به نام خدا

Performance Evaluation of Computer Systems

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Performance Modeling and Design of Computer Systems

8- DISCRETE-TIME MARKOV CHAINS

1- Introduction

- Interarrivals and service times: exponential
  - Exponential distribution is memoryless
    - Markovian process
  - Can be modeled by Markov chains
- Markov chains:
  - Discrete-time Markov chains
    - Event can only occur at the end of a time step
  - Continuous-time Markov chains
    - Event can occur at any moment in time

2- DTMC

Definition 8.1 A ***DTMC*** (discrete-time Markov chain) is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$, where X_n denotes the state at (discrete) time step n and such that, $\forall n \geq 0$, $\forall i, j$, and $\forall i_0, \ldots, i_{n-1}$,

$$\begin{aligned} \mathbf{P}\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} &= \mathbf{P}\{X_{n+1} = j \mid X_n = i\} \\ &= P_{ij} \text{ (by stationarity)}, \end{aligned}$$

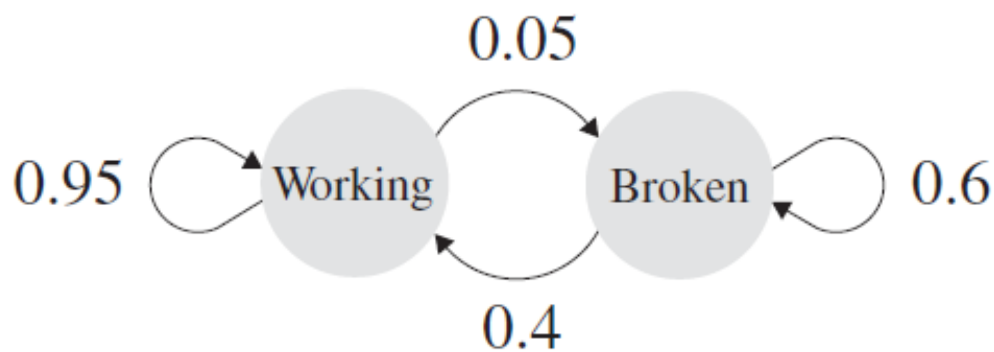
where P_{ij} is independent of the time step and of past history.

2- DTMC

- Markovian property
- Stationary property
  - Transition probability is independent of time
  - Transition probability matrix
    - Is a matrix P, whose (i,j)th entry, P_{ij}, represents the probability of moving to state j on the next transition, given that the current state is i.
    - $\sum_j P_{ij} = 1, \forall i$

3- Finite state DTMCs

- Example: Repair facility problem
  - Machine is either working or in the repair center



$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} W & B \end{array} \\ \begin{array}{c} W \\ B \end{array} & \begin{bmatrix} 0.95 & 0.05 \\ 0.40 & 0.60 \end{bmatrix} \end{array}$$

4- n-step transition probability

- Example: Repair facility problem
  - By induction:

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \quad \mathbf{P}^n = \begin{bmatrix} \frac{b+a(1-a-b)^n}{a+b} & \frac{a-a(1-a-b)^n}{a+b} \\ \frac{b-b(1-a-b)^n}{a+b} & \frac{a+b(1-a-b)^n}{a+b} \end{bmatrix}$$

  - What is the meaning of each entry?

$$P_{ij}^{n} = \sum_{k=0}^{M-1} P_{ik}^{n-1} P_{kj}$$

4- n-step transition probability

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

– Limiting probability

$$\lim_{n\to\infty} P_{ij}^n = \left(\lim_{n\to\infty} \mathbf{P}^n\right)_{ij}$$

– Why rows are same?

- Starting state does not matter

4- n-step transition probability

Definition 8.4 Let

$$\pi_j = \lim_{n \to \infty} P_{ij}^n.$$

π_j represents the ***limiting probability*** that the chain is in state j (independent of the starting state i). For an M-state DTMC, with states $0, 1, \ldots, M-1$,

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \text{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

represents the ***limiting distribution*** of being in each state.

- In this chapter we assume that the limiting probabilities exist

5- Stationary equations

Definition 8.5 A probability distribution $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is said to be ***stationary*** for the Markov chain if

$$\vec{\pi} \cdot \mathrm{P} = \vec{\pi} \quad \text{and} \quad \sum_{i=0}^{M-1} \pi_i = 1.$$

6- Equality of stationary distribution and limiting distribution

Theorem 8.6 (Stationary distribution = Limiting distribution) *Given a finite-state DTMC with M states, let*

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$$

be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \textit{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and no other *stationary distribution exists.*

6- Equality of stationary distribution and limiting distribution

- Proof
  - Assume $\{\pi_j, j = 0,1,2,\dots,M-1\}$ is the limiting distribution.
  - Part 1: $\{\pi_j, j = 0,1,2,\dots,M-1\}$ is a stationary distribution

$$\pi_j = \lim_{n\to\infty} P_{ij}^{n+1} = \lim_{n\to\infty} \sum_{k=0}^{M-1} P_{ik}^{n} \cdot P_{kj}$$

$$= \sum_{k=0}^{M-1} \lim_{n\to\infty} P_{ik}^{n} P_{kj} = \sum_{k=0}^{M-1} \pi_k P_{kj}$$

6- Equality of stationary distribution and limiting distribution

- Part 2: Any stationary distribution must equal the limiting distribution
  - Let $\vec{\pi}'$ be any stationary probability distribution
  - We will prove that $\vec{\pi}' = \vec{\pi}$
    - Assume that we start at time 0 with distribution $\vec{\pi}'$

$$\pi'_j = \mathbf{P}\{X_0 = j\} = \mathbf{P}\{X_n = j\}$$

6- Equality of stationary distribution and limiting distribution

$$\begin{aligned}
\pi'_j &= \mathbf{P}\{X_n = j\}, \quad \forall n \\
&= \sum_{i=0}^{M-1} \mathbf{P}\{X_n = j \mid X_0 = i\} \cdot \mathbf{P}\{X_0 = i\}, \quad \forall n \\
&= \sum_{i=0}^{M-1} P_{ij}^n \pi'_i, \quad \forall n.
\end{aligned}$$

$$\pi'_j = \lim_{n\to\infty} \pi'_j = \lim_{n\to\infty} \sum_{i=0}^{M-1} P_{ij}^n \pi'_i = \sum_{i=0}^{M-1} \lim_{n\to\infty} P_{ij}^n \pi'_i$$

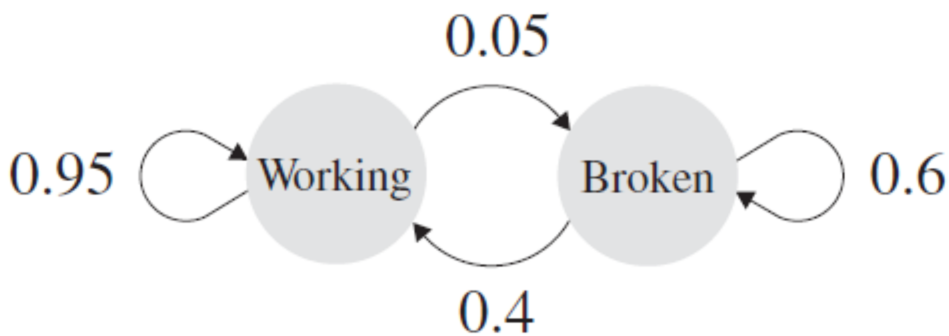
$$= \sum_{i=0}^{M-1} \pi_j \pi'_i = \pi_j \sum_{i=0}^{M-1} \pi'_i = \pi_j$$

6- Equality of stationary distribution and limiting distribution

- **Definition 8.7:** A Markov chain for which the limiting probabilities exist is said to be stationary or in steady state if the initial state is chosen according to the stationary probabilities
- So we can obtain the limiting distribution by solving the stationary equations.

7- Examples

- Repair facility problem



- It costs $300 every day of repair
- What will be the annual repair bill?

7- Examples

- Answer:
  - We should derive the limiting distribution $\vec{\pi} = (\pi_W, \pi_B)$
  - To get $\vec{\pi}$ we solve the stationary equations
    - $\pi_W = \pi_W \cdot .95 + \pi_B \cdot .4$
    - $\pi_B = \pi_W \cdot .05 + \pi_B \cdot .6$
    - $\pi_W + \pi_B = 1$
    - If $\vec{\pi} = \vec{\pi} \cdot P$ results in M equations, only M-1 equations are linearly independent.
  - $\pi_B = 1/9$, expected daily cost=$33.33
  - Annual cost of more than $12000

8- Infinite state DTMCs

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$$

$$\text{where} \quad \pi_j = \lim_{n \to \infty} P_{ij}^n \quad \text{and} \quad \sum_{j=0}^{\infty} \pi_j = 1$$

9- Infinite state stationary result

Theorem 8.8 (Stationary distribution = Limiting distribution) *Given an infinite-state DTMC, let*

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$$

be the limiting probability of being in state j and let

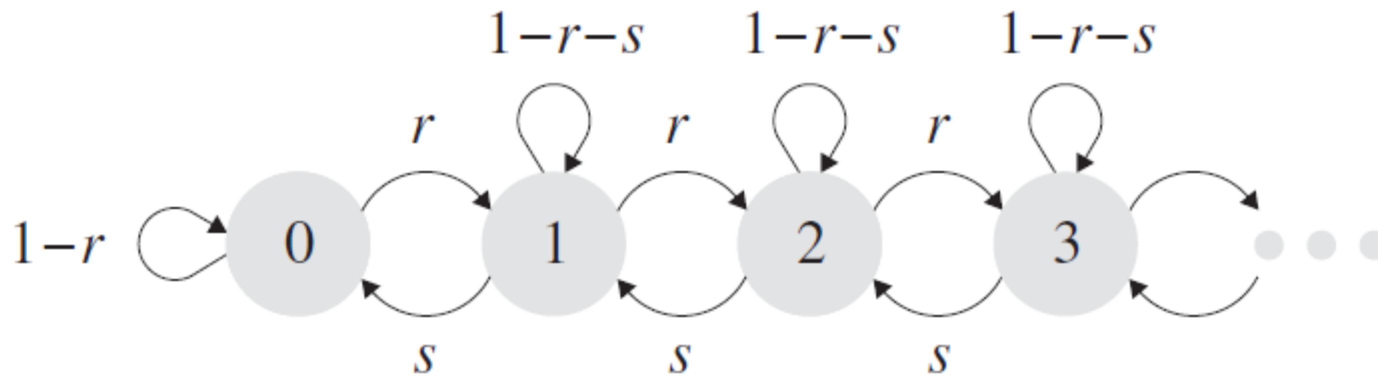
$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \quad \textit{where} \quad \sum_{i=0}^{\infty} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and no other *stationary distribution exists.*

10- Solving stationary equations in infinite state DTMCs

- Example: Queuing system with unbounded queue
  - With probability p one job arrives
  - With probability q one job departs
  - What is the average number of jobs in the system?

10- Solving stationary equations in infinite state DTMCs



- Let r = p(1-q) and s = q(1-p)

10- Solving stationary equations in infinite state DTMCs

- The transition probability matrix is infinite

$$\mathbf{P} = \begin{pmatrix} 1-r & r & 0 & 0 & \dots \\ s & 1-r-s & r & 0 & \dots \\ 0 & s & 1-r-s & r & \dots \\ 0 & 0 & s & 1-r-s & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

10- Solving stationary equations in infinite state DTMCs

- Stationary equations

$$\pi_1 = \frac{r}{s}\pi_0$$

$$\pi_2 = (\frac{r}{s})^2\pi_0$$

$$\dots$$

$$\pi_i = (\frac{r}{s})^i\pi_0$$

  - $\sum_i \pi_i = 1$ so $\pi_0 = 1 - \frac{r}{s}$
  - $\pi_i = \left(\frac{r}{s}\right)^i \cdot \left(1 - \frac{r}{s}\right)$

10- Solving stationary equations in infinite state DTMCs

- Let N denote the number of jobs in the system
  - $E[N] = \pi_0 \cdot 0 + \pi_1 \cdot 1 + \cdots$
  - $\rho = \frac{r}{s}$

$$\mathbf{E}[N] = 1\rho(1-\rho) + 2\rho^2(1-\rho) + 3\rho^3(1-\rho) + \ldots$$

$$= (1-\rho)\rho\frac{d}{d\rho}\left(1 + \rho + \rho^2 + \rho^3 + \rho^4 + \ldots\right)$$

$$= \frac{\rho}{1-\rho}$$