SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 2

MARCH 2017

Mathematics Extension 2

Name ....

Teacher ....

Total marks 65

- Attempt Questions 1-9.

General Instructions:

- Working Time – 90 min
- Write a using BLUE or BLACK pen.
- Board approved calculators may be used.
- The BOSTES reference sheet is provided.
- In Questions 6-9, show relevant mathematical reasoning and/or calculations.
- **Begin each question on a new page.**
- Full marks may not be awarded for careless and illegible writing.

Multiple Choice		5
Question 6		15
Question 7		15
Question 8		15
Question 9		15
TOTAL		/65

Section 1 - Multiple choice

1. If $\alpha, \beta\ and\ \gamma$ are the roots of the equation $x^3 - 3x + 4 = 0$,

   then the cubic equation with roots $\alpha^2, \beta^2 and\ \gamma^2$ is:

   (A) $8x^3 - 9x + 4 = 0$ (B) $x^3 + 9x^2 - 12x + 4 = 0$

   (C) $x^3 - 6x^2 + 9x - 16 = 0$ (D) $8x^3 + 4x^2 - 9x + 16 = 0$

2. The foci and the directrices of the ellipse with equation $4x^2 + y^2 = 4$ are:

   (A) $(\pm\sqrt{3}, 0)$ and $x = \pm\frac{4\sqrt{3}}{3}$ (B) $(0, \pm\sqrt{3})$ and $y = \pm\frac{4\sqrt{3}}{3}$

   (C) $(0, \pm\sqrt{3})$ and $x = \pm\frac{4\sqrt{3}}{3}$ (D) $(\pm\sqrt{3}, 0)$ and $y = \pm\frac{4\sqrt{3}}{3}$

3. The complex number z lies on the curve $|z - (1 + i)| = 1$

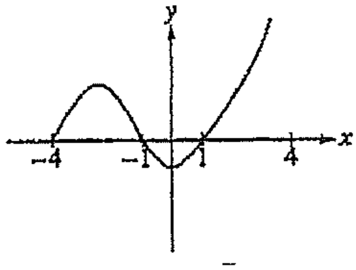
   What is the minimum value of $|z|$?

   (A) 1 (B) $\sqrt{2}$ (C) $\sqrt{2} - 1$ (D) $\sqrt{2} + 1$

4. ω is a non real root of the equation $z^5 + 1 = 0$. Which of the following is not a root of the equation?

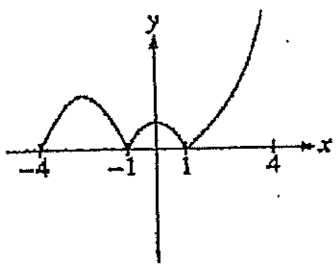
   (A) $\overline{\omega}$ (B) ω^2 (C) $\frac{1}{\omega}$ (D) ω^3

5.

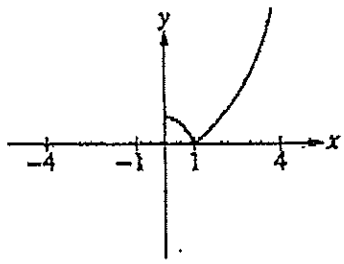


The graph of $y = f(x)$ is shown above. Which of the following could be the graph of $y = f(|x|)$?

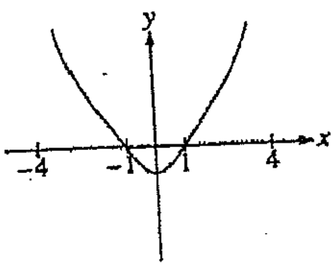
(A)



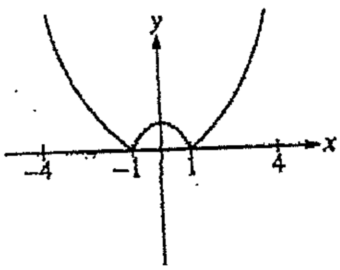
(B)



(C)



(D)



Section 2 - (Start a new page)

Question 6 — Marks

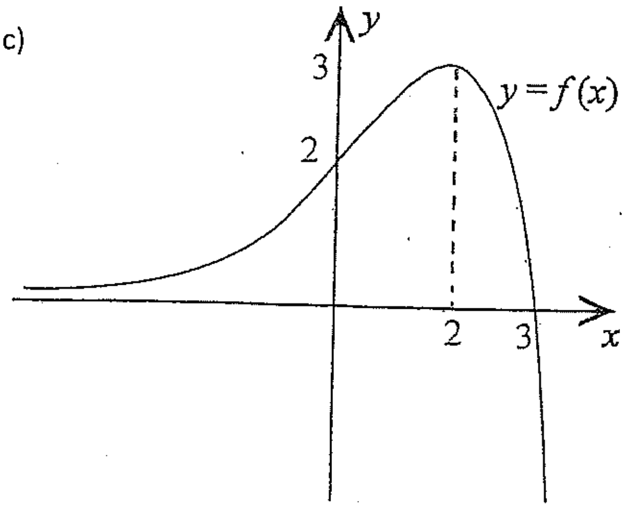
a) Find real numbers a, b and c such that — 4

$$\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

b) Sketch the locus of — 2

Arg $(z + i - 1) =$ Arg z

c)



Shown is a sketch of the function $y = f(x)$. On separate diagrams, showing all main features, sketch

(i) $y = \frac{1}{f(x)}$ — 2

(ii) $y = [f(x)]^2$ — 2

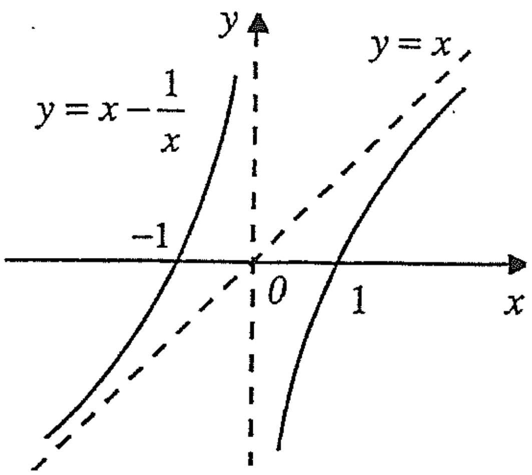
(iii) $y = f\,'(x)$ — 2

d) Consider the curve $2x^2 + xy - y^2 = 0$. — 3

At the point (2, 4) on the curve find the value of $\frac{dy}{dx}$

Question 7 (Start a new page) Marks

a)



The diagram shows the graph of the function $f(x) = x - \frac{1}{x}$. On separate diagrams sketch the following curves, showing any intercepts on the coordinate axes and equations of any asymptotes:

(i) $y = |f(x)|$ 2

(ii) $y^2 = f(x)$ 2

b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$, has eccentricity $e = \frac{1}{2}$. The point $P(2, 3)$ lies on the ellipse.

(i) Find the values of a and b. 3

(ii) Sketch the ellipse showing intercepts, coordinates of the foci and equations of the directrices. 3

c) The polynomial $P(z)$ is defined by $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$

(i) Given that $z = 2 - i$ is a root of $P(z)$ write down another root giving a reason for your answer. 1

(ii) Hence express $P(z)$ as a product of real quadratic factors. 2

d) Find $\int \frac{3x}{\sqrt{2x^2-1}}\,dx$ by using the substitution $v = 2x^2 - 1$ 2

Question 8 (Start a new page) Marks

a) Use the substitution $v = x^3 + 3x - 2$ to evaluate 3

$$\int_0^1 (x^2+1)\sqrt[3]{x^3+3x-2}\ dx$$

b) If $P(2\cos\theta,\ 3\sin\theta)$ lies on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ 3

(i) Show that the equation of the normal at P is

$y - 3\,sin\theta = \frac{2sin\theta}{3cos\theta}\,(x - 2\cos\theta)$

(ii) Find the value of θ (acute) to the nearest degree if the normal passes through 2

the point$(-2, 0)$.

c) The complex number ω is given by $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$

(i) Show $\omega^2 = \overline{\omega}$ 1

(ii) Evaluate $|\ \omega\ |$ and arg ω 2

(iii) Show that w is a root of $\omega^3 - 1 = 0$ 2

d) Give that $1, \omega, \omega^2$ are the cube roots of unity, ie: roots of $z^3 = 1$, simplify

$(1-\omega)(1-\omega^2)(1-\omega^7)(1-\omega^{11})$. 2

Question 9 (Start a new page) Marks

a) Find values for a and b if $(x-1)^2$ is a factor of $P(x) = x^5 + 2x^4 + ax^3 + bx^2$ 3

b) Let the roots of $x^3 - 2x^2 - x + 4 = 0$ be α, β, γ. 2

Find the cubic equation whose roots are $\alpha - 1, \beta - 1, \gamma - 1$.

c) α, β *and* γ are the roots of the equation $x^3 - px^2 + qx - r = 0$

(i) Write down expressions in terms of p, q, r for $\alpha + \beta + \gamma$ and 2

$\alpha\beta + \beta\gamma + \alpha\gamma$

(ii) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q$ and 4

$\alpha^3 + \beta^3 + \gamma^3 = p^3 - 3pq + 3r$

(iii) Hence solve the equations 4

$\alpha + \beta + \gamma = -1$

$\alpha^2 + \beta^2 + \gamma^2 = 5$

$\alpha^3 + \beta^3 + \gamma^3 = -7$

Suggested mark break up - Staff Copy

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2017 Extension 2 Task 2 Solutions

1. C 2. B 3 C 4. B 5. C

6. a) $\dfrac{5-3x}{(x+1)(x^2+1)} = \dfrac{a}{x+1} + \dfrac{bx+c}{x^2+1}$

$5-3x = a(x^2+1) + (bx+c)(x+1)$

$5-3x = ax^2+a + bx^2+bx+cx+c$

$-3x+5 = (a+b)x^2 + (b+c)x + (a+c)$ ①

$a+b=0$ ① $\therefore a=-b$

① $c+b=-3$ ② sub. into ③

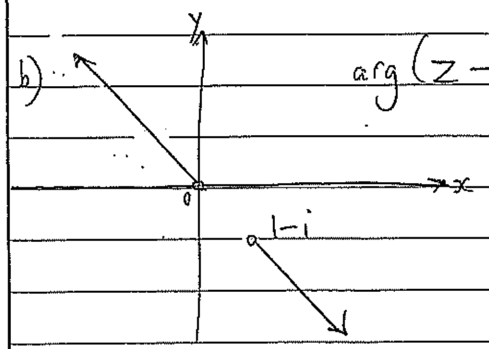
$a+c=5$ ③

$-b+c=5$ ④

② + ④ gives

$2c=2$

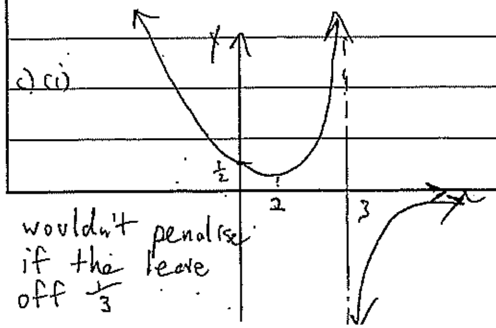
$c=1 \quad \therefore a=4, \quad b=-4$ ②

b) 

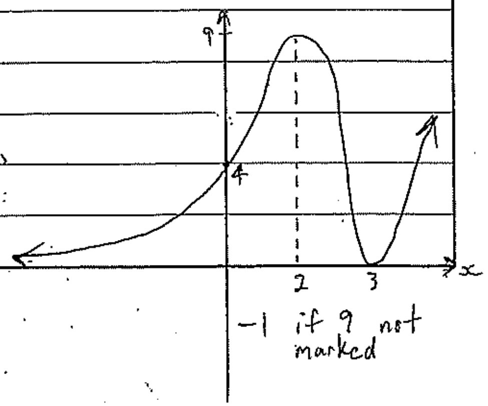
$\arg(z-(1-i)) = \arg z$

1 mark for either branch

-1 mark for closed circles.

c) (i) 

wouldn't penalise if the leave off $\frac{1}{3}$

(ii) 

-1 if 9 not marked

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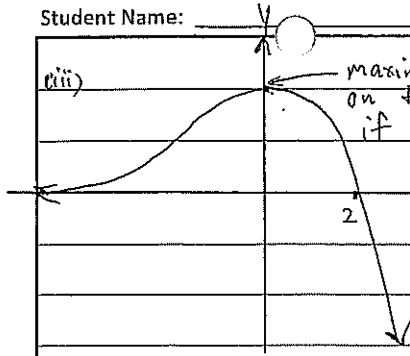
(iii) 

maximum must be on the y axis, -1 if not.

Must be clear on graph that $y=0$ is an asymptote

d) $2x^2+xy-y^2=0$

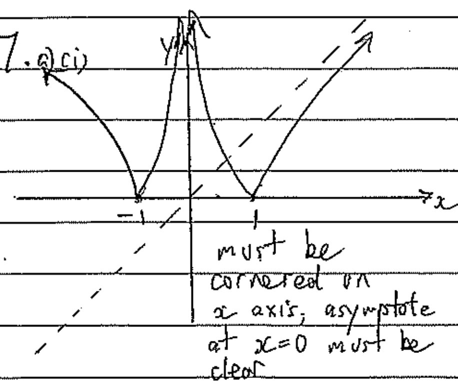
$4x + y + x\dfrac{dy}{dx} - 2y\dfrac{dy}{dx} = 0$ ①

$\dfrac{dy}{dx}(x-2y) = -4x-y$

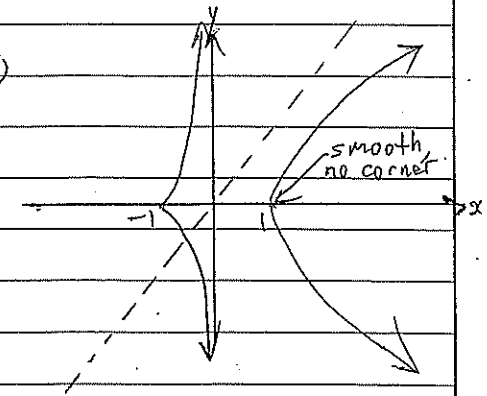
$\dfrac{dy}{dx} = \dfrac{4x+y}{2y-x}$ ①

$\therefore$ At $(2,4)$, $\dfrac{dy}{dx} = \dfrac{12}{8-2}$

$= 2$ ①

7. a) (i) 

must be cornered on x axis, asymptote at $x=0$ must be clear

(ii) 

smooth, no corner

b) (i) $b^2 = a^2(1-e^2)$

$b^2 = a^2\left(1-\frac{1}{4}\right)$

$b^2 = \dfrac{3a^2}{4}$ ①

$\dfrac{4}{a^2} + \dfrac{9}{b^2} = 1$

$\therefore \dfrac{4}{a^2} + \dfrac{9}{\frac{3a^2}{4}} = 1$

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$\frac{4}{a^2} + \frac{12}{a^2} = 1$

$4 + 12 = a^2$

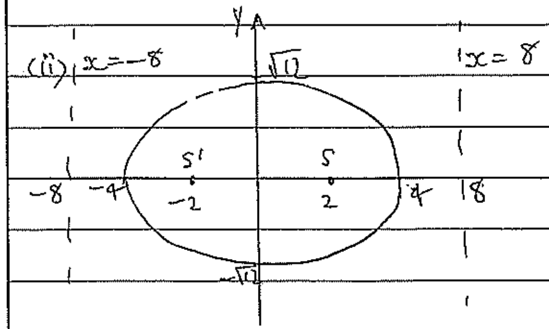
$a = \pm 4$

$a = 4$ as $a > 0$

$\therefore b^2 = \frac{3 \times 16}{4} = 12$

$\therefore a = 4, \ b = \sqrt{12}$ 1 + 1

(ii)



intercepts 1
focii 1
directrices 1

c) (i) $2+i$. Since coefficients are real, complex roots occur in conjugate pairs. 1

c(ii) $z^2 - (\text{sum of roots})z + \text{product}$

$z^2 - 4z + 5$ 1

Now

$z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 - 4z + 5)(z^2 + 2z + 2)$ 1

by inspection.

d) $\int \frac{3x}{\sqrt{2x^2-1}}\, dx$ $\qquad u = 2x^2 - 1$

$\frac{3}{4}\int u^{-\frac{1}{2}}\, du$ 1 $\qquad du = 4x\, dx$

$\frac{3}{4} \times 2u^{\frac{1}{2}} + c$ $\qquad \therefore \frac{3}{4} du = 3x\, dx$

$\frac{3}{2}\sqrt{2x^2-1} + c$

1

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8. a) $\int_0^1 (x^2+1)\sqrt[3]{x^3+3x-2}\, dx$

$u = x^3 + 3x - 2$ $\qquad x = 0, \ u = -2$

$du = 3x^2 + 3\, dx$ $\qquad x = 1, \ u = 2$

$\frac{1}{3} du = (x^2+1)\, dx$

$\frac{1}{3}\int_{-2}^{2} u^{\frac{1}{3}}\, du$ 1

$\frac{1}{3} \times \frac{3}{4}\left[u^{\frac{4}{3}}\right]_{-2}^{2}$ 1

$\frac{1}{4}\left[2^{\frac{4}{3}} - (-2)^{\frac{4}{3}}\right] = 0$ 1 must have 0 for 3rd mark.

b) (i) $\frac{x^2}{4} + \frac{y^2}{9} = 2$

$\frac{2x}{4} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{9x}{4y}$ 1

At the pt. $(2\cos\theta, \ 3\sin\theta)$

$\frac{dy}{dx} = -\frac{9}{4} \times \frac{2\cos\theta}{3\sin\theta}$

$= -\frac{3}{2}\frac{\cos\theta}{\sin\theta}$ 1

$\therefore$ gradient of normal is $\frac{2\sin\theta}{3\cos\theta}$

$\therefore$ Equation is

$y - 3\sin\theta = \frac{2\sin\theta}{3\cos\theta}(x - 2\cos\theta)$ 1

(ii) $(-2, 0)$ satisfies equation

$0 - 3\sin\theta = \frac{2\sin\theta}{3\cos\theta}(-2 - 2\cos\theta)$

$-9\sin\theta\cos\theta = -4\sin\theta - 4\sin\theta\cos\theta$

$0 = 5\sin\theta\cos\theta - 4\sin\theta$

$0 = \sin\theta(5\cos\theta - 4)$ 1

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$\cos\theta = \frac{4}{5}$

$\theta = 37^\circ$. (1)

c) (i) $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\omega^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2$

$= \frac{1}{4} - \frac{3}{4} - \frac{i\sqrt{3}}{4} - \frac{i\sqrt{3}}{4}$

$= -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ (1)

$= \overline{\omega}$ as req'd

(ii) $|\omega|$

$= \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right|$

$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}}$

$= 1$ (1)

$\arg\omega$

$= \tan^{-1}(-\sqrt{3})$

$= \frac{2\pi}{3}$ (1)

(iii) $\omega = \text{cis}\,\frac{2\pi}{3}$

$\therefore \omega^3 = \text{cis}\,\frac{6\pi}{3}$ by De Moivre's

$= \text{cis}(2\pi)$ (1)

$= \cos 2\pi + i\sin 2\pi$

$= 1$ as req'd (1)

or $\omega^3 = \omega^2\omega$

$= \overline{\omega}\omega$ (1) from (i)

$= |\omega|^2$

$= 1$ as req'd (1)

d) $(1-\omega)(1-\omega^2)(1-\omega^7)(1-\omega^{11})$

$(1-\omega)(1-\omega^2)(1-\omega^3 \times \omega^3 \times \omega)(1-\omega^3\omega^3\omega^3\omega^2)$

$(1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$(1-\omega)^2(1-\omega^2)^2$

$(1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$ (1)

$(1+\omega+\omega^2-3\omega)(1+\omega+\omega^2-3\omega^2)$ as $\omega^4 = \omega \times \omega^3 = \omega$

$(-3\omega) \times -3\omega^2$ sum of roots $= 0$

9 (1)

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9. $P(x) = x^5 + 2x^4 + ax^3 + bx^2$

$P'(x) = 5x^4 + 8x^3 + 3ax^2 + 2bx$

Now both $P(1) = P'(1) = 0$ (1)

$1 + 2 + a + b = 0 \Rightarrow a + b = -3 \therefore 3a + 3b = -9$ (1)

$5 + 8 + 3a + 2b = 0 \Rightarrow 3a + 2b = -13$ (2)

(1) $-$ (2) gives

$b = 4$ (1) $\therefore a = -7$ (1)

b) Replace x with $x+1$

$(x+1)^3 - 2(x+1)^2 - (x+1) + 4 = 0$ (1)

$x^3 + 3x^2 + 3x + 1 - 2x^2 - 4x - 2 - x - 1 + 4 = 0$

$x^3 + x^2 - 2x + 2 = 0$ (1)

c) (i) $\alpha + \beta + \gamma = p$ (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = q$ (1)

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ (1)

$= p^2 - 2q$ (1)

$\alpha^3 = p\alpha^2 - q\alpha + r$

$\beta^3 = p\beta^2 - q\beta + r$

$\gamma^3 = p\gamma^2 - q\gamma + r$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = p(\alpha^2 + \beta^2 + \gamma^2) - q(\alpha+\beta+\gamma) + 3r$ (1)

$= p \times (p^2 - 2q) - q(p) + 3r$

$= p^3 - 3pq + 3r$ (1)

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(iii) $\alpha+\beta+\gamma=-1 \Rightarrow p=-1$ ①

$\alpha^2+\beta^2+\gamma^2=5 \Rightarrow p^2-2q=5$ ②

$\alpha^3+\beta^3+\gamma^3=-7 \Rightarrow p^3-3pq+3r=-7$ ③

1 mark

Sub. ① into ②

$1-2q=5$

$q=-2$ sub. into ③

$(-1)^3-3\times-1\times-2+3r=-7$

$-1-6+3r=-7$

$r=0$ ①

$\therefore x^3+x^2-2x=0$ has roots α,β,γ

$x(x^2+x-2)=0$

$x(x-1)(x+2)=0$ ①

$x=0,+1,-2$ ①

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Multiple Choice Answer Working

1. Replace x with $\sqrt{x}$

$(\sqrt{x})^3 - 3\sqrt{x} + 4 = 0$

$x^{3/2} - 3x^{\frac{1}{2}} + 4 = 0$

$x^{\frac{1}{2}}(x-3) + 4 = 0$

$(x^{\frac{1}{2}})(x-3) = -4$

$x(x-3)^2 = 16$

$x(x^2-6x+9) = 16$

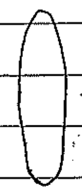
$x^3 - 6x^2 + 9x - 16 = 0$

2. $4x^2 + y^2 = 4$

$\frac{x^2}{1} + \frac{y^2}{4} = 1$

$a^2 = 1 < b^2 = 4$

$\therefore b = 2$



Focii : $(0, \mp be)$

Directrices $y = \pm \frac{b}{e}$

$a^2 = b^2(1-e^2)$

$1 = 4(1-e^2)$

$\frac{1}{4} = 1 - e^2$

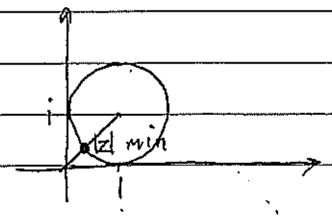
$e^2 = \frac{3}{4}$

$e = \frac{\sqrt{3}}{2}$

$\therefore$ Focii $(0, \pm\sqrt{3})$ Directrices $y = \pm \frac{2}{\sqrt{3}/2}$

$= \pm \frac{4}{\sqrt{3}}$

$y = \pm \frac{4\sqrt{3}}{3}$

3. 

$\sqrt{2} - 1$

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4. $(\omega^2)^5 + 1 = 0$

$\omega^{10} = -1$

$(\omega^5)^2 = -1$

$(-1)^2 = -1$

$1 = -1$ NO

5. Graphs of $y = f(|x|)$ form have the section where $x > 0$ on $y = f(x)$, reflected across the y axis.