SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION 1

YEAR 11 YEARLY EXAMINATION

2002

Time allowed: 90 minutes

Instructions:

- Show all necessary working
- Start each question on a new page
- All questions are of equal value
- Marks will not be awarded for careless or badly arranged work
- Non-programmable calculators may be used
- This paper must be handed in with your answer sheets
- Answers must be written in blue or black pen

Name:______________________________

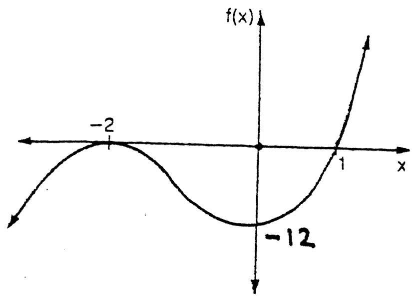
Class: _______________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	TOTAL

Question 1 **Marks**

a) If $(x+1)$ is a factor of $P(x) = x^3 - ax + 3$. Find the value of a. **1**

b) **2**



Write down the equation of the polynomial function (in factored form)

c) A parabola is symmetrical about the line $y = 2$ it has a focal length 3 units and the equation of the directrix is $x = 1$. **3**

i) How many parabolas satisfy these conditions?

ii) If the vertex is (4, 2) find the equation of the parabola

d) Solve $\frac{x+1}{x-1} \leq 0$ **2**

e) The roots of the quadratic equation $(k+2)x^2 - 4x + k^2 = 0$ are reciprocals. Find the value/s of k. **3**

Question 2 **Marks**

a) A polynomial of degree 7 is divided by the polynomial Q(x), the remainder is $x^2 + x + 2$. What is the least degree of Q(x). **1**

b) For the quadratic equation $x^2 + (k-3)x + 2 - k = 0$ **3**

i) Find the value of the discriminant in the terms of k

ii) Explain why the roots of this quadratic equation are real for all values of k

c) If $a + b = 1$ and $a^2 + b^2 = 2$ **3**

i) Find the value of ab

ii) Hence find the value of $a^3 + b^3$

d) i) Write $x^{-\frac{1}{2}}$ with a positive index **4**

ii) Solve $x^{\frac{1}{2}} + 10x^{-\frac{1}{2}} = 7$

Question 3 **Marks**

a) For the function $y = \sqrt{x^2 - 4}$ **3**

Write down

i) the domain

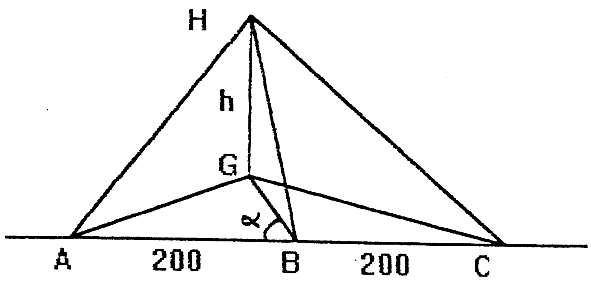
ii) the range

b) The points $P(12t, 6t^2)$ and Q (36, 54) are points on a parabola **3**

i) Find the cartesian equation of the parabola

ii) If PQ is a focal chord find the value of t

c)



A cyclist riding along a straight flat road passes by three stop signs A, B and C spaced 200m apart. From these three signs the respective angles of elevation to the top of a mobile phone tower are 45°,45° and 30°. If 'h' is the height of the tower GH. **5**

i) Show that $CG = \sqrt{3}\,h$.

ii) If $\angle GBA = \alpha$. Find two different expressions for $\cos\alpha$ in terms of h.

iii) Hence find the height of the tower.

Question 4 **Marks**

a) i) Simplify
$(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ **3**

ii) The roots of $x^3-4x^2-8=0$ are α, β and γ. Use the result in part (i) to find The value of $\alpha^2+\beta^2+\gamma^2$.

b) i) Show that $\dfrac{1+\cos 2A}{\sin 2A}=\cot A$ **4**

ii) Hence find the exact value of cot 15°

c) T $(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus S. **4**
P is the point which divides ST internally in the ratio 1 : 2.

i) Write down the coordinates of P in terms of t.

ii) Hence show that as T moves on the parabola $x^2=4y$ that the locus of P is the parabola $9x^2=12y-8$

Question 5

a) The roots of the equation $x^3-6x^2+5x+8=0$ are α, β, γ **3**

The roots of the equation $x^3+ax^2+bx+512=0$ are $k\alpha$, $k\beta$, $k\gamma$

i) Find the value of k

ii) Hence find the value of b.

b) Consider the points A(-2, 3) B(6, 5) the point P(x, y) moves so that the angle APB = 90° **4**

i) Write down an expression for the gradient of AP

ii) Show that the locus of P is a circle

iii) Find its centre and radius.

c) i) Expand tan $(A+B)$

4

ii) The roots of $x^2-2x-1=0$ are $\tan A$ and $\tan B$. If A and B are acute find the size of $A+B$.

Question 6 **Marks**

a) i) Derive the equation of the tangent to the parabola $x^2 = 4ay$ at the point P $(2at, at^2)$ **7**

ii) The tangent cuts the y axis at R. Find the co-ordinates of R.

iii) If S is the focus of the parabola. Find the length of PS.

iv) Prove that the triangle PSR is isosceles

v) If $< PSR = 120°$. Find the numerical value of t.

b) If $P(x) = 4x^3 + 9x - 4$ **4**

i) Find $P(\alpha + 1)$

ii) If α is a root of $P(x)$ use part (i) to help show that $P(\alpha + 1) > 0$

Question 1

a) $P(-1)=0 \quad \therefore -1+a+3=0$

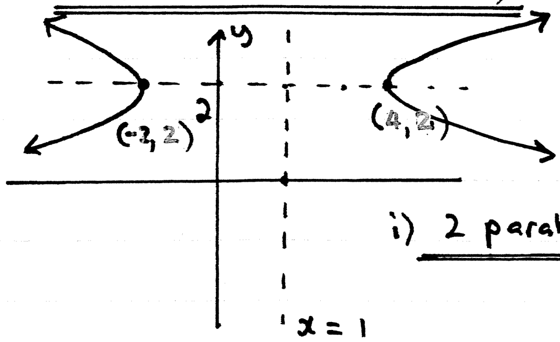
$\underline{a=-2}$

b) $P(x) = A(x+2)^2(x-1)$

sub $(0,-12) \quad \therefore A = 3$

$\underline{\underline{P(x) = 3(x+2)^2(x-1)}}$

c)



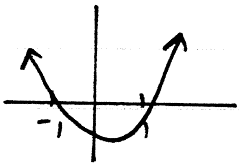
i) $\underline{\underline{2 \text{ parab.}}}$

ii) $\underline{\underline{(y-2)^2 = 12(x-4)}}$

d) $\dfrac{x+1}{x-1} \leq 0$

$\dfrac{\cancel{(x-1)^2}(x+1)}{\cancel{(x-1)}} \leq 0.(x-1)^2$

$(x-1)(x+1) \leq 0$



$\underline{\underline{x: \ -1 \leq x < 1}}$

e) $\alpha, \dfrac{1}{\alpha}$ roots

$\therefore$ prod $= 1$

$\dfrac{k^2}{k+2} = 1$

$k^2 = k+2$

$k^2 - k - 2 = 0$

$(k-2)(k+1) = 0$

$\underline{\underline{\therefore k = 2, \ k = -1}}$

$\therefore \quad \sqrt{x} = 5 \qquad \sqrt{x} = 2$

$\underline{\underline{x = 25 \qquad x = 4}}$

Question 2

a) $Q(x)$ deg 3

b) i) $\Delta = (k-3)^2 - 4.1(2-k)$

$= k^2 - 6k + 9 - 8 + 4k$

$\underline{\underline{\Delta = k^2 - 2k + 1 = (k-1)^2}}$

ii)

since $\Delta \geq 0$ for all values of k

$\underline{\underline{\therefore \text{roots real}}}$

c) i) $(a+b)^2 = a^2+b^2 + 2ab$

$(a+b)^2 - (a^2+b^2) = 2ab$

$1 - 2 = 2ab$

$\underline{\underline{\therefore ab = -\frac{1}{2}}}$

ii)

a^3+b^3

since $(a+b)^3 =$

$a(a^2+2ab+b^2) + b(a^2+2ab+b^2)$

$a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore \ a^3+b^3 = (a+b)^2 - 3a^2b - 3ab^2$

$= (a+b)^2 - 3ab(a+b)$

$= 1 - 3 \times -\frac{1}{2} . 1$

$\underline{\underline{= 2\tfrac{1}{2}}}$

d) i) $x^{-1/2} = \dfrac{1}{\sqrt{x}}$

ii) $x^{1/2} + 10x^{-1/2} = 7$

$\sqrt{x} + \dfrac{10}{\sqrt{x}} = 7 \qquad$ Let $u = \sqrt{x}$

$u + \dfrac{10}{u} = 7$

$u^2 + 10 = 7u$

$u^2 - 7u + 10 = 0$

$(u-5)(u-2) = 0$

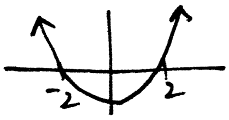
$u = 5 \qquad u = 2$

Question 3

a) $y = \sqrt{x^2 - 4}$

$x^2 - 4 \geqslant 0$

$(x-2)(x+2) \geqslant 0$



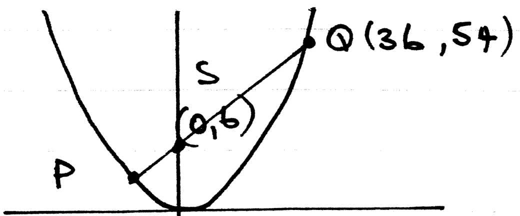
i) D: $\underline{\underline{x \geqslant 2 \; , \; x \leqslant -2}}$ (2)

ii) R: $\underline{\underline{y \geqslant 0}}$ (1)

b) i) $P(12t, 6t^2)$

$x = 12t \quad \therefore t = \dfrac{x}{12} \qquad y = 6\left(\dfrac{x}{12}\right)^2$

$\therefore y = \dfrac{x^2}{24}$ OR $x^2 = 24y$ (1)



find eqn of SQ

$m_{SQ} = \dfrac{4}{3}$

$\therefore$ eqn SQ $\quad y = \dfrac{4}{3}x + 6$

sub $P(12t, 6t^2)$

$6t^2 = \dfrac{4}{3} \cdot 12t + 6$

$6t^2 - 16t - 6 = 0$

$3t^2 - 8t - 3 = 0$

$(3t+1)(t-3) = 0$

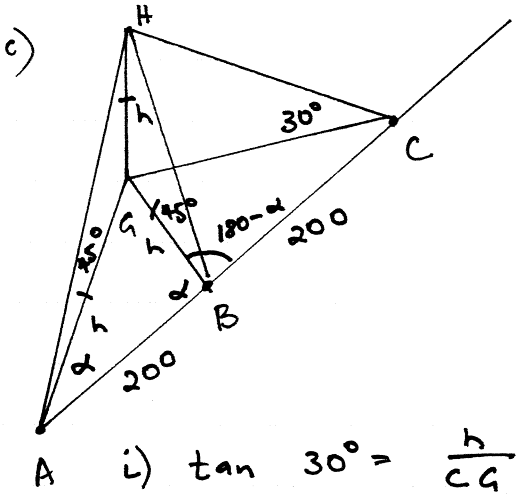
$t = 3 \qquad t = -1/3$ (1)

$t = 3 \Rightarrow Q(36, 54)$

$\therefore \underline{\underline{t = -\tfrac{1}{3}}} \Rightarrow P(12t, 6t^2)$

$\underline{\underline{P\left(-4, \tfrac{2}{3}\right)}}$ (1)

c)



i) $\tan 30° = \dfrac{h}{CG}$

$\therefore \underline{\underline{CG = \sqrt{3}\, h}}$ (1)

ii)

In $\Delta ABG \Rightarrow \cos\alpha = \dfrac{h^2 + 200^2 - h^2}{2 \cdot h \cdot 200}$

$\cos\alpha = \dfrac{200^2}{400h}$

(1) — $\cos\alpha = \dfrac{100}{h}$ — A

In ΔGBC

$\Rightarrow \cos(180 - \alpha) = \dfrac{h^2 + 200^2 - (\sqrt{3}h)^2}{2 \times h \times 200}$

$-\cos\alpha = \dfrac{20{,}000 - h^2}{200h}$ — B

In $\Delta AGC \Rightarrow \cos\alpha = \dfrac{h^2 + 400^2 - (\sqrt{3}h)^2}{800h}$

(2)

$\cos\alpha = \dfrac{80{,}000 - h^2}{400h}$ — C

(3)

any comb of A, B, C.

$-\dfrac{100}{h} = \dfrac{20{,}000 - h^2}{200h}$

$-20000 = 20{,}000 - h^2$

$h^2 = 40{,}000$

$\underline{\underline{h = 200\text{ m}}}$ (1)

Question 4

a) i) $(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= \alpha(\alpha+\beta+\gamma)+\beta(\alpha+\gamma+\beta)+\gamma(\alpha+\beta+\gamma)$
$\quad -2\alpha\beta-2\beta\gamma-2\gamma\alpha$

$= \underline{\underline{\alpha^2+\beta^2+\gamma^2}}$

ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\left(\begin{matrix}\text{roots}\\ \text{2 at time}\end{matrix}\right)$

$a=1 \quad b=-4 \quad c=0 \quad d=-8$

$=(\ 4\ )^2 - 2(\ 0\ )$

$\underline{\underline{=16}}$

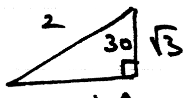
b) i) LHS $= \dfrac{1+\cos^2 A-\sin^2 A}{2\sin A.\cos A}$

$= \dfrac{\cancel{2}\cos^{\cancel{2}} A}{\cancel{2}\sin A.\cancel{\cos} A}$

$= \dfrac{\cos A}{\sin A}$

$= \cot A$

$\underline{\underline{= \text{RHS}}}$



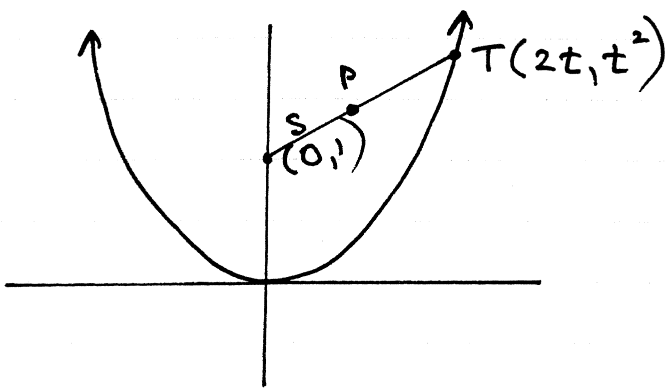
ii) $\cot 15^\circ = \dfrac{1+\cos 30^\circ}{\sin 30^\circ}$

$= \left(1+\dfrac{\sqrt{3}}{2}\right) \div \left(\dfrac{1}{2}\right)$

$= \left(\dfrac{2+\sqrt{3}}{\cancel{2}}\right) \times \dfrac{\cancel{2}}{1}$

$\underline{\underline{= 2+\sqrt{3}}}$

c)



i) $S(0,1) \qquad T(2t, t^2)$

$1:2$ internally

$P\left(\dfrac{0+2t}{3}, \dfrac{2+t^2}{3}\right)$

$\therefore P\left(\dfrac{2t}{3}, \dfrac{2+t^2}{3}\right)$

ii) $T(2t, t^2)$

$x = 2t \qquad y = t^2$

$\dfrac{x}{2} = t \quad \longrightarrow \quad y = \left(\dfrac{x}{2}\right)^2$

$\underline{\underline{\text{locus of } T \text{ is } 4y = x^2}}$

$P\left(\dfrac{2t}{3}, \dfrac{2+t^2}{3}\right)$

$x = \dfrac{2t}{3} \qquad y = \dfrac{2+t^2}{3}$

$\dfrac{3x}{2} = t \qquad 3y = 2 + \left(\dfrac{3x}{2}\right)^2$

$4\times 3y = 8 + 9x^2$

$\underline{12y-8 = 9x^2}$

Question 5

a) i) $\alpha+\beta+\gamma = 6$

$k\alpha + k\beta + k\gamma = -a$

$k(\alpha+\beta+\gamma) = -a$

$\therefore \quad 6k = -a$

$\alpha\beta\gamma = -8$

$k^3(\alpha\beta\gamma) = -512$

$-8k^3 = -512$

$k^3 = 64$

$\underline{\underline{k=4}} \qquad \therefore a = -24$

$\alpha\beta + \alpha\gamma + \beta\gamma = 5$

$k^2\alpha\beta + k^2\alpha\gamma + k^2\beta\gamma = b$

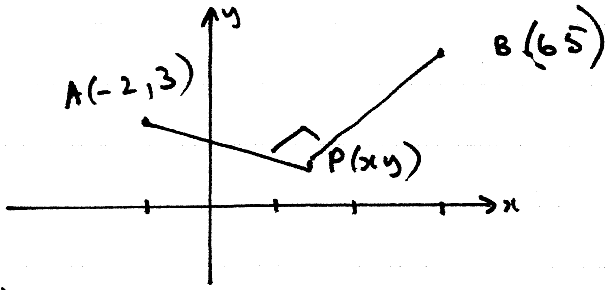
$k^2(\alpha\beta + \alpha\gamma + \beta\gamma) = b$

$k^2 . 5 = b$

$16 \times 5 = b$

$\underline{\underline{b = 80}}$

b)



i) $m_{AP} = \dfrac{y-3}{x+2}$

ii) $\left(\dfrac{y-3}{x+2}\right) . \left(\dfrac{y-5}{x-6}\right) = -1$

$y^2 - 8y + 15 = -1(x^2 - 4x - 12)$

$y^2 - 8y + 15 = -x^2 + 4x + 12$

$x^2 + y^2 - 4x - 8y - 3 = 0$

$(x^2 - 4x + 4) + (y^2 - 8y + 16) = 17$

$(x-2)^2 + (y-4)^2 = \sqrt{17}^2$

centre $(2,4)$ radius $\sqrt{17}$

c) i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$

ii) Root $\tan A$, $\tan B$

Sum $\tan A + \tan B = 2$

$\tan A . \tan B = -1$

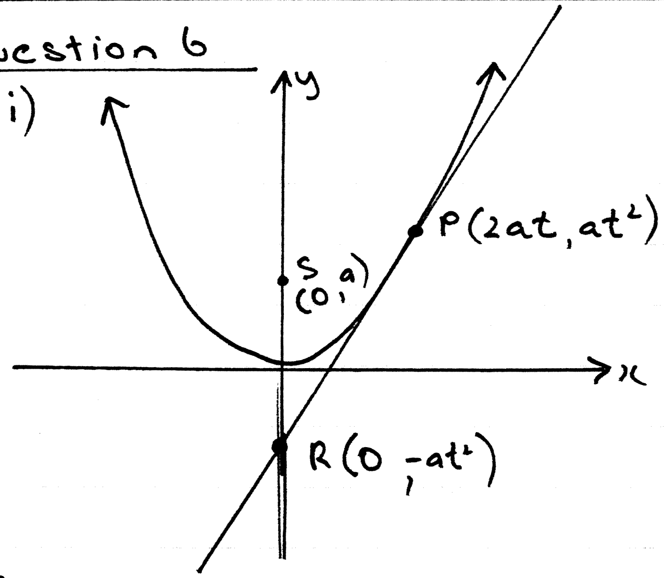
$\tan(A+B) = \dfrac{2}{1 - -1}$

$\tan(A+B) = 1$

$A+B) = 45°$

Question 6

a) i)



i) $x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{2x}{4a} = \dfrac{x}{2a}$

at P : $m_T = \dfrac{2at}{2a} = t$

$\therefore$ eqn tang at P $y - at^2 = t(x - 2a$

$\therefore \underline{\underline{y = tx - at^2}}$

ii) $\underline{\underline{R(0, -at^2)}}$

iii) $PS = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$

$= \sqrt{a^2(t^2-1)^2 + 4a^2t^2}$

$= a\sqrt{t^4 - 2t^2 + 1 + 4t^2}$

$= a\sqrt{t^4 + 2t^2 + 1}$

$= a\sqrt{(t^2+1)^2}$

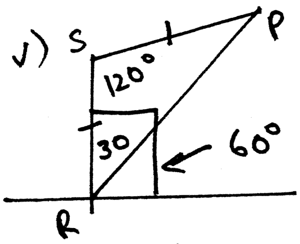
$\underline{\underline{PS = a(t^2+1)}}$

iv) $SR = a + at^2$

$= a(1+t^2)$

$SR = PS$ $\therefore$ ΔPSR isosceles

v)



$\therefore \tan 60° = t$ (grad of tang)

$\therefore \underline{\underline{\sqrt{3} = t}}$

b) $p(x) = 4x^3 + 9x - 4$

i) $P(\alpha+1) = 4(\alpha+1)^3 + 9(\alpha+1) - 4$

ii) α a root $\therefore$

$p(\alpha) = 0$

$4\alpha^3 + 9\alpha - 4 = 0$

$$P(\alpha+1) = 4(\alpha^3 + 3\alpha^2 + 3\alpha + 1) + 9\alpha + 9 - 4$$

$$= 4\alpha^3 + 12\alpha^2 + 12\alpha + 4 + 9\alpha + 9 - 4$$

$$= 4\alpha^3 + 12\alpha^2 + 21\alpha + 9$$

$$= (4\alpha^3 + 9\alpha - 4) + 12\alpha^2 + 12\alpha + 13$$

$$\therefore P(\alpha+1) = \underbrace{12\alpha^2 + 12\alpha + 13}_{\substack{\text{+ve def} \\ \text{since } a>0 \ \Delta<0}}$$

$\therefore P(\alpha+1) > 0$ for all α