SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC ASSESSMENT TASK 2

MARCH 2007

MATHEMATICS

Extension 1

Time Allowed: 70 minutes

Instructions:

Attempt all questions

Start each question on a new page

Show all necessary working

The marks for each question are indicated next to the question

Marks may be deducted for careless or badly arranged work

Marks indicated are a guide only and may be varied if necessary

Name:______________________________ Teacher:__________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total

QUESTION 1 - (9 marks)

a) You are given $\int_0^a f(x)dx = A$. Evaluate $\int_{-a}^{a} f(x)\,dx$ if **2**

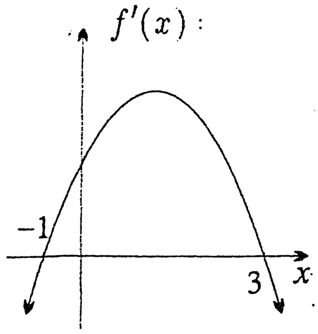
i) $f(x)$ is an even function

ii) $f(x)$ is an odd function

b) Evaluate $\int_0^2 (4-2x)^3\,dx$ **2**

c) For what values of x is $f(x) = x^5 - 5x^4$ concave down? **2**

d) The diagram shows the graph of $f^1(x)$ which is the derivative of a certain function $f(x)$ **3**

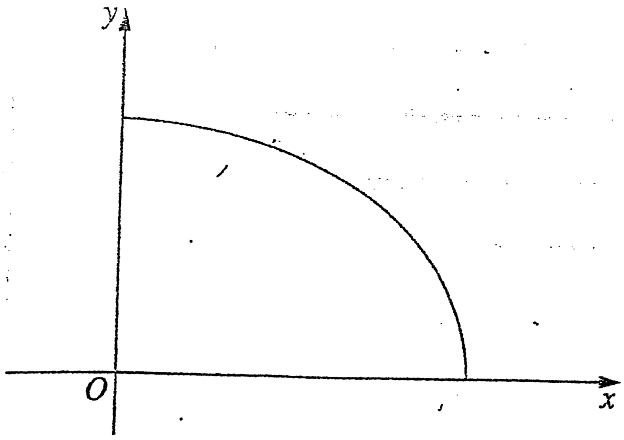


Given that $f(0) = 0$, sketch the graph of $f(x)$

QUESTION 2 - (9 Marks)

a) Find a primitive of $\dfrac{1}{2x^2}$ **1**

b) **4**



Part of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ is shown above

The following table gives values for the graph

x	0	1	2	3	4
y	3	2.90	2.60	1.98	0

i) Use Simpsons rule and all 5 function values to find an approximation to the area under the curve shown above (2 dec).

ii) If the area of the whole ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

is πab, use this result and your answer above to find an approximate value of π to 2 decimal places.

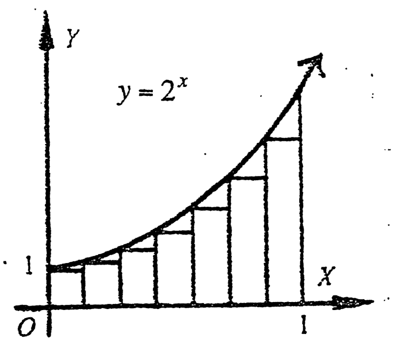
c) Find $\displaystyle\int\frac{x}{\sqrt{4-x}}\,dx$ using the substitution $x=4-u$ **4**

QUESTION 3 - (8 Marks)

a) i) Show that the sum of 3

$$1+2^{a}+2^{2a}+\ldots\ldots\ldots\ldots+2^{(n-1)a}=\frac{2^{na}-1}{2^{a}-1}$$

ii)

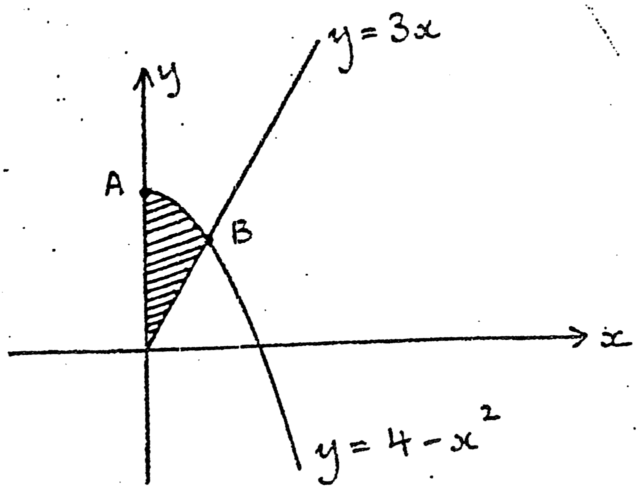


Using 100 inscribed rectangles as shown above, find an approximation for

$$\int_0^1 2^x\,dx$$

you may use the result in part (i)

b) 5



The sketch above shows $y=4-x^2$ and $y=3x$ for $x\geq 0$

i) Find the co-ordinates of A and B

ii) The shaded area is rotated around the y axis. Find the volume of the solid formed.

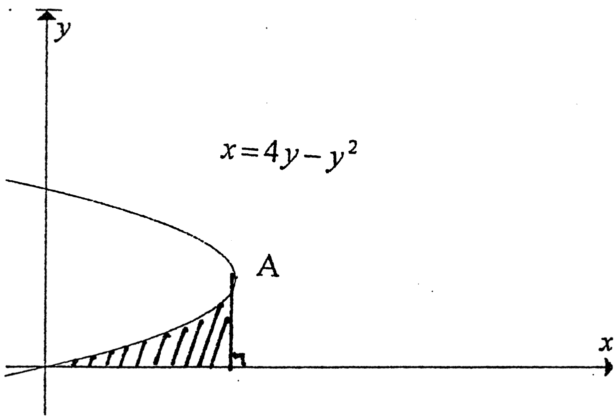
QUESTION 4 - (8 Marks)

a) Suppose the cubic $f(x)=x^3+ax^2+bx+c$ has a relative maximum at $x=\alpha$ and a relative minimum at $x=\beta$. 3

i) Prove that $\alpha+\beta=-\frac{2}{3}a$

ii) Deduce that the point of inflexion occurs at $x=\frac{\alpha+\beta}{2}$

b) 5



i) Find the co-ordinates of A, the vertex of the parabola.

ii) By completing the square, make y the subject of $x=4y-y^2$

iii) Hence or otherwise find the shaded area

QUESTION 5 - (8 Marks)

a) The number of unemployed people u at time t was studied over a period of time. At the start of this period, the number of unemployed was 800 000. **2**

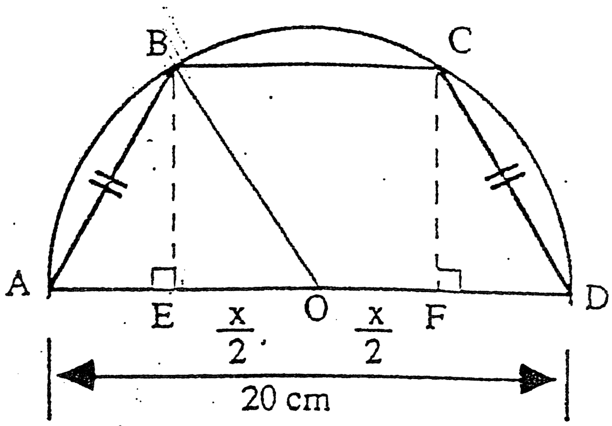
i) Throughout the period, $\frac{du}{dt} < 0$.

What does this say about the number of unemployed during the period?

ii) It is also observed that, throughout the period, $\frac{d^2u}{dt^2} > 0$.

Sketch a graph of u against t.

b) An isosceles trapezium ABCD is drawn with its vertices on a semicircle centre O and diameter 20cm (see diagram). **6**



i) If $EO = OF = \frac{x}{2}$, show that:

$$BE = \frac{1}{2}\sqrt{400 - x^2}$$

ii) Show that the area (A cm^2) of the trapezium ABCD is given by:

$$A = \frac{1}{4}(x + 20)\sqrt{400 - x^2}$$

iii) Show that $\frac{dA}{dx} = \frac{1}{4}\left[\frac{400 - 20x - 2x^2}{\sqrt{400 - x^2}}\right]$

iv) Hence find the length of BC so that the area of trapezium ABCD is a maximum.

QUESTION 6 (8 Marks)

a) i) Solve $\dfrac{x+1}{(x-1)^2} > 0$

For the curve $y = \dfrac{x+1}{(x-1)^2}$

ii) Write down the equations of the asymptotes

iii) Find the co-ordinates of the stationary point and determine its nature.

iv) Sketch the curve showing the stationary point, the asymptotes and any intercepts.

v) Mark on your graph, labelling clearly, the approximate position of any points of inflexion.

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QUESTION 1

a) i) 2A

ii) 0

b) $\int_0^2 (4-2x)^3 \, dx = \left[\frac{(4-2x)^4}{-8}\right]_0^2$

$= 0 - \left(\frac{4^4}{-8}\right)$

$= 32$

c) concave ↓ $f''(x) < 0$

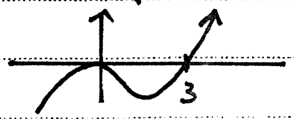
$f(x) = x^5 - 5x^4$

$f'(x) = 5x^4 - 20x^3$

$f''(x) = 20x^3 - 60x^2$

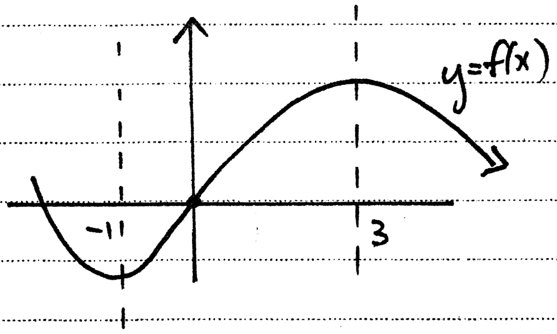
$20x^3 - 60x^2 < 0$

$20x^2(x-3) < 0$



$x < 0$, $0 < x < 3$

d)



QUESTION 2

a) $\frac{1}{2}\int x^{-2} \, dx = \frac{1}{2}\left[\frac{x^{-1}}{-1}\right] + c$

$= \frac{-1}{2x} + c$

b)

x	0	1	2	3	4
y	3	2.9	2.6	1.98	0
	F	y_1	y_2	y_3	L

i) $A = \frac{1}{3}\left[3 + 0 + 4(2.9 + 1.98) + 2(2.6)\right]$

$= 9.24$

ii) $\pi ab = 9.24 \times 4$ $a = 4$ $b = 3$

$12\pi = 9.24 \times 4$

$\pi = 3.08$

c) $x = 4 - u$ $u = 4 - x$

$\frac{dx}{du} = -1$

$dx = -du$

$\int \frac{x}{\sqrt{4-x}} \, dx = \int \frac{4-u}{\sqrt{u}} \cdot -du$

$= -\int (4-u)u^{-1/2} \, du$

$= -\int 4u^{-1/2} - u^{1/2} \, du$

$= -\left[\frac{4u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right] + c$

$= -8\sqrt{u} + \frac{2\sqrt{u^3}}{3} + c$

$= -8\sqrt{4-x} + \frac{2}{3}\sqrt{(4-x)^3} + c$

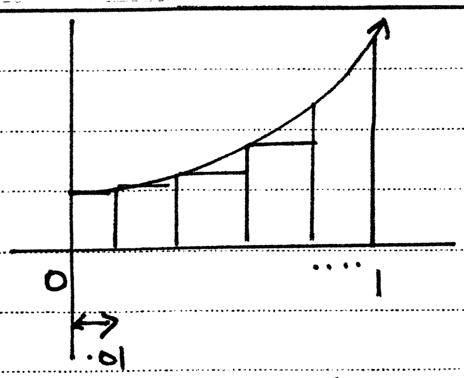
QUESTION 3

a) $a = 1$ $r = 2^a$

i) $1 + 2^a + 2^{2a} + \ldots 2^{(n-1)a}$

T_1 T_2 T_3 T_n

$S_n = \frac{1(2^{na} - 1)}{2^a - 1}$ as required ii)



$$= \frac{1}{100}\left[2^{0} + 2^{.01} + 2^{.02} + \cdots 2^{0.99}\right]$$

$$= \frac{1}{100}\frac{\left[2^{100\times .01} - 1\right]}{2^{.01}-1}$$

$$= \underline{1.44}$$

b) i) $A(0,4)$ $B(1,3)$

$3x = 4 - x^2$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4$ $\underline{x = 1}$

ii)
$$V = \pi\int_0^3 \left(\frac{y}{3}\right)^2 dy + \pi\int_3^4 (4-y)\,dy$$

$$= \pi\left[\frac{y^3}{27}\right]_0^3 + \pi\left[4y - \frac{y^2}{2}\right]_3^4$$

$$= \pi\left[1 + (16-8) - \left(12 - \frac{9}{2}\right)\right]$$

$$= \frac{3\pi}{2} \text{ unit}^3$$

QUESTION 4

a) $f(x) = x^3 + ax^2 + bx + c$

i) max $x = \alpha$

min $x = \beta$

$f'(x) = 3x^2 + 2ax + b$

$f'(\alpha) = 0$ $3\alpha^2 + 2a\alpha + b = 0$ — ①

$f'(\beta) = 0$ $3\beta^2 + 2a\beta + b = 0$ — ②

① − ②

$3\alpha^2 - 3\beta^2 + 2a\alpha - 2a\beta = 0$

$3(\alpha^2 - \beta^2) + 2a(\alpha - \beta) = 0$

$3(\alpha-\beta)(\alpha+\beta) + 2a(\alpha-\beta) = 0$

$3(\alpha+\beta) + 2a = 0$

$3(\alpha+\beta) = -2a$

$$\alpha + \beta = -\frac{2a}{3}$$

ii) Inflexion $f''(x) = 0$

$6x + 2a = 0$

$x = \frac{-2a}{6}$

$x = \frac{-a}{3}$ *

from i) $\alpha + \beta = \frac{-2}{3}a$

$\frac{3}{-2}(\alpha+\beta) = a$ sub into *

$x = -\frac{1}{3}\left(\frac{-3}{2}(\alpha+\beta)\right)$

$$= \frac{\alpha+\beta}{2}$$

b) i) $\underline{A(4,2)}$

ii) $x = 4y - y^2$

$y^2 - 4y + 4 = -x + 4$

$(y-2)^2 = 4 - x$

$$y-2 = \pm\sqrt{4-x}$$

$$y = 2 \pm \sqrt{4-x}$$

use $y = 2 - \sqrt{4-x}$

$$A = \int_0^4 2 - (4-x)^{1/2} \, dx$$

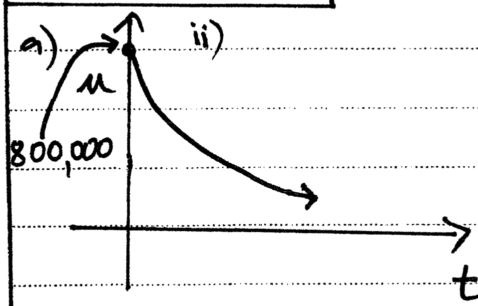
$$= \left[2x + \frac{2(4-x)^{3/2}}{3}\right]_0^4$$

$$= 8 - \left(0 + \frac{16}{3}\right)$$

$$= \frac{8}{3} \text{ unit}^2$$

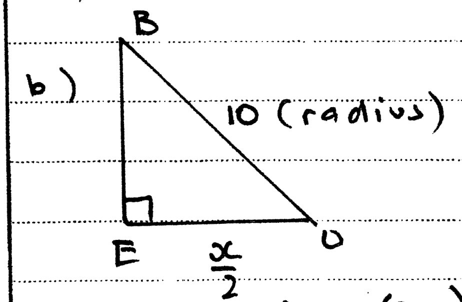
QUESTION 5

a) ii)



i) no. unemployed decreasing

b)



$$10^2 = \left(\frac{x}{2}\right)^2 + (BE)^2$$

$$(BE)^2 = 100 - \frac{x^2}{4}$$

$$BE = \sqrt{\frac{400 - x^2}{4}}$$

$$BE = \frac{1}{2}\sqrt{400 - x^2}$$

ii) $A_{\text{Trap}} = \frac{1}{2} h (a+b)$

$$= \frac{1}{2}\left(\frac{1}{2}\sqrt{400-x^2}\right)(20+x)$$

$$= \frac{1}{4}(x+20)\left(\sqrt{400-x^2}\right)$$

iii) $u = \frac{1}{4}(x+20)$ $\quad v = (400-x^2)^{1/2}$

$u' = \frac{1}{4}$ $\quad v' = \frac{1}{2} \cdot -2x(400-x^2)^{-1/2}$

$$v' = \frac{-x}{\sqrt{400-x^2}}$$

$$\frac{dA}{dx} = \frac{\sqrt{400-x^2}}{4} - \frac{x(x+20)}{4\sqrt{400-x^2}}$$

$$= \frac{400-x^2 - x(x+20)}{4\sqrt{400-x^2}}$$

$$= \frac{400 - x^2 - x^2 - 20x}{4\sqrt{400-x^2}}$$

$$= \frac{1}{4}\left[\frac{400 - 2x^2 - 20x}{\sqrt{400-x^2}}\right]$$

iv) st pt $A' = 0$

$$400 - 2x^2 - 20x = 0$$

$$200 - x^2 - 10x = 0$$

$$(x-10)(x+20) = 0$$

$$x = 10$$

x	10^-	10	10^+
A'	+	0	−

$\therefore$ Max $x = 10$

BC = 10 cm

QUESTION 6

i) $\dfrac{(x+1)}{(x-1)^2} > 0$ $\quad x+1>0$

$x > -1$

but $x \neq 1$

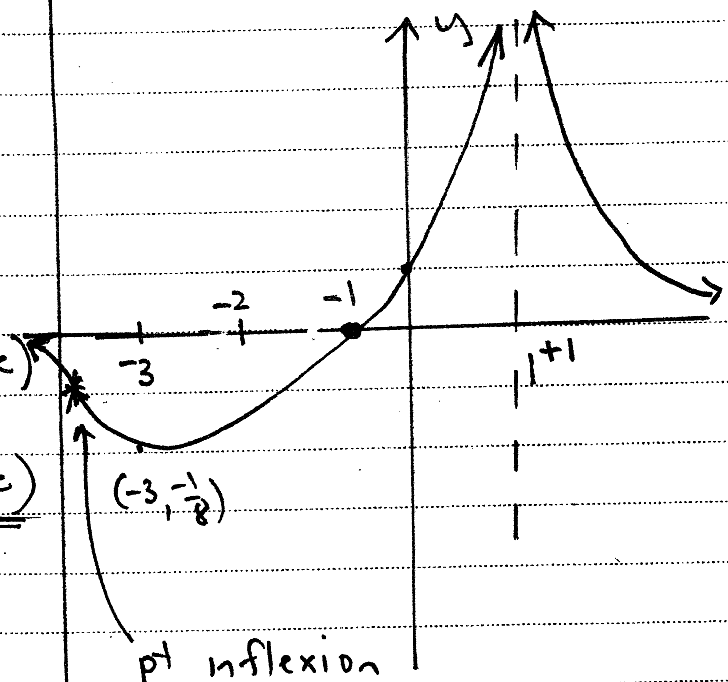
ii) asymptotes $x = 1$

$x \to \infty$ $\quad y = 0^+$

(from +ve side)

$x \to -\infty$ $\quad y = 0^-$

(from −ve side)



iii) $u = x+1$ $\quad v = (x-1)^2$

$u' = 1$ $\quad v' = 2(x-1)^1$

$$y' = \frac{(x-1)^2 - 2(x+1)(x-1)}{(x-1)^4}$$

st pt.

$(x-1)^2 - 2(x+1)(x-1) = 0$

$(x-1)\left[(x-1) - 2(x+1)\right] = 0$

$x = 1$ $\quad x - 1 - 2x - 2 = 0$

$-x - 3 = 0$

$x = -3$.

$x \neq 1$ ∴

st pt $\quad x = -3$

x	-4	-3	-2
y'	$-$	0	$+$

$- \setminus \;\; \underset{0}{\_} \;\; / +$

min

at $\left(-3, \frac{-1}{8}\right)$