Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics

HSC Course

Assessment 2

March, 2017

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A Reference Sheet is attached to the last page of this booklet. You may detach it.

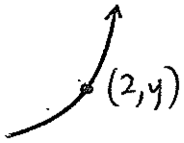
Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-9
57 Marks

QUESTION 1

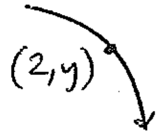
Which curve has $f'(2) > 0$ <u>and</u> $f''(2) < 0$?

A.

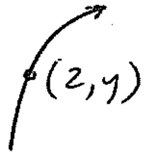


B.

C.



D.

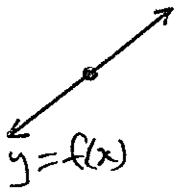


QUESTION 2

Which does <u>not</u> show $f''(x) = 0$ at the indicated point?

C

A.



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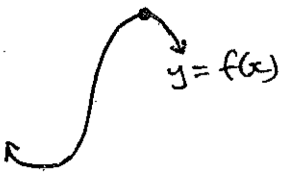
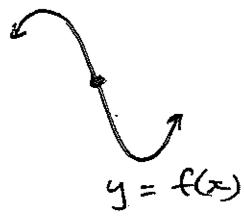


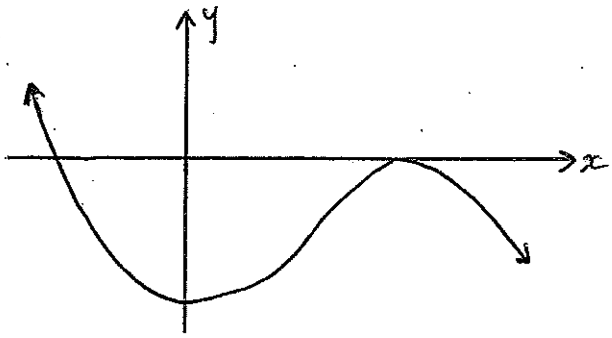
C.

D.



QUESTION 3

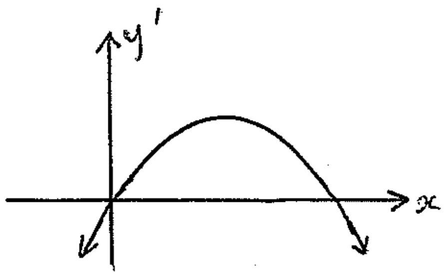
The graph of a function is shown.



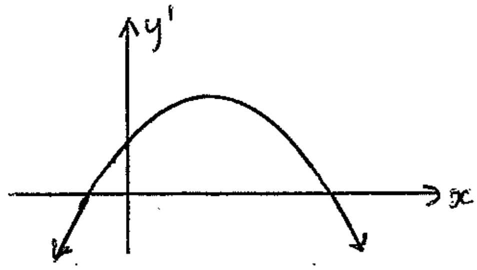
C

Which is the graph of its derivative?

A.



C.



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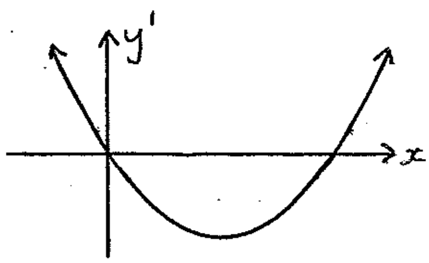

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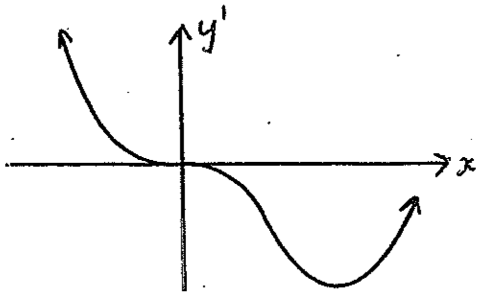


QUESTION 4

In an arithmetic sequence, the sum of the 6th and 12th terms is 250. What is the 9th term?

A. 100 B. 125 C. 150 D. 175

QUESTION 5

With annual compounding interest, $1000 doubles in value after 18 years. What is the approximate rate of compound interest?

A. 3.9% B. 0.039% C. 5.6% D. 0.56%

SECTION II

C

QUESTION 6 (14 marks) Start a new page.

a) The general term of a certain sequence is given by $T_n = 3n - 2$.

i) Determine whether 348 belongs in this sequence. 1

ii) Find the sum of the first 100 terms. 1

b) The sum to n terms of a certain series is given by $S_n = 2n + n^2$.

i) Find the first and second terms, T_1 *and* T_2 2

ii) Find T_n , simplifying your answer. 2

iii) How many terms are required for the sum S_n to first exceed 80? 1 C

c) Evaluate : i) $\sum_{n=1}^{5}(n^2 - n)$ 1

ii) $\sum_{n=100}^{200}(80 + n)$ 1

d) For the infinite geometric series $1 + (x - 1) + (x - 1)^2 + \ldots\ldots,$

i) find x such that a limiting sum exists. 1

ii) find the limiting sum when $x = \frac{3}{4}$ 1

e) If $2, a, b$ are in arithmetic progression and $a, b, 9$ are in geometric progression, find the values of a and b. 3

QUESTION 7 (14 marks) Start a new page.

a) For a geometric sequence, $T_3 = 21$ and $T_7 = 336$. Find the common ratio. 2

b) Express the series $1 + 3 + 5 + 7 + \ldots + 101$ using sigma notation. 2

Do not evaluate the sum.

c) Find the equation of the tangent to the curve $y = x^2 - x$ at the point where $x = 3$. 2

Leave your answer in general form.

d) Differentiate: i) $y = \frac{3x+2}{2x-1}$ 1

ii) $y = x^2(3x+2)^5$ 2

e) For the curve $f(x) = 2x - \sqrt{x}$, find: i) $f'(x)$ 1

ii) $f''(4)$ 2

iii) the x value for which the curve is stationary. 2

QUESTION 8 (14 marks) Start a new page.

a) For the curve $y = x^3 - x^2$,

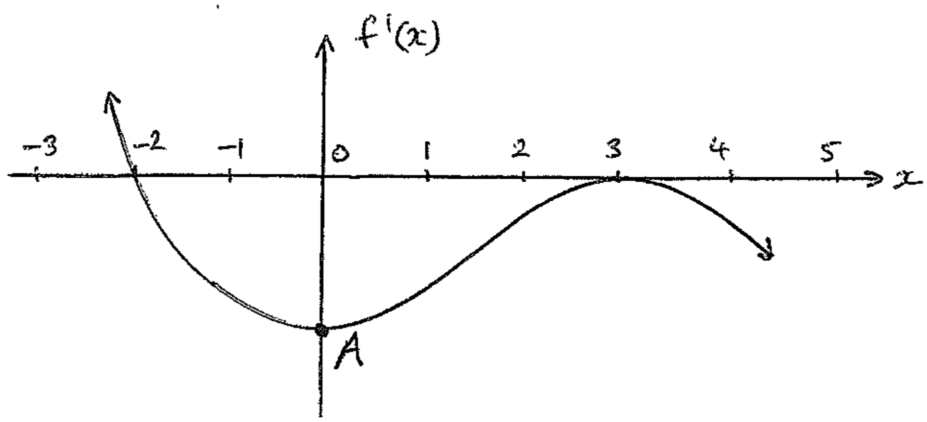
i) Find stationary points and determine their nature. 3

ii) Find, and prove, that a point of inflexion exists. 2

iii) Sketch the curve for $-1 \leq x \leq 2$. 2

iv) Find the maximum value of $x^3 - x^2$ in the domain $-1 \leq x \leq 2$. 1

b)



The diagram shows the graph of $y = f'(x)$, i.e. the derivative function of $y = f(x)$.

i) At which x value(s) are there stationary points on the original $y = f(x)$ curve? 1 C

ii) For the x value(s) in part i), determine the nature of each stationary point on $y = f(x)$. 2

iii) For what x value(s) is the curve $y = f(x)$ decreasing? 2

iv) Which feature on the curve $y = f(x)$ is indicated by the point A? 1

QUESTION 9 (15 marks) Start a new page.

a) A man plans to invest \$200 per month into a superannuation fund. It is assumed that he will do this for 40 years and that the fund earns 0.6% interest per month.

i) To what amount will the first \$200 invested grow? 1

ii) Find the predicted total value of his superannuation after 40 years (nearest \$). 2 C

iii) The man's wife wants him to achieve a total superannuation value of one million dollars after 40 years. Assuming the same interest rate, how much should she tell him to invest every month so that this goal is achieved? (Nearest \$) 2

b) A piece of wire 20 cm long is cut into two sections, each of which is then bent to form a square. Let x cm be the length of one of the sections.

i) Show that the combined area of the two squares is given by $A = \frac{x^2-20x+200}{8}$ 1

ii) Find the smallest possible total area enclosed by the two squares. 3

c) A car loan of \$30,000 is to be repaid over 5 years. Interest is charged at the rate of 10% p.a. and repayments made <u>every 3 months</u>. Immediately after the quarterly interest is charged, a repayment is to be made.

Let A_n be the loan balance remaining after n repayments, and R be the amount of each repayment.

i) Write an expression for A_1 and <u>derive</u> that $A_2 = 30000 \times 1.025^2 - R(1.025 + 1)$ 2

ii) Find the amount of each quarterly repayment R (nearest \$). 2

iii) Find the equivalent rate of annual simple interest for this loan (answer to 2 dec. places). 2

END OF TEST

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2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Factorisation

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Angle sum of a polygon

$S = (n-2) \times 180°$

Equation of a circle

$(x-h)^2 + (y-k)^2 = r^2$

Trigonometric ratios and identities

$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

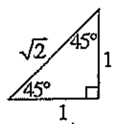
$\sec\theta = \dfrac{1}{\cos\theta}$

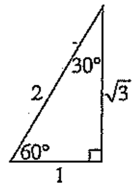
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$

Exact ratios





Sine rule

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos C$

Area of a triangle

$\text{Area} = \dfrac{1}{2}ab\sin C$

Distance between two points

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Perpendicular distance of a point from a line

$d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Slope (gradient) of a line

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point-gradient form of the equation of a line

$y - y_1 = m(x - x_1)$

***n*th term of an arithmetic series**

$T_n = a + (n-1)d$

Sum to *n* terms of an arithmetic series

$S_n = \dfrac{n}{2}[2a + (n-1)d]$ or $S_n = \dfrac{n}{2}(a + l)$

***n*th term of a geometric series**

$T_n = ar^{n-1}$

Sum to *n* terms of a geometric series

$S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$

Limiting sum of a geometric series

$S = \dfrac{a}{1 - r}$

Compound interest

$A_n = P\left(1 + \dfrac{r}{100}\right)^n$

Differentiation from first principles

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivatives

If $y=x^n$, then $\frac{dy}{dx}=nx^{n-1}$

If $y=uv$, then $\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$

If $y=\frac{u}{v}$, then $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

If $y=F(u)$, then $\frac{dy}{dx}=F'(u)\frac{du}{dx}$

If $y=e^{f(x)}$, then $\frac{dy}{dx}=f'(x)e^{f(x)}$

If $y=\log_e f(x)=\ln f(x)$, then $\frac{dy}{dx}=\frac{f'(x)}{f(x)}$

If $y=\sin f(x)$, then $\frac{dy}{dx}=f'(x)\cos f(x)$

If $y=\cos f(x)$, then $\frac{dy}{dx}=-f'(x)\sin f(x)$

If $y=\tan f(x)$, then $\frac{dy}{dx}=f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha+\beta=-\frac{b}{a}$ $\qquad$ $\alpha\beta=\frac{c}{a}$

Equation of a parabola

$(x-h)^2=\pm 4a(y-k)$

Integrals

$$\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+C$$

$$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$$

$$\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+C$$

$$\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$$

$$\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$$

$$\int\sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)dx\approx\frac{b-a}{2}\left[f(a)+f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

Logarithms – change of base

$$\log_a x=\frac{\log_b x}{\log_b a}$$

Angle measure

$180^\circ=\pi$ radians

Length of an arc

$l=r\theta$

Area of a sector

$\text{Area}=\frac{1}{2}r^2\theta$

Angle sum identities

$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$

$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$

$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

***t* formulae**

If $t=\tan\frac{\theta}{2}$, then

$$\sin\theta=\frac{2t}{1+t^2}$$

$$\cos\theta=\frac{1-t^2}{1+t^2}$$

$$\tan\theta=\frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta=a,\quad \theta=n\pi+(-1)^n\sin^{-1}a$

$\cos\theta=a,\quad \theta=2n\pi\pm\cos^{-1}a$

$\tan\theta=a,\quad \theta=n\pi+\tan^{-1}a$

Division of an interval in a given ratio

$$\left(\frac{mx_2+nx_1}{m+n},\ \frac{my_2+ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2=4ay$,

$$x=2at,\quad y=at^2$$

At $(2at,\ at^2)$,

tangent: $y=tx-at^2$

normal: $x+ty=at^3+2at$

At $(x_1,\ y_1)$,

tangent: $xx_1=2a(y+y_1)$

normal: $y-y_1=-\frac{2a}{x_1}(x-x_1)$

Chord of contact from $(x_0,\ y_0)$: $xx_0=2a(y+y_0)$

Acceleration

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$x=b+a\cos(nt+\alpha)$

$\ddot{x}=-n^2(x-b)$

Further integrals

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\frac{x}{a}+C$$

$$\int\frac{1}{a^2+x^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

Sum and product of roots of a cubic equation

$\alpha+\beta+\gamma=-\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$

$\alpha\beta\gamma=-\frac{d}{a}$

Estimation of roots of a polynomial equation

Newton's method

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

2 UNIT MARCH 2017

SECTION I

① D

②. B

③. A

④ $a+5d+a+11d=250$

$2a+16d=250$

$T_9=a+8d$

$=125$ B

⑤ $2000=1000(1+r)^{18}$

$2=(1+r)^{18}$

$r=2^{\frac{1}{18}}-1$

$r=3.9\%$ A

SECTION II

⑥ a) $T_n=3n-2$

i. $348=3n-2$

$\frac{350}{3}=n$

no

ii. $T_1=3\times1-2$

$=1$

$T_2=3\times2-2$

$=4$

$T_{100}=3\times100-2$

$=298$

$S_{100}=\frac{100}{2}(1+298)$

$=14\,950$

b) $S_n=2n+n^2$

i. $S_1=2\times1+1^2$

$=3$

$\therefore T_1=3$

and $S_2=8$

$\therefore T_2=8-3=5$

ii) $T_n=S_n-S_{n-1}$

$=2n+n^2-[2(n-1)+(n-1)^2]$

$=2n+\cancel{n^2}-\cancel{2n}+2-\cancel{n^2}+\cancel{2n}-1$

$=2n+1$

OR

$T_n=a+(n-1)d$

$=3+(n-1)\times2$

$\therefore T_n=2n+1$

iii. $n^2+2n>80$

$n^2+2n-80>0$

$(n+10)(n-8)>0$

$-10 \quad 8$

$n>8$

$\therefore$ 9 terms

c) i. $\sum_{n=1}^{5}(n^2-n)=0+2+6+12+20$

$=40$

ii. $\sum_{n=100}^{200}(80+n)=180+181+\dots+280$

$=\frac{101}{2}(180+280)$

$=23\,230$

d) $1 + (x-1) + (x-1)^2 + \dots$

i. limiting sum $-1 < r < 1$

$-1 < x-1 < 1$

$\underline{\underline{0 < x < 2}}$

ii. $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-(x-1)}$

$= \frac{1}{2-x}$

$= \frac{1}{2-\frac{3}{4}}$

$= \frac{4}{5}$

e) $a-2 = b-a$ & $\frac{b}{a} = \frac{9}{b}$

$2a = b+2$ $9a = b^2$

$a = \frac{b+2}{2}$ $a = \frac{b^2}{9}$

$\frac{b+2}{2} = \frac{b^2}{9}$

$9b + 18 = 2b^2$

$2b^2 - 9b - 18 = 0$

$(b-6)(2b+3) = 0$

$\therefore \underline{\underline{b = 6, a = 4}}$ or

$\underline{\underline{b = -\frac{3}{2}, a = \frac{1}{4}}}$

⑦ a) $ar^2 = 21$ —①

$ar^6 = 336$ —②

② ÷ ①

$r^4 = 16$

$\underline{\underline{r = \pm 2}}$

b) $1 + 3 + 5 + 7 + \dots + 101$

$a = 1$

$d = 2$

$101 = 1 + 2(n-1)$

$100 = 2n - 2$

$n = 51$

$T_n = 1 + 2(n-1)$

$= 2n - 1$

$\therefore \underline{\underline{\sum_{n=1}^{51} 2n-1}}$

c) $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$

at $x = 3$, $y = 6$

$m_{tangent} = 2 \times 3 - 1$

$= 5$

$y - 6 = 5(x-3)$

$y - 6 = 5x - 15$

$\underline{\underline{5x - y - 9 = 0}}$

d) i. $u = 3x+2$ $v = 2x-1$

$u' = 3$ $v' = 2$

$\frac{dy}{dx} = \frac{3(2x-1) - 2(3x+2)}{(2x-1)^2}$

$= \frac{-7}{(2x-1)^2}$

ii. $y = x^2(3x+2)^5$

$u = x^2 \qquad v = (3x+2)^5$

$u' = 2x \qquad v' = 15(3x+2)^4$

$\frac{dy}{dx} = 2x(3x+2)^5 + 15x^2(3x+2)^4$

$= x(3x+2)^4[2(3x+2)+15x]$

$= x(3x+2)^4(21x+4)$

(not necessary to factorise)

e) $f(x) = 2x - x^{\frac{1}{2}}$

i. $f'(x) = 2 - \frac{1}{2}x^{-\frac{1}{2}}$

$= 2 - \frac{1}{2\sqrt{x}}$

ii. $f''(x) = \frac{1}{4}x^{-\frac{3}{2}}$

$= \frac{1}{4\sqrt{x^3}}$

$f''(4) = \frac{1}{4\sqrt{4^3}}$

$= \frac{1}{32}$

iii. Stat when $f'(x) = 0$

$0 = 2 - \frac{1}{2\sqrt{x}}$

$2 = \frac{1}{2\sqrt{x}}$

$\sqrt{x} = \frac{1}{4}$

$\therefore x = \frac{1}{16}$

(8) a) $y = x^3 - x^2$

$\frac{dy}{dx} = 3x^2 - 2x$

$\frac{d^2y}{dx^2} = 6x - 2$

i. Stat pts: $\frac{dy}{dx} = 0$

$3x^2 - 2x = 0$

$x(3x-2) = 0$

when $x = 0,\ y = 0 \qquad x = \frac{2}{3},\ y = -\frac{4}{27}$

$\frac{d^2y}{dx^2} < 0 \qquad \frac{d^2y}{dx^2} > 0$

$\therefore$ max $(0,0) \qquad \therefore$ min $\left(\frac{2}{3}, -\frac{4}{27}\right)$

ii. Inflexion: $\frac{d^2y}{dx^2} = 0$ & concavity changes

$0 = 6x - 2$

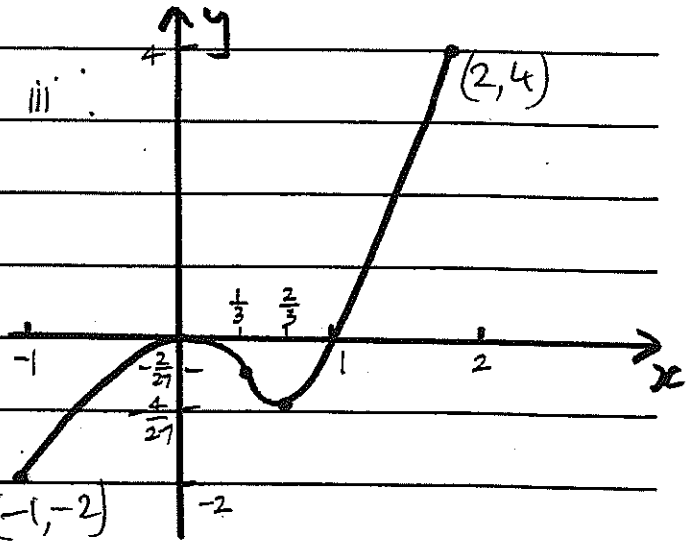
$x = \frac{1}{3},\ y = -\frac{2}{27}$

Verify that concavity changes

x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$\frac{d^2y}{dx^2}$	$-$	0	$+$

$\therefore$ since concavity changes $\left(\frac{1}{3}, -\frac{2}{27}\right)$ is inflexion point.

iii.

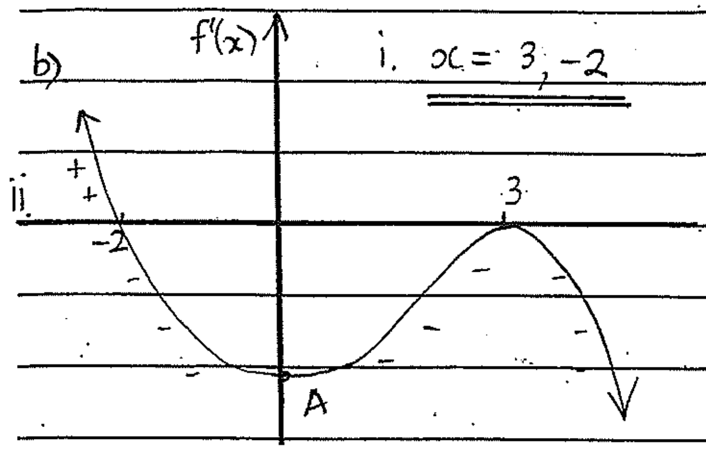


iv. 4

b)

i. $x = 3, -2$

ii.

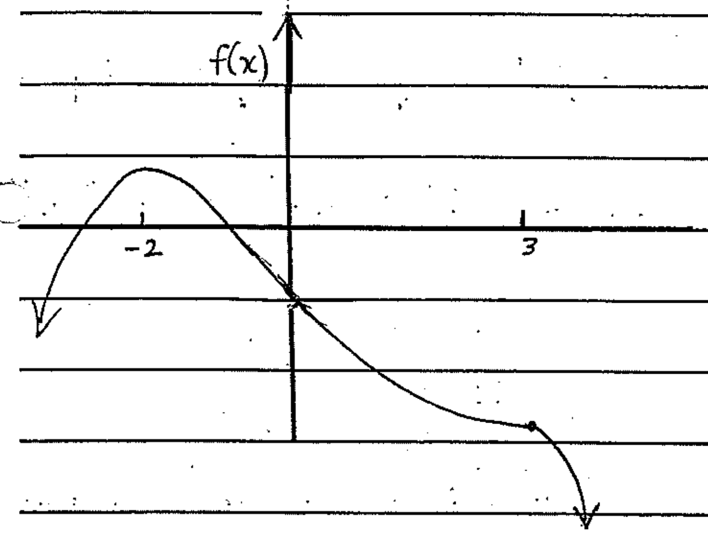


$= 200\left[1.006^1 + 1.006^2 + \dots + 1.006^{480}\right]$

$a = 1.006$
$r = 1.006$
$n = 480$

$= 200 \times 1.006 \dfrac{\left[1.006^{480} - 1\right]}{1.006 - 1}$

$= \$558,720$



$x = -2$ is max tp

$x = 3$ is horizontal pt of inflexion

iii. $x > -2,\ x \neq 3$

iv. inflexion

iii. $1000000 = P \times 1.006 \dfrac{\left[1.006^{480} - 1\right]}{1.006 - 1}$

$P = \dfrac{1000\,000 \times 0.006}{1.006(1.006^{480} - 1)}$

$\therefore P = \$358$

b) 20 cm

x $\quad 20 - x$

i. $\dfrac{x}{4}$ $\quad \dfrac{20-x}{4}$

$A = \left(\dfrac{x}{4}\right)^2 + \left(\dfrac{20-x}{4}\right)^2$

$= \dfrac{x^2}{16} + \dfrac{400 - 40x + x^2}{16}$

$= \dfrac{2x^2 - 40x + 400}{16}$

$\therefore A = \dfrac{x^2 - 20x + 200}{8}$ as required.

ii. $\dfrac{dA}{dx} = \dfrac{x}{4} - \dfrac{20}{8}$

$\dfrac{d^2A}{dx^2} = \dfrac{1}{4}$

(9) a) i. $A_1 = 200(1 + 0.006)^{480}$

$= \$3532$

ii. $A_2 = 200(1 + 0.006)^{479}$

$\vdots$

$A_{480} = 200(1 + 0.006)^1$

Total $= 200(1.006)^1 + 200(1.006)^2 + \dots + 200(1.006)^{480}$

min A when $\frac{dA}{dx} = 0$

$0 = \frac{x}{4} - \frac{20}{8}$

$\frac{x}{4} = \frac{20}{8}$

$x = 10$

When $x = 10$, $\frac{d^2A}{dx^2} > 0 \therefore \min$

$A = \left(\frac{10}{4}\right)^2 + \left(\frac{10}{4}\right)^2$

$\therefore A = \underline{\underline{12.5 \text{ cm}^2}}$

c) 10% p.a $= 2.5\%$ p quarter

i. $A_1 = 30\,000 \times 1.025 - R$

$= 30000\,(1.025) - R$

$A_2 = A_1\,(1.025) - R$

$= [30\,000\,(1.025) - R] \times 1.025 - R$

$= 30\,000\,(1.025)^2 - 1.025R - R$

$= 30\,000\,(1.025)^2 - R(1 + 1.025)$

ii. $A_{20} = 30\,000\,(1.025)^{20} - R\,(\underbrace{1 + 1.025 + \ldots + 1.025^{19}})$

but $A_{20} = 0$

$a = 1$
$r = 1.025$
$n = 20$

So $R = \dfrac{30\,000\,(1.025)^{20}}{\dfrac{1\,(1.025^{20} - 1)}{1.025 - 1}}$

$= \underline{\underline{\$\,1924}}$

iii. Repayments $= 20 \times 1924$

$= 38\,480$

$38\,480 - 30\,000 = 8\,480$

$\therefore$ S I for 1 year $= \frac{8480}{5}$

$= 1696$

$\therefore$ rate of S I $= 1696 / 30000$

$= 0.0565$

$\therefore r = 5.65\%$ p.a.