Name:____________________________ Class:____________________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 ASSESSMENT TASK

EXTENSION 2

MATHEMATICS

MARCH 2006

Instructions

- Attempt all questions
- Answers to be written on the paper provided
- Start each question on a new page
- Marks may not be awarded for careless or badly arranged working
- Indicated marks are a guide and may be changed slightly if necessary
- These questions must be handed in attached to the top of your solutions.

Q1	Q2	Q3	Total
/16	/17	/17	

Question 1 (16 marks)

a) If $z = -1 + i$ find, (4)

(i) $\bar{z}$

(ii) $|z|$

(iii) $\arg(iz)$

b) i. Express z $= -\sqrt{3} - i$ in modulus – argument form (3)

ii. Hence write z^{12} in the form $x + iy$, where x and y are real

c) Find all complex numbers z, such that $z^3 = 64\ i$ (3)

d) Given that a and b are real numbers, find a and b if, $\dfrac{5+2i}{a+bi}=1+i$ (3)

e) On an argand diagram shade the region containing all points representing complex numbers z such that $2 \le \text{Re}(z) \le 5 \quad and \quad -2 \le \text{Im}(z) \le 4$

Question 2 (17 marks) **(Start a new page)**

a) Evaluate $\int_0^1 x\sqrt{1-x}\,dx$ using a suitable substitution. (3)

b) Consider the ellipse $3x^2+4y^2=12$ (6)

i. Determine the eccentricity of the ellipse

ii. Find the coordinates of the foci S and S^1 and also the equation of of the directrices.

iii. Sketch the ellipse showing all important information.

c)

i. Show that the equation of the tangent to the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ at the point P $(3\cos\theta, 2\sin\theta)$ is given by $\dfrac{x\cos\theta}{3}+\dfrac{y\sin\theta}{2}=1$ (3)

The ellipse above cuts the y axis at the points A and B (5)

The tangents to the ellipse at A and B meet the tangent to the ellipse at $P(3\cos\theta, 2\sin\theta)$ at the points C and D respectively.

ii. Draw a neat diagram showing the positions of P, A, B, C and D.

iii. Show that AC $\times$ BD = 9

Start a new page

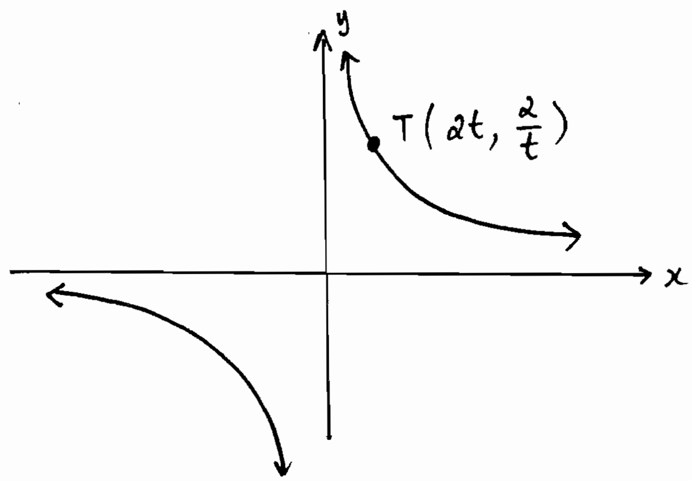
Question 3 (17 marks)

a) Show that the locus specified by $3|z-(4+4i)| = |z-(12+12i)|$ is a circle, (4)

Write down its radius and the coordinates of its centre

b) Find the equation of the tangent to the curve $x^2 + xy^2 - 6y = 0$ at the point (2,1) (4)

c) Consider the diagram below



i. Show that the tangent to the hyperbola $xy = 4$ at the point T $(2t, \frac{2}{t})$ has equation $x + t^2 y = 4t$. (2)

ii. This tangent cuts the x – axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2 x - y = 4t^3$. (2)

iii. This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates M $(2t, -2t^3)$. (2)

iv. Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. (3)

End of Test

March 2006 Ext II

Solutions

Question 1

a) i. $\overline{z} = -1 - i$ ✓

ii. $\sqrt{2}$ ✓

iii. $\arg(z) = \frac{3\pi}{4}$ $3\pi/4$

$\arg(iz) = \pi/2 + 3\pi/4$

$= 5\pi/4$ ✓

$= -\frac{3\pi}{4}$ ✓

b) i. $z = -\sqrt{3} - i$ $-\sqrt{3}$ 1 θ $(-\sqrt{3}, -1)$

$\angle = -\frac{5\pi}{6}$

$|z| = 2$

$\therefore z = 2\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)$ ✓

ii. $z^{12} = 2^{12}\,\mathrm{cis}\left(12 \times -\frac{5\pi}{6}\right)$ ✓

$= 4096(1 + i0)$

$= 4096$ ✓ ie 2^{12}

c) $z^3 = 64i$

$z^3 - 64i = 0$

$z^3 + (4i)^3 = 0$

$(z + 4i)(z^2 - 4iz - 16) = 0$ ✓

$z = -4i$ ✓, $z = 2i \pm 2\sqrt{3}$

d) $5 + 2i = (1+i)(a+bi)$ ✓

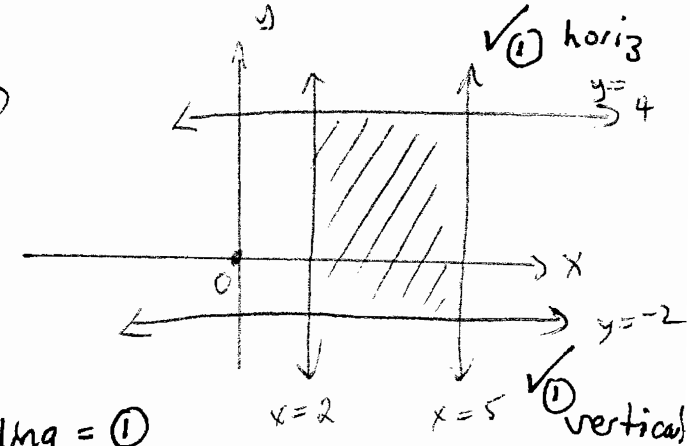
$= a + bi + ai - b$

$\therefore a - b = 5$ ✓

$a + b = 2$

$2a = 7$

e)



shading = ①

Question 2

a) $u = 1 - x$ $\qquad x = 1 - u$

$\frac{du}{dx} = -1$ $\qquad x = 1 \quad u = 0$

$\qquad x = 0 \quad u = 1$

$-du = dx$ ①

$\therefore \int_1^0 (1-u)\sqrt{u}\,.\,-du$ ①

$-\int_1^0 \sqrt{u} - u^{3/2}\,du$

$= \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$= \left(2/3 - 2/5\right) - (0)$

$= \frac{4}{15}$ ①

b) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ ✓

$a = 2 \quad b = \sqrt{3}$

$\therefore \quad b^2 = a^2(1 - e^2)$

$3/4 = 1 - e^2$

$e^2 = \frac{1}{4}$ ✓

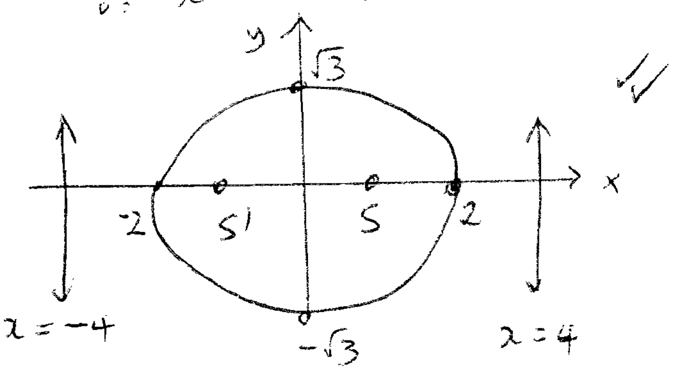
$e = \frac{1}{2}$

$S(ae, 0) \quad S'(-ae, 0)$

$S = (1, 0) \quad S'(-1, 0)$ ✓

directrices $x = \pm a/e$

$\therefore \quad x = \pm 4$ ✓



c) $\frac{2x}{9} + \frac{2y}{4} \cdot \frac{dy}{dx} = 0$ ✓

$\frac{dy}{dx} = -\frac{2x}{9} \times \frac{4}{2y}$

$= -\frac{4}{9}\frac{x}{y}$

$M_T = \frac{-4 \cdot 3\cos\theta}{9 \cdot 2\sin\theta}$

$= \frac{-2\cos\theta}{3\sin\theta}$ ✓

✓

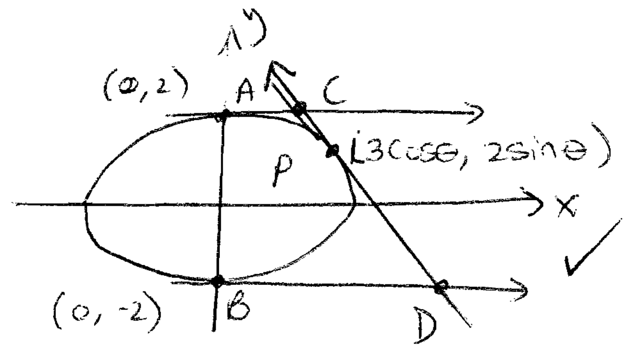
$y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

$3\sin\theta\, y - 6\sin^2\theta = -2\cos\theta\, x + 6\cos^2\theta$

$2\cos\theta\, x + 3\sin\theta\, y = 6$

$\therefore \quad \frac{\cos\theta\, x}{3} + \frac{\sin\theta\, y}{2} = 1$

d)



✓

tangent at A $y = 2$

at B $y = -2$

tangent at P

$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$ (1)

Finding C sub $y = 2$ into (1)

$\frac{x\cos\theta}{3} + \sin\theta = 1$

$\frac{x\cos\theta}{3} = 1 - \sin\theta$

✓ $x = \frac{3(1 - \sin\theta)}{\cos\theta}$

$C = \left[\frac{3(1 - \sin\theta)}{\cos\theta}, 2\right]$

Finding D sub $y = -2$ into (1)

✓ $\frac{x\cos\theta}{3} - \sin\theta = 1$

$D = \left[\frac{3(1 + \sin\theta)}{\cos\theta}, -2\right]$

$KC \times BD = \frac{3(1-\sin\theta)}{\cos\theta} \times \frac{3(1+\sin\theta)}{\cos\theta}$

$= 9\frac{(1-\sin^2\theta)}{\cos^2\theta}$

$= 9$

<u>Question 3</u>

i) $3|x+iy-4-4i| = |x+iy-12-12i|$

$3|(x-4)+(y-4)i| = |(x-12)+(y-12)i|$

$9[(x-4)^2+(y-4)^2] = (x-12)^2+(y-12)^2$

$9[x^2-8x+16+y^2-8y+16] = x^2-24x+144 + y^2-24y+144$

$8x^2-48x+8y^2-48y=0$

$x^2-6x+y^2-6y=0$

$(x-3)^2+(y-3)^2 = 18$

$\therefore$ centre $(3,3)$ radius $= 3\sqrt{2}$.

b) Implicit diff

$\cdots + 1.y^2 + x.2y.\frac{dy}{dx} - 6\frac{dy}{dx} = 0$ 1

$2x + y^2 + \frac{dy}{dx}(2xy-6) = 0$

$\frac{dy}{dx} = \frac{-2x-y^2}{2xy-6}$ 1

$\therefore$ at $(2,1)$ $M_T = \frac{-2(2)-1}{2(2)(1)-6}$

$= \frac{5}{2}$ 1

$\therefore$ eq tangent

$y-1 = 5/2(x-2)$ ← 1

c) $y = 4/x$ $y' = \frac{-4}{x^2}$ at $x = 2t$

I. $M_T = \frac{-4}{4t^2}$

$= -1/t^2$ 1

$\therefore$ eq $y - \frac{2}{t} = -\frac{1}{t^2}(x-2t)$ 1

$t^2y - 2t = -x+2t$

$\therefore x + t^2y = 4t$ QED

II. x-int $y=0$ $x=4t$ $\therefore Q(4t,0)$

$M_\perp = t^2$ 1

eq: $y - 0 = t^2(x-4t)$

$y = t^2x - 4t^3$

$\therefore t^2x - y = 4t^3$

III. Solve $t^2x - y = 4t^3$ and $xy = 4$ simultaneously

$y = t^2x - 4t^3$ & $xy = 4$

$\therefore x(t^2x-4t^3) = 4$

$t^2x^2 - 4t^3x - 4 = 0$ 1

as the 2 x values for this give R & S let them be α & β

$\alpha+\beta = \frac{4t^3}{t^2} = 4t$

$\therefore$ midpt $\frac{\alpha+\beta}{2} = 2t$ 1

and y value $= -2t^3$

iv. Locus of M

$x = 2t \longrightarrow \frac{x}{2} = t$

Sub into $y = -2t^3$

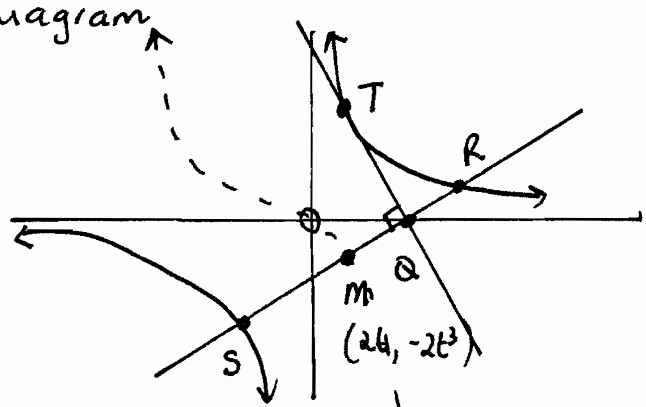
$$y = -2\left(\frac{x}{2}\right)^3$$

$$y = -2 \cdot \frac{x^3}{8}$$

$$y = \frac{x^3}{-4}$$

$$-4y = x^3 \quad (1)$$

diagram



Locus of M moves on the cubic equation in the 2nd & 4th quadrants (1)

restriction $\rightarrow$ excluding the origin as $t \neq 0$. (1)

Alternative Sol^n to Q3 (c) iii

$$t^2x^2 - 4t^3x - 4 = 0$$

$$x = \frac{4t^3 \pm 4t\sqrt{t^4+1}}{2t^2}$$

$$= 2t \pm \frac{2}{t}\sqrt{t^4+1}$$

$$R\left[2t + \frac{2}{t}\sqrt{t^4+1},\ -2t^3 + 2t\sqrt{t^4+1}\right]$$

$$S\left[2t - \frac{2}{t}\sqrt{t^4+1},\ -2t^3 - 2t\sqrt{t^4+1}\right]$$

Midpt $M = \left[\frac{4t}{2},\ \frac{-4t^3}{2}\right]$

$$= (2t, -2t^3)$$