S O L U T I O N S

Introduction to University Mathematics 2018

MATLAB Worksheet VII

Complete the following tasks and hand in the sheet by the end of the lab session.

This week we will be exploring *symbolic* mathematics on the computer. This includes, for instance, solving an equation in x, differentiation, integration *etc*.

We will use MATLAB's *Symbolic Math Toolbox* to perform these tasks. By the way, we could use a dedicated symbolic environment called *MuPAD*. However, the *MuPAD* will be removed in a future release and using it is now deprecated! The *Symbolic Math Toolbox* includes functions for symbolic math expressions that parallel MATLAB functionality for numeric values. Unlike *MuPAD* functionality, the *Symbolic Math Toolbox* functions enable you to work in familiar interfaces, such as the Live Editor or MATLAB Command Window, which offer a smooth workflow and are optimized for usability. Therefore, use the equivalent *Symbolic Math Toolbox* functionality to work with symbolic math expressions. To have a look at the full list of the available symbolic functions, search *Symbolic Math Toolbox* documentation.

1. Symbolic computation basics. Basic MATLAB operations (`+,-,*,/,^`) carry over into symbolic computation.

   To create the symbolic variables x and y, you can either type

   ```
   x = sym('x');
   y = sym('y');
   ```

   or simply type

   ```
   syms x y
   ```

   Type `doc sym` and `doc syms` to read the introduction of the symbolic functions `sym` and `syms`.

   (a) Write down the command that produces the fraction `3/7` (not the numerical value). *Ans*: **sym(3/7) or sym('3/7')**

   (b) Write down the command that creates the following symbolic array.
   Ans: **M=sym('b%d', [2, 3])**

$$M = \begin{pmatrix} b11 & b12 & b13 \\ b21 & b22 & b23 \end{pmatrix}$$

The symbolic computation in MATLAB assumes that you want to do symbolic mathematics and so it keeps expressions in *exact* forms as far as possible (otherwise you could have just used numerical computation in MATLAB instead). This is also true for symbolic functions like `sin, cos, log` *etc.* Depending on their arguments, these functions return exact symbolic results or floating-point ones. For example, below lines use the symbolic `sin` function.

```
symVector = sym([-3, 2*pi, 3*pi/7]);
symSineValues = sin(symVector);
```

To reiterate this difference, try to evaluate 2^{1000} in both symbolic computation and numerical computation. You will see that MATLAB numerical computation is designed for numerical tasks, and its behaviour is governed by the same rules as other programming languages (essentially just binary arithmetic). On the other hand, MATLAB symbolic computation is designed for symbolic/exact calculations. It is more accurate, but slower and requires a lot more computer memory.

But what if you want to know the numerical value of exact expressions like `sin(pi/7)`? You can use *vpa* function to approximate symbolic results with floating-point numbers.

Exercise: Define variable a as $7\pi/8$. Find the exact values of $\sin(a)$ and $\cos(a)$.

Ans: $\sin a = \frac{\sqrt{2-\sqrt{2}}}{2}$ $\cos a = -\frac{\sqrt{2+\sqrt{2}}}{2}$

2. Solve. Suppose we want to solve the equation $x^2 - 3x - 154 = 0$, type

```
syms x
solve(x^2-3*x-154==0, x)
```

You should find two solutions: -11 and **14**. The `solve(A,B)` function takes two arguments: `A` is the equation you want to solve, `B` is the variable to solve for.

You could also have done the above calculation by defining the equation first.

```
syms x
eq = x^2-3*x-154==0
solve(eq,x)
```

(a) Write down the solutions to the equation $x^3 - 6x^2 - 19x + 24 = 0$.

Ans: **-3, 1, 8**

Now see what happens if we try to solve

$$x^3 - 3x + 1 = 0.$$

You should find that MATLAB symbolic computation can't tell you the *exact* answer. If this happens, we can use the `vpa` again to get approximate symbolic results with floating-point numbers. You will find $x \approx -1.88, 0.347$, and **1.53** (to 3 SF).

(b) Write down the numeric solution of the equation $e^{-u} = u$ (to 4 SF).

Ans: $u = 0.5671$

(c) Find all solution(s) to the equation $|x| = 2 - \cosh x$ (to 4 SF).

Ans: $x = \pm 0.7253$ **MATLAB can only find one answer because it uses numerical methods like bisection or Newton Raphson. Remember: Computers are not mathematicians. You are!**

3. Algebra. Let's do some basic algebraic manipulation. Let's first look at `expand` and `factor` commands. Try these and see if you understand the results.

- `factor(x^2-6*x+5)`
- `expand((x-3)^5)`
- `expand(sin(x+y))`

(a) Use `factor` to simplify the expression $\dfrac{x^2-1}{x^3+5x^2-x-5}$. Ans: $\underline{\dfrac{1}{x+5}}$

(b) The coefficient of x^2 in the expansion of $(2x-3)^5\left(1-\dfrac{2}{x}\right)$ is **-2520**.

(c) Write down the command which will express $\cos 4x$ in terms of $\cos x$.

expand($\cos(4 * x)$**)**

Algebraic simplifications can also be done with `simplify, combine, collect` amongst many other functions. Read the help documentations and experiment with these commands in your own time, but let's move on for now.

4. Differentiation. Suppose we want to differentiate $\sin(x)$. The command is

```
diff(sin(x),x)
```

The syntax is similar to the `solve` command: in `diff(A,B)`, `A` is the expression to differentiate, `B` is the variable to differentiate with respect to.

We can also do multiple differentiations: to differentiate $\dfrac{1}{x}$ five times, type

```
diff(1/x , x, 5)
```

Sometimes the gradient at a specific point might be needed. Use the function `subs` to substitute in a value into an expression. For instance, here's how to calculate the gradient of the tangent to the curve $y = x^4 - 10$ at $x = 3$:

```
y=x^4-10
D1=diff(y,x)
subs(D1,x,3)
```

You should find the answer = 108. Of course you could have combined everything into a single line, but you'll probably make fewer mistakes if you break it down into small steps.

(a) If $y = \ln(\ln 5x)$, then $\dfrac{dy}{dx} = \underline{\dfrac{1}{x\ln(5x)}}$.

(b) If $u = \tanh^{-1}(\tan v)$, then $\dfrac{du}{dv} = \underline{\sec(2v)}$. (try `simplify`)

(c) If $s(t) = e^{-t^2}\cos t$, then $s''(0) = \underline{\mathbf{-3}}$.

(d) (slightly tricky!) Let's define a function $H_n(x)$ by

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

This function appears frequently in probability and quantum mechanics. Try evaluating this function with $n = 1$. You should find $H_1(x) = x$. Now evaluate it with $n = 2, \ldots$ They should look like simple *polynomials* with no exponentials around.

$$H_2(x) = x^2 - 1$$

Calculate $H_{16}(x = 0) = \mathbf{2027025}$

[*Hint: to minimise typing, you could first define* `n=1` *and define* `H` *in terms of* n *and* x*, then change* `n` *and rerun. Use* `simplify` *to tidy up the output.*]

5. Integration. Symbolic integration can be done using the function `int` with the same syntax as `diff`. For instance, to integrate x^2 wrt x, type:

```
int(x^2,x)
```

(note that MATLAB doesn't provide the constant of integration). To evaluate definite integrals such as $\int_0^3 x^2 dx$, we use

```
int(x^2, x, 0, 3)
```

MATLAB can deal with a huge range of nasty integrals, even improper integrals with infinite limits or singularities.

However, if no exact answer can be found by MATLAB, use the `vpa` function to get a floating-point approximation in the same way we did with `solve`.

Use MATLAB to evaluate the following integrals.

$$\int \sqrt{1-x^2}\ \mathrm{d}x = \frac{1}{2}\left[\sin^{-1}x + x\sqrt{1-x^2}\right] + C$$

$$\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\ \mathrm{d}t = \frac{22}{7} - \pi$$

$$\int_0^\infty \frac{\sin x}{x}\ \mathrm{d}x = \frac{\pi}{2}$$

$$\int_{-1}^1 \sec(\theta^2)\ \mathrm{d}\theta = \mathbf{2.2654\ (4\ d.p.)}$$

6. Functions. *syms f(var1,...,varN)* creates the symbolic function f and the symbolic variables *var1,...,varN* representing the input arguments of *f*.

   To create multiple symbolic functions in one call, for example, *syms f(x) g(t)* creates two symbolic functions (f and g) and two symbolic variables (x and t).

   To define $f(x) = x^2 + 5$, type

   ```
   f(x)=x^2+5
   ```

   This means *"f is defined such that $x \to x^2 + 5$ "* (*i.e. x is mapped to* $x^2 + 5$).

   You can now try using f, for example, by typing $f(2)$.

   Here is a little exercise combining functions with integration.

   Consider the following function.

$$G(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t.$$

   (note that t is just a dummy variable which is integrated away. You can try a different symbolic variable and you will get the same result.)

   Define this function as `G(x)`. When done correctly, you should see $G(x) = gamma(x)$ and you can find that $G(1/2) = \sqrt{\pi}$.

   Use your function `G(x)` (*not the `gamma` function*) to calculate the following.

   $G(3/2) = \dfrac{\sqrt{\pi}}{2}$ $\qquad$ $G(5) = \mathbf{24}$

7. Plot. Plotting graphs of symbolic functions is not hard in MATLAB symbolic computation for common household functions. For example, to plot $y = \sin(x)$ from 0 to 2π, type

```
syms x
fplot(sin(x),[0, 2*pi])
```

Plot the graph of the function $y = G(x)$ where $-4 \leq x \leq 4$. Sketch it below.