Name: ______________________________ Teacher: ______________

Sydney Technical High School
Year 12 Ext.2 Mathematics HSC Assessment Task 1 March 2004

Instructions:
Start each question on a new page.
Show all necessary working. Single column of work only.
Staple these questions to the front of your answers.
Full marks may not be awarded for careless* or incomplete work.
Indicated marks are a guide and may change slightly during the marking process.
* Be careful when writing "z" so that is distinguishable from "2".

Time allowed: ***70 mins***

Q1	Q2	Q3	TOTAL
/14	/17	/16	/47

Question 1

3 a) Given that a and b are real numbers, find a and b if

$$\frac{3+4i}{a+bi} = 1+i$$

8 b) If $z = -1+\sqrt{3}\,i$ and $w = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$

i) Find $|z|$
ii) Write z in mod-arg form
iii) Evaluate the following in <u>*simplest*</u> mod-arg form
α) zw
β) $\dfrac{z}{w}$
γ) w^7
iv) Show w and $\sqrt{w}$ on a number plane diagram and on it write the values of $\sqrt{w}$ in mod-arg form.

3 c) For a complex number z, $\text{Arg}(z+2) = \frac{1}{2}\text{Arg}(z)$.

i) Find, giving reasons, the value of $|z|$.

ii) Give an expression for $\text{Arg}(z-2)$ in terms of $\text{Arg}(z)$.

Question 2 (Begin a new page)

8 a) For the ellipse $4x^2 + 9y^2 = 36$

i) find the co-ordinates of the foci

ii) find the equations of the directrices

iii) Sketch the ellipse showing x & y intercepts, foci and directrices.

iv) Sketch the following ellipses, explaining the relationship to $4x^2 + 9y^2 = 36$ for each one.

α) $9x^2 + 4y^2 = 36$

β) $c^2x^2 + 9y^2 = 36$ where $c^2 > 4$

9 b) i) Differentiate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ implicitly to show that $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

ii) Hence prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

iii) The tangent at P cuts the directrix in the first quadrant at D. Find the co-ordinates of D.

iv) If S is the focus associated with the directrix in iii), prove that $\angle PSD = 90°$

Question 3 (Begin a new page)

4 a) Sketch the following loci

i) $|z-2| = 2,\ 0 \le \arg z \le \frac{\pi}{2}$

ii) $\arg(z+1) = {}^{\pi}/_{4}$, $Re(z) \le 2$

3 b) i) The locus of the point P (x, y) which represents the complex number z is given by the equation $Im(z) = |z - 2i|$. Find the Cartesian equation and sketch the locus of P.

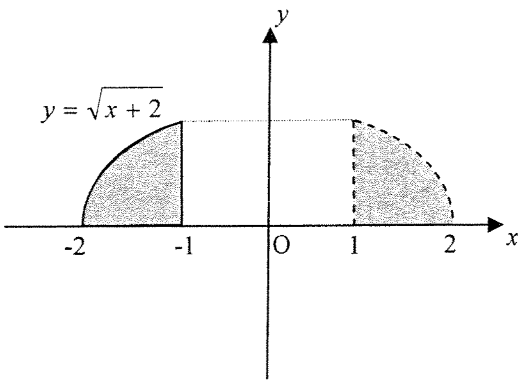
ii) Find the least value of $\arg z$ in part b (i)

5 c) i) Show on an Argand diagram the positions of the roots of $z^3 = -1$.

ii) Explain algebraically why the roots of $z^3 = -1$ are among the roots of $z^6 = 1$.

iii) By referring to the roots of $z^6 = 1$, find the roots of $z^4 + z^2 + 1 = 0$ in mod-arg form.

4 d)



The area under the curve $y = \sqrt{x+2}$ between $x = -2$ and $x = -1$ is rotated about the y axis to form a kind of "donut". Find the volume of the donut in terms of π.

End of Examination

SUGGESTED SOLUTIONS AND MARKING SCHEME
EXT. 2 EXAM MARCH 2004.

Q1 a) $\dfrac{3+4i}{a+bi} = 1+i$

$3+4i = (a+bi)(1+i)$

$= a+(a+b)i - b$

$\therefore \left.\begin{array}{l} a+b=4 \\ a-b=3 \end{array}\right\}$ (1)

$\therefore 2a = 7$

$\left.\begin{array}{l} a=\frac{7}{2} \\ \therefore b = \frac{1}{2} \end{array}\right\}$ (2)

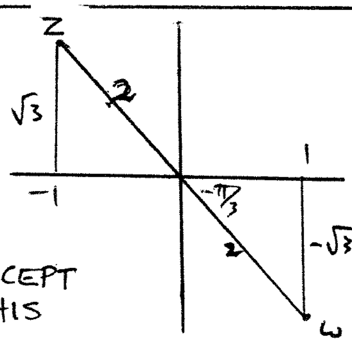
Q1 b) i) $|z| = 2$ (1)

ii) $z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ (1)

iii) α) $zw = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (1)

β) $\dfrac{z}{w} = 1(\cos\pi + i\sin\pi)$ ← ACCEPT THIS

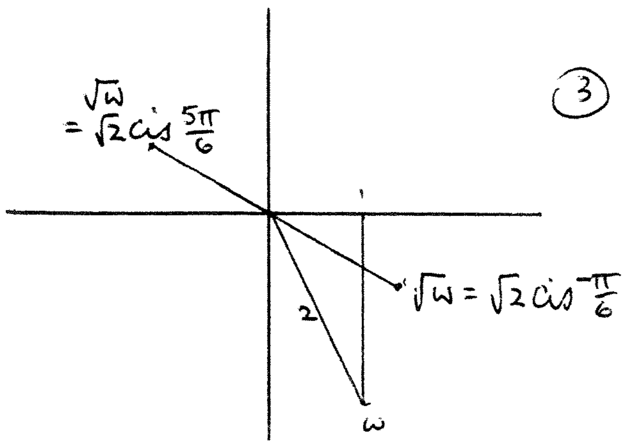
$= -1$ (1)



γ) $w^7 = 2^7\left(\cos\frac{-7\pi}{3} + i\sin\frac{-7\pi}{3}\right)$

$= 128\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$ (1) MUST BE IN SIMPLEST FORM

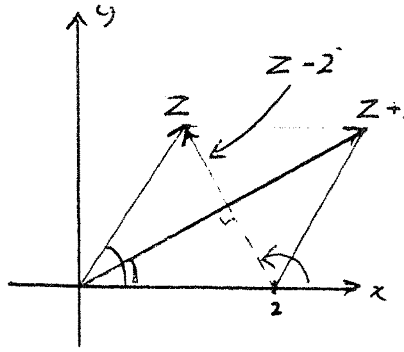
iv)



(3)

1 for $\sqrt{w}$ approx bisecting $\arg w$. 1 each for correct values of $\sqrt{w}$

Q1 c)



i) $|z| = 2$, vectors form rhombus since $\arg z$ is bisected. (2)

1 for value

1 for reason

ii) $\frac{\pi}{2} + \frac{1}{2}\arg(z)$. (1)

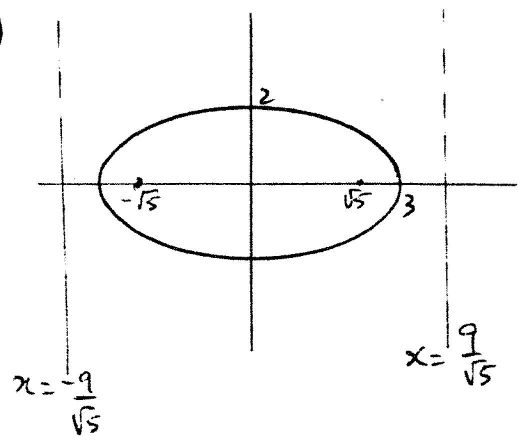
1 - no reason req^d.

Q2(a) $4x^2 + 9x^2 = 36$

i) foci $(\pm\sqrt{5}, 0)$ (1)

ii) $x = \pm \frac{9}{\sqrt{5}}$ (1)

iii)



(1)

Be lenient. eg OK if they forget ± etc.

We're looking for correct orientation here / or allow (1) if the curve matches the data from i) & ii) (even if incorrect)

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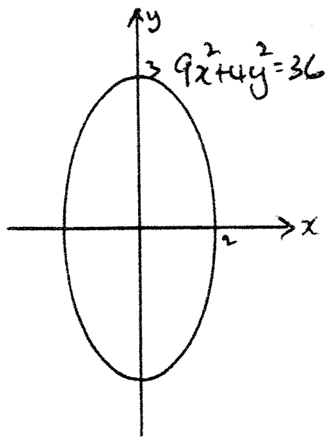
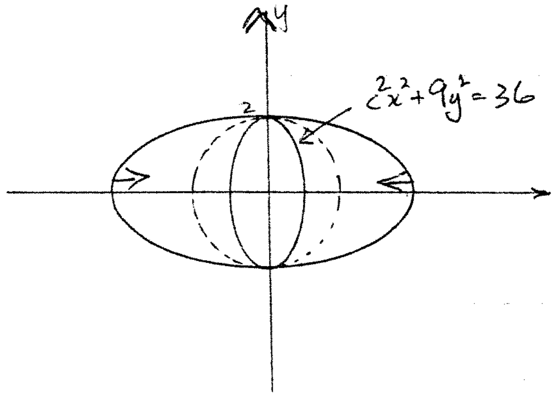
2(a) (iv) α)
cont



(1)

$4x^2+9y^2=36$ is rotated 90° about centre etc (1)

β)



(1)

Major axis shortens until (ellipse becomes the circle $x^2+y^2=4$ when $c^2=9$ then) orientation changes and y axis becomes the major axis

(1) for correct idea of "squashing" ellipse towards y axis

(1) for specific mention of circle when $c^2=9$

OR

specific mention of reorientation so that majoraxis/foci now lie on y axis.

Q2 b) i) $\frac{d}{dx}\frac{x^2}{a^2} + \frac{d}{dx}\frac{y^2}{b^2} = \frac{d}{dx}1$

ie $2x + 2y\frac{dy}{dx} = 0$ ← 1 for this step.

$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

ii) at $P(x_1, y_1)$ slope of tangent is $-\frac{b^2x_1}{a^2y_1}$ 1

$\therefore$ Eqn of tangent is

$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$ 1

↓

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

But $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ ← 1 for recognising this

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

c) at D, $x = \frac{a}{e}$

$\therefore \frac{x_1}{ea} + \frac{yy_1}{b^2} = 1$

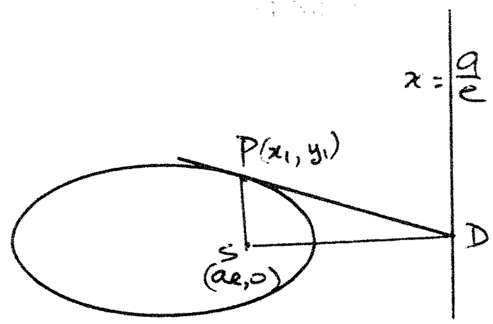
↓

$y = \frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)$ 1

$\therefore D\left(\frac{a}{e}, \frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)\right)$

Q2 cont.

b(iv) Now $\text{slope}_{PS} = \dfrac{y_1}{x_1 - ae}$ (1)



$$\text{Slope}_{DS} = \frac{\frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)}{\frac{a}{e} - ae}$$ (1)

And $\left(\dfrac{y_1}{x_1 - ae}\right) \times \left(\dfrac{\frac{b^2}{y_1}\left(1 - \frac{x_1}{ae}\right)}{\frac{a}{e} - ae}\right)$

$$= \frac{-b^2/ae}{a\left(\frac{1}{e} - e\right)}$$ (1)

$$= \frac{-b^2}{a^2(1-e^2)}$$

$$= -\frac{b^2}{b^2}$$ (1)

$$= -1$$

$\therefore \angle DSP = 90^\circ$

2 marks for working.

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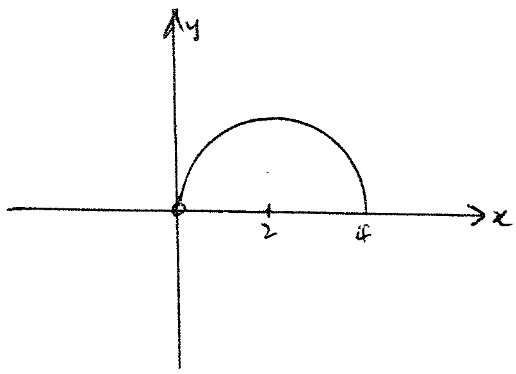
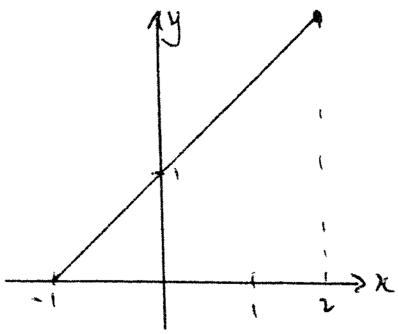
QUESTION 3.

Q3(a) i)



① for circle at $x=2$
① for top half.
No penalty for (0,0) if included.

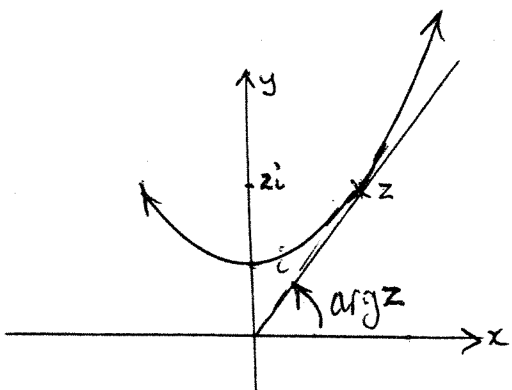
ii)



① For line from $x=-1$.

① For truncating at $x=2$.

3(b)



$x^2 = 4(y-1)$

or $y = \frac{x^2}{4} + 1$

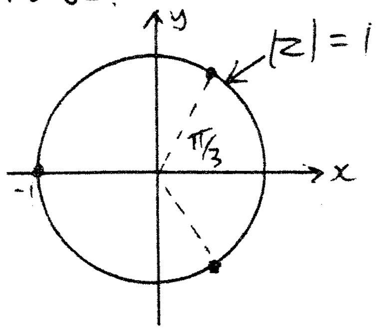
① for equation

Min arg z $= \pi/4$

① for $\arg z = \pi/4$

QUESTION 3.

Q3(c) i)



1 for roots of $z^3 = -1$

ii) Since $z^6 = 1$ can be factorized $(z^3+1)(z^3-1) = 0$ — 1

some of the roots of $z^6 = 1$ are given by $z^3 + 1 = 0$ which — 1

are the roots of $z^3 = -1$ as well.

iii) Since $z^6 - 1 = 0$ can be factorized as

$(z^2)^3 - 1 = 0$

ie $(z^2-1)(z^4+z^2+1) = 0$ — 1

when $z \neq \pm 1$, the roots of $z^4 + z^2 + 1 = 0$ are the 4 complex roots of $z^6 = 1$

ie $\text{cis} \pm \frac{\pi}{3}$, $\text{cis} \pm \frac{2\pi}{3}$ — 1

2 marks
- reference to $z^6 = 1$ must be made.

d) $V = \pi \int_0^1 x^2 \, dy - \pi \, 1^2 . 1$ — 1

$= \pi \int_0^1 (y^2 - 2)^2 \, dy - \pi$ — 1

$= \pi \int_0^1 y^4 - 4y^2 + 4 \, dy - \pi$ — 1

$= \frac{28\pi}{15}$ — 1