5.9 *For an "ergodic" system (see Definition 5.10), the ensemble average exists and, with probability 1,*

$$\overline{N}^{\text{Time Avg}} = \overline{N}^{\text{Ensemble}}.$$

That is, for (almost) all sample paths, the time average along that sample path converges to the ensemble average.