

SYDNEY TECHNICAL HIGH SCHOOL



Mathematics-Extension 1

2013

H.S.C ASSESSMENT TASK 1

Name

Teacher

General Instructions

- Working Time - 70 minutes.
- **Write using a blue or black pen.**
- Board Approved calculators may be used.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (53)

Section 1

5 marks

- Attempt Questions 1-5.
- Allow about 10 minutes for this section.

Section 1

48 marks

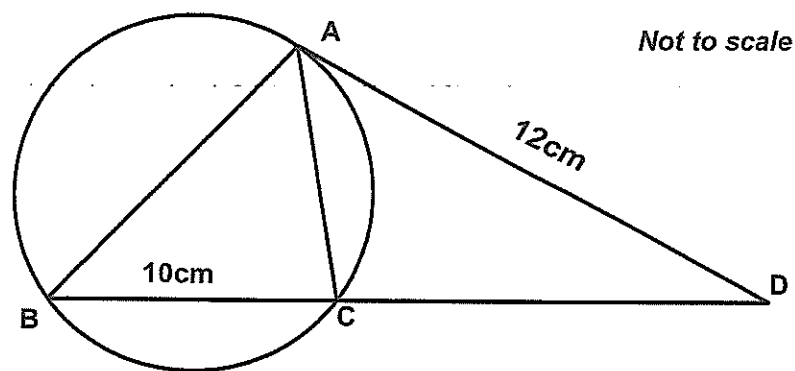
- Attempt Questions 6-11.
- Allow about 60 minutes for this section.

Question 1

A crystal measured 12.0 cm in length at the beginning of a chemistry experiment. Each day it increased in length by 3%.

The length of the crystal after 14 days growth is closest to

- A. 12.4 cm B. 16.7 cm C. 17.6 cm D. 18.2 cm

Question 2

ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced to D where $BC = 10\text{cm}$ and $AD = 12\text{cm}$. What is the length of CD?

- A. 6 cm B. 7 cm C. 8 cm D. 9 cm

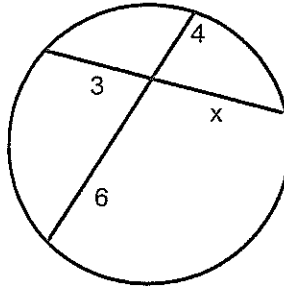
Question 3

The equation $2x^3 + x^2 - 13x + 6 = 0$ has roots $\alpha, \frac{1}{\alpha}$ and β . What is the value of β ?

- A. 3 B. 2 C. -3 D. -6

Question 4

Given the circle at the right with two intersecting chords.
Find the length represented as x .



A. 2

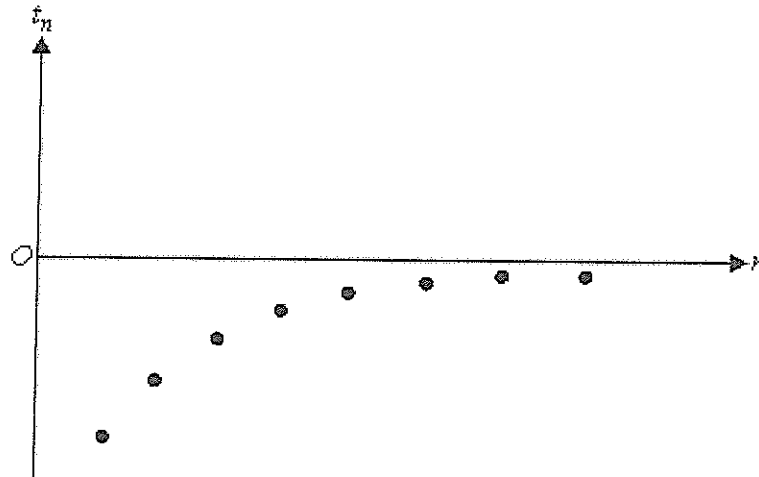
B. 6

C. 8

D. 10

Question 5

The graph below shows consecutive terms of a sequence.
The sequence could be



- A. Geometric with common ratio r , where $r < 0$
- B. Geometric with common ratio r , where $0 < r < 1$
- C. Geometric with common ratio r , where $r > 1$
- D. Arithmetic with common difference d , where $d < 0$

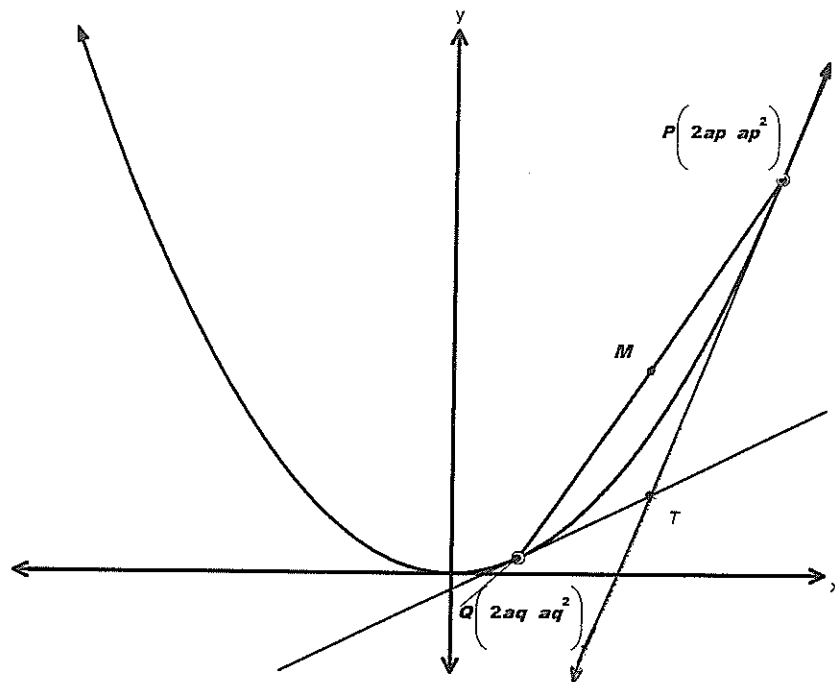
End of Multiple Choice questions

Question 6 (8 Marks)

Use a Separate Sheet of paper

- a) Find the equation of the normal to the parabola $x^2 = 8y$ at the point $(4p, 2p^2)$. 3

- b) The diagram below shows the tangents drawn at the point $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$. The tangents at P and Q intersect at T .



You may assume that the equation of the tangent at P is $y = px - ap^2$ and that the point T has coordinates $T[a(p + q), apq]$.

- (i) Suppose that point T lies on the line $y = a$, show that $pq = 1$ 1
- (ii) Find the Cartesian equation of the locus of the midpoint, M of the chord PQ . 3
- (iii) State any restrictions on the x -coordinates of the locus of M . 1

End of Question 6

Question 7 (8 Marks)

Use a Separate Sheet of paper

- a) i) Use the method of mathematical induction to prove for $n \geq 2$ 3

$$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \dots \times \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

- ii) Hence evaluate $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \dots \times \frac{9999}{10000}$ 1

- b) Harrison has started an exercise program to lose weight. When he started the program he weighed 105kg. In the first month he lost 5 kg, in the second he lost 4 kg and in the third month he lost 3.2 kg. If this weight loss trend continues

- i) how much will Harrison lose in the fourth month? 1

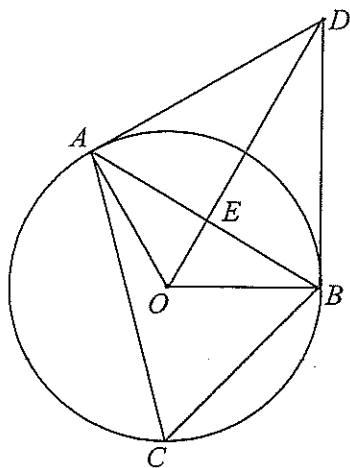
- ii) what will be his ultimate weight? 3

End of Question 7

Question 8 (8 Marks)

Use a Separate Sheet of paper

- a) The diagram shows points A , B and C on a circle centre O . Tangents are drawn from A and B which meet at D . O is joined to D and the interval OD intersects AB at E .

Not to
Scale

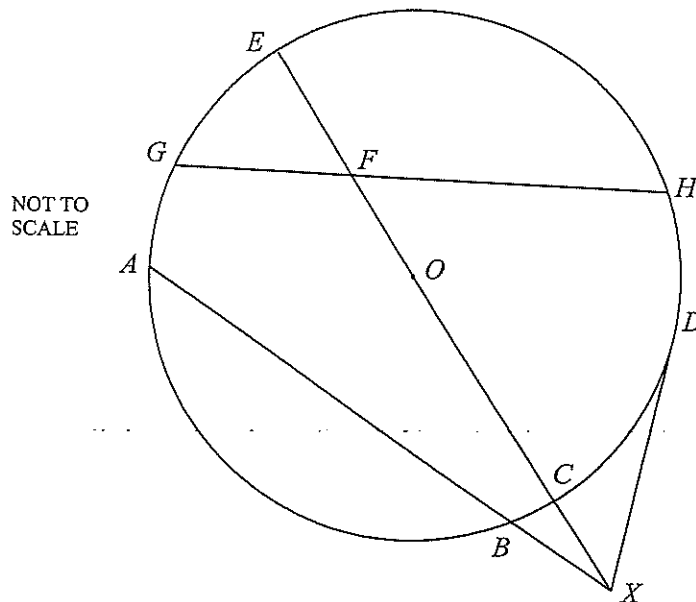
- | | |
|--|---|
| i) Prove that $\angle AOB = 2 \times \angle DAB$. | 2 |
| ii) Prove that AOBD is a cyclic quadrilateral | 1 |
| iii) Prove that E is the midpoint of AB . | 2 |
- b) The polynomial $p(x) = x^3 + ax + b$ has $(x - 5)$ as one of its factors and has a remainder of -60 when divided by $(x + 5)$. Find the values of a and b .
- 3

End of Question 8

Question 9 (8 Marks)

Use a Separate Sheet of paper

- a) The circle with center O has radius 6 cm. From an external point X , a tangent is drawn with a point of contact D . From X the secants XA and XE are also drawn.



- i) If $DX = 8$ cm calculate the distance CX . 2
- ii) If F is the bisector of EO and $GF = 4.5$ cm, calculate the distance GH . 2
- b) In a geometric Series, the 3rd term is -8 and the 6th term is 216. Find the 1st term and the common ratio. 2
- c) The chord of contact of the tangents to the parabola $x^2 = 4ay$ from an external point $A(x_1, y_1)$ passes through the point $B(0, 2a)$. Find the equation of the locus of the midpoint of AB . 2

End of Question 9

Question 10 (8 Marks)

Use a Separate Sheet of paper

- a) Find the cartesian equation of the curve represented by the following parametric equations;

2

$$x = 3t - 4$$

$$y = 2t^2 - t$$

- b) Stephanie borrows \$50 000 at the beginning of 2013 from his local Building Society. The loan is to be repaid in equal monthly repayments of \$900, with interest charged at 7.2% p.a. at the end of each month, just before repayment.

Let A_n be the amount owing after the n th repayment.

- i) Find an expression for A_1 and A_2

2

- ii) **Show that**

$$A_n = 50000(1.006)^n - 900(1 + 1.006 + 1.006^2 + \dots + 1.006^{n-1}).$$

1

- iii) After how many months will Stephanie have halved her loan?

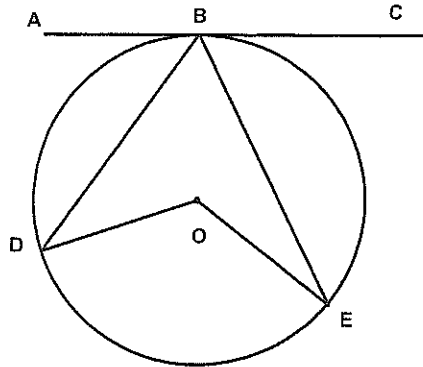
3**End of Question 10**

Question 11 (8 Marks)

Use a Separate Sheet of paper

- a) In the diagram below, O is the centre of the circle, AC is a tangent at B and D and E are points on the circumference. If $\angle ABD = 80^\circ$ and $\angle DBE = 40^\circ$, find the size of $\angle BEO$, giving reasons.

3



- b) John wants to save for a holiday in 10 years' time. The interest rate is fixed at 9.08% p.a. compounding yearly for the first 5 years and will change to 10% p.a. compounding yearly for the next 5 years. How much should John invest now so that he has \$5000 in 10 years from now? Give your answer to the nearest dollar.

3

- c) The height of a tree was 10 metres and it increased by 2m during the next year. If in each succeeding year the growth is $\frac{2}{3}$ of that in the previous year, find the limiting height?

2

End of Examination

Extension 1 Mathematics 2013

H.S.C. Assessment Task 1

Multiple Choice

- 1 D
- 2 C
- 3 C
- 4 C
- 5 B

Question 6

a) $x^2 = 8y$
 $y = \frac{x^2}{8}$
 $y' = \frac{x}{4}$

When $x = 4p$, $m_1 = p$, $m_2 = -\frac{1}{p}$
 Equation of the normal
 $y - 3p^2 = -\frac{1}{p}(x - 4p)$
 $4p - 3p^3 = -x + 4p$
 $x + 4p - 3p^3 - 4p = 0$

b) i) $apq = a$
 $\therefore pq = 1$

ii) Let $x = a(p+q)$
 $y = \frac{a}{x}(p^2 + q^2)$
 $x = a(p+q)$
 $\frac{x}{a} = p+q$

$$y = \frac{a}{x}(p^2 + q^2)$$

$$xy = a(p^2 + q^2)$$

$$\frac{xy}{a} = p^2 + q^2$$

$$\frac{xy}{a} = (p+q)^2 - 2pq$$

$$\frac{xy}{a} = \left(\frac{x}{a}\right)^2 - 2 \times 1$$

$$\frac{xy}{a} = \frac{x^2}{a^2} - 2a^2$$

$$xay = x^2 - 2a^2$$

$$xay + 2a^2 = x^2$$

$$2a(y+a) = x^2$$

or

$$y+a = \frac{x^2}{2a}$$

$$y = \frac{x^2}{2a} - a$$

iv) M and T share the same

x-co-ordinate, the extreme case will be when P and Q are NOT distinct points i.e.

$$P=Q$$

When this happens T (and M) will be $(2a, a)$. Since T must lie outside the parabola then $|x| > 2a$

or

$$\frac{1}{2a}x^2 - a > a$$

$$\frac{1}{2a}x^2 > 2a$$

$$x^2 > 4a$$

$$x < -2a \quad x > 2a$$

Question 7

a) $\left(1 - \frac{1}{2^n}\right) \times \left(1 - \frac{1}{3^n}\right) \times \left(1 - \frac{1}{4^n}\right) \dots \times \left(1 - \frac{1}{n^n}\right) = \frac{n!}{n!}$

Step 1 when $n=2$

$$L.H.S = 1 - \frac{1}{2^2} = \frac{3}{4}$$

$$R.H.S = \frac{2!}{2 \times 2} = \frac{2}{4}$$

$$\therefore L.H.S = R.H.S$$

Step 2

Assume true for $n=k$

$$\left(1 - \frac{1}{2^k}\right) \times \left(1 - \frac{1}{3^k}\right) \times \left(1 - \frac{1}{4^k}\right) \dots \times \left(1 - \frac{1}{k^k}\right)$$

$$= \frac{k+1}{2k}$$

Step 3.

Prove the formula is true for $n=k+1$ i.e.

$$\left(1 - \frac{1}{2^{k+1}}\right) \times \left(1 - \frac{1}{3^{k+1}}\right) \times \left(1 - \frac{1}{4^{k+1}}\right) \dots \times \left(1 - \frac{1}{(k+1)^{k+1}}\right)$$

$$= \frac{k+2}{2(k+1)}$$

$$L.H.S = \frac{k+1}{2k} \times \left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k} \times \frac{(k+1)^2 - 1}{(k+1)^2}$$

$$= \frac{(k+1)^2 - 1}{2k(k+1)}$$

$$= \frac{k^2 + 2k}{2k(k+1)} = \frac{k+2}{2(k+1)}$$

as required

We know that the formula is true for $n=2$, so it must be true for $n=3$. If it is true for $n=3$, then it is true for $n=4$ and so on
 \therefore it is true for all integers $n \geq 2$

a) ii) When $n=100$

$$\frac{3 \times 8 \times 15 \times \dots \times 9999}{4 \times 7 \times 16 \times \dots \times 10000}$$

$$\frac{9999}{10000} = 1 - \frac{1}{n^2}$$

$$n = 100$$

$$\therefore \frac{100+1}{2 \times 100} = \frac{101}{200}$$

b.2)

$$5, 4, 3.2$$

$$r = \frac{3.2}{4} = \frac{4}{3}$$

$$3.2 \times \frac{4}{3} = 2.56 \text{ kg}$$

$$\text{ii)} \quad a = \frac{5}{4}$$

$$r = \frac{4}{3}$$

$$S_{\text{circ}} = \frac{a}{1-r}$$

$$= \frac{5}{1-\frac{4}{3}}$$

$$= 25 \text{ kg}$$

$$\text{Ultimate weight} = 105 - 25$$

$$= 80 \text{ kg}$$

Question 8

$$\text{ai)} \quad \angle DAB = \angle ACB$$

(angle between tangent and chord is equal to the angle in the alternate segment)

$$\angle AOB = 2 \times \angle ACB$$

(angle at the centre is twice the angle at the circumference on the same arc)

$$\angle AOB = 2 \times \angle DAB \quad (\angle DAB = \angle ACB)$$

$$\text{aii)} \quad \angle DAO = \angle DBO = 90^\circ$$

(tangent is perpendicular to the radius)

$$\angle DAO + \angle DBO = 180^\circ$$

(sum of two right angles)

\therefore opposite angles of AOB

are supplementary

\therefore AOB is a cyclic quadrilateral

$$\text{aiii)} \quad AO = BO \text{ (equal radii)}$$

$$AO = BO \text{ (tangents from an external point are equal in length)}$$

\therefore AOB is a kite

OD bisects AB (symmetry of a kite)

\therefore E is the midpoint of AB.

$$\text{b)} \quad p(x) = x^2 + ax + b$$

$$p(5) = 0$$

$$5^2 + 5a + b = 0$$

$$125 + 5a + b = 0$$

$$p(-5) = -60$$

$$(-5)^2 - 5a + b = -60$$

$$-125 - 5a + b = -60$$

$$125 + 5a + b = 0$$

$$-125 - 5a + b = -60$$

$$2b = -60$$

$$b = -30$$

$$5a + b + 125 = 0$$

$$5a - 30 + 125 = 0$$

$$5a = -95$$

$$a = -19$$

$$b = -30$$

Question 9

ai)

$$DX^2 = CX \times EX$$

$$8^2 = CX(CX + EX)$$

$$8^2 = CX(CX + 12)$$

$$64 = CX^2 + 12CX$$

$$0 = (CX)^2 + 12CX - 64$$

$$0 = (CX + 16)(CX - 4)$$

$$CX = -16 \quad CX = 4 \text{ (only solution)}$$

aii)

$$EF = FO = 3 \text{ cm}$$

$$GF = 4.5 \text{ cm}$$

$$GF \times FH = EF \times FE$$

$$4.5 \times FH = 3 \times 9$$

$$4.5 \times FH = 27$$

$$FH = 6$$

$$\text{Distance of FH} = 6 + 4.5$$

$$= 10.5 \text{ cm}$$

b.i)

$$T_3 = -8$$

$$T_6 = 216$$

$$ar^2 = -8$$

$$ar^5 = 216$$

$$r^3 = -27$$

$$r = -3$$

$$a = -\frac{8}{9}$$

c) Equation of the chord of

$$\text{contact } x, y = 2a(y+y_1)$$

$$B(0, 2a)$$

$$0 \times x = 2a(y + 2a)$$

$$0 = 2ay + 4a^2$$

$$-4a^2 = 2ay$$

$$y = -2a$$

Locus of the midpoint AB

$$A(x, -2a) \quad B(0, 2a)$$

$$y = \frac{-2a + 2a}{2}$$

$$x = \frac{0 + x}{2} = \frac{x}{2}$$

$\therefore y = 0$ is the equation of the locus of the midpoint AB.

Question 10

$$x = 3t - 4$$

$$x + 4 = 3t$$

$$t = \frac{x + 4}{3}$$

$$y = 2t^2 - t$$

$$y = 2\left(\frac{x+4}{3}\right)^2 - \left(\frac{x+4}{3}\right)$$

$$y = 2\frac{(x^2 + 8x + 16)}{9} - \frac{x+4}{3}$$

$$y = \frac{2x^2 + 16x + 32 - 3x - 12}{9}$$

$$y = \frac{2x^2 - 3x + 20}{9}$$

$$b) \quad A_1 = 50\,000 (1 + 0.006) - 900 \\ = 50\,000 (1.006) - 900$$

$$A_2 = A_1 (1.006) - 900 \\ = [50\,000 (1.006 - 900)] 1.006 - 900 \\ = 50\,000 (1.006)^2 - 900 (1.006) - 900 \\ = 50\,000 (1.006)^2 - 900 (1 + 1.006)$$

$$A_3 = 50\,000 (1.006)^3 - 900 (1 + 1.006 + 1.006^2)$$

$$\therefore A_n = 50\,000 (1.006)^n - 900 (1 + 1.006 + 1.006^2 + \dots + 1.006^{n-1})$$

$$b) \text{ iv) } \text{ let } n = 25\,000$$

$$25\,000 = 50\,000 (1.006)^n - 900 (1 + 1.006 + 1.006^2 + \dots + 1.006^{n-1})$$

$$25\,000 = 50\,000 (1.006)^n - 900 \left(\frac{1.006^n - 1}{0.006} \right)$$

$$25\,000 = 50\,000 (1.006)^n - 150\,000 (1.006^n - 1)$$

$$1 = 2 (1.006)^n - 6 (1.006^n - 1)$$

$$1 = 2 (1.006)^n - 6 (1.006^n) + 6$$

$$-5 = -4 (1.006)^n$$

$$\frac{5}{4} = (1.006)^n$$

$$\ln \left(\frac{5}{4} \right) = \ln (1.006)^n$$

$$n = \frac{\ln \left(\frac{5}{4} \right)}{\ln (1.006)}$$

$$n = 37.3 \text{ months}$$

* allow for guess and check method.

Question 11

$$a) \quad \angle BED = \angle AED = 80^\circ$$

Construct line DE
(angle between tangent and chord is equal to the angle in alternate segment)

$$\angle DOE \times 2 = \angle DBE = 80^\circ$$

(angle at the centre is twice the angle at the circumference of the same arc)

DO = OE (equal radii)

$$\angle OED = \angle ODE = 50^\circ$$

(angles opposite equal sides are equal)

$$\angle BED = \angle BED = \angle OED$$

$$= 80^\circ - 50^\circ$$

$$= 30^\circ$$

$$11b) \quad 5000 = x (1.0908)^5 \times (1.1)^5$$

$$x = \frac{5000}{(1.0908)^5 (1.1)^5}$$

$$x = \$2010.39$$

$$x = \$2010 \text{ nearest dollar}$$

$$11c) \quad 10 + 2 + \frac{4}{3} + \frac{8}{3} + \dots$$

$$= 10 + S_\infty$$

$$= 10 + \frac{a}{1-r}$$

$$= 10 + \frac{2}{1-\frac{2}{3}}$$

$$= 10 + 6$$

$$= 16$$

