Balancing Straight-Line Programs

Moses Ganardi, Artur Jeż and Markus Lohrey

Universität Siegen

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An SLP over the alphabet Γ is a sequence of productions (grammar rules)

$$\mathcal{G} = \langle X_i \to \alpha_i \mid 1 \leq i \leq n \rangle,$$

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The depth of $\mathcal{G}$ (depth($\mathcal{G}$)) is the height of the derivation tree of $\mathcal{G}$.

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$\mathcal{G} = \langle X_i \to X_{i+1}X_{i+2}$ for $1 \leq i \leq 4, \quad X_5 \to b, \quad X_6 \to a\rangle.$

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The main result

Theorem (Charikar et al. 2002, Rytter 2004)

From a given SLP $\mathcal{G}$ of size n such that $[\![\mathcal{G}]\!]$ has length N, one can construct in time $\mathcal{O}(n \cdot \log N)$ an SLP $\mathcal{G}'$ such that:

- $[\![\mathcal{G}']\!] = [\![\mathcal{G}]\!]$
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Our main result

From a given SLP $\mathcal{G}$ of size n such that $[\![\mathcal{G}]\!]$ has length N, one can construct in time $\mathcal{O}(n)$ an SLP $\mathcal{G}'$ such that:

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Random access queries

For a string $w \in \Sigma^*$ of length N, a random access query gets a position $i \in [1, N]$ and returns the i-th symbol of w.

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From a given SLP $\mathcal{G}$ of size n such that $w := [\![\mathcal{G}]\!]$ has length N, one can compute in time $\mathcal{O}(n)$ a data structure of size $\mathcal{O}(n)$ that allows to answer random access queries for w in time $\mathcal{O}(\log N)$.

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- Size of the data structure is measures in number of words of bit length $\log N$.
- Random access queries can be answered in time $\mathcal{O}(\text{depth}(\mathcal{G}))$ and space $\mathcal{O}(n)$.

Symmetric centroid path decomposition

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\begin{aligned}
X_0 &\to X_1 X_{14} \\
X_1 &\to X_{13} X_2 \\
X_2 &\to X_{12} X_3 \\
X_3 &\to X_4 X_{12} \\
X_4 &\to X_{11} X_5 \\
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X_6 &\to X_7 X_{10} \\
X_7 &\to X_{10} X_8 \\
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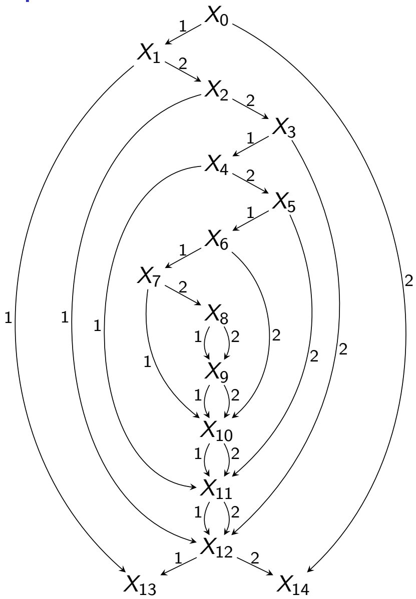

Symmetric centroid path decomposition

Label X_i with the pair consisting of

- the number of paths from the root X_0 to X_i, and
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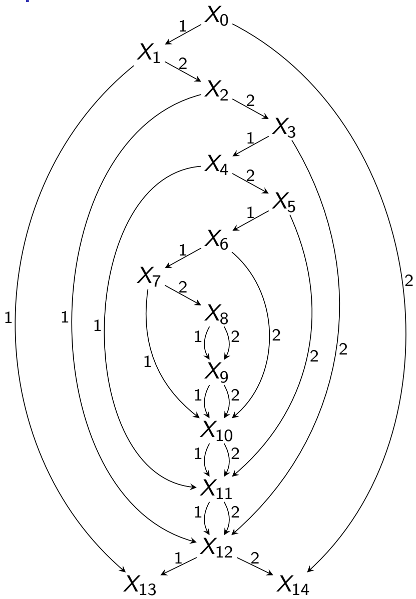

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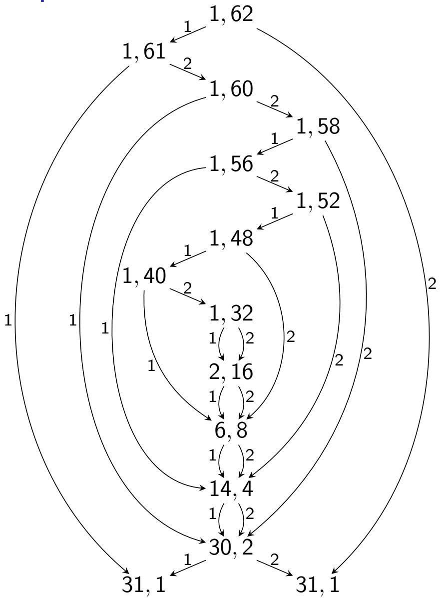

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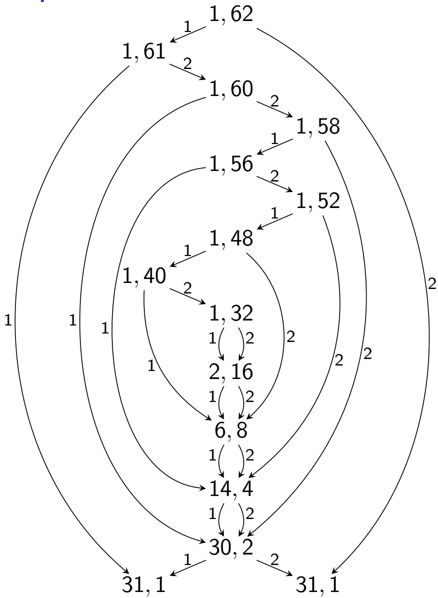

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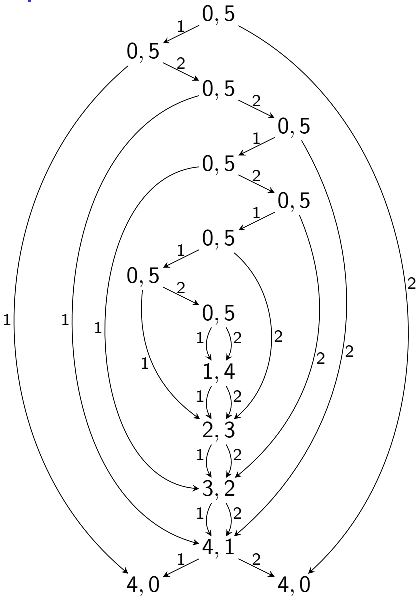

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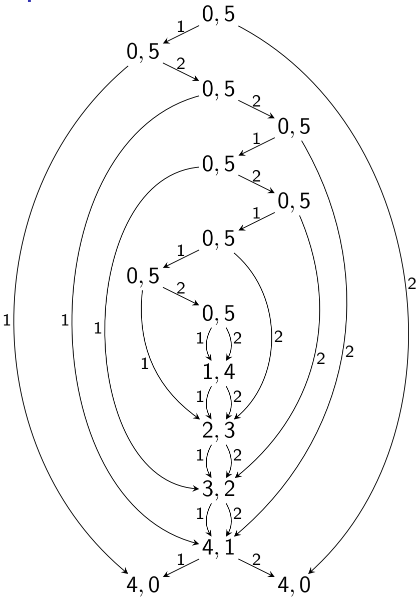
The symmetric centroid path decomposition consists of those edge connecting nodes that are labelled with the same pair.



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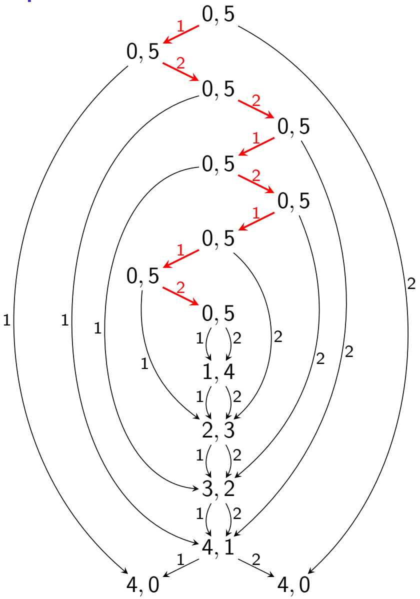

Symmetric centroid path decomposition

Properties of the symmetric centroid path decomposition

- For every variable X_i at most one outgoing edge belongs to the symmetric centroid path decomposition.
- For every variable X_i at most one incoming edge belongs to the symmetric centroid path decomposition.
- Every path from the root X_0 to a terminal variable X_j contains a most $2 \cdot \log_2 N$ edges that do not belong to the symmetric centroid path decomposition. Here N is the length of the string $[\![\mathcal{G}]\!]$.

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By the first two properties the symmetric centroid path decomposition is indeed a decomposition of the DAG into paths (possibly of length zero).

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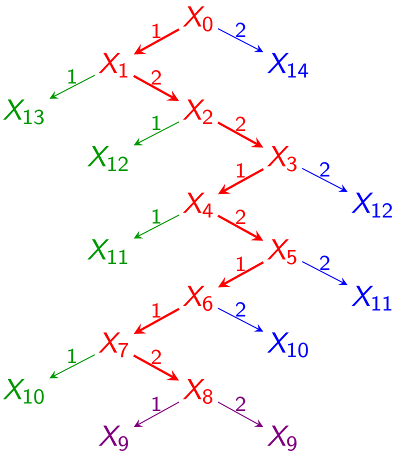
Consider a symmetric centroid path $X_s \to \cdots \to X_t$ (may consist only of X_t).



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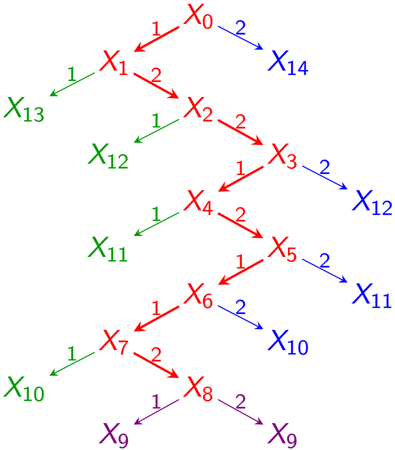

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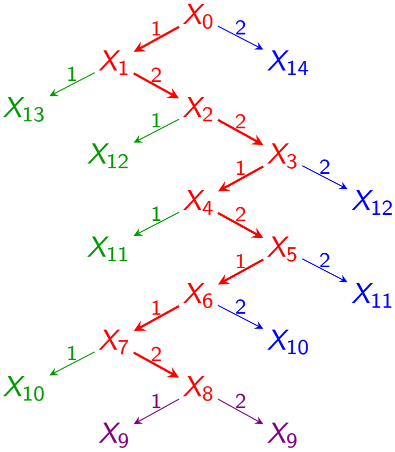

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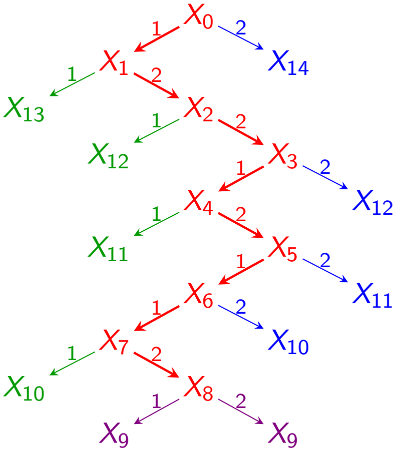
Add productions such that every L_i/R_i derives the appropriate suffix/prefix of $X_{13}X_{12}X_{11}X_{10}/X_{10}X_{11}X_{12}X_{14}$.



Construction of the balanced SLP $\mathcal{G}'$ from $\mathcal{G}$, Part II

The remaining problem

Given: word $y = Y_1 Y_2 \cdots Y_m$ of variables ($X_{13}X_{12}X_{11}X_{10}$ in our example)

Goal: construct an SLP $\mathcal{S}$ of size $\mathcal{O}(m)$ and “small depth” that contains for every suffix $s_i := Y_i Y_{i+1} \cdots Y_m$ a variable S_i that produces s_i.

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What means "small depth"?

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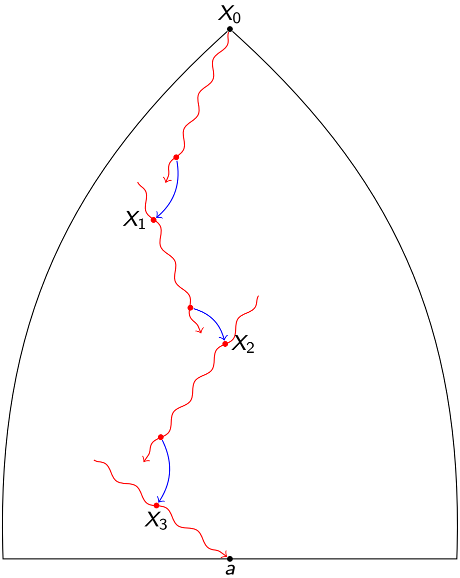
Every path in the derivation tree of $\mathcal{S}$ from variable S_i to a terminal Y_j ($j \geq i$) has length $4 + 2 \log \|s_i\| - 2 \log_2 \|Y_j\|$.

The global picture

- symmetric centroid paths
- edges not on symmetric centroid paths



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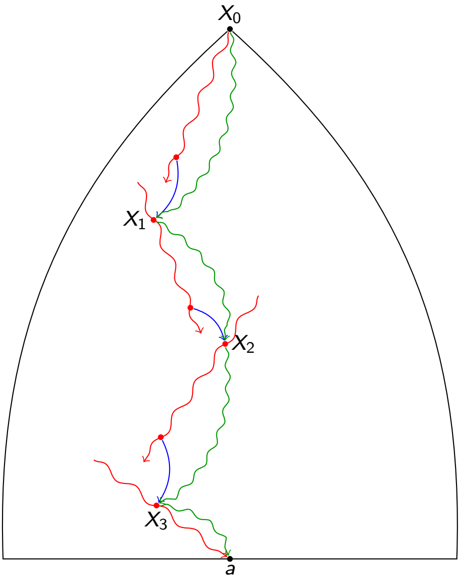

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  $5+2\log\|X_0\|-2\log_2\|X_1\|$

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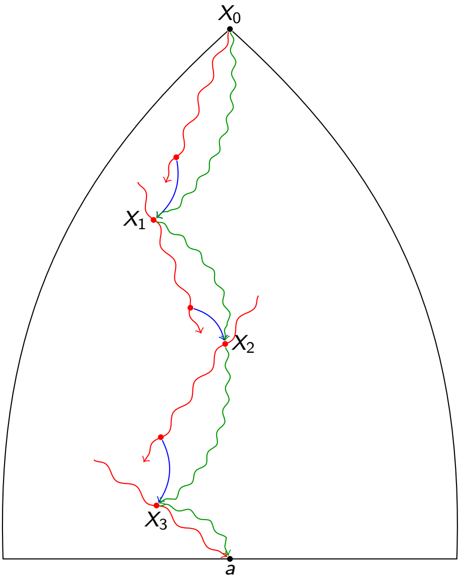

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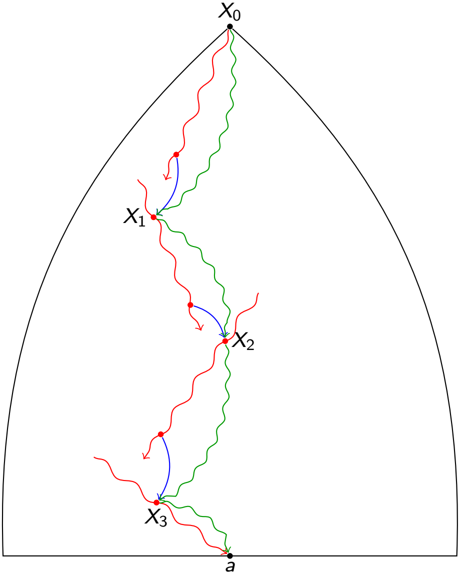
- total path length:
  $\leq \mathcal{O}(\log N) + 2\log\|X_0\|$
  $= \mathcal{O}(\log N)$



Suffixes of weighted strings: the main lemma

We consider strings over an alphabet Σ, where every $a \in \Sigma$ has a weight $\|a\|$, which is additively extended to strings over Σ.

Main lemma

For every weighted string s of length n one can construct in linear time an SLP $\mathcal{G}$ with the following properties:

- $\mathcal{G}$ contains at most $3n$ variables,
- all right-hand sides of $\mathcal{G}$ have length at most four,
- $\mathcal{G}$ contains variables $S_1, \ldots, S_n$ that produce the suffixes of s, and
- every path from a variable S_i to some terminal symbol a in the derivation tree has length $3 + 2(\log \|S_i\| - \log \|a\|)$.

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- Recursive call for suffix v.
- Factorize u into block of length 2 and possibly one block of length 1.

  Introduce nonterminals $X_1, \ldots, X_k$ ($k = \lceil |u|/2 \rceil$) for the blocks.

  Recursive call for $X_1 X_2 \cdots X_k$, where the weight of X_i is the weight of the corresponding block.

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SLPs have been extended to trees:

- tree straight-line programs (TSLPs)
- forest straight-line programs (FSLPs)
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Balancing TSLPs/FSLPs/top dags

From a given TSLP/FSLP/top dag $\mathcal{G}$ of size n producing a tree t of size N, one can construct in time $\mathcal{O}(n)$ a TSLP/FSLP/top dag $\mathcal{G}'$ such that:

- $\mathcal{G}'$ produces t as well,
- $|\mathcal{G}'| \in \mathcal{O}(n)$
- $\text{depth}(\mathcal{G}') \in \mathcal{O}(\log N)$

Balancing algebraic circuits: the meta theorem

Meta theorem for balancing circuits

Let $\mathcal{A}$ be an algebra with a certain finiteness property. From an algebraic circuit $\mathcal{G}$ over $\mathcal{A}$ one can compute an algebraic circuit $\mathcal{G}'$ such that:

- $\mathcal{G}'$ and $\mathcal{G}$ produces the same element of $\mathcal{A}$.
- $|\mathcal{G}'| \in \mathcal{O}(n)$
- $\text{depth}(\mathcal{G}') \in \mathcal{O}(\log N)$, where N is the size of the unravelling (unfolding) of $\mathcal{G}$.

An open problem

Is it possible to solve the random access problem for an SLP of size n producing a string of length N in

- time $\mathcal{O}(\log N / \log\log N)$
- using a data structure of size $\mathcal{O}(n)$?