Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

TRIAL HIGHER SCHOOL CERTIFICATE

August 2012

TIME ALLOWED: 180 minutes

READING TIME: 5 minutes

General Instructions:

- Write your name and class at the top of this page, and on your answer booklet.
- Hand in all of your answers and this question sheet.
- Use only blue or black pen
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- START ALL QUESTIONS ON A NEW PAGE
- Approved calculators may be used.
- A table of *Standard Integrals* is attached. You may detach this page now.

Section I Pages 1 to 5

10 marks

- Colour in the circle on your Section I answer sheet corresponding to the correct answer
- There is only one correct answer for all questions in this section
- Allow about 15 minutes for this section

Section II Pages 6 to 16

90 marks

- Allow about 2 hours 45 minutes for this section

SECTION I

1 If $z = 1 + \sqrt{3}i$, then $z^4 =$

A $8 + 8\sqrt{3}i$

B $8 - 8\sqrt{3}i$

C $-8 + 8\sqrt{3}i$

D $-8 - 8\sqrt{3}i$

2 $\int sin^3 x dx =$

A $\frac{1}{4} sin^4 x + k$

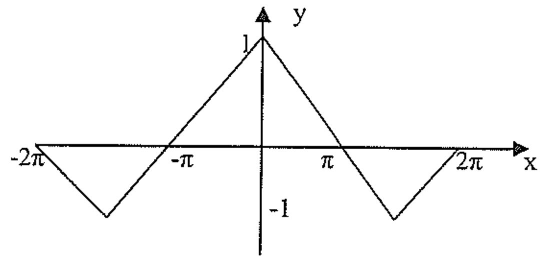
B $-cosx + \frac{1}{3} cos^3 x + k$

C $-cosx - \frac{1}{3} cos^3 x + k$

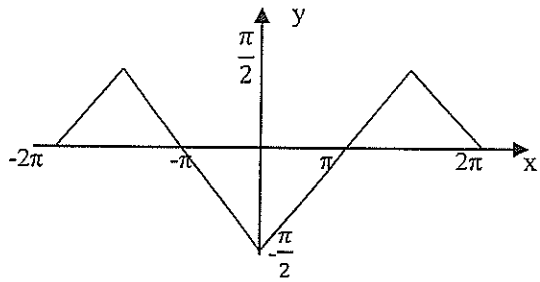
D $cosx - \frac{1}{3} cos^3 x + k$

3 Which of the curves below represents the curve $y = sin^{-1}(cosx)$ for $-2\pi \leq x \leq 2\pi$

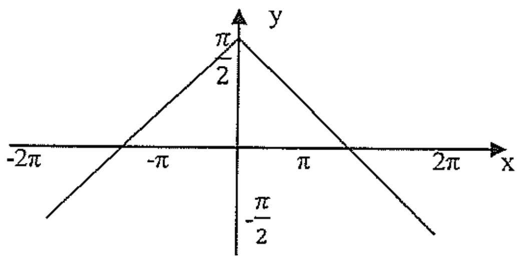
A



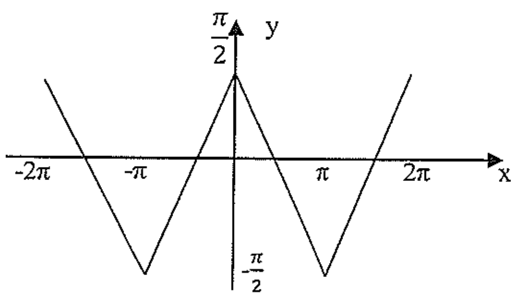
B



C



D



4 Given that $tanx + \cot x = \dfrac{1}{sinxcosx}$ then a Primitive of $\dfrac{1}{sinxcosx}$ is

A $\dfrac{1}{cos^2x} logsinx$

B $logsinxcosx$

C $log|tanx|$

D $logcotx$

5 A quadratic expression with zeros of $4 + i$ and $4 - i$ is:

A $x^2 - 8x + 17$

B $x^2 + 8x + 17$

C $x^2 - 8x - 17$

D $x^2 + 8x - 17$

6 The derivative of the curve

$$x^3 + 9x^2 - y^2 + 27x - 4y + 23 = 0 \quad \text{is:}$$

A $\dfrac{dy}{dx} = \dfrac{x^2+6x+9}{2y}$

B $\dfrac{dy}{dx} = \dfrac{x^2+6x+9}{-2y}$

C $\dfrac{dy}{dx} = \dfrac{3x^2+18x+27}{-2y-4}$

D $\dfrac{dy}{dx} = \dfrac{3x^2+18x+27}{2y+4}$

7 For the curve given by:

$$f(x) = x^2, \qquad 0 \le x \le 1$$

$$f(x) = 2x - 1, \qquad x > 1$$

Which of the following statements is correct?

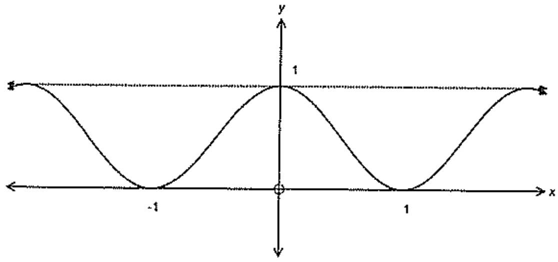
A $f(x)$ has a discontinuity at $x = 1$

B $f(x)$ is differentiable at $x = 1$

C $f(x)$ has an asymptote at $x = 1$

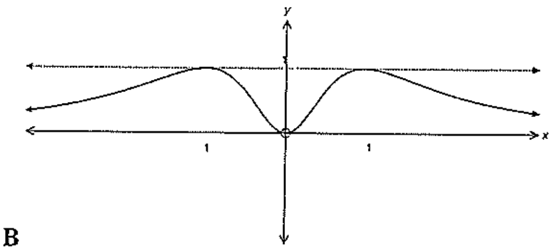
D $f(x)$ has a stationary point at $x = 1$

8 The graph of $y = f(x)$ is given below.

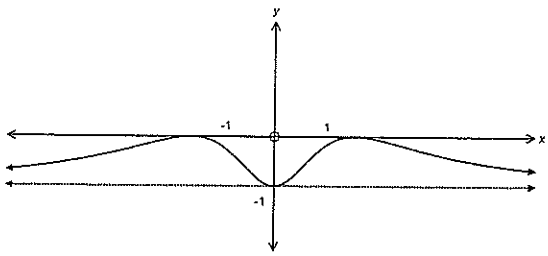


Which of the following represents $y = \frac{1}{f(x)}$?

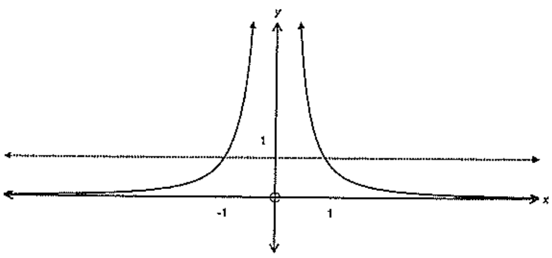
A



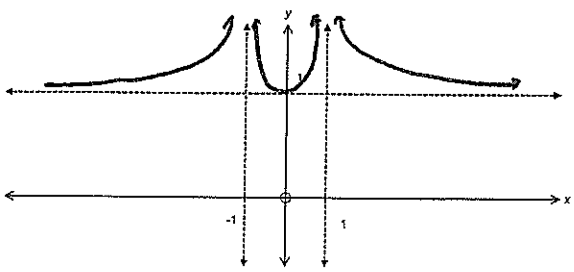
B



C



D



9 It is known that $x = 2 - 3i$ is a solution to $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$

Another solution is $x =$

A $1 - 2i$

B $-1 - 2i$

C $-2 - i$

D $-2 + i$

10 A particle moves in a straight line so that its velocity at any particular time is given by $v = k(a - x)$, where x is its displacement from a given point O.

The particle is initially at O.

Which of the following gives an expression for x:

A $x = a(1 - e^{kt})$

B $x = a(1 + e^{kt})$

C $x = a(1 - e^{-kt})$

D $x = a(1 + e^{-kt})$

SECTION II

QUESTION 11:

Marks

2 (a) Find $\int \frac{dx}{x^2-6x+13}$

4 (b) (i) Find values of A, B and C so that

$$\frac{2x^2+x+9}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

(ii) Hence find $\int_0^2 \frac{2x^2+x+9}{(x^2+4)(x+1)}dx$ giving your answer in exact form

1 (c) (i) On an Argand Diagram, draw and shade the region $\boldsymbol{R}$ given by

$$|z-2-2i| \leq 2$$

2 (ii) P is a point in $\boldsymbol{R}$, representing the complex number z. What is the maximum value of $|z|$?

2 (iii) The tangent to the curve at P cuts the x-axis at the point T.

By using the nature of ΔOPT, or otherwise, find the exact area of ΔOPT.

(d) Let $x = \alpha$ be a root of the polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$ Where $(B+2)^2 \neq 4A^2$

1 (i) Show that α cannot be 0, 1 or -1

1 (ii) Show that $\frac{1}{\alpha}$ is a root of $P(x) = 0$

2 (iii) Deduce that if α is a multiple root of $P(x)=0$, then its multiplicity is 2

QUESTION 12: (Start a new page)

Marks

2 (a) Find the value of $\int_0^1 tan^{-1}x\ dx$

(b) Let $f(x) = ln\,(1 + x) - ln\,(1 - x)$ where $-1 < x < 1$

1 (i) Show that $f'(x) > 0$ for all x in the given Domain

3 (ii) On the same diagram, sketch

$y = \{ln\,(1{+}x)$ for $x > -1$
$y = \{ln\,(1{-}x)$ for $x < 1$
$y = \{f(x)$ for $-1 < x < 1$

clearly labelling all 3 graphs

1 (iii) Find an expression for the inverse function $y = f^{-1}(x)$

1 (c) (i) Show that $(1 + t^{2})^{n-1} + t^2(1 + t^2)^{n-1} = (1 + t^2)^n$

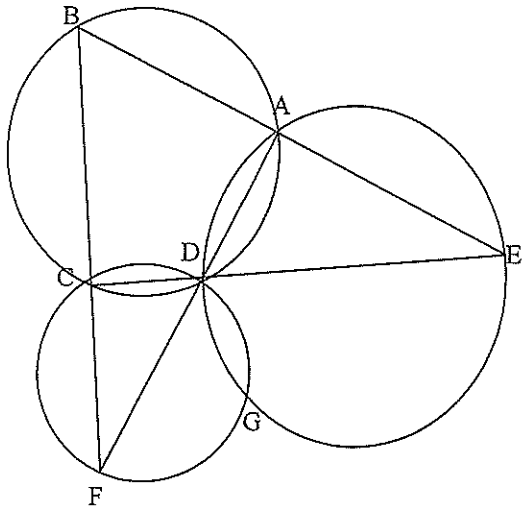
3 (ii) $I_n = \int_0^x (1 + t^2)^n\, dt$ for $n = 1, 2, 3, \ldots\ldots$

Use integration by parts, and part (i) above, to show that

$$I_n = \frac{1}{2n+1}(1 + x^2)^n x + \frac{2n}{2n+1} I_{n-1}$$

Question 12 continues overpage.......)

4 (d)



In the diagram, ABCD is a cyclic quadrilateral.
BA and CD are produced to meet at E
Similarly BC and AD are produced to meet at F
Circles are then drawn through A, D and E, and C, D and F
These two circles intersect at D and G as shown

A copy of this diagram is included after your table of standard integrals.
Detach it, put it with your answer sheets, and then join

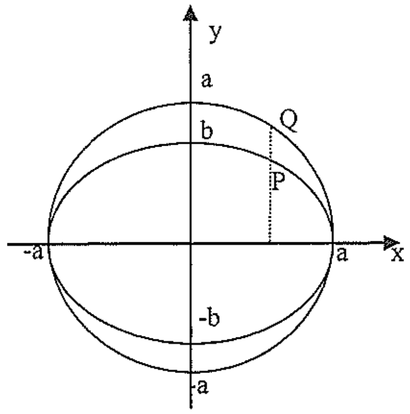
F to G
G to E and
D to G

Prove that E, G and F are collinear, clearly stating all geometric reasoning.

QUESTION 13: (Start a new page)

Marks

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=a^2$ where $a>b$, are drawn below.



P is a point on the ellipse with co-ordinates $(acos\theta, bsin\theta)$.

A line perpendicular to the major axis is drawn through P to meet the circle at the point Q

1 (i) Find the co-ordinates of the point Q

2 (ii) Show that the equation of the tangent to the ellipse at P is given by

$$\frac{xcos\theta}{a}+\frac{ysin\theta}{b}=1$$

1 (iii) Find the equation of the tangent to the circle at Q

2 (iv) The tangents at P and Q meet at the point T.

Show that the point T lies on the x-axis.

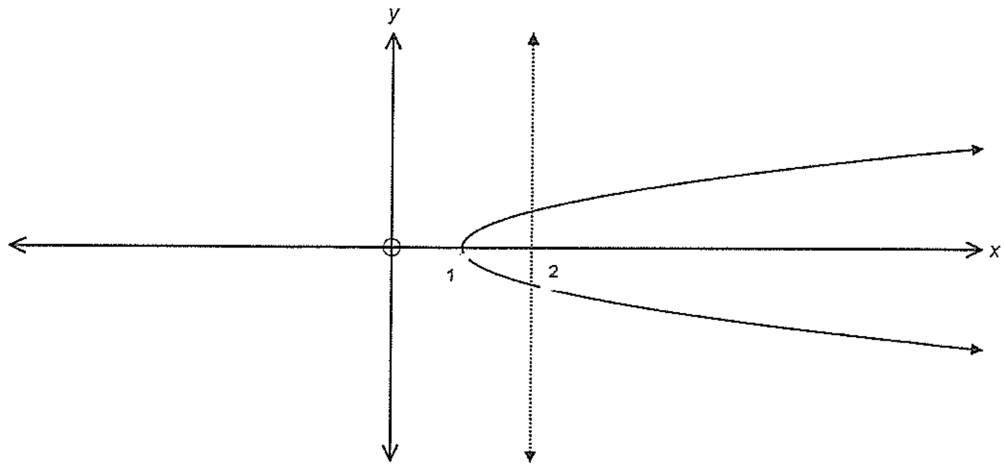
3 (b) $x^3+px^2+qx+r=0$ has roots of α, β, and γ, where $\alpha=\beta+\gamma$

Show that $p^3-4pq+8r=0$

Question 13 continues overpage…….)

2 (c) (i) Show that $\int x\sqrt{x-1}dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c$

4 (ii) The area between the curve $y^2 = x - 1$ and the line $x = 2$, is rotated through 2π radians about the *y axis*



Using the method of cylindrical shells, taken parallel to the y-axis, show that the volume of the solid so formed is $\frac{64\pi}{15}$ cubic units.

QUESTION 14: **(Start a new page)**

Marks

(a) The roots of the equation $z^5 + 1 = 0$ are -1, ω_1, ω_2, ω_3, ω_4 in cyclic order, antic-clockwise around the Argand Diagram.

2 (i) Show that $\omega_1 = \overline{\omega_4}$

2 (ii) Find values of a, b and c so that $(z+1)(z^4 + az^3 + bz^2 + cz + 1) = z^5 + 1$

and hence show that if ω is a root of $z^5 + 1 = 0$, not equal to -1, then

$$\omega^4 + \omega^2 + 1 = \omega^3 + \omega$$

1 (iii) Show that ${\omega_1}^3 = \omega_3$

(For the rest of this question you may also assume the other results:
$${\omega_2}^3 = \omega_1\,, \quad {\omega_4}^3 = \omega_2 \;\; \textit{and} \;\; {\omega_3}^3 = \omega_4 \,\textit{)}$$

1 (iv) Deduce that ${\omega_1}^3 + {\omega_2}^3 + {\omega_3}^3 + {\omega_4}^3 = 1$

3 (v) By using the sum of the roots of $z^5 + 1 = 0$ in pairs, or otherwise, prove that

$$\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = -\frac{1}{2}$$

(b) $T_1\ (t, \frac{1}{t})$ and $T_2\ (3t, \frac{1}{3t})$ are two points on the hyperbola xy=1

2 (i) Show that, as t varies, the the midpoint of T_1T_2 lies on $3xy = 4$

1 (ii) Show that equation of the normal to the hyperbola $xy = 1$ at T_1 is given by

$$t^4 - t^3x + ty - 1 = 0$$

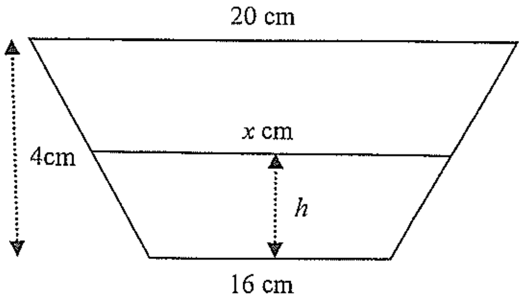
3 (iii) R*(0, h)* is a point on the y-axis ($h \neq 0$). Show that there are exactly two points on xy = 1 with normals which pass through R.

QUESTION 15: (Start a new page)

Marks

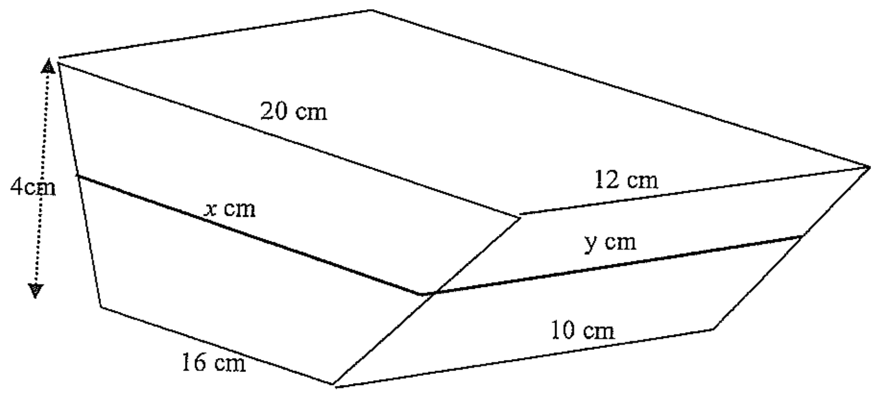
(a) An isosceles trapezium has parallel sides of 20cm and 16cm and a height of 4cm.

A line, parallel to the base, is taken h cm above the 16 cm side, and has length x cm.



2 (i) By considering the ***areas*** of the three trapezia thus formed, or otherwise, prove that $x = 16 + h$

4 (ii) A cake tin is made using the shape above as its two ends, and two more equal trapezia as shown for its two sides.



The strip corresponding to x cm along the sides is of length y cm and you may assume the result $y = 10 + \frac{h}{2}$

Find the volume of the cake tin. (Show all appropriate working.)

Question 15 continues overpage.......)

QUESTION 15 continued.......)

(b) A particle of mass 1 Kg is projected vertically upwards from the ground with a speed of 20m/s. The particle is under the effect of both gravity(g) and an air resistance of magnitude $\frac{1}{40}v^2$ where v is the current velocity of the particle at any time. The effects of gravity and air resistance remain the same no matter which direction the particle is going.

1 (i) Explain why the acceleration of the particle at any time whilst travelling upwards is given by:

$$\ddot{x} = -g - \frac{1}{40}v^2$$

(For the remainder of this question you may use g = 10 m/s²)

4 (ii) Calculate the greatest height reached by the particle

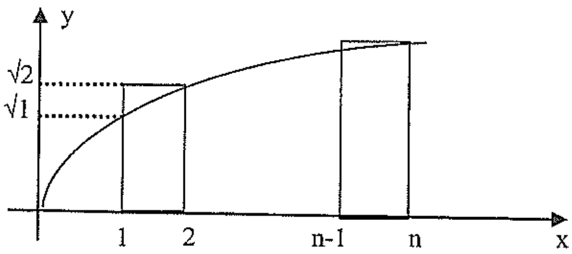
1 (iii) Write an expression for the acceleration of the particle as it returns to earth.

3 (iv) Find the speed of the particle *just before* it strikes the ground.

QUESTION 16: (Start a new page)

Marks

(a) The figure below is of the curve $y = \sqrt{x}$. It is not drawn to scale.



1 (i) Show that the curve is increasing for all $x \geq 0$

2 (ii) Referring to the diagram above, show that $\sqrt{1} + \sqrt{2} + \ldots\ldots\sqrt{n} > \frac{2}{3}n\sqrt{n}$

for all finite values of $n \geq 1$

1 (iii) Prove, by expansion, or otherwise, that:

$$(4n+3)^2 n < (4n+1)^2(n+1)$$

4 (iv) Use Mathematical Induction to show that

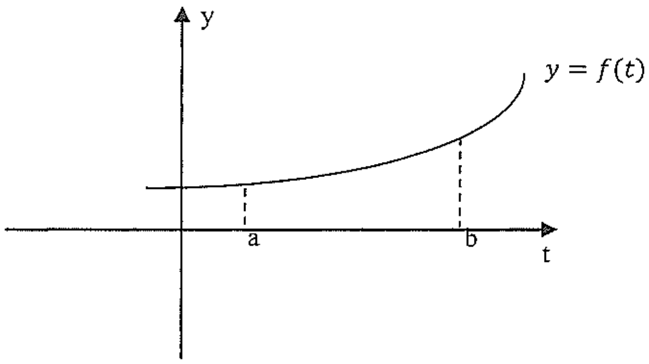
$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} < \frac{4n+3}{6}\sqrt{n} \qquad \text{for all integers } n \geq 1$$

1 (v) Using parts (ii) and (iv) estimate

$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{10\,000}$ to the nearest hundred.

Question 16 continues overpage.......)

2 (b) (i) Let m and M be the smallest and greatest values of the integrable function $f(t)$ in the Domain $a \leq t \leq b$, as shown in the diagram below:



Explain carefully why

$$m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$$

3 (ii) Using part (i), or otherwise, deduce that,

$$\text{if } x > 0, \quad \frac{x}{1+x} \leq \log(1+x) \leq x$$

1 (iii) Hence show that $1 \leq ln4 \leq 2$

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

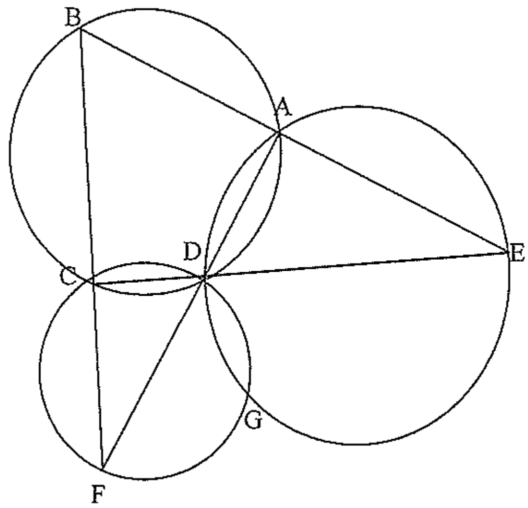
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

THIS DIAGRAM IS FOR QUESTION 12 (d)

It should be removed from the question booklet and placed with your answer booklet.

12 (d)



In the diagram, ABCD is a cyclic quadrilateral.
BA and CD are produced to meet at E
Similarly BC and AD are produced to meet at F
Circles are then drawn through A, D and E, and C, D and F
These two circles intersect at D and G as shown

Join:

F to G
G to E and
D to G

STHS EXTENSION 2 TRIAL HSC

AUGUST 2012

QUESTION

1	D
2	B
3	D
4	C
5	A
6	D
7	B
8	D
9	A
10	C

QUESTION 11:

MARKING

(a) $$\int \frac{dx}{x^2-6x+13} = \int \frac{dx}{(x-3)^2+4}$$

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + k$$

2 marks

1 for inverse tan

1 for ½. No penalty for

(b) (i) $(Ax+B)(x+1) + C(x^2+4) = 2x^2+x+9$

$x=-1$ $\quad 5C = 10$

$C = 2$

Coefficients of x^2 $\quad A + C = 2$

$\therefore A = 0$

Constants $\quad B + 4C = 9$

$B = 1$

$$\begin{cases} A = 0 \\ B = 1 \\ C = 2 \end{cases}$$

2 marks

1 off for each of A, B, C incorrect

(ii) $$\int_0^2 \frac{2x^2+x+9}{(x^2+4)(x+1)}\,dx = \int_0^2 \frac{1}{x^2+4}\,dx + \int_0^2 \frac{2}{x+1}\,dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_0^2 + 2\ln(x+1)\Big]_0^2$$

$$= \frac{1}{2}\tan^{-1}(1) + 2\ln 3$$

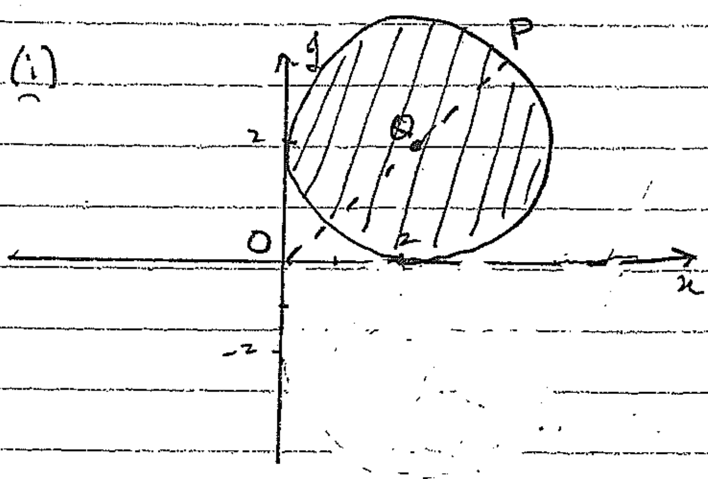
$$= 2\ln 3 + \pi/8$$

2 marks

1 for each part of answer.

MARKING

Q 11 (c) (i)



1 MARK

(ii)

$OQ = \sqrt{8}$ (by Pythagoras' Theorem) — 1 MARK

$\therefore OP = 2 + 2\sqrt{2}$

$\therefore \max |z| = 2 + 2\sqrt{2}$ — 1 MARK

(iii) There is a right angle at P

$\angle TOP = \angle OTP = 45^\circ \Rightarrow$ isosceles $\triangle OPT$

$\therefore PT = OP$

1 MARK for realising this (OR OTHERWAYS)

$\therefore$ Area $\triangle OPT = \frac{1}{2} \cdot OP \cdot PT$

$= \frac{1}{2}(2+2\sqrt{2})^2$

$= 2(3+2\sqrt{2})$ — 1 MARK

(d)

(i) $x = \alpha$ is a root of $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$

$P(0) \neq 0 \Rightarrow \alpha$ cannot be 0

$P(1) = 2 + 2A + B = 0$ only if $(B+2)^2 = 4A^2$

$P(-1) = 2 - 2A + B = 0$ " " " "

1 MARK

(ii) $P\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + B/\alpha^2 + A/\alpha + 1$ — 1 MARK

$= \frac{1}{\alpha^4}\left(1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4\right)$

$= 0$ since $P(\alpha) = 0$

(iii) If α has multiplicity N then so does $\frac{1}{\alpha}$ (because of (ii) above)

Because $P(x)$ is of degree 4 it has at most 4 roots.

$\therefore$ MAX value of N is 2

2 for reasoning

QUESTION 12 | MARKING

(a) $\int_0^1 \tan^{-1}x\,dx = \int_0^1 \frac{d}{dx}(x)\tan^{-1}x\,dx$

$= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$= \tan^{-1}(1) - \frac{1}{2}\ln(1+x^2)\Big]_0^1$

$= \pi/4 - \frac{1}{2}\ln 2$ — 2 MARKS

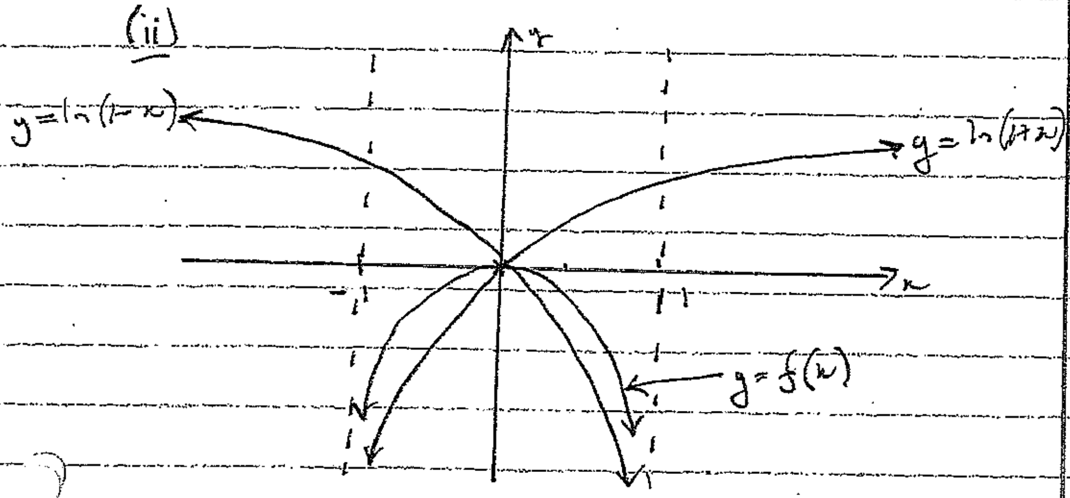
(b) $f(x) = \ln(1+x) - \ln(1-x),\ -1<x<1$

(i) $f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$

$= \frac{1-x+1+x}{1-x^2}$ — 1 for this

$= \frac{2}{1-x^2} > 0\ \forall\ -1<x<1$

(ii)



1 MARK for each graph $= 3$

(iii) $y = \ln\left(\frac{1+x}{1-x}\right)$

becomes $x = \ln\left(\frac{1+y}{1-y}\right)$

$\frac{1+y}{1-y} = e^x$

$1+y = e^x - ye^x$

$y(1+e^x) = e^x - 1$

$y = \frac{e^x-1}{e^x+1}$

$\therefore f^{-1}(x) = e^x-1/e^x+1$ — 1 MARK

MARKING

(c) $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1}$

$= (1+t^2)^{n-1}\left[1+t^2\right]$

$= (1+t^2)^n$ — 1 MARK

(ii) $\int_0^x (1+t^2)^n \, dt = t(1+t^2)^n \Big]_0^x - \int_0^x 2t^2(n)(1+t^2)^{n-1} \, dt$ — 1 for this

Now $\int_0^x 2t^2 n (1+t^2)^{n-1} \, dt$

$= 2n \int_0^x t^2 (1+t^2)^{n-1} \, dt$

$= 2n \int_0^x (1+t^2)^n \, dt - 2n \int_0^x (1+t^2)^{n-1} \, dt$ — 1 for "seeing" this

from part (i)

$\therefore I_n = t(1+t^2)^n \Big]_0^x - 2n I_n + 2n I_{n-1}$ — 1 for completion — 3

$\therefore I_n(2n+1) = x(1+x^2)^n + 2n I_{n-1}$

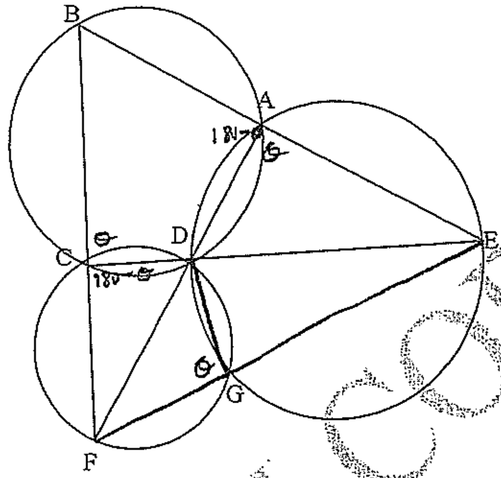
$\therefore I_n = \dfrac{x(1+x^2)^n}{2n+1} + \dfrac{2n}{2n+1} I_{n-1}$

(d) See attachment

THIS DIAGRAM IS FOR QUESTION 12 (d)

It should be removed from the question booklet and placed with your answer booklet.

12 (d)



In the diagram, ABCD is a cyclic quadrilateral.
BA and CD are produced to meet at E
Similarly BC and AD are produced to meet at F
Circles are then drawn through A, D and E, and C, D and F
These two circles intersect at D and G as shown

Join:

F to G
G to E and
D to G

Let $\angle FGD = \theta^\circ$.

$\therefore \angle FCD = (180-\theta)^\circ$ (opposite angles of cyclic quadrilateral FCDG)

(1) $\therefore \angle BCD = \theta^\circ$ (straight angle BCF)

(1) $\therefore \angle BAD = (180-\theta)^\circ$ (opposite angles of cyclic quadrilateral CDAB)

$\therefore \angle EAD = \theta^\circ$ (straight angle BAE)

(1) $\therefore \angle DGE = (180-\theta)^\circ$ (opposite angles of cyclic quadrilateral AEGD)

(1) $\longrightarrow$ $\therefore \angle DGE + \angle FGD = 180^\circ$

$\therefore$ FGE is a straight line.

MARKING

QUESTION 13:

(a) (i) Q is $(a\cos\theta, a\sin\theta)$ ← 1 MARK

(ii) $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\cdot\frac{b^2}{2y}$$

$$= \frac{-b^2x}{a^2y}$$

At P $\quad m_T = \dfrac{-b\cos\theta}{a\sin\theta}$ ← 1 MARK

Equation of tangent:

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$ay\sin\theta + bx\cos\theta = ab. \qquad (1)$$

$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

1 for correct working

(iii) At Q $\quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

$$m_T = \frac{-\cos\theta}{\sin\theta}$$

Equation of tangent:

$$y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$$

$$y\sin\theta - a\sin^2\theta = -x\cos\theta + a\cos^2\theta$$

$$x\cos\theta + y\sin\theta = a \qquad (2)$$ ← 1 MARK

(iv) $(1) - (2)$

$$\frac{ay\sin\theta}{b} - y\sin\theta = 0$$

$$\therefore \sin\theta\, y\left(\frac{a}{b} - 1\right) = 0$$ ← 1 MARK

since $a \neq b$. and $\theta \neq 0°$

$\therefore y = 0$

1 MARK

(b)

$$x^3 + px^2 + qx + r = 0 \qquad \alpha = \beta + \gamma.$$

Sum of roots $= 2\alpha = -p$

$$\therefore \alpha = -\frac{p}{2}$$ ← 1 for α

Sum of roots $\times 2$ $\quad \alpha\beta + \alpha\gamma + \beta\gamma = q$

$$\therefore -\frac{p}{2}(\beta + \gamma) + \beta\gamma = q$$

$$\frac{p^2}{4} + \beta\gamma = q$$ ← 1 for $\beta\gamma$

Product of roots $\quad \alpha\beta\gamma = -r$

$$\therefore -\frac{p}{2}\left(q - \frac{p^2}{4}\right) = -r$$

$$\frac{p^3}{8} - \frac{pq}{2} = -r$$ 1 for completion

$$\therefore p^3 - 4pq + 8r = 0$$

Q13 CONTINUED

MARKING

(c) (i) $\int x\sqrt{x-1}\,dx = \int (x-1)\sqrt{x-1}\,dx + \int \sqrt{x-1}\,dx$ } ①

$= \int (x-1)^{3/2}\,dx + \int (x-1)^{1/2}\,dx$

$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c$ } ①

OR // Let $u = \sqrt{x-1} \Rightarrow x = u^2+1$ OR

$dx = 2u\,du$

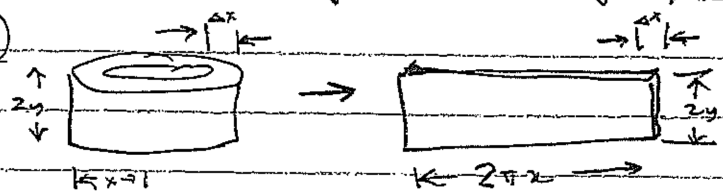
$\therefore$ Integral $= \int (u^2+1)\,u\,2u\,du$ ← ①

$= \int 2u^4\,du + \int 2u^2\,du$

$= \frac{2}{5}u^5 + \frac{2}{3}u^3 + c$ ← ①

$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c$

(ii)



$\Delta V = 2\pi x \cdot 2y\,\Delta x$

$= 4\pi xy\,\Delta x$

$\therefore$ VOL $= 4\pi \int_1^2 xy\,dx$ ← 2 MARKS to here

Since $y = \sqrt{x-1}$

VOL $= 4\pi \int_1^2 x\sqrt{x-1}\,dx$

$= 4\pi \left[\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2}\right]_1^2$ from (i) ① for seeing this

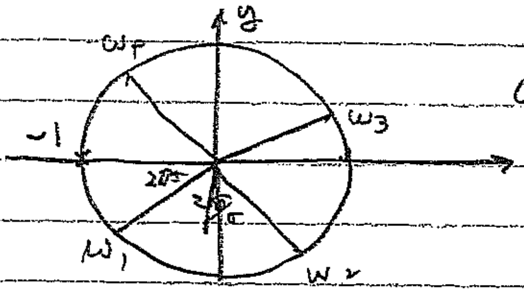
$= 4\pi \left(\frac{2}{5} + \frac{2}{3}\right]$ } 1 MARK

$= \frac{64\pi}{15}$ cu units

QUESTION 14:

MARKING

(a)



Angles between roots are $\left(\frac{2\pi}{5}\right)^c$

$\therefore \omega_1 = cis\ \frac{7\pi}{5}$

$\omega_2 = cis\ \frac{9\pi}{5}$

$\omega_3 = cis\ \frac{11\pi}{5}$ or $cis\ \frac{\pi}{5}$

$\omega_4 = cis\ \frac{3\pi}{5}$

1 for these or similar.

PLUS

(i) $\overline{\omega_4} = \overline{\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}}$

$= \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$= \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \omega_1$

1 for this or similar.

= 2 MARKS

(ii) $a = -1 \quad b = 1 \quad c = -1$ ← 1 MARK

Roots of $z^5 + 1 = 0$

are roots of $(z+1)(z^4 - z^3 + z^2 - z + 1) = 0$

Since $z \neq -1$ ω solves $z^4 - z^3 + z^2 - z + 1 = 0$

1 for this step

$\therefore \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$

$\therefore \omega^4 + \omega^2 + 1 = \omega^3 + \omega$.

no marks here

(iii) $\omega_1^3 = \left(cis\ \frac{7\pi}{5}\right)^3$

$= cis\ \frac{21\pi}{5}$

$= cis\ \frac{\pi}{5}$ because of unit circle

$= \omega_3$

1 MARK. Or MUST carry some form of reasoning

QUEST 15 CONT...)

$\therefore\ x = -20\log(40g+v^2) + 20\log(40g+400)$

Since $g = 10$

$x = 20\log\left(\frac{800}{400+v^2}\right)$

At greatest height $v = 0$ — 1 MARK

$\therefore\ x = 20\log 2$ — 1 MARK (or equivalent)

(iii) EARTHBOUND $\ddot{x} = g - \frac{1}{40}v^2$ — 1 MARK.

(iv) Restarting the motion with $v=0,\ x=0$

$v\frac{dv}{dx} = g - \frac{1}{40}v^2$

$\therefore\ \frac{dv}{dx} = \frac{40g - v^2}{40v}$

$dx/dv = \frac{40v}{400-v^2}$ since $g = 10$

$\therefore\ x = -20\log(400-v^2) + c_2$ ← 1 MARK

At $v=0,\ x=0$

$\therefore\ c_2 = 20\log 400$ ← 1 MARK

$\therefore\ x = 20\log\left(400/400-v^2\right)$

At $x = 20\log 2$ from pt (ii)

$20\log 2 = 20\log\left(400/400-v^2\right)$

$\therefore\ 2 = 400/400-v^2$

$800 - 2v^2 = 400$

$\therefore\ v^2 = 200$

$\therefore\ v = 10\sqrt{2}$ m/s — 1 MARK

MARKING

QUESTION 15:

(i) $A_{large} = \frac{1}{2}4(36)$

$A_1 = \frac{1}{2}h(x+16) \qquad A_2 = \frac{1}{2}(20+x)(4-h)$ — ①

$\therefore h(x+16) + (20+x)(4-h) = 144$

$\therefore hx + 16h + 80 - 20h + 4x - xh = 144$

$-4h + 80 + 4x = 144$

$4x = 64 + 4h$

$x = 16 + h$ — ①

(ii) Volume of the "strip"

$= xy\Delta h$ — ① using "h"

$\therefore$ vol tin $= \lim\limits_{\Delta h \to 0} \sum\limits_{0}^{4} xy\Delta h$

$= \int_0^4 xy\,dh$ — ①

Now $x = 16+h$ and $y = 10 + h/2$

$\therefore$ vol $= \int_0^4 160 + 8h + 10h + h^2/2 \, dh$

$= 160h + 9h^2 + h^3/6 \Big]_0^4$

$= 640 + 144 + \frac{64}{6}$

$= 794\frac{2}{3}$ cm^3 — ② 1 for limits

(b) (i) $t=0, x=0, m=1, v=20$

Two forces are acting against its upwards motion gravity (mg) and air resistance ($\frac{1}{40}v^2 m$) since $m=1$ — ① should mention it is against the direction of motion

$\ddot{x} = -g - \frac{1}{40}v^2$

(ii) $v\frac{dv}{dx} = -g - v^2/40$

$\frac{dv}{dx} = \frac{-40g - v^2}{40v}$

$\therefore \frac{dx}{dv} = \frac{-40v}{40g + v^2}$

$x = -20\log(40g + v^2) + C_1$ — ①

At $x=0, v=20$

$\therefore C_1 = +20\log(40g+400)$ — ①

MARKING

Q 14 (iv) In $z^5+1=0$

Sum of roots $=0$ ← 1

$\therefore \omega_1+\omega_2+\omega_3+\omega_4+-1=0$

$\therefore \omega_2^3+\omega_4^3+\omega_1^3+\omega_3^3=1$

(v) Sum of roots in pairs $=0$

i.e.

$-\omega_1-\omega_2-\omega_3-\omega_4$

$+\omega_1\omega_2+\omega_1\omega_3+\omega_1\omega_4$

$+\omega_2\omega_3+\omega_2\omega_4$

$+\omega_3\omega_4 \quad =0$

~~i.e.~~ $-1+\text{cis}\,\frac{7\pi}{5}\,\text{cis}\,\frac{9\pi}{5}+\text{cis}\,\frac{7\pi}{5}\,\text{cis}\,\frac{\pi}{5}+\text{cis}\,\frac{7\pi}{5}\,\text{cis}\,\frac{3\pi}{5}$

$+\text{cis}\,\frac{9\pi}{5}\,\text{cis}\,\frac{11\pi}{5}+\text{cis}\,\frac{9\pi}{5}\,\text{cis}\,\frac{3\pi}{5}$

$+\text{cis}\,\frac{11\pi}{5}\,\text{cis}\,\frac{3\pi}{5}=0$

$\therefore -1+\text{cis}\,\frac{6\pi}{5}+\text{cis}\,\frac{3\pi}{5}+\text{cis}\,2\pi$

$+\text{cis}\,2\pi+\text{cis}\,\frac{2\pi}{5}+\text{cis}\,\frac{4\pi}{5}=0$

$\therefore 1+\text{cis}\left(-\frac{4\pi}{5}\right)+\text{cis}\,\frac{3\pi}{5}+\text{cis}\,\frac{2\pi}{5}+\text{cis}\,\frac{4\pi}{5}=0$

$\therefore 2\cos\frac{4\pi}{5}+2\sin\frac{2\pi}{5}=-1$

$\therefore \cos\frac{4\pi}{5}+\sin\frac{2\pi}{5}=-\frac{1}{2}$

3 MARKS

1 for getting angles to be between 0 and π

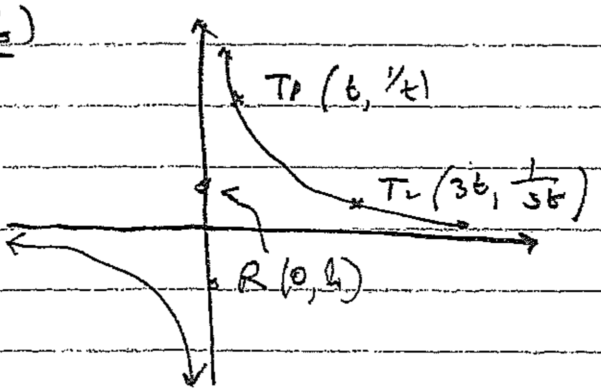
1 for getting all to be in terms of $\frac{2\pi}{5}$ and $\frac{4\pi}{5}$

1 for "cancelling" cos and sin's

QUESTION 14 CONT...

MARKING.

(b)



(i) midpoint is $M = \left(\frac{t+3t}{2}, \frac{\frac{1}{t}+\frac{1}{3t}}{2}\right)$

$\therefore M = \left(2t, \frac{4}{6t}\right)$

$x = 2t \quad y = \frac{2}{3t}$ ← ①

$\therefore y = \frac{2}{3\left(\frac{x}{2}\right)}$

$\frac{xy}{2} = \frac{2}{3}$

$3xy = 4$ } 1 for eliminating t

(ii) $\frac{dy}{dx} = -\frac{1}{x^2}$

At T_1 $\quad m_T = -\frac{1}{t_1^2}$

Equation of normal $\quad y - \frac{1}{t_1} = t_1^2(x - t_1)$

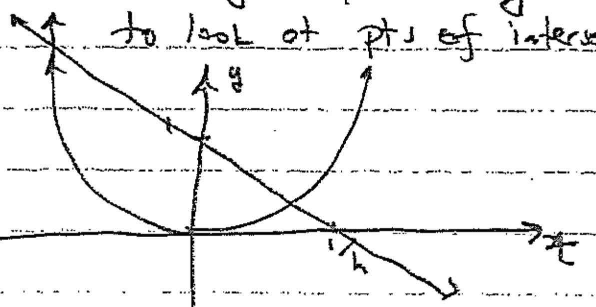
$t_1 y - 1 = t_1^3 x - t_1^4$

ie $\quad t_1^4 - t_1^3 x + t_1 y - 1 = 0$ } 1

(iii) If R is on the normal, R satisfies the equation above.

$\therefore \quad t_1^4 + t_1 h - 1 = 0$ 1

Graphing $\quad y = t_1^4$ and $y = 1 - t_1 h$ to look at pts of intersection



This only ever has 2 solutions ($\frac{1}{h}$ cannot be on y-axis as $\frac{1}{h} \neq 0$ } 2 or similar

QUESTION 16

(a) (i) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}} > 0 \quad \forall x > 0$

OR

At $x = n$, $y = \sqrt{n}$

At $x = n+1$ $y = \sqrt{n+1} > \sqrt{n}$

$\therefore$ increasing

either method for a MARK

(ii) The rectangles drawn are all 1 unit wide

$\therefore$ Area of the "big" rectangles

$\simeq 1 \times [\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}]$ — 1 for realising this concept

The "exact" area is given by

$\int_0^n \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_0^n$

$= \frac{2}{3}n^{\frac{3}{2}}$

and is less than the rectangles

$\therefore \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} > \frac{2}{3}n\sqrt{n}$ — 1 MARK

(iii) $16n^3 + 24n^2 + 9n > 16n^3 + 24n^2 + 9n + 1$

$= (16n^2 + 8n + 1)(n+1)$

$= (4n+1)^2(n+1)$ — 1 for any correct method

(iv) For $n = 1$

LHS $= \sqrt{1}$ RHS $= \frac{7}{6} >$ LHS

$\therefore$ the formula is true for $n = 1$ ← 1 for testing $n = 1$

Assume the formula is true for $n = k$

$\therefore \sqrt{1} + \sqrt{2} + \ldots + \sqrt{k} < \frac{4k+3}{6}\sqrt{k}$

For $n = k+1$

$\sqrt{1} + \sqrt{2} + \ldots + \sqrt{k} + \sqrt{k+1}$

$< \frac{4k+3}{6}\sqrt{k} + \sqrt{k+1}$

(the result from part (iii) can be rewritten as:

$(4n+3)\sqrt{n} < (4n+1)\sqrt{n+1}$

$\therefore \sqrt{1} + \sqrt{2} + \ldots + \sqrt{k} + \sqrt{k+1} < \frac{1}{6}(4k+1)\sqrt{k+1} + \sqrt{k+1}$ — 1 for realising the connection

$= \frac{1}{6}\sqrt{k+1}\,(4k+7)$ ← 2 to get here

~~which is of the same form as for $n = k$~~

$\therefore$ If the formula is true for $n = k$ it is true for $n = k+1$

But it is true for $n = 1$

$\therefore$ " " " " $n = 2$ and so on

ie true $\forall n$

QUESTION 16 CONTD....

(v) $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{10000} < \frac{40003}{6}\sqrt{10000}$

from part (iv)

and from part (ii)

$\sqrt{1}+\sqrt{2}+\ldots+\sqrt{10,000} > \frac{2}{3}\,10000\sqrt{10,000}$

$\therefore \frac{4000300}{6} > \text{EXP}^N > \frac{2000000}{3}$

$\therefore \text{EXP}^N \approx 666,700$

1 MARK

(b)(i) Area of small rectangle $= (b-a)m$.

which is less than the exact area $= \int_a^b f(t)\,dt$

which is less than the area of the large rectangle $= (b-a)M$.

(2)

(ii) Let $y = \frac{1}{1+t}$ be the function above

while $a = 0$ and $b = x$

1 MARK

$\therefore$ The smallest value of y is $\frac{1}{1+x}$

" largest " " " is 1

$\therefore (x-0)\frac{1}{1+x} \leq \int_0^x \frac{1}{1+t}\,dt \leq (x-0)\,1$

1 for here

$\therefore \frac{x}{1+x} \leq \ln(1+t)\Big]_0^x \leq x$

$\therefore x/_{1+x} \leq \ln(1+x) \leq x.$

1 for this

(iii) Set $x = 1$

$\therefore$ from above $\frac{1}{2} \leq \ln 2 \leq 1$

1 MARK.

Doubling all terms

$1 \leq 2\ln 2 \leq 2$

ie $1 \leq \ln 4 \leq 2$