Name:____________________ Maths Class Teacher:____________________

SYDNEY TECHNICAL HIGH SCHOOL

MANNERS MAKYTH MAN

Extension 1 Mathematics

HSC Assessment Task 1

Dec 2010

General Instructions

- Working time – 70 minutes
- Write using **black or blue pen**
- Board-approved calculators may be used
- **All necessary working** should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Start each question on a **new page**.
- Place your papers **in order** with the question paper on top and staple or pin them.

Total Marks - 50

- Attempt Questions 1 – 6
- Mark values are shown with the questions.

(For markers use only)

Q1	Q2	Q3	Q4	Q5	Q6	Total
10	8	8	8	8	8	50

Question 1 **Marks**

a) The sum of the first two terms of a geometric progression is *-4* and the sum of the fourth and fifth terms is *108*. Calculate:

(i) the common ratio **2**

(ii) the seventh term. **2**

b) For the geometric progression *3, 9, 27, …*

(i) Find the sum of n terms **1**

(ii) Determine how many terms must be taken for the sum to exceed *10000*. **2**

c) Prove that, if x is positive, the sum of **3**

$$1+\frac{x}{1+x}+\frac{x^2}{(1+x)^2}+\frac{x^3}{(1+x)^3}+\ \dots$$

never exceeds $1+x$.

Question 2 **Marks**

a) *AB* is a diameter and *AC* is a chord of a circle whose centre is *O*.

D is the midpoint of the arc *BC*.

(i) Construct a diagram showing the above information. **1**

(ii) Prove that *OD* is parallel to *AC*. **3**

b) Prove by mathematical induction that **4**

$$\sum_{r=1}^{n} r(r+1)=\frac{n(n+1)(n+2)}{3}$$

Question 3 **Marks**

a) Two circles intersect in X and Y and P is a point on one of them. PX and PY, when produced, meet the other circle in M and N respectively.

(i) Construct a diagram showing all relevant information. **1**

(ii) Prove that the tangent at P is parallel to MN. **3**

b) Find, without deriving, the locus of a point $P(x,y)$ which moves so that its distance from the fixed point $(0,4)$ is always equal to its perpendicular distance from the fixed line $y = -4$. **1**

c) Show that the equation of the chord joining the points where $x = x_1$ and $x = x_2$ on the parabola $x^2 = y$ is **3**

$$y = xx_1 + xx_2 - x_1x_2.$$

Question 4 **Marks**

A woman is considering borrowing *$24 000* to finance renovations to her house. The interest rate is *9%* per annum compounded monthly on the balance owing.

a) If A_n represents the amount owing after n months and, using M to represent the monthly repayment, write an expression to show the amount owing after *1* month. **1**

b) Write another expression showing the amount owing after *2* months. **1**

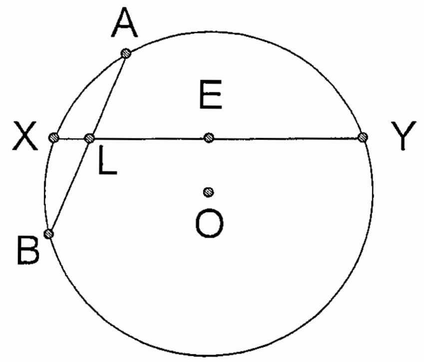
c) Construct an expression to express the amount owing after n months. **2**

d) Calculate the monthly instalment (to the nearest dollar) if the loan is to be repaid in *8* years. **3**

e) What is the full amount of interest paid (to the nearest dollar)? **1**

Question 5 **Marks**

a) AB and XY are chords in a circle with centre O. XY cuts AB in L, which is the midpoint of AB. E is the midpoint of XY. Prove that XY is greater than AB. [Hint: Construct OL and OE.] **3**



b) In a proof by mathematical induction, it is assumed that $8^k - 5^k$ is divisible by 3 for a positive integer value of k. **3**
Using this assumption, show that this must also be true for $k+1$.

c) A circle is drawn with one of the equal sides of an isosceles triangle as diameter. **2**
Show that the circle passes through the midpoint of the base of the isosceles triangle.

Question 6 **Marks**

a) On the parabola $x^2 = 4ay$, the point P has coordinates $(2ap, ap^2)$. **2**
Show that the locus of the midpoints of chords PO where O is the vertex is another parabola, $x^2 = 2ay$.

b) Tangents are drawn to a parabola $x^2 = 4y$ from an external point $A(x_1, y_1)$, touching the parabola at P and Q.

(i) Write the equation of the chord of contact. **1**

(ii) Prove that the midpoint, M, of PQ is the point $(x_1, \frac{1}{2}x_1^2 - y_1)$. **3**

(iii) If A moves along the straight line $y = x - 1$, find the equation of the locus of M. **2**

End of Exam

HSC Assessment One — Extension 1 — 2010

Q1 a) $T_1 + T_2 = -4$ — (i)
$T_4 + T_5 = 108$ — (ii)

(i)
From (i) $a + ar = -4$ — (iii)
From (ii) $ar^3 + ar^4 = 108$ — (iv)
$\therefore ar^3(1+r) = 108$ — (v)
from (iii) $a(1+r) = -4$
sub into (v) $ar^3 \times \frac{-4}{a} = 108$
$\therefore r^3 = \frac{108}{-4}$
$= -27$
$\therefore r = -3$

(ii)
sub into (iii) $a - 3a = -4$
$\therefore -2a = -4$
$\therefore a = 2$
$\therefore T_7 = 2 \times (-3)^6$
$= 1458$

b) $3, 9, 27, \ldots$
$a = 3,\ r = 3$

(i) $S_n = \frac{a(r^n - 1)}{r-1}$
$= \frac{3(3^n - 1)}{2}$

(ii)
Now $\frac{3(3^n-1)}{2} > 10000$
$3(3^n - 1) > 20000$
$\therefore 3^n - 1 > \frac{20000}{3}$
$\therefore 3^{n+1} - 3 > 20000$
$\therefore 3^{n+1} > 20003$
$\therefore \log(3^{n+1}) > \log 20003$
$\therefore (n+1)\log 3 > \log 20003$
$\therefore n+1 > \frac{\log 20003}{\log 3}$
$\approx 9.01467..$
$\therefore n > 8.01467$
So $n = 9$ for $S_n > 10000$

c) $1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \ldots$

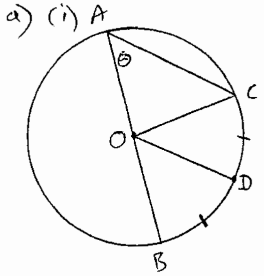
GP where $a = 1$, $r = \frac{x}{1+x}$

If $x > 0$, then $0 < \frac{x}{1+x} < 1$

$\therefore |r| < 1$
$\therefore S_\infty$ exists.

$\therefore S_\infty = \frac{a}{1-r}$
$= \frac{1}{1 - \frac{x}{1+x}}$
$= \frac{1}{\frac{1+x-x}{1+x}}$
$= 1 + x$ QED

Q2 a) (i)



(ii)
Let $B\hat{O}C = \theta$
Now, $C\hat{O}B = 2\theta$ (Angle at centre is twice that at circumference)
$C\hat{O}D = B\hat{O}D$ (angles on equal arcs)
$\therefore B\hat{O}D = \theta$
$\therefore AC \parallel OD$ (corresponding angles)

b) RTS $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$

ie $1.2 + 2.3 + \ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$

(i) For $n = 1$, LHS $= 1.2$
$= 2$
RHS $= \frac{1 \times 2 \times 3}{3}$
$= 2$
$\therefore$ LHS = RHS
$\therefore$ true for $n = 1$.

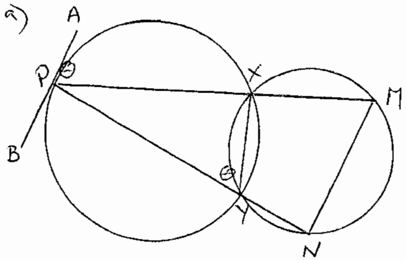
(ii) Assume true for $n = k$
$\therefore 1.2 + 2.3 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$

For $n = k+1$,
RHS $= \frac{(k+1)(k+2)(k+3)}{3}$

LHS $= 1.2 + 2.3 + \ldots + k(k+1) + (k+1)(k+2)$
$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$
$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$
$= \frac{(k+1)(k+2)(k+3)}{3}$
$=$ RHS
$\therefore$ if true for $n = k$, then true for $n = k+1$

(iii) If true for $n = k = 1$, then true for $n = k+1 = 2$
If true for $n = k = 2$, then true for $n = k+1 = 3$
etc.
$\therefore$ true for all positive integral values of n.

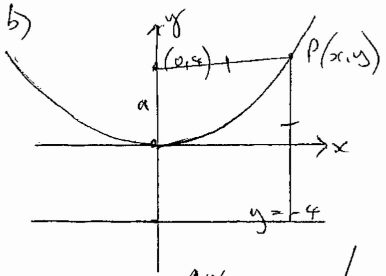
Q3 a)



Let tangent at P be AB.
Let $\theta = A\hat{P}M$
Now, $P\hat{Y}X = \theta$ (angle in alternate segment)

$\therefore X\hat{Y}N = 180 - \theta$ (supplementary angles)
$\therefore X\hat{M}N = \theta$ (opposite angles in cyclic quadrilateral are supplementary)
$\therefore AB \parallel MN$ (alternate angles are equal)

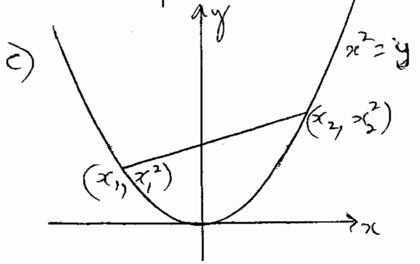
b)



Locus of P is a parabola, vertex $(0,0)$ with focal length, 4.

$\therefore x^2 = 4 \times 4 \times y$
$\therefore x^2 = 16y$

c)



$$\frac{y - x_1^2}{x - x_1} = \frac{x_2^2 - x_1^2}{x_2 - x_1}$$

$\therefore (y - x_1^2)(x_2 - x_1) = (x_2^2 - x_1^2)(x - x_1)$

$\therefore yx_2 - x_1 y - x_1^2 x_2 + \cancel{x_1^3} = x_2^2 x - x_2^2 x_1 - x_1^2 x + \cancel{x_1^3}$

$\therefore y(x_2 - x_1) = x_1^2 x_2 - x_2^2 x_1 + x x_2^2 - x x_1^2$
$= x_1 x_2 (x_1 - x_2) + x(x_2^2 - x_1^2)$
$= x_1 x_2 (x_1 - x_2) + x(x_2 - x_1)(x_2 + x_1)$
$= -x_1 x_2 (x_2 - x_1) + x(x_2 + x_1)(x_2 - x_1)$

$\therefore y = -x_1 x_2 + x(x_2 + x_1)$
$= -x_1 x_2 + x x_2 + x x_1$
$\therefore y = x x_1 + x x_2 - x_1 x_2$

QED

Q4 $P = \$24000$ $\quad r = 0.09\%$ pa
$= 0.0075\%$ pm

a) $A_1 = 24000 \times 1.0075 - M$

b) $A_2 = (24000 \times 1.0075 - M) \times 1.0075 - M$
$= 24000 \times 1.0075^2 - M(1 + 1.0075)$

c) $A_n = 24000 \times 1.0075^n - M(1 + 1.0075 + 1.0075^2 + \cdots + 1.0075^{n-1})$

d) $A_{96} = 24000 \times 1.0075^{96} - M\left(1\ \dfrac{(1.0075^{96} - 1)}{1.0075 - 1}\right)$
$= 24000 \times 1.0075^{96} - M \times 139.85562$

$\therefore 139.8562M = 24000 \times 1.0075^{96}$
$= 49174.10948$
≈ 351.60
$\therefore M = \$352$

e) Interest $= 352 \times 96 - 24000$
$= 33792 - 24000$
$= \$9792$

$\therefore y = x - 1$ on line $y = y_1$, $x = x_1$

Now $y_1 = x_1 - 1$

For M, $y = \frac{x_1^2}{2} - (x_1 - 1)$

$\therefore y = \frac{x_1^2 - 2x_1 + 2}{2}$

Also for M, $x = x_1$

$\therefore y = \frac{x^2 - 2x + 2}{2}$

$\therefore 2y = x^2 - 2x + 2$
$= (x-1)^2 + 1$

$\therefore (x-1)^2 = 2y - 1$
$= 2(y - \frac{1}{2})$

b) Assume $3 \mid 8^k - 5^k$

$\therefore 8^k - 5^k = 3M$, $M \in \mathbb{N}$

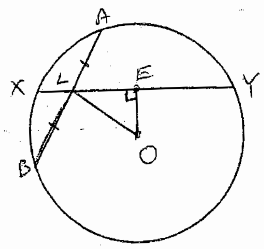
For $n = k+1$,

$8^{k+1} - 5^{k+1} = 8 \times 8^k - 5 \times 5^k$
$= 5(8^k - 5^k) + 3 \cdot 8^k$
$= 5 \times 3M + 3 \cdot 8^k$
$= 3(5M + 8^k)$ where $5M + 8^k \in \mathbb{N}$

$\therefore$ if true for k, then true for $k+1$.

QED

Q5 a)



Construct OL and OE.

L is midpoint of AB (given)

$\therefore OL$ is perpendicular distance from O (perpendicular from midpoint passes through centre)

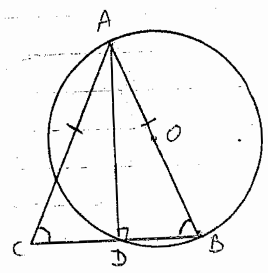
E is midpoint XY (given)

$\therefore \hat{LEO} = 90°$ (line from O to midpoint meets chord at $90°$)

Now $LO > EO$ (hypotenuse is longest side)

$\therefore XY > AB$ (longer chord is closer to centre)

c)



Construct AD

$\therefore \hat{ADB} = 90°$ (angle in semicircle is $90°$)

$\therefore \hat{ADC} = 90°$ (supplementary angles)

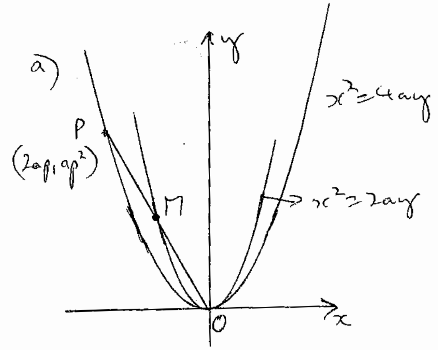
$\hat{ABC} = \hat{ACD}$ (base angles of isosceles Δ)

AD is common to Δs ABD and ACD.

$\therefore \Delta ABD \equiv \Delta ACD$ (AAS)

$\therefore CD = DB$ (corresponding sides in congruent Δs)

Q6 a)



$O \Rightarrow (0, 0)$ $P \Rightarrow (2ap, ap^2)$

$\therefore M = (ap, \frac{ap^2}{2})$

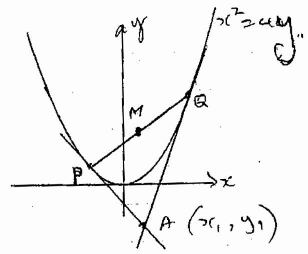
$\therefore x = ap$ and $y = \frac{ap^2}{2}$

$\therefore p = \frac{x}{a}$

$\therefore y = \frac{a}{2}\left(\frac{x}{a}\right)^2$
$= \frac{x^2}{2a}$

$\therefore x^2 = 2ay$ QED

b)



(i) chord of contact $\Rightarrow xx_1 = 2a(y + y_1)$
$xx_1 = 2(y + y_1)$ — (i)

$a = 1$

(ii) $x^2 = 4y$ — (ii)

$\therefore y = \frac{x^2}{4}$

Sub into (i) $xx_1 = 2\left(\frac{x^2}{4} + y_1\right)$
$= \frac{x^2}{2} + 2y_1$

$\therefore x^2 - 2xx_1 + 4y_1 = 0$

$\therefore x = \frac{2x_1 \pm \sqrt{4x_1^2 - 4 \times 4y_1}}{2}$
$= \frac{2x_1 \pm 2\sqrt{x_1^2 - 4y_1}}{2}$
$= x_1 \pm \sqrt{x_1^2 - 4y_1}$

$\therefore x$ value of $M = x_1$ (by symmetry)

$\therefore x_1^2 = 2(y - y_1)$
$= 2y + 2y_1$

$\therefore 2y = x_1^2 - 2y_1$

$\therefore y = \frac{x_1^2}{2} - y_1$

$\therefore M \Rightarrow \left(x_1, \frac{x_1^2}{2} - y_1\right)$ QED

Q6 b (iii) on previous page