Sydney Technical High School

TRIAL HIGHER SCHOOL CERTIFICATE

2006

MATHEMATICS EXTENSION 2

General Instructions

- Reading time - 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplies at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________

Class: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

Question 1 **Marks**

a) Find:

(i) $\int \frac{x\,dx}{(1+x^2)^2}$ **2**

(ii) $\int \sin^3 x dx$ **2**

(iii) $\int x\sqrt{1-x}\,dx$ **3**

b) (i) Find real numbers a and b such that

$$\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$ **2**

(ii) Hence find $\int \frac{5-3x}{(x+1)(x^2+1)}\,dx$ **2**

c) Evaluate $\int_0^{\pi} \frac{dx}{\cos\frac{x}{2} + \sin\frac{x}{2} + 3}$ using the substitution $t = \tan\left(\frac{x}{4}\right)$ **4**

Question 2

a) (i) Express $w=-1-i$ in modulus – argument form. **2**

(ii) Hence express w^{12} in the form $x+iy$ where x and y are real numbers. **2**

b) Find the equation, in Cartesian form, of the locus of the point z if **2**

$$|z-i|=|z+3|.$$

c) Sketch the region in the Argand diagram that satisfies the inequality **3**

$$\mathrm{Re}\left(\frac{1}{z}\right)\leq\frac{1}{2}$$

d) (i) On the Argand diagram draw a neat sketch of the locus specified by **1**

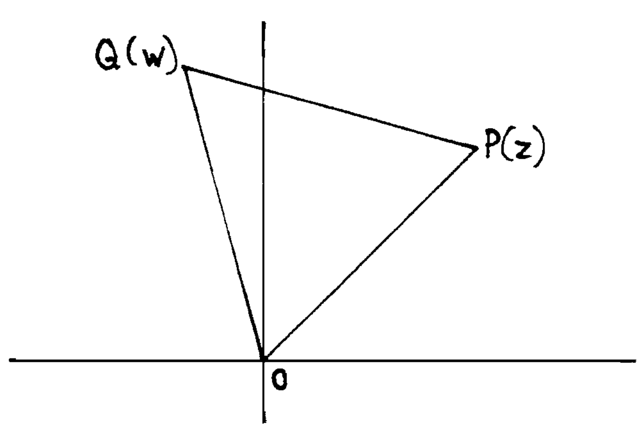
$$\arg(z+1)=\frac{\pi}{3}$$

(ii) Hence find z so that $|z|$ is a minimum. **2**

e) Points P and Q represent the complex numbers z and w respectively in the Argand Diagram. If ΔOPQ (where O is the origin) is an equilateral triangle

(i) Show why $wz=z^2 cis\frac{\pi}{3}$ **1**

(ii) Prove that $z^2+w^2=zw$ **2**



Question 3

a) The hyperbola, H, has a Cartesian equation $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

(i) Find the coordinates of the foci S and S′ **1**

(ii) Show that any point, P, on H can be represented by the coordinates $(5\sec\theta,\ 4\tan\theta)$ and hence, or otherwise, prove that PS − PS′ is a constant. **3**

(iii) Show that the equation of the normal at the point P on the hyperbola is **3**

$$\frac{5x}{\sec\theta} + \frac{4y}{\tan\theta} = 41$$

(iv) If this normal meets the x axis at M and the y axis at N, prove that **3**

$$\frac{PM}{PN} = \frac{16}{25}$$

b) Consider the function $y = \cos^{-1}(\cos x)$. Given the domain and range are

D: all real x

R: $0 \le y \le \pi$

(i) State whether the function is even, odd or neither and find its period. **2**

(ii) Hence sketch the graph of the function over $-4\pi \le x \le 4\pi$ **1**

c) Solve for x: **2**

$$\tan^{-1}(3x) - \tan^{-1}(2x) = \tan^{-1}\left(\frac{1}{5}\right)$$

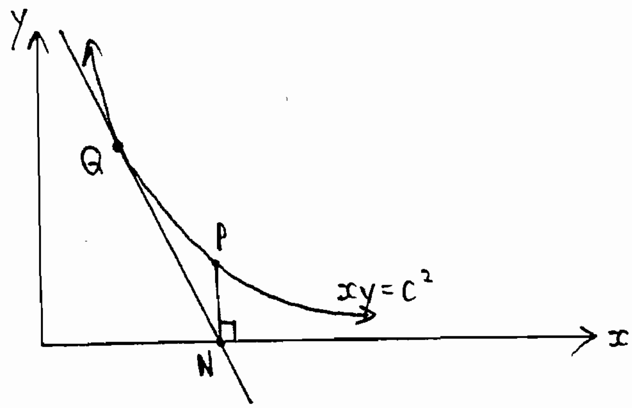
Question 4

a) Find Q which is rational where **2**

$$\sqrt{Q} = \sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$$

b) If $f(x) = f(x-1) + x^2$ and $f(3) = 7$, evaluate $f(1)$. **2**

c)



In the diagram above, P $(ct_1, \frac{c}{t_1})$ and Q $(ct_2, \frac{c}{t_2})$ are distinct variable points on the rectangular hyperbola $xy = c^2$. PN is the perpendicular from P to the x axis and the tangent at Q passes through N.

(i) Show that $t_1 = 2t_2$ **3**

(ii) Find the Cartesian equation of the locus of T, the point of intersection of the tangents at P and Q. **3**

d) (i) By solving the equation $z^3 = 1$, find the 3 cube roots of 1. **2**

(ii) Let w be a cube root of 1 where w is not real. **1**

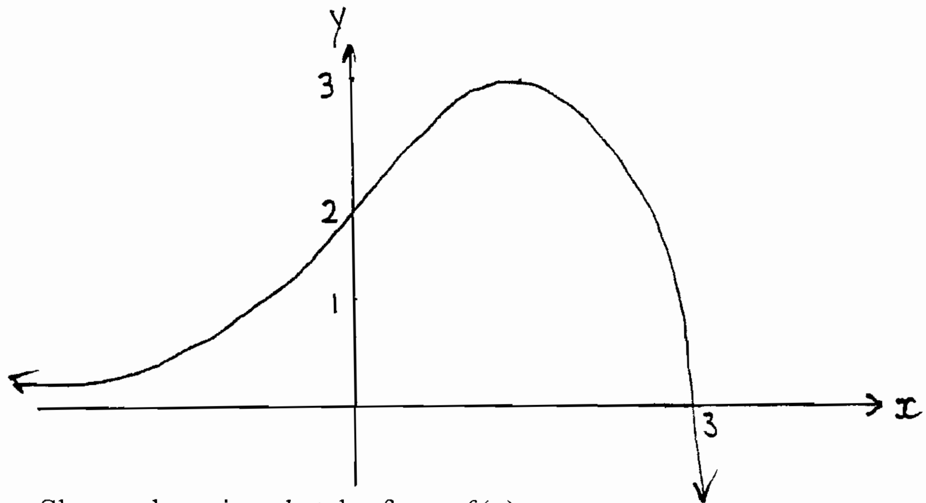
Show that $1 + w + w^2 = 0$

(iii) Find the quadratic equation, with integer coefficients, that has roots **2**

$4 + w$ and $4 + w^2$

Question 5

a)



Shown above is a sketch of $y = f(x)$.

On separate diagrams draw sketches of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^3$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $y = \log_e[f(x)]$ **2**

b) The deck of a ship was 3m below the level of a wharf at low tide and 1m above the wharf level at high tide. Low tide was at 9:30am and high tide at 4:00pm. Find the first time after low tide when the deck was level with the wharf, if the motion of the tide was simple harmonic. **4**

c) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Question 6

a) Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$ **2**

b) None of the roots α, β and γ of the equation $x^3 + 3px + q = 0$ is zero.

(i) Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\alpha\beta}{\gamma}$ **4**

expressing its coefficients in terms of p and q.

(ii) Show that if $\gamma = \alpha\beta$ then $(3p-q)^2 + q = 0$. **2**

c) For the equation $x^3 - 6x^2 + 9x - 5 = 0$

(i) By considering stationary points, show that the equation has only one real root α. **3**

(ii) Determine the two consecutive integers between which α lies. **1**

(iii) By considering the product of the roots of the equation, express the modulus of each of the complex roots in terms of α and deduce that the value of this modulus lies between 1 and $\frac{\sqrt{5}}{2}$. **3**

Question 7

a) (i) Let $I_n = \int_1^e x(\ln x)^n \, dx$, $n = 0, 1, 2, 3 \ldots$ **2**

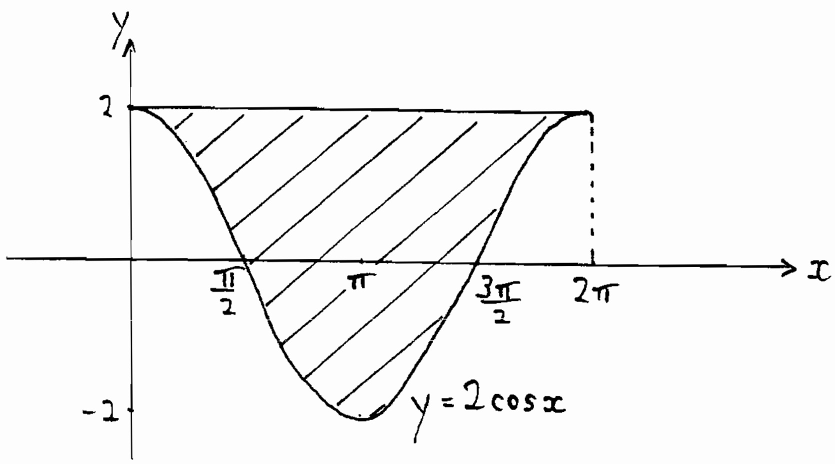
Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, $n = 1, 2, 3\ldots$

(ii) The area bounded by the curve $y = \sqrt{x}(\ln x)^2$, the x axis **3**

and the lines $x=1$ and $x = e$ is rotated about the x axis.

Find the exact value of the volume of the solid of revolution so formed.

b)



The shaded region is rotated about the y axis to obtain a solid of revolution.

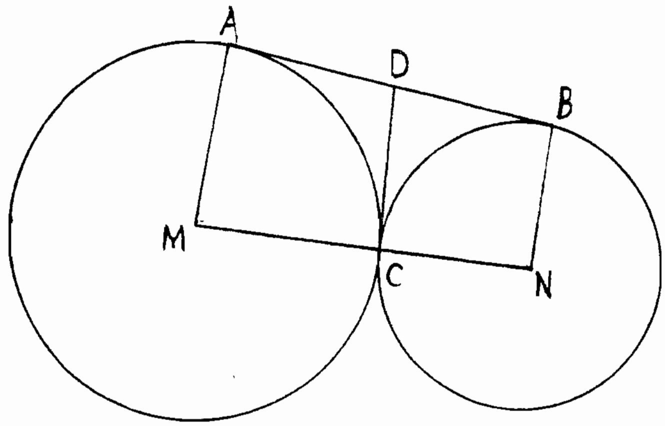
(i) Use the method of cylindrical shells to show that the volume of this solid is given by **2**

$$4\pi \int_0^{2\pi} x(1-\cos x)\, dx \ .$$

(ii) Hence calculate this volume. **2**

Question 7 (cont.)

c)



In the diagram MCN is a straight line. Circles are drawn with centre M, radius MC and centre N, radius NC. AB is a common tangent to the two circles with points of contact at A and B respectively. CD is the common tangent at C, and meets AB at D.

(i) Explain why AMCD and BNCD are cyclic quadrilaterals. **2**

(ii) Show that Δ ACD $|||$ Δ CBN **2**

(iii) Show that MD $||$ CB **2**

Question 8

A particle of mass m is projected vertically upwards under gravity. The air resistance to the motion is $\frac{1}{100}mgv^2$ where v is the speed of the particle.

(a) (i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t, then **1**

$$\ddot{x} = \frac{-1}{100}g\left(100 + v^2\right)$$

(ii) If the initial speed of projection is u, show that the greatest height (above the projection point) reached by the particle is **5**

$$\frac{50}{g}\ln\left(\frac{100 + u^2}{100}\right).$$

(iii) Show that during the downward motion of the particle, if x is the downward vertical displacement of the particle from its highest position at a time t after it begins the downward motion, then **1**

$$\ddot{x} = \frac{1}{100}g(100 - v^2)$$

(iv) Show that the speed of the particle on return to its point of projection is **5**

$$\frac{10u}{\sqrt{100 + u^2}}$$

(v) Find the terminal velocity V of the particle for the downward motion. **1**

(vi) If the initial speed of projection of the particle is V, as found in part (v), show that the speed on return to the point of projection is $\frac{1}{\sqrt{2}}V$. **2**

End of Paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Teacher's Name: ____________________ Student's Name/Nº: ____________________

STHS 2006 Ext 2 Trial Solutions

Question 1.a.(i) $\displaystyle\int \frac{x\,dx}{(1+x^2)^2}$

$= \frac{1}{2}\int (1+x^2)^{-2}\,2x\,dx$ (1)

$= \frac{1}{2}\,\frac{(1+x^2)^{-1}}{-1} + C$

$= \frac{-1}{2(1+x^2)} + C$ (1)

(ii) $\displaystyle\int \sin^3 x\,dx$

$\int \sin x\,(1-\cos^2 x)\,dx$ (1)

$= \int \sin x\,dx + \int \cos^2 x(-\sin x)\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$ (1)

(iii) $\displaystyle\int x\sqrt{1-x}\,dx$ $\qquad u = 1-x$, $du = -dx$

$= \int (1-u)\sqrt{u}\,du$ (1)

$\int u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$ (1)

$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C$

$= \frac{2}{3}(1-x)^{\frac{3}{2}} - \frac{2}{5}(1-x)^{\frac{5}{2}} + C$ (1)

b.(i) $\displaystyle\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$

$5-3x = a(x^2+1) + (bx+c)(x+1)$

$= ax^2 + a + bx^2 + bx + cx +$

$= (a+b)x^2 + (b+c)x + (a+c)$

$a+b = 0 \Rightarrow b = -a$

$b+c = -3 \Rightarrow -a+c = -3$

$a+c = 5$

$2c = 2$

$c = 1$

$b = -4$ (1) $\quad a = 4$ (1)

(ii) $\displaystyle\int \frac{4}{x+1} - \frac{4x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= 4\ln|x+1| - 2\ln|x^2+1| + \tan^{-1}x + C$ (1)

$= 2\ln\left|\frac{(x+1)^2}{x^2+1}\right| + \tan^{-1}x + C$ (1)

c) $\displaystyle\int_0^{\pi}\frac{dx}{\cos\frac{x}{2} + \sin\frac{x}{2} + 3} = \int_0^1 \frac{\frac{4dt}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 3}$

$t = \tan\frac{x}{4}$ (1)

$dt = \frac{1}{4}\sec^2\frac{\pi}{4}\,dx$

$4dt = (1+\tan^2\frac{\pi}{4})\,dx$

$dx = \frac{4dt}{1+t^2}$

$\displaystyle = \int_0^1 \frac{4dt}{1-t^2+2t+3+3t^2}$ (1)

Teacher's Name: Student's Name/N°:

$$\int_0^1 \frac{2dt}{t^2+t+2}$$

$$= \int_0^1 \frac{2dt}{(t+\frac{1}{2})^2+\frac{7}{4}} \quad 1$$

$$= \frac{4}{\sqrt{7}}\left[\tan^{-1}\frac{2t+1}{\sqrt{7}}\right]_0^1$$

$$= \frac{4}{\sqrt{7}}\left(\tan^{-1}\frac{3}{\sqrt{7}} - \tan^{-1}\frac{1}{\sqrt{7}}\right) \quad 1$$

Question 2. a) (i) $\omega = -1-i$

$$= \sqrt{2}\,cis\left(-\frac{3\pi}{4}\right) \quad 2$$

(ii) $\omega^{12} = \left(\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)\right)^{12}$

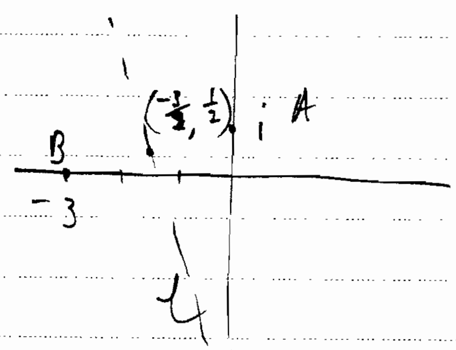
$$= 2^6\,cis\,\frac{-36\pi}{4}$$

$$= 64\,cis(-9\pi)$$

$$= 64\,cis\,\pi$$

$$= -64 \quad 2$$

(b) $|z-i| = |z+3|$



$m_{AB} = \frac{1}{3}$

$m_l = -3$

$\therefore\ y - \frac{1}{2} = -3\left(x+\frac{3}{2}\right)$

$y - \frac{1}{2} = -3x - \frac{9}{2}$

$y = -3x - 4 \quad 2$

(c) $Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$

$Re\left(\frac{1}{x+iy}\right) \leq \frac{1}{2}$

$Re\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right) \leq \frac{1}{2}$

$Re\left(\frac{x-iy}{x^2+y^2}\right) \leq \frac{1}{2} \quad 1$

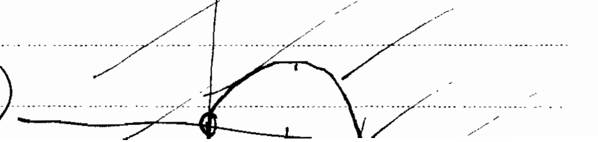
$\frac{x}{x^2+y^2} \leq \frac{1}{2}$

$2x \leq x^2+y^2$

$x^2-2x+1+y^2 \geq 1$

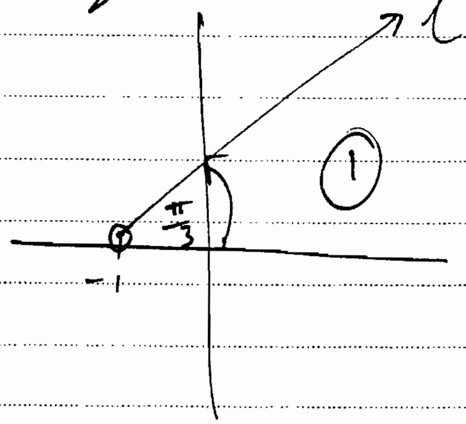
$1 \quad (x-1)^2+y^2 \geq 1 \quad (z \neq 0)$

1



Teacher's Name: ____________ Student's Name/Nº: ____________

d (i) $\arg(z+1) = \frac{\pi}{3}$



(ii) Equation of l is

$y - 0 = \tan\frac{\pi}{3}(x - -1)$

$y = \sqrt{3}(x+1)$

$|z|$ is a minimum at A where $OA \perp l$

A is $(a, \sqrt{3}(a+1))$

$m_{OA} = \frac{\sqrt{3}(a+1)}{a} = \frac{-1}{\sqrt{3}}$ 1

$\therefore 3(a+1) = -a$

$4a + 3 = 0$

$a = -\frac{3}{4}$

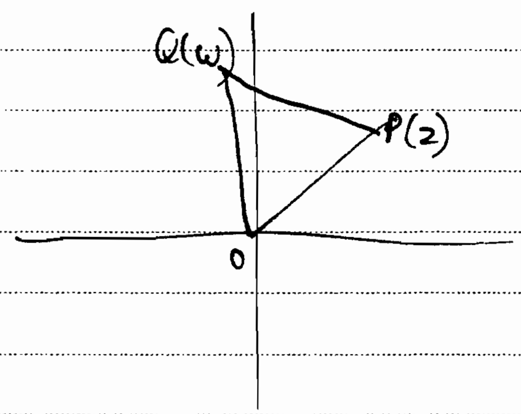
$\therefore A$ is $\left(-\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$

$\therefore z$ is $-\frac{3}{4} + \frac{\sqrt{3}}{4}i$ 1

(e) $\omega = z \operatorname{cis}\frac{\pi}{3}$

as $|\omega| = |z|$ and

$\angle QOP = \frac{\pi}{3}$ (equilateral)



$\therefore \omega z = \left(z \operatorname{cis}\frac{\pi}{3}\right) z$

$= z^2 \operatorname{cis}\frac{\pi}{3}$ 1

(ii) $z^2 + \omega^2 = z^2 + z^2 \operatorname{cis}\frac{2\pi}{3}$

$= z^2\left(1 + \operatorname{cis}\frac{2\pi}{3}\right)$

$= z^2\left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= z^2\left(1 - \frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)$

$= z^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ 1

$= z^2 \operatorname{cis}\frac{\pi}{3}$

$= \omega z$ from part (i) 1

Question 3. a.(i) $a^2 = 25$ $b^2 = 16$

$b^2 = a^2(e^2-1)$

$16 = 25(e^2-1)$

$\frac{16}{25} = e^2 - 1$

$e^2 = \frac{41}{25}$

$e = \frac{\sqrt{41}}{5}$

$\therefore$ Focii $= (\pm ae, 0)$

are $(\pm\sqrt{41}, 0)$ 1

(ii) Sub $(5\sec\theta, 4\tan\theta)$ into H

$\frac{\cancel{25}\sec^2\theta}{\cancel{25}} - \frac{\cancel{16}\tan^2\theta}{\cancel{16}} = 1$

$\sec^2\theta - \tan^2\theta = 1$

is true $\forall\,\theta \therefore$

$P(5\sec\theta, 4\tan\theta)$ 1

always satisfies H.

(ii) cont'd.

From the def'n of a Hyperbola,

$|PS - PS'|$

$= e\,P_{directrix_1} - e\,P_{directrix_2}$ 1

$= e\left(5\sec\theta - \frac{5}{\sqrt{41}/5}\right) - e\left(5\sec\theta + \frac{5}{\sqrt{41}/5}\right)$

$= \cancel{\frac{\sqrt{41}}{5} \times 5\sec\theta} - \cancel{\frac{\sqrt{41}}{5}} \times \frac{5}{\cancel{\sqrt{41}/5}} - \cancel{\frac{\sqrt{41}}{5} \times 5\sec\theta} - \cancel{\frac{\sqrt{41}}{5}} \times \frac{5}{\cancel{\sqrt{41}/5}}$

$= 10 \therefore$ a constant. 1

(iii) Differentiating implicitly,

$\frac{2x}{25} - \frac{2y\frac{dy}{dx}}{16} = 0$

$\frac{dy}{dx} = \frac{+2x}{25} \times \frac{16}{2y} = \frac{16x}{25y}$ 1

$\therefore$ At P, $\frac{dy}{dx}$ of tangent

$$= \frac{80\sec\theta}{100\tan\theta}$$

$$= \frac{4}{5} \times \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$= \frac{4}{5\sin\theta}$$

$\therefore$ $m_{normal} = -\frac{5\sin\theta}{4}$ (1)

$\therefore$ Equation of normal is

$$y - 4\tan\theta = -\frac{5\sin\theta}{4}(x - 5\sec\theta)$$

$$4y - 16\tan\theta = -5\sin\theta x + 25\tan\theta$$

$$5\sin\theta x + 4y = 41\tan\theta$$

$$\frac{5\sin\theta x}{\frac{\sin\theta}{\cos\theta}} + \frac{4y}{\tan\theta} = 41$$

$$5\cos\theta x + \frac{4y}{\tan\theta} = 41$$

$$\frac{5x}{\sec\theta} + \frac{4y}{\tan\theta} = 41. \quad (1)$$

(iv) M is $\left(\frac{41\sec\theta}{5}, 0\right)$

N is $\left(0, \frac{41\tan\theta}{4}\right)$ (1)

(iv) cont'd.

using $x = \frac{kx_2 + lx_1}{k+l}$

$$5\sec\theta = \frac{k \times 0 + l \times \frac{41\sec\theta}{5}}{k+l}$$

$$5\sec\theta k + 5\sec\theta l = \frac{41\sec\theta}{5} l$$

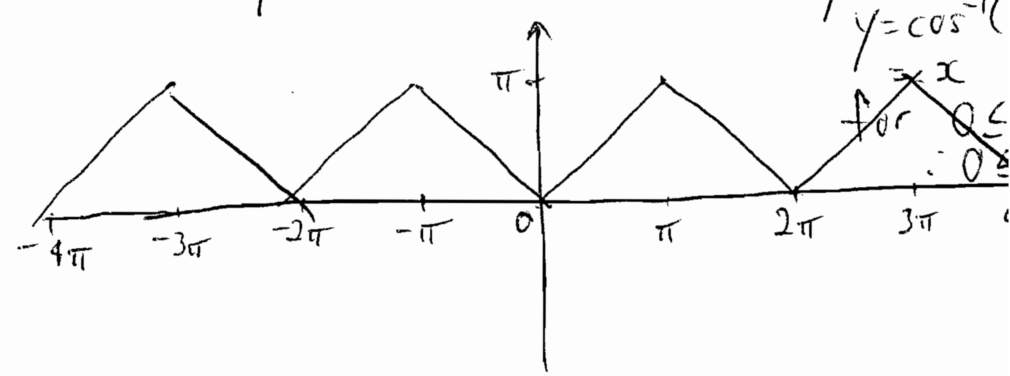
$$5\cancel{\sec\theta} k = \frac{16\cancel{\sec\theta}}{5} l$$

$$\frac{k}{l} = \frac{16}{25}$$

$$\therefore \frac{PM}{PN} = \frac{16}{25} \checkmark$$

b)(i) Period of $y = \cos^{-1}(\cos x)$ is s…
as period of $\cos x$ ie: 2π…
$y = \cos^{-1}(\cos x)$ is even as cos…
(1) is even. ie: $\cos^{-1}(\cos x) = \cos^{-1}(c…$
$= \cos^{-1}(c…$

(ii) Symmetrical across the y axis.



Teacher's Name: ____________ Student's Name/Nº: ____________

3c. $\tan^{-1} 3x - \tan^{-1} 2x = \tan^{-1}\frac{1}{5}$

$$\tan\left[\tan^{-1} 3x - \tan^{-1} 2x\right] = \tan\left[\tan^{-1}\frac{1}{5}\right]$$

$$\frac{\tan[\tan^{-1}3x] - \tan[\tan^{-1}2x]}{1+\tan[\tan^{-1}3x]\tan[\tan^{-1}2x]} = \frac{1}{5}$$

$$\frac{3x-2x}{1+3x.2x} = \frac{1}{5} \quad 1$$

$$5x = 1+6x^2$$

$$6x^2 - 5x + 1 = 0$$

$$(3x-1)(2x-1) = 0$$

$$x = \frac{1}{3} \text{ or } \frac{1}{2} \quad 1$$

Question 4. a) $\sqrt{Q} = \sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$

$$Q = 2+\sqrt{3} + (2-\sqrt{3}) + 2\sqrt{2+\sqrt{3}}\sqrt{2-\sqrt{3}} \quad 1$$

$$= 4 + 2\sqrt{4-3}$$

$$= 4+2 = \underline{6} \quad 1$$

b) $f(x) = f(x-1) + x^2$

$$f(3) = f(2) + 3^2 = 7$$

$$\therefore f(2) = -2 \quad 1$$

$$f(2) = f(2-1) + 2^2$$

$$-2 = f(1) + 4$$

$$\therefore f(1) = -6 \quad 1$$

Teacher's Name: ______________ Student's Name/Nº: ______________

c. (i) $x = ct \qquad y = ct^{-1}$

$\frac{dx}{dt} = c \qquad \frac{dy}{dt} = \frac{-c}{t^2}$

$\frac{dy}{dx} = \frac{-c}{t^2} \times \frac{1}{c}$

$= \frac{-1}{t^2}$ (1)

tangent:

$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$x + t^2 y = 2ct$ (1)

tangent at Q:

$x + t_2^2 y = 2ct_2$

N $(ct, 0)$:

$ct_1 + 0 = 2ct_2$

$t_1 = 2t_2$ (1)

(ii) tangent at Q

$x + t_2^2 y = 2ct_2$ (1)

tangent at P

$x + t_1^2 y = 2ct_1$

ie: $x + 4t_2^2 y = 4ct_2$ (2)

since $t_1 = 2t_2$ (1) mar

(2) − (1)

$3t_2^2 y = 2ct_2$

$y = \frac{2c}{3t_2}$

sub. in (1)

$x + \frac{2ct_2}{3} = 2ct_2$

$x = \frac{4ct_2}{3}$ (1) mark

$xy = \frac{2c}{3\cancel{t_2}} \cdot \frac{4c\cancel{t_2}}{3}$

$xy = \frac{8c^2}{9}$ (1) mark.

d. (i) $z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$ (1)

$z = 1$ or $\frac{-1 \pm \sqrt{1-4}}{2}$

$= 1$ or $\frac{-1 \pm i\sqrt{3}}{2}$ (1)

(ii) $(\omega - 1)(\omega^2 + \omega + 1) = 0$ from (i)

Now $\omega \neq 1$ $\therefore$

$\omega^2 + \omega + 1 = 0$ (1)

(iii) $\alpha + \beta = 4 + \omega + 4 + \omega^2$

$= 7 + 1 + \omega + \omega^2$ (1)

$= 7$

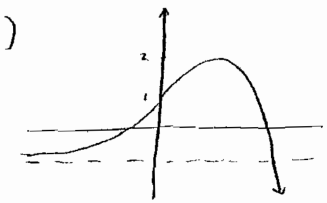
$\alpha\beta = 16 + 4\omega + 4\omega^2 + \omega^3$

$= 12 + 4(1 + \omega + \omega^2) + 1$

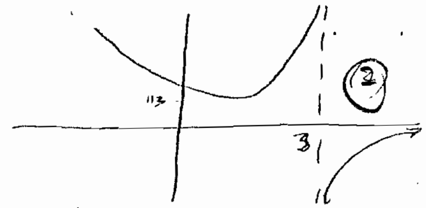
$= 13$

$\therefore z^2 - 7z + 13 = 0$ (1)

(i)

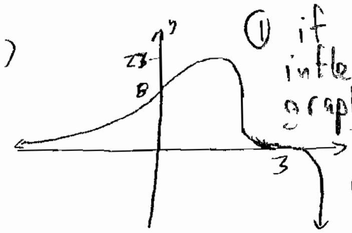


(1)



2

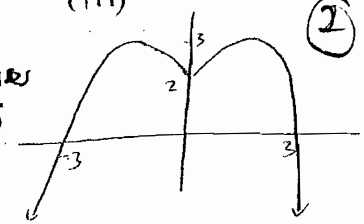
(ii)



1 if no pt. of inflexion but graph still passes through y=8

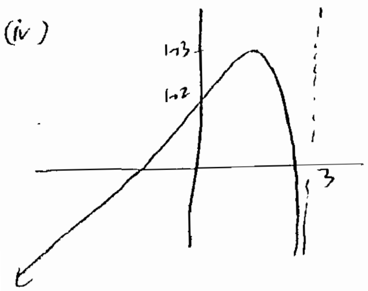
2

(iii)



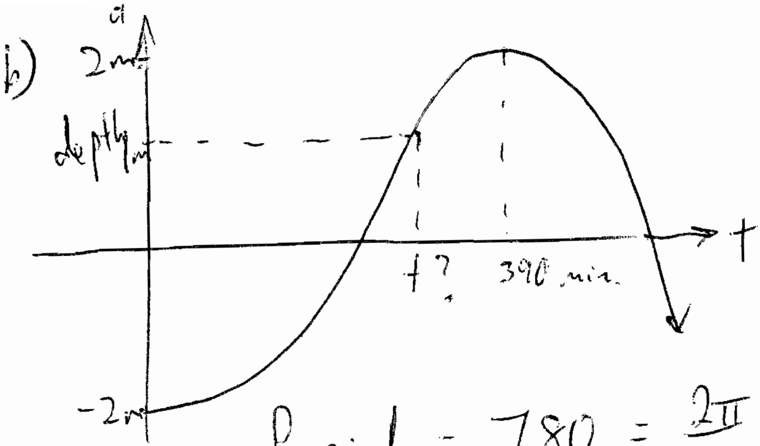
2

(iv)



2

b)



Period = 780 = $\frac{2\pi}{n}$

$\therefore n = \frac{2\pi}{780}$ 1

$\therefore d = -2\cos\frac{2\pi}{780}t$ 1

$\therefore$ when $d=1$ solve for t

$-\frac{1}{2} = \cos\frac{2\pi}{780}t$

$\frac{2\pi}{780}t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\therefore t = \frac{780}{3} = 260$ minutes 1

ie. 4 hrs 20 minutes after 9:30 am

ie 1:50 pm is when deck is level with wharf. 1

QUESTION 6

(a) Let $P(x) = x^5 + 2x^2 + mx + n$; $\therefore P'(x) = 5x^4 + 4x + m$.
$(x+1)^2$ is a factor of $P(x)$; $\therefore P(-1) = 0$ and $P'(-1) = 0$. (2)
$P'(-1) = 0$, $\therefore m = -1$. $P(-1) = 0$, $\therefore n = -2$.

(b) (i) $x^3 + 3px + q = 0$ has roots α, β, γ $(\alpha \neq 0, \beta \neq 0, \gamma \neq 0)$;
$\therefore \alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = 3p$, $\alpha\beta\gamma = -q$ $(q \neq 0)$. (1)

$$\frac{\beta\gamma}{\alpha} + \frac{\alpha\gamma}{\beta} + \frac{\alpha\beta}{\gamma} = \frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha\beta + \alpha\gamma + \beta\gamma)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma} = \frac{-9p^2}{q}.$$ (1)

$$\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta} + \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\beta}{\gamma} + \frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} = \gamma^2 + \beta^2 + \alpha^2$$

$$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = -6p.$$ (1)

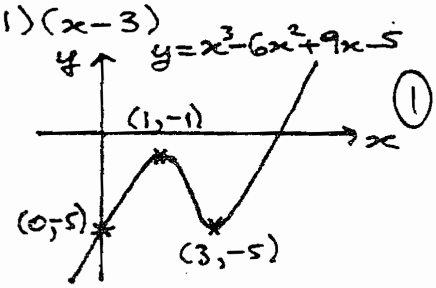
$$\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} = \alpha\beta\gamma = -q.$$

$\therefore$ the required equation is
$$x^3 - \left(\frac{-9p^2}{q}\right)x^2 + (-6p)x - (-q) = 0, \therefore x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0.$$ (1)

(ii) $\gamma = \alpha\beta$ if and only if $x = 1$ is a root of this equation
i.e. if and only if $(3p - q)^2 + q = 0$. (2)

(c) (i) Let $f(x) = x^3 - 6x^2 + 9x - 5$
$\therefore f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

sign of $f'(x)$: +ve, $-$ve, +ve (at x = 1, 3) (1)



(1)

$(1,-1)$ is a maximum turning point, $(3,-5)$ is a minimum turning point.
$\therefore x^3 - 6x^2 + 9x - 5 = 0$ has only one real root α. (1)

(ii) $f(4) = -1$, $f(5) = 15$; $\therefore 4 < \alpha < 5$ (1)

(iii) Let the complex roots be $p + iq$ and $p - iq$;
$\therefore$ product of roots $(p+iq)(p-iq)\alpha = 5$
$\therefore p^2 + q^2 = \frac{5}{\alpha}$, $\therefore \sqrt{p^2+q^2} = \frac{\sqrt{5}}{\sqrt{\alpha}}$ (1)

Now $4 < \alpha < 5$, $\therefore \frac{1}{5} < \frac{1}{\alpha} < \frac{1}{4}$, $\therefore \frac{1}{\sqrt{5}} < \frac{1}{\sqrt{\alpha}} < \frac{1}{2}$, (1)

$\therefore 1 < \frac{\sqrt{5}}{\sqrt{2}} < \frac{\sqrt{5}}{2}$, $\therefore 1 < \sqrt{p^2+q^2} < \frac{\sqrt{5}}{2}$ (1)

Teacher's Name: ______ Student's Name/Nº: ______

Question 7.

(a) (i) $I_n = \int_1^e x(\ln x)^n \, dx$

$= \left[\frac{x^2}{2} (\ln x)^n \right]_1^e - \int_1^e \frac{x^2}{2} n (\ln x)^{n-1} \frac{1}{x} \, dx$ (1)

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ (1) as req'd.

(ii) $V = \pi \int_1^e y^2 \, dx$

$= \pi \int_1^e x (\ln x)^4 \, dx$

$= \pi I_4$ (1)

$I_4 = \left(\frac{e^2}{2} - 2I_3 \right)$

$= \frac{e^2}{2} - 2\left(\frac{e^2}{2} - \frac{3}{2} I_2 \right)$

$= \frac{-e^2}{2} + 3I_2$ (1)

$= -\frac{e^2}{2} + 3\left(\frac{e^2}{2} - I_1 \right)$

$= e^2 + 3I_2$

$= e^2 - 3I_1$

$= e^2 - 3\left(\frac{e^2}{2} - \frac{1}{2} I_0 \right)$

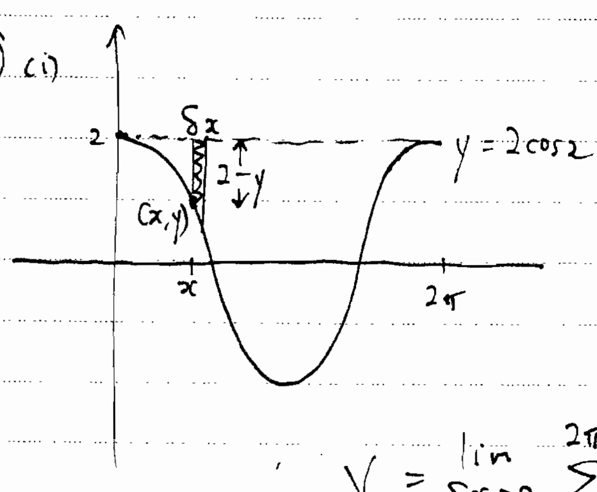
$= -\frac{e^2}{2} + \frac{3}{2} I_0$

$= \frac{-e^2}{2} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2} \right)$

$= \frac{e^2}{4} - \frac{3}{4}$

$\therefore$ Volume $= \frac{\pi}{4}(e^2 - 3)$ units³ (1)

(b) (i)



gives a shell with volume

$\delta V = 2\pi x (2 - y) \, \delta x$

$= 2\pi x (2 - 2\cos x) \, \delta x$

$= 4\pi x (1 - \cos x) \, \delta x$ (1)

$\therefore V = \lim_{\delta x \to 0} \sum^{2\pi} 4\pi x (1 - \cos x) \, \delta x$

Teacher's Name: Student's Name/N°:

$$\therefore V = 4\pi\int_0^{2\pi} x(1-\cos x)\,dx \quad (1)$$

(ii) $$= 4\pi\left\{\left[x(x-\sin x)\right]_0^{2\pi} - \int_0^{2\pi} x - \sin x\,dx\right\}$$

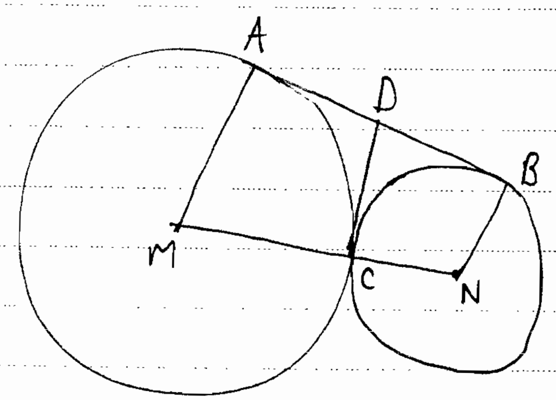
$$= 4\pi\left\{2\pi \times (2\pi) - \left[\frac{x^2}{2} + \cos x\right]_0^{2\pi}\right\} \quad (1)$$

$$= 4\pi\left\{4\pi^2 - \left[\left(\frac{4\pi^2}{2}+1\right) - (1)\right]\right.$$

$$= 4\pi\left\{4\pi^2 - 2\pi^2\right)$$

$$= 8\pi^3 \text{ units}^3 \quad (1)$$

(c)



(i) $\angle MAD = \angle MCD = 90^\circ$ (tangent $\perp$ radius at pt. of contact)
AMCD is cyclic (opposite angles supplementary) (1)

$\angle NBD = \angle NCD = 90^\circ$ (tangent $\perp$ radius at pt. of contact)
BNCD is cyclic (opposite angles supplementary) (1)

(ii) $\angle CDA = \angle BNC$ ($= 2\theta$ say) (ext. angle of cyclic quad BNCD equals interior opposite angle)

$DA = DC$ (1) (tangents drawn from external point)

$\therefore \angle DAC = \angle ACD = 90 - \theta$ (equal angles opposite equal sides, angle sum of triangle is 180°)

$NB = NC$ (equal radii of circle)

$\therefore \angle NCB = \angle CBN = 90 - \theta$ (equal angles opposite equal sides

Teacher's Name: ____________ Student's Name/Nº: ____________

c(iii) $\angle NCB = \angle DAC$ $(= 90 - \theta)$

$\angle BAC = \angle CMD$ (angles in circle through AMCD standing on arc CD are equal) 1

$\therefore \angle CMD = \angle NCB$ ($\angle CMD$ and $\angle NCB$ are a pair of equal corresponding angles on transversal MCN) 1

$\therefore MD \parallel CB$

Teacher's Name: ____________ Student's Name/N°: ____________

Question 8

a) (i) $m\ddot{x} = -mg - \frac{1}{100}mgv^2$

$\ddot{x} = -g - \frac{1}{100}gv^2$

$\ddot{x} = \frac{-1}{100}g(100 + v^2)$ as req'd. 1

(ii) $v\frac{dv}{dx} = \frac{-g(100+v^2)}{100}$

$\frac{dv}{dx} = \frac{-g(100+v^2)}{100v}$

$\frac{dx}{dv} = \frac{-100v}{g(100+v^2)}$ 1

$x = \int \frac{-100v}{g(100+v^2)}\,dv$

$x = \frac{-100}{g}\frac{1}{2}\ln(100+v^2) + c$ 1

when $x = 0$ $v = U$

$\therefore 0 = \frac{-50}{g}\ln(100+U^2) + C$

$\therefore C = \frac{50}{g}\ln(100+U^2)$ 1

$\therefore x = \frac{50}{g}\ln(100+U^2) - \frac{50}{g}\ln(100+v^2)$

$x = \frac{50}{g}\ln\left(\frac{100+U^2}{100+v^2}\right)$ 1

Greatest height reached when $v=0$

ie: $x = \frac{50}{g}\ln\left(\frac{100+U^2}{100}\right)$ as required 1

(iii) $m\ddot{x} = mg - \frac{1}{100}mgv^2$

$\ddot{x} = g - \frac{1}{100}gv^2$

$\ddot{x} = \frac{1}{100}g(100 - v^2)$ as req'd 1

(iv) $v\frac{dv}{dx} = \frac{1}{100}g(100 - v^2)$

$\frac{dv}{dx} = \frac{g}{100v}(100 - v^2) = \frac{g(100-v^2)}{100v}$ 1

$$\therefore \frac{dx}{dv} = \frac{100v}{g(100-v^2)}$$

$$x = \frac{100}{g} \times -\frac{1}{2}\log_e(100-v^2) + c \quad (1)$$

$$x = -\frac{50}{g}\ln(100-v^2) + c$$

when $x = 0$, $v = 0$

$$\therefore 0 = -\frac{50}{g}\ln 100 + c$$

$$\therefore c = \frac{50}{g}\ln 100 \quad (1)$$

$$x = \frac{50}{g}\ln 100 - \frac{50}{g}\ln(100-v^2)$$

$$x = \frac{50}{g}\ln\left(\frac{100}{100-v^2}\right) \quad (1)$$

Now find v when $x = \frac{50}{g}\ln\left(\frac{100+U^2}{100}\right)$ from part (ii)

$$\cancel{\frac{50}{g}}\cancel{\ln}\left(\frac{100+U^2}{100}\right) = \cancel{\frac{50}{g}}\cancel{\ln}\left(\frac{100}{100-v^2}\right)$$

$$\frac{100+U^2}{100} = \frac{100}{100-v^2}$$

$$100 - v^2 = \frac{10000}{100+U^2}$$

$$v^2 = 100 - \frac{10000}{100+U^2}$$

$$v^2 = \frac{100(100+U^2) - 10000}{100+U^2}$$

$$= \frac{\cancel{10000} + 100U^2 - \cancel{10000}}{100+U^2}$$

$$\therefore v = \frac{10U}{\sqrt{100+U^2}} \quad \text{as req'd.} \quad (1)$$

Teacher's Name: ____________ Student's Name/Nº: ____________

(v) Terminal velocity is when $\ddot{x} = 0$. $\ddot{x}$ for downward motion is

$$\ddot{x} = \frac{1}{100} g (100 - v^2) = 0$$

when $v = 10$

$\therefore$ $V = 10$ is terminal velocity

1

(vi) If the initial speed of projection is $V = 10$, then speed upon return to the point of projection will be

$$\frac{10 \times 10}{\sqrt{100 + 10^2}} \quad \text{from (iv)}$$

1

$$= \frac{100}{\sqrt{200}}$$

$$= \frac{100}{10\sqrt{2}}$$

$$= \frac{10}{\sqrt{2}}$$

which equals $\frac{1}{\sqrt{2}} \times 10$

$= \frac{1}{\sqrt{2}} V$ as required

1