SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC ASSESSMENT TASK 2

MARCH 2008

MATHEMATICS

Extension 1

Time Allowed: 70 minutes

Instructions:

- Attempt all questions
- Start each question on a new page
- Show all necessary working
- The marks for each question are indicated next to the question
- Marks may be deducted for careless or badly arranged work
- Marks indicated are a guide only and may be varied if necessary

Name: ______________________________ Teacher: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Total

QUESTION 1 (10 Marks)

a) Find the gradient of the tangent to the curve $y = \cos^3 x$ at $x = \frac{\pi}{6}$ **2**

b) Evaluate $\lim_{x \to 0} \frac{\sin 5x}{4x}$ **1**

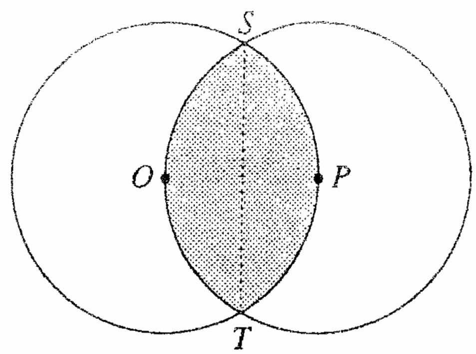
c) Write a primitive for $(5 - 2x)^4$ **2**

d) Find $\int \frac{x\ dx}{(1+x^2)^2}$ by first differentiating $\frac{x^2}{1+x^2}$ **2**

e) Evaluate $\int_0^{\sqrt{3}} \frac{x\ dx}{\sqrt{1+x^2}}$ using the substitution $u = 1 + x^2$ **3**

QUESTION 2 (10 Marks)

a) The points O and P in the plane are d cm apart. A circle centre O is drawn to pass through P, and another circle centre P is drawn to pass through O. The two circles meet at S and T, as in the diagram.

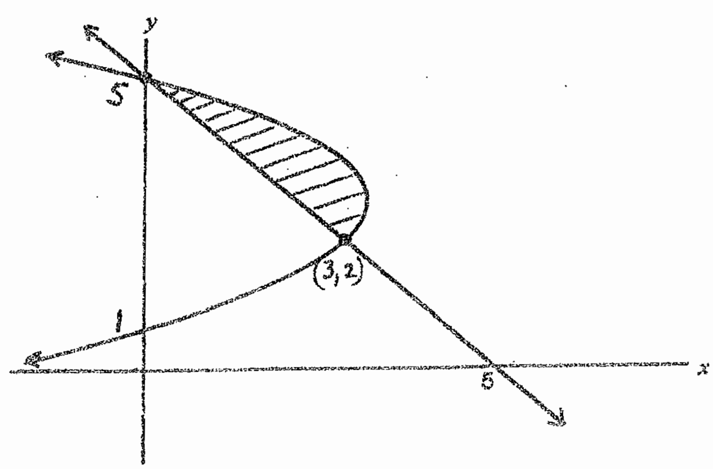


i) Explain why angle SOT is $\frac{2\pi}{3}$ **2**

ii) Hence find the exact area of the shaded region in terms of d **2**

b) The diagram shows the curve $x = 6y - 5 - y^2$ and the line $x + y = 5$ **3**

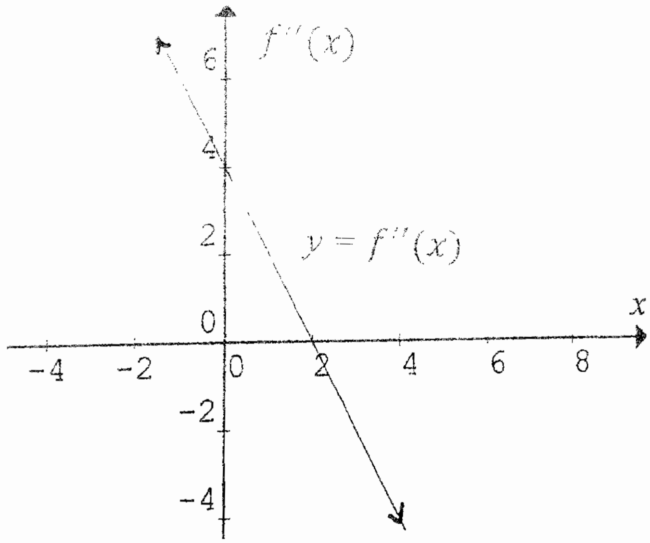
The two graphs intersect at (0, 5) and (3, 2)



Determine the magnitude of the shaded area

c) For the curve $y = x^5 - 80x$, $\frac{d^2y}{dx^2} = 0$ at (0,0). Is (0,0) a point of inflexion? Justify your answer. **3**

QUESTION 3 (10 Marks)



This is the graph of $y = f''(x)$

i) Find the equation of $f'(x)$ if there is a stationary point at (1,4) **2**

ii) What is the nature of the stationary point at (1,4)? Give a reason. **1**

b)

i) Sketch on the same diagram the graphs of $y = 2\sin x$ and $y = \cos 2x$ for $0 \leq x \leq 2\pi$ **3**

ii) Use your graph or otherwise determine a value for d, where d is an integer, so that the equation $2\sin x - \cos 2x = d$ has 4 solutions in the interval $0 \leq x \leq 2\pi$ **1**

c) Find $\int x^3 (x^2 + 1)^2 \, dx$ by using the substitution $u = x^2 + 1$ **3**

QUESTION 4 (10 Marks)

a) For the curve $y = \frac{x^2}{1+x}$

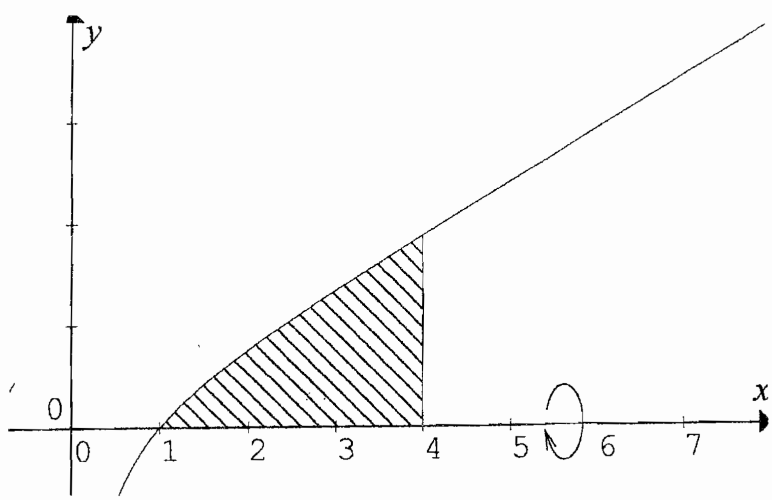
i) Find the co-ordinates of the stationary points and determine their nature. **3**

ii) Given $y = \frac{x^2}{1+x}$ can be written as $y = x - 1 + \frac{1}{x+1}$

Write down the equations of any asymptotes **2**

iii) Sketch the curve showing the stationary points and the asymptotes **2**

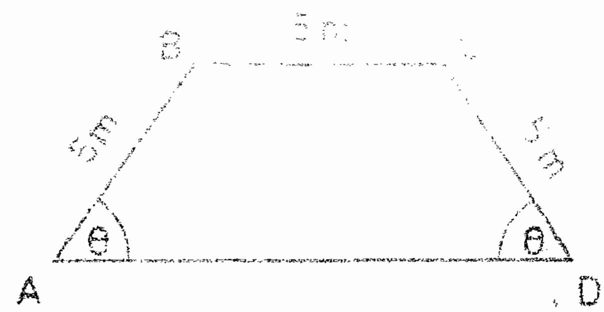
b) **3**



The shaded region in the diagram is bounded by the curve $y = x - \frac{1}{x}$, the x - axis and the line $x = 4$. Find the volume of the solid of revolution formed when the shaded region is rotated about the x – axis.

QUESTION 5 (10 Marks)

a)



In a quadrilateral *ABCD*, *BC* is parallel to *AD*, the sides *AB*, *BC*, *CD* are each 5m long and the angles *BAD*, *ADC* each have size θ, as shown in the diagram:

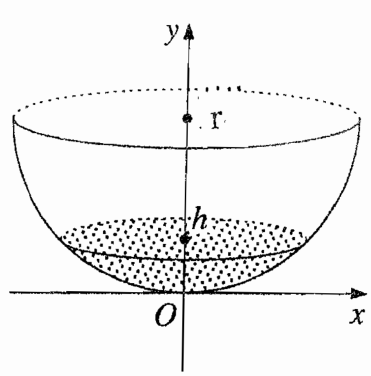
i) Show that the area of the trapezium is given by the formula **2**

Area = $25 \sin\theta\,(1 + \cos\theta)$

ii) Hence find the value of θ for which this area is a maximum **4**

b) A hemi-spherical bowl is formed by rotating the semi-circle **4**

$y = r - \sqrt{r^2 - x^2}$ about the y – axis. The bowl contains water up to the height h where $0 < h < r$.



Show that the volume of water in the bowl is $\dfrac{\pi h^2(3r-h)}{3}$

Question 1

a) $y = \cos^3 x$

$y' = -3\sin x \cos^2 x$

at $x = \pi/6$

$y' = -3 \cdot \frac{1}{2} \cdot \left(\frac{\sqrt{3}}{2}\right)^2$

$= -9/8$

b) $\lim_{x \to 0} \frac{\sin 5x}{4x} = 5/4$

c) $-\frac{(5-2x)^5}{10} + c$

d) $\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right) = \frac{(1+x^2).2x - x^2.2x}{(1+x^2)^2}$

$= \frac{2x}{(1+x^2)^2}$

$\therefore \int \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\frac{x^2}{1+x^2} + c$

e) $u = 1+x^2 \longrightarrow du = 2x\,dx$

$x = \sqrt{3}$ $\quad u = 4$

$x = 0$ $\quad u = 1$

$\therefore = \frac{1}{2}\int_0^4 \frac{du}{u^{1/2}}$

$= \frac{1}{2}\int_0^4 u^{-1/2}\,du$

$= \frac{1}{2}\left[2u^{1/2}\right]_1^4$

$= 1$

Question

b) $\text{Area} = \int_2^5 6y - 5 - y^2 \, dy - \frac{1}{2} \times 3 \times 3$

$= \left[3y^2 - 5y - \frac{y^3}{3}\right]_2^5 - \frac{9}{2}$

$= \left[75 - 25 - \frac{125}{3}\right] - \left[12 - 10 - \frac{8}{3}\right] - \frac{9}{2}$

$= 8\frac{1}{3} - (-\frac{2}{3}) - \frac{9}{2}$

$= 4\frac{1}{2}$

c) $y' = 5x^4 - 80$

$y' = 0$ for stationary pt.

$\therefore 5x^4 - 80 = 0$

$5[x^4 - 16] = 0$

$\therefore x = \pm 2$

$\therefore$ stationary pts at $x = \pm 2$

$\therefore (0,0)$ is a point of inflection as it is between two stationary points on a continuous curve.

or

$y'' = 20x^3$

x	-1	0	1
y''	$-$	0	$+$

curve changes concavity on either side of $(0,0)$

$= r^2 - [r^2 - 2ry + y^2]$

$= 2ry - y^2$

$$\text{Volume} = \pi \int_0^h 2ry - y^2 \, dy$$

$$= \pi \left[ry^2 - \frac{y^3}{3} \right]_0^h$$

$$= \pi \left[rh^2 - \frac{h^3}{3} \right]$$

$$= \pi \left[\frac{3rh^2 - h^3}{3} \right]$$

$$= \frac{\pi h^2 [3r - h]}{3}$$

Teacher's Name　　　Student's Name

Question 3

a) i) $f''(x) = \ldots$

$f'(x) = x^2 + \ldots x + c$

at $(1,4)$ $0 = \ldots + c$

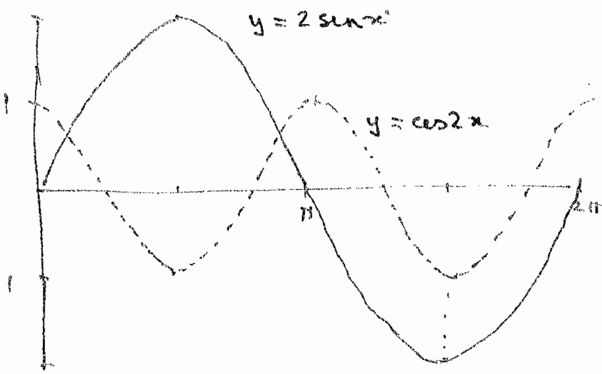
$c = -3$

$\therefore f'(x) = -x^2 + 4x - 3$

ii at $x = 1$ $f''(x) > 0$

$\therefore$ minimum turning point

i)



ii) $2\sin x = \cos 2x + d$

$\therefore d = -1$

c) $u = x^2 + 1 \Rightarrow du = 2x\,dx$

$x^2 = u - 1$

$$\therefore \int x^3 (x^2+1)^2 dx = \int x^2 \cdot x \cdot (x^2+1)^2 dx$$

$$= \int (u-1) u^2 \cdot \frac{du}{2}$$

$$= \frac{1}{2} \int u^3 - u \, du$$

$$= \frac{1}{2} \left[\frac{u^4}{4} - \frac{u^3}{3} \right] + c$$

$$= \frac{(x^2+1)^4}{8} - \frac{(x^2+1)^3}{6} + c$$

Question 4

a) i) $y' = \frac{(1+x).2x - x^2}{(1+x)^2}$

$= \frac{2x + x^2}{(1+x)^2}$

$y' = 0$ for stationary point

$\therefore 2x + x^2 = 0$

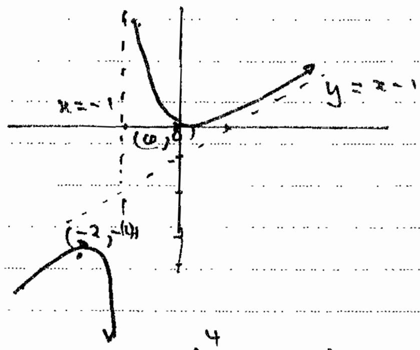
$x(2+x) = 0$

$x = -2$ or 0

x	-3	-2	$-1\frac{1}{2}$	$-\frac{1}{2}$	0	1
y'	$+$	0	$-$	$-$	0	$+$

$\therefore$ maximum at $(-2,-4)$ minimum at $(0,0)$

ii) $y = x - 1$ and $x = -1$



b) Volume $= \pi \int_1^4 \left(x - \frac{1}{x}\right)^2 dx$

$= \pi \int_1^4 x^2 - 2 + \frac{1}{x^2}\, dx$

$= \pi \left[\frac{x^3}{3} - 2x - \frac{1}{x} \right]_1^4$

$= \pi \left[\left(\frac{64}{3} - 8 - \frac{1}{4}\right) - \left(\frac{1}{3} - 2 - 1\right) \right]$

$= \frac{63\pi}{4}$

Question 5

a) i)

$h = 5\sin\theta$ and $d = 5\cos\theta$

$\therefore$ Area $= \frac{5\sin\theta}{2}\left[5 + 5 + 10\cos\theta\right]$

$= 25\sin\theta\left[1 + \cos\theta\right]$

ii) $A' = 25\left[(1+\cos\theta)\cos\theta + \sin\theta . \sin\theta\right]$

$= 25\left[\cos\theta + \cos^2\theta - \sin^2\theta\right]$

$= 25\left[\cos\theta + \cos^2\theta - (1 - \cos^2\theta)\right]$

$= 25\left[2\cos^2\theta + \cos\theta - 1\right]$

For maxim $A' = 0$

$\therefore 25\left[2\cos^2\theta + \cos\theta - 1\right] = 0$

$25\left[(2\cos\theta - 1)(\cos\theta + 1)\right] = 0$

$\therefore \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$ (out of domain)

$\theta = \frac{\pi}{3}$

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	π
A'	$+$	0	0

$\therefore$ max area when $\theta = \frac{\pi}{3}$