3u Mathematics

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HSC 2001

Geometry

Division of intervals to specific ratio:
k, l correspond to ratio towards points 1 and 2 respectively

$$\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$$

Perpendicular distance:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Angle between 2 lines:

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Line between 2 points:

$$\Delta Yx - \Delta Xy = s$$

(s found through substitution of a point)

Gradient-point form:

$$y - y_1 = m(x - x_1)$$

Sequences & Series

Arithmetic Progressions	Geometric Progressions
$T_n = a + (n-1)d$	$T_n = ar^{n-1}$
$S_n = \frac{n}{2}(a+l) \quad (l = T_n)$	$S_n = \frac{a(1-r^n)}{1-r}$
Mean: $\frac{a+b}{2}$	$\sqrt{ab}$
	$n \to \infty, S_n = \frac{a}{1-r}, \lvert r\rvert \le 1$

Equations & Inequalities

- useful to put all on LHS
- for $\frac{f(x)}{g(x)} > 0$, etc, state ”LHS has same sign as $y = f(x)g(x)$”, plot graph or solve
  Note that $g(x) = 0$ cannot be in solution; beware of =
- substitute to form known inequalities (like below)
- square both sides if necessary (make b.s. easier to square by not putting 2 surds on one side)
- for $\lvert$absolute values$\rvert$, use graphs (or $|a| = b, \therefore\ a = \pm b$)

Common inequalities:

$$(a-b)^2 \ge 0$$
$$a^2 + b^2 \ge 2ab$$

replace $a \to \sqrt{a}$, $b \to \sqrt{b}$ to get: $a + b \ge 2\sqrt{ab}$ (AM/GM inequality)

$$a^2 + b^2 + c^2 \ge ab + ac + bc$$

$$x + \frac{1}{x} \geq 2 \text{ (sub into AM/GM)}$$

Functions

A function $f(x)$ exists if there is only one solution for $f(a)$ where a is a value in the domain

$f^{-1}(x)$ exists if there is only one value in the domain of $f(x)$ for each value in its range, ie. for any k, there is only one possible x for $f(x) = k$

Simple operations on graphs:

$y = f\left(\frac{x}{a}\right)$ scales x axis $1 : a$ compared to $f(x)$
$y = f\left(\frac{x}{a}\right)$ scales x axis $1 : a$ compared to $f(x)$
y coordinate is known as *ordinate*
x coordinate is known as *abcissa*

Odd and even

odd: $f(-x) = -f(x)$
point symmetry about origin
even: $f(-x) = f(x)$
symmetrical about y axis

if E is an even function and O is an odd function then:

$$-E \to E$$
$$-O \to O$$
$$E + E \to E$$
$$O + O \to O$$
$$OE \to O$$
$$EE \to E$$
$$OO \to E$$

Trigonometry

Basic rules

Triangle ABC:

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

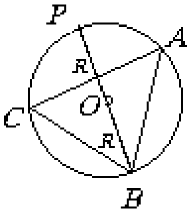
[note: ambiguous case]

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

area of $\Delta ABC = \frac{1}{2}ab\sin C$

$abc = 4AR$ (where R is radius of circumcircle)

[Proof: Construct diameter BOP



$\therefore\ \angle BOP = 90^\circ$ (angle in a semicircle)

$\angle BPC = \angle BAC$ (angles subtended by same arc, BC, are euqal)

$= \angle A$

$\therefore\ \sin\angle A = \dfrac{a}{2R}$

$= \dfrac{c}{\sin\angle C}$

$\sin\angle C = \dfrac{c}{2R}$

Area of $\Delta ABC = \dfrac{1}{2}\dfrac{abc}{2R}$

$abc = 4AR$]

Graphs and calculus

$y = \sin x$	$y = \cos x$	$y = \tan x$
$T = 2\pi$	$T = 2\pi$	$T = \pi$
$-1 \le y \le 1$	$-1 \le y \le 1$	$y \in R$

$$\frac{dy}{dx} = \cos x \quad \frac{dy}{dx} = -\sin x \quad \frac{dy}{dx} = \sec^2 x$$

derivatives come from small angle theory and definition of derivative

$y = \sin^{-1}x$	$y = \cos^{-1}x$	$y = \tan^{-1}x$
$-1 \le x \le 1$	$-1 \le x \le 1$	$x \in R$
$-\frac{\pi}{2} < y \le \frac{\pi}{2}$	$0 \le y \le \pi$	$-\frac{\pi}{2} < y \le \frac{\pi}{2}$

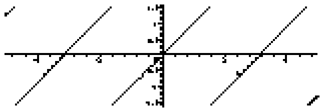
$$\frac{d}{dx}\sin^{-1}(ax+b) = \frac{a}{\sqrt{1-(ax+b)^2}} \quad \frac{d}{dx}\cos^{-1}(ax+b) = -\frac{a}{\sqrt{1-(ax+b)^2}} \quad \frac{d}{dx}\tan^{-1}(ax+b) = \frac{a}{1+(ax+b)^2}$$

$$\int\frac{dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\sin^{-1}\frac{bx}{a} + c \quad \int\frac{dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\cos^{-1}\frac{bx}{a} + c \quad \int\frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + c$$

Note vertical tangents for inv. sin & cos at x=-1,1 and the $\pm\frac{\pi}{4}$ gradient at x=0 for the 3 functions.

Other graphs

$y = \tan^{-1}\tan x$



$y = \tan\tan^{-1}x$

$y = x, x \in R$

$y = \sin\sin^{-1}x$

$y = \sin^{-1}\sin x$

$y = x, -1 \leq x \leq 1$

$y = \cos\cos^{-1}x$

$y = \cos^{-1}\cos x$

$y = x, -1 \leq x \leq 1$

Proof of derivative of inv. trig.:

$x = \sin y$

$\frac{dx}{dy} = \cos y$

$\frac{dy}{dx} = \frac{1}{\cos\sin^{-1}x}$

From Δ:

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Trig identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \csc^2 x$$

the following are proven through unit circle, equating distance formula with cos rule:

$\sin(a+b) = \sin a\cos b + \cos a\sin b$ $\quad \sin(a-b) = \sin a\cos b - \cos a\sin b$ $\quad \sin 2x = 2\sin x\cos x$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$ $\quad \cos(a-b) = \cos a\cos b + \sin a\sin b$

$$\cos 2x = \begin{array}{l} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{array}$$

$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$ $\quad \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$ $\quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

from above:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$
$$\cos 3x = 4\cos^3 x - 3\cos x$$

Solutions of trig equations

$$\sin\theta = k \quad \theta = n\pi + (-1)^n \sin^{-1}k$$
$$\cos\theta = k \quad \theta = 2n\pi \pm \cos^{-1}k$$
$$\tan\theta = k \quad \theta = n\pi + \tan^{-1}k$$
$$n \in Z$$

note that in [trig] equations, squaring both sides can add additional solutions, so check the solutions

dividing by the highest power of cos puts equations in terms of tan

Auxiliary angle

$$f(x) = a\sin x + b\cos x = R\sin(x+\theta)$$
$$a\sin x + b\cos x = R\sin x\cos\theta + R\cos x\sin\theta$$
$$R\cos\theta = a, \qquad R\sin\theta = b$$
$$R^2\left(\sin^2\theta + \cos^2\theta\right) = a^2 + b^2$$
$$\frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

– Therefore –

$$R = \sqrt{a^2+b^2}$$
$$\theta = \tan^{-1}\frac{b}{a} \text{ (depending on function)}$$
$$a\sin x + b\cos x = \sqrt{a^2+b^2}\,\sin\left(x + \tan^{-1}\frac{b}{a}\right)$$

t-formulae

$$\text{let } t = \tan\frac{\theta}{2}$$
$$\sin\theta = \frac{2t}{1+t^2} \quad \cos\theta = \frac{1-t^2}{1+t^2} \quad \tan\theta = \frac{2t}{1-t^2}$$
$$\frac{dt}{d\theta} = \frac{2}{1+t^2} \quad \text{(not required for 3u)}$$

NOTE: problematic for solutions with $\theta = \pi$ (tan $\frac{\pi}{2}$ undefined)

these equns identified by coefficent of $\cos x$ is the opposite of te constant term

terms in t^2 cancel out (no quadratic equn)

extension: **products to sums**, **sums to products**

derive when needed. not needed often

Products to sums	Sums to products
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\sin S + \sin T = 2\sin\frac{1}{2}(S+T)\cos\frac{1}{2}(S-T)$
$2\cos A\sin B = \sin(A+B) - \sin(A-B)$	$\sin S - \sin T = 2\cos\frac{1}{2}(S+T)\sin\frac{1}{2}(S-T)$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	$\cos S + \cos T = 2\cos\frac{1}{2}(S+T)\cos\frac{1}{2}(S-T)$
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$	$\cos S - \cos T = -2\sin\frac{1}{2}(S+T)\sin\frac{1}{2}(S-T)$

Log Laws

$$\text{if } x = \log_a k,\ a^x = k \quad (k > 0)$$
$$k = e^{\ln k}$$
$$\log ab = \log a + \log b$$
$$\log_a b = \frac{\log b}{\log a}$$
$$\log a^b = b\log a$$
$$\log_a a = 1 \text{ (ie. } \ln e = 1)$$
$$\log 1 = 0$$

The Calculus

Volume of solid of revolution ($y = f(x)$ revolved around x axis)

$$V = \pi\int y^2 dx$$

Approximate definite integrals

$$\text{Trapezium rule: } A \doteqdot \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n]$$
$$\text{Simpson's rule: } A \doteqdot \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_3 + \ldots + 4y_{n-1} + y_n]$$

[n must be even for Simpson's rule]

Exponentials and Natural Logs

$\frac{d}{dx}k^x \propto k^x$ Therefore e is defined as k for which $\frac{d}{dx}e^x = e^x$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$
$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)}$$

Proof:

$y = \log_e x$

$e^y = x$

$\frac{dx}{dy} = e^y$

$\frac{dy}{dx} = \frac{1}{e^y}$

$= \frac{1}{x}$

Exponential growth and decay:
$f(x)$ required such that $\frac{dN}{dt} = kN$, N is growing/decaying at a constant rate, k

$$\frac{dN}{dt} = kN$$
$$= Ake^{kt}$$

$N = Ae^{kt}$, where A is the initial amount

growth	decay
$f(t) = e^t$	$f(t) = e^{-t}$

note that $f(x) \neq 0$

General

To find $\frac{dy}{dx}$ from a parametric representation:
ie. $x = f(t), y = g(t)$

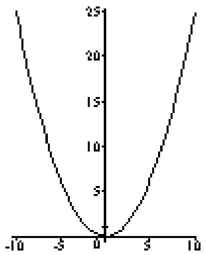
$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$= \frac{g'(t)}{f'(t)}$$

or put $y = f(g^{-1}(x))$

if $f(x)$ is odd, $\int_{-a}^{a} f(x)dx = 0$
if $f(x)$ is even, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Parabolas

$$x^2 = 4ay$$
$$\left[2at, at^2\right]$$



Focus $S(0, a)$
Directrix $y = -a$
Vertex $(0, 0)$
Gradient ($\frac{dy}{dx}$) at $P(2at, at^2) = t$
Tangent:

$$y = px - ap^2$$
$$xx_1 = 2a(y + y_1)$$

Normal:

$$x + py = 2ap + ap^3$$

Point of intersection of 2 tangents:

$(a(p+q), apq)$

Chord:

$y = \left(\frac{p+q}{2}\right)x - apq$

focal chords have $pq = -1$

Chord of contact from $T(x_0, y_0)$

$xx_0 = 2a(y + y_0)$

Latus rectum passes through focus, parallel to directrixm: $y = a$

$P(x)$ has real roots if $\Delta \geq 0$

$P(x)$ has one unique if $\Delta = 0$

$P(x)$ has imaginary roots if $\Delta \leq 0$

Δ, the *discriminant* $= b^2 - 4ac$, where $P(x) = ax^2 + bx + c$

Two polynomials are equivalent, or congruent, shown $P(x) \equiv Q(x)$

To determine unknowns,

- equate coefficients
- sub values (1,0,etc)

Limits

$\lim_{x \to a} f(x)$

if $f(a)$ is a finite number, $lim = f(a)$

if $f(a) = \infty$, $\lim \;\; = \infty$

if it's $\frac{0}{0}$, factorise

if it's $\frac{\infty}{\infty}$, divide through by highest power of x

Small angles

$$\lim_{x \to 0} \sin x = 0$$
$$\lim_{x \to 0} \cos x = 1$$
$$\lim_{x \to 0} \tan x = 0$$
$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ (check the coefficient of x)}$$
$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$
$$\lim_{x \to \infty} \frac{\cos x - 1}{x} = 0$$

Motion

basic ideas

average velocity $= \frac{\Delta x}{\Delta t}$

displacement is Δx, while the distance travelled is the sums of $|\Delta x|$ between each extremum

$$v = \dot{x} = \frac{dx}{dt} \qquad \ddot{x} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$
$$\Delta x = \int_{t_1}^{t_2} v\,dt \qquad \Delta v = \int_{t_1}^{t_2} \ddot{x}\,dt$$

simple harmonic motion

$$x = x_0 + a\sin(nt + \alpha)$$
$$\text{amplitude (maximum displacement)} = a, \qquad -a \leq x \leq a$$
$$\text{period, } T = \frac{2\pi}{n}$$

$$v = an\cos(nt + \alpha)$$
$$\ddot{x} = -an^2\sin(nt + \alpha)$$

for motion centred in the origin, as functions of x:

$$\ddot{x} = -n^2 x$$
$$v^2 = n^2\left(a^2 - x^2\right) \quad \text{(derive each time)}$$

(Note: $x = b\sin nt + c\cos nt$ can be used for ease if shm starts out of phase)

motion as a function of x

$$\text{if } v = f(x) = \frac{dx}{dt}, \text{ use } \frac{dt}{dx} = \frac{1}{v}$$
$$\ddot{x} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

proof:

$$\begin{aligned}\frac{d}{dx}\left(\frac{v^2}{2}\right) &= \frac{d}{dv}\left(\frac{v^2}{2}\right) \times \frac{dv}{dx}\\ &= v\frac{dv}{dx}\\ &= \frac{dx}{dt} \times \frac{dv}{dx}\\ &= \frac{dv}{dt}\\ &= \ddot{x}\end{aligned}$$

projectile motion

projectile is fired at $V\,\mathrm{m\,s^{-1}}$ at an angle of θ to the horizontal

let the starting position be the origin

Horizontal component	Vertical component
$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = C_1$	$\dot{y} = -gt + C_3$
at $t = 0,\ \dot{x} = V\cos\theta$	at $t = 0,\ \dot{y} = V\sin\theta$
$C_1 = V\cos\theta$	$C_3 = V\sin\theta$
$\therefore\ \dot{x} = V\cos\theta$	$\therefore\ \dot{y} = V\sin\theta - gt$
$x = Vt\cos\theta + C_2$	$y = Vt\sin\theta - \frac{1}{2}gt^2 + C_4$
at $t = 0,\ x = 0,\ C_2 = 0$	at $t = 0,\ y = 0,\ C_4 = 0$
$x = Vt\cos\theta$	$y = Vt\sin\theta - \frac{1}{2}gt^2$

$$t = \frac{x}{V\cos\theta}, \text{ sub into } y$$
$$\begin{aligned}y &= V\frac{x}{V\cos\theta}\sin\theta - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2\\ &= \frac{x\sin\theta}{\cos\theta} - \frac{gx^2}{2V^2\cos^2\theta}\\ &= x\tan\theta - \frac{gx^2}{2V^2}\left(1 + \tan^2\theta\right)\end{aligned}$$

thus the cartesian equation of the projectile motion is a quadratic in $x, \tan\theta, V$; linear in g, y

Radian measure

$$\text{if } \theta \text{ is in radians, then } \theta^\circ = \frac{180^\circ\theta}{\pi}$$

for a circle radius r, and an angle at the centre of the circle, θ:
length of arc: $r\theta$
area of sector: $\frac{1}{2}r^2\theta$
area of triangle: $\frac{1}{2}r^2\sin\theta$
$\therefore$ area of segment: $\frac{1}{2}r^2(\theta - \sin\theta)$