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Performance Evaluation of Computer Systems

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Performance Modeling and Design of Computer Systems

11- EXPONENTIAL DISTRIBUTION AND THE POISSON PROCESS

Exponential Distribution

- Random Variable $X \sim \mathrm{Exp}(\lambda)$
  - Probability density function of X

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

  - Cumulative density function of X

$$F(x) = \int_{-\infty}^{x} f(y) dy = \begin{cases} 1 - e^{-\lambda x} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

  - Mean value of X

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\lambda}.$$

Memoryless Property of the Exponential

- Memoryless Property

$$\mathbf{P}\{X > s + t \mid X > s\} = \mathbf{P}\{X > t\}, \quad \forall s, t \geq 0.$$

- $X \sim \mathrm{Exp}(\lambda)$
  - X is Memoryless

$$\mathbf{P}\{X > s + t \mid X > s\} = \frac{\mathbf{P}\{X > s + t\}}{\mathbf{P}\{X > s\}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbf{P}\{X > t\}.$$

Failure Rate Function r(t)

$$r(t) \equiv \frac{f(t)}{\overline{F}(t)}.$$

- a.k.a hazard rate function
- The probability that a t-year-old item will fail during the next (dt) seconds

$$\begin{aligned}
\mathbf{P}\{X \in (t, t+dt) \mid X > t\} &= \frac{\mathbf{P}\{X \in (t, t+dt)\}}{\mathbf{P}\{X > t\}} \\
&\approx \frac{f(t) \cdot dt}{\overline{F}(t)} \\
&= r(t) \cdot dt
\end{aligned}$$

Failure Rate Function r(t)

- Increasing Failure Rate

  $\mathbf{P}\{X > s + t \mid X > s\}$ goes down as s goes up

- Decreasing Failure Rate

  $\mathbf{P}\{X > s + t \mid X > s\}$ goes up as s goes up

- Constant Failure Rate
  - Exponential Dist.

More Properties of the Exponential

Theorem 11.3 *Given* $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2,$

$$\mathbf{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

- There are two potential failure points for our server: the power supply and the disk. The lifetime of the power supply is Exponentially distributed with mean 500 days, and the lifetime of the disk is independently Exponentially distributed with mean 1,000 days.
- What is the probability that the system failure, when it occurs, is caused by the power supply?

Answer: $\frac{\frac{1}{500}}{\frac{1}{500} + \frac{1}{1000}}$.

More Properties of the Exponential

Theorem 11.4 *Given* $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2$. *Let*

$$X = \min(X_1, X_2).$$

Then

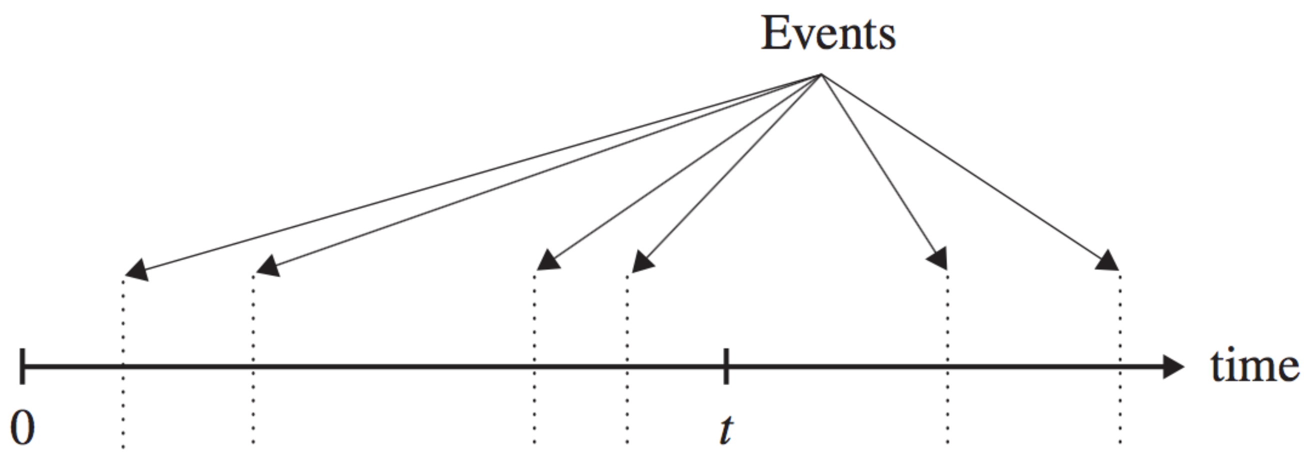
$$X \sim Exp(\lambda_1 + \lambda_2).$$

- In the server from the previous example, what is the time until there is a failure of either the power supply or the disk?

Answer: Exponential with rate $\left(\frac{1}{500} + \frac{1}{1{,}000}\right)$.

Poisson Process

Consider a sequence of events:



Define $N(t), t \geq 0$ as the number of events that occurred by time t.

Independent Increments

Definition 11.5 An event sequence has ***independent increments*** if the numbers of events that occur in disjoint time intervals are independent. Specifically, for all $t_0 < t_1 < t_2 < \ldots < t_n$, the random variables

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$$

are independent.

- Let us look at three event processes
  1. Births of Children
  2. People entering a building
  3. Goals scored by a particular soccer player
- **Question:** Do these event processes have independent increments?
- **Answer:**
  1. No. Birth rate depends on population, which increases with births.
  2. Yes.
  3. Maybe. Depends on whether we believe in slumps!

Stationary Increments

Definition 11.6 The event sequence has ***stationary increments*** if the number of events during a time period depends only on the length of the time period and not on its starting point. That is, $N(t+s) - N(s)$ has the same distribution for all s.

Definition 1 of the Poisson Process:

A **Poisson process having rate** λ is a sequence of events such that

1. $N(0) = 0$.
2. The process has independent increments.
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, $\forall s, t \geq 0$,

$$\mathbf{P}\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, \ldots$$

Poisson Process

<u>Definition 2 of the Poisson Process</u>:

A **Poisson process with rate** λ is a sequence of events such that the interarrival times are i.i.d. Exponential random variables with rate λ and $N(0) = 0$.

Definition 1 <=> Definition 2

Poisson Process

Definition 1 $\Rightarrow$ Definition 2

Let $T_1, T_2, \ldots, T_n, \ldots$ be the *interarrival* times of a sequence of events. We need to show that $T_i \sim \text{Exp}(\lambda),\ \forall i$. By Definition 1,

$$\mathbf{P}\{T_1 > t\} = \mathbf{P}\{N(t) = 0\} = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}.$$

Next,

$$\begin{aligned}
\mathbf{P}\left\{T_{n+1} > t \,\middle|\, \sum_{i=1}^{n} T_i = s\right\} &= \mathbf{P}\left\{0 \text{ events in } (s, s+t) \,\middle|\, \sum_{i=1}^{n} T_i = s\right\} \\
&= \mathbf{P}\left\{0 \text{ events in } (s, s+t)\right\}, \\
&\quad \text{by independent increments} \\
&= e^{-\lambda t}, \text{ by stationary increments.}
\end{aligned}$$

Poisson Process

Definition 3 of the Poisson Process:

A **Poisson process having rate** λ is a sequence of events such that

1. $N(0) = 0$.
2. The process has stationary and independent increments.
3. $\mathbf{P}\{N(\delta) = 1\} = \lambda\delta + o(\delta)$.
4. $\mathbf{P}\{N(\delta) \geq 2\} = o(\delta)$.

Definition 1 <=> Definition 3

Merging Independent Poisson Processes

Theorem 11.7 *Given two independent Poisson processes, where process 1 has rate λ_1 and process 2 has rate λ_2, the merge of process 1 and process 2 is a single Poisson process with rate $(\lambda_1 + \lambda_2)$.*

Alternative Proof Let $N_i(t)$ denote the number of events in process i by time t.

$$N_1(t) \sim \text{Poisson}(\lambda_1 t)$$
$$N_2(t) \sim \text{Poisson}(\lambda_2 t)$$

Yet the sum of two independent Poisson random variables is still Poisson with the sum of the means, so

$$\underbrace{N_1(t) + N_2(t)}_{\text{merged process}} \sim \text{Poisson}(\lambda_1 t + \lambda_2 t).$$

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Poisson splitting

Theorem 11.8 *Given a Poisson process with rate λ, suppose that each event is classified "type A" with probability p and "type B" with probability $1-p$. Then type A events form a Poisson process with rate $p\lambda$, type B events form a Poisson process with rate $(1-p)\lambda$, and these two processes are independent. Specifically, if $N_A(t)$ denotes the number of type A events by time t, and $N_B(t)$ denotes the number of type B events by time t, then*

$$\begin{aligned}\mathbf{P}\{N_A(t)=n, N_B(t)=m\} &= \mathbf{P}\{N_A(t)=n\}\cdot\mathbf{P}\{N_B(t)=m\}\\ &= e^{-\lambda tp}\frac{(\lambda tp)^n}{n!}\cdot e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!}.\end{aligned}$$

Uniformity

Theorem 11.9 *Given that one event of a Poisson process has occurred by time t, that event is equally likely to have occurred anywhere in $[0, t]$.*

Proof Let T_1 denote the time of the one event.

$$\begin{aligned}
\mathbf{P}\{T_1 < s \mid N(t) = 1\} &= \frac{\mathbf{P}\{T_1 < s \text{ and } N(t) = 1\}}{\mathbf{P}\{N(t) = 1\}} \\
&= \frac{\mathbf{P}\{1 \text{ event in } [0, s] \text{ and } 0 \text{ events in } [s, t]\}}{\frac{e^{-\lambda t}(\lambda t)^1}{1!}} \\
&= \frac{\mathbf{P}\{1 \text{ event in } [0, s]\} \cdot \mathbf{P}\{0 \text{ events in } [s, t]\}}{e^{-\lambda t} \cdot \lambda t} \\
&= \frac{e^{-\lambda s} \cdot \lambda s \cdot e^{-\lambda(t-s)} \cdot (\lambda(t-s))^0}{e^{-\lambda t} \cdot \lambda t} \\
&= \frac{s}{t} \qquad \blacksquare
\end{aligned}$$