SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION 1

YEAR 11 YEARLY EXAMINATION
SEPTEMBER 2003

Time allowed: 90 minutes

Instructions:

- Show all necessary working
- Start each question on a new page
- Marks will not be awarded for careless or badly arranged work
- Non-programmable calculators may be used
- This paper must be handed in with your answer sheets
- Answers must be written in blue or black pen

Name:________________________________

Teacher:____________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/11	/10	/10	/10	/10	/12	/63

Question 1 (11 marks)

a) If $f(n) = n(n+1)(n+2)$

i) Simplify $\dfrac{f(n)}{f(n+1)}$ (1)

ii) Express $f(n) - f(n+1)$ in factored form (2)

b) i) Find the gradient of the normal to the curve

$y = x^4 + x^{3/2}$ at A (1, 5) (2)

ii) Find the acute angle to the nearest degree between the normal and the line $2x + 3y = 7$ (2)

c) If α, β *and* γ are the roots of

$2x^3 + 12x^2 - 6x + 1 = 0$ find

i) $\alpha + \beta + \gamma$ (1)

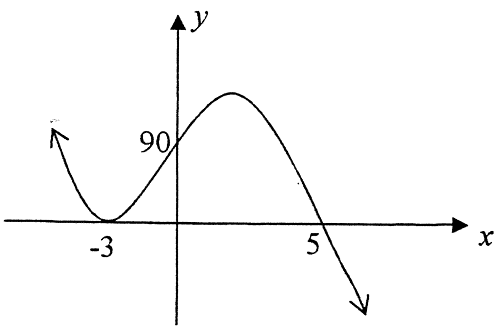
ii) $\alpha\beta\gamma$ (1)

iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ (2)

Question 2 (10 marks)

Start a new page

a) Find the equation of the cubic polynomial in the sketch below (2)



b) Solve $\dfrac{1}{x-1} - \dfrac{1}{x} > 0$ (3)

c) Find the equation of the parabola with vertex (2, -1), axis of symmetry parallel to the y axis and passing through the point (8 , 2) (2)

d) Prove that $3kx^2 - (2k+3n)x + 2n = 0$ has rational roots if k and n are rational (3)

Question 3 (10 marks) Start a new page

a) Solve $2\sin x - 3\cos x = 2$ for $0° \leq x \leq 360°$ (3)

b) Prove $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$ (2)

Hence find the exact value of $\tan 15°$ in simplest form (2)

c) The monic polynomial P (x) has a degree of 2. When P (x) is divided by x the remainder is -6. If P (3) = P (-5), find the polynomial. (3)

Question 4 (10 marks) Start a new page

a)

i) The polynomial equation P $(x) = 0$ has a double root at $x = a$. By putting $P(x) = (x-a)^2 Q(x)$, show that $P'(a) = 0$ (2)

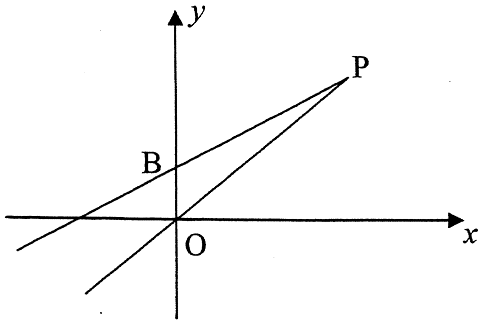
ii) You are told that the equation $mx^4 + nx^3 - 6x^2 + 22x - 12 = 0$ has a double root at $x = 1$. Find the values of m and n. (3)

b) Find the values of m for which the line y = m x + 4 intersects the curve $y = x^2 - 4x + 5$ at two distinct points (3)

c) Find the co-ordinates of the point that divides the interval joining A(3,2) to B(-1,1) externally in the ratio 3:2. (2)

Question 5 **(10 marks)** **Start a new page**

a)



B is the point (0, 3). The gradient of BP is m and the gradient of OP is $2m$.

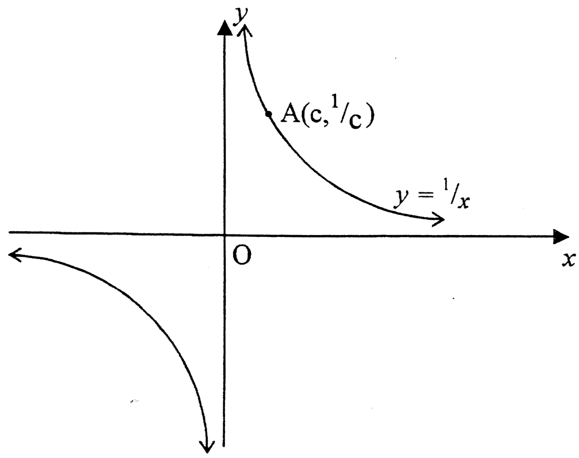
O is the origin.

i) Write the equations of the lines BP and OP (2)

ii) Find the co-ordinates of P (1)

iii) Find the locus of P as the value of m varies. (1)

b) The point A $(c, \frac{1}{c})$ lies on the curve $y = \frac{1}{x}$



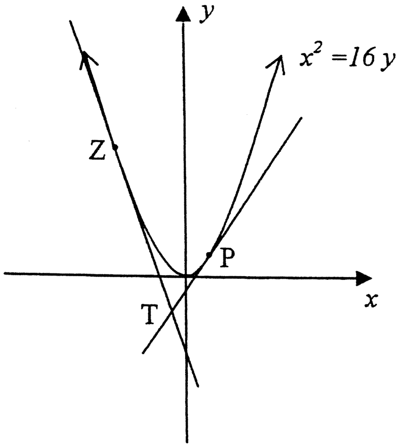
i) Find the equation of the tangent at A (2)

ii) The tangent at A cuts the x axis at B and the y axis at C. Find the coordinates of B and C. (2)

iii) Show that the area of triangle BOC is a constant (2)

Question 6 (12 marks) Start a new page

a)



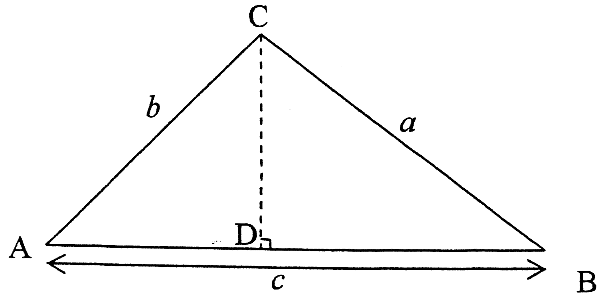
i) Find the equation of the tangent to the curve $x^2 = 16y$ at $P(8p, 4p^2)$ (2)

ii) If this tangent at P also passes through T (-2, -2) and TZ is another tangent to the parabola at Z, find the coordinates of P and Z (2)

iii) Let RQ be a chord on the same parabola with $R(8r, 4r^2)$ and $Q(8q, 4q^2)$. Prove that the equation of RQ is $y = (\frac{r+q}{2})x - 4rq$ (2)

iv) If RQ is a focal chord show that $rq = -1$. (1)

b)



The triangle ABC has sides of length a, b and c as shown in the diagram.

The point D lies on AB and CD is perpendicular to AB

i) Show that $a \sin B = b \sin A$ (1)

ii) Show that $c = a \cos B + b \cos A$ (1)

iii) Hence given that $c^2 = 4ab \cos A \cos B$ show that $a = b$ (3)

End of Paper.

Question 1

a) i) $\dfrac{f(n)}{f(n+1)} = \dfrac{n\cancel{(n+1)}\cancel{(n+2)}}{\cancel{(n+1)}\cancel{(n+2)}(n+3)}$

$= \dfrac{n}{n+3}$

ii) $f(n) - f(n+1)$

$= n(n+1)(n+2) - (n+1)(n+2)(n+3)$

$= (n+1)(n+2)(n-(n+3))$

$= -3(n+1)(n+2)$

b) i) $y = x^4 + x^{3/2}$

$\dfrac{dy}{dx} = 4x^3 + \dfrac{3}{2}x^{1/2}$

at $A(1,5)$ $m_T = \dfrac{11}{2}$ $\therefore m_{normal} = \dfrac{-2}{11}$

ii) $\tan\theta = \left|\dfrac{\frac{-2}{11} - \frac{-2}{3}}{1+\left(\frac{-2}{11}\times\frac{-2}{3}\right)}\right|$

$\left(m_1 = \dfrac{-2}{11} \quad m_2 = \dfrac{-2}{3}\right)$

$\tan\theta = \left|\dfrac{16/33}{37/33}\right|$

$\theta = 23^\circ$

c) $a = 2$ $b = 12$ $c = -6$ $d = 1$

i) $\alpha + \beta + \gamma = \dfrac{-12}{2} = -6$

ii) $\alpha\beta\gamma = \dfrac{-1}{2}$

iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

$\dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \dfrac{-6/2}{-1/2}$

$= 6$

Question 2

a) $P(x) = A(x+3)^2.(x-5)$

sub $x = 0$ $P(0) = 90$

$90 = A.\,3^2 \times -5$

$90 = -45A$

$\therefore A = -2$

$P(x) = -2(x+3)^2.(x-5)$

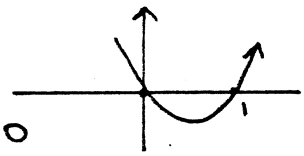
b) $\dfrac{1}{x-1} - \dfrac{1}{x} > 0$

$\dfrac{x-(x-1)}{x(x-1)} > 0$

$\dfrac{1}{x(x-1)} > 0$

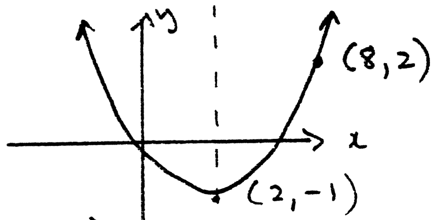
$\dfrac{x^2(x-1)^2}{x(x-1)} > 0$

$x(x-1) > 0$



$x > 1,\ x < 0$

c)



$(x-2)^2 = 4a(y+1)$ sub $(8,2)$

$6^2 = 4a.3$

$\therefore a = 3$

$\therefore$ eqn $(x-2)^2 = 12(y+1)$

d) Roots rational if Δ square

$\Delta = (2k+3n)^2 - 4.3k.2n$

$= 4k^2 + 12kn + 9n^2 - 24kn$

$= 4k^2 - 12kn + 9n^2$

$\Delta = (2k-3n)^2$ if k, n rational

Question 3

a) $R = \sqrt{4+9} = \sqrt{13}$

$2\sin x - 3\cos x = \sqrt{13}\left[\frac{2}{\sqrt{13}}\sin x - \frac{3}{\sqrt{13}}\cos x\right]$

place in form $\sqrt{13}(\sin(x-\alpha))$

ie $\sqrt{13}[\sin x\cos\alpha - \cos x \sin\alpha]$

$\therefore \quad \cos\alpha = \frac{2}{\sqrt{13}} \qquad \sin\alpha = \frac{3}{\sqrt{13}}$

α is acute $\quad \alpha = 56.31^\circ$

or $56^\circ 19'$

$\sqrt{13}\sin(x - 56^\circ 19') = 2$

$\sin(x - 56^\circ 19') = \frac{2}{\sqrt{13}}$

S ✓	A ✓
T	C

$x - 56^\circ 19' = 33^\circ 41', \ 146^\circ 19'$

$\therefore \ x = 90^\circ, \ 202^\circ 38'$

b) LHS $= \dfrac{1-\cos 2x}{\sin 2x}$

$= \dfrac{1 - [\cos^2 x - \sin^2 x]}{2\sin x\cdot\cos x}$

$= \dfrac{\cancel{2}\sin^2 x}{\cancel{2}\sin x\cos x}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$

$=$ RHS

$\therefore \ \tan 15^\circ = \dfrac{1-\cos 30^\circ}{\sin 30^\circ}$

$= \left(1 - \frac{\sqrt{3}}{2}\right) \div \frac{1}{2}$

$= \dfrac{2-\sqrt{3}}{2} \times \dfrac{2}{1}$

$= 2-\sqrt{3}$

c) $P(x) = x^2 + bx + c$

$P(0) = -6 \quad \therefore \quad c = -6$

$P(3) = 9 + 3b - 6 = 3 + 3b$

$P(-5) = 25 - 5b - 6 = 19 - 5b$

$\therefore \ 3 + 3b = 19 - 5b$

$8b = 16$

$b = 2$

$\therefore \ P(x) = x^2 + 2x - 6$

Question 4

a) i) $P(x) = (x-a)^2 Q(x)$

$P'(x) = 2(x-a).Q(x) + (x-a)^2 Q'(x)$

$P'(a) = 2(\cancel{a-a})Q(a) + (\cancel{a-a})^2 Q'(a)$

$\therefore \ P'(a) = 0$

ii) double root at $x = 1$

$\therefore \ P(1) = 0 \quad P'(1) = 0$

$P(x) = mx^4 + nx^3 - 6x^2 + 22x - 12$

$P'(x) = 4mx^3 + 3nx^2 - 12x + 22$

$P(1) = m + n - 6 + 22 - 12 = 0$

$\therefore \ m + n = -4$ —— ①

$P'(1) = 4m + 3n - 12 + 22 = 0$

$4m + 3n = -10$ —— ②

$n = -6 \quad m = 2$

b) $y = mx + 4$

$y = x^2 - 4x + 5$

$x^2 - 4x + 5 = mx + 4$

$x^2 - 4x - mx + 1 = 0$

$x^2 - x(4+m) + 1 = 0$

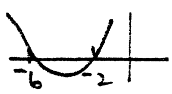
2 pts intersection $\therefore$ 2 distinct solutions

$\Delta > 0$

$(4+m)^2 - 4 > 0$

$m^2 + 8m + 12 > 0$

$(m+6)(m+2) > 0$



$\therefore \ m > -2$ and $m < -6$

c) $A(3, 2)$ $B(-1, 1)$

$-3 : 2$

$x: \dfrac{(2\times 3)+(-3\times -1)}{-3+2} = -9$

$y: \dfrac{(2\times 2)+(-3\times 1)}{-3+2} = -1$

$P(-9, -1)$

Question 5

a) i) BP: $y = mx + 3$

OP: $y = 2mx$

ii) $mx + 3 = 2mx$

$3 = mx$

$x = \dfrac{3}{m}$

$\therefore P\left(\dfrac{3}{m}, 6\right)$

iii) line $y = 6$

b) i) $y = x^{-1}$

$\dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

$m_T = -\dfrac{1}{c^2}$ at $A\left(c, \dfrac{1}{c}\right)$

$\therefore$ eqn tang: $y - \dfrac{1}{c} = -\dfrac{1}{c^2}(x - c)$

ii) $B(2c, 0)$

$C\left(0, \dfrac{2}{c}\right)$

iii) $\triangle BOC = \dfrac{1}{2} \times \dfrac{2}{\cancel{c}} \times 2\cancel{c}$

$= 2$ units $\therefore$ constant

Question 6

a) i) $y = \dfrac{x^2}{16}$ $P(8p, 4p^2)$

$\dfrac{dy}{dx} = \dfrac{2x}{16} = \dfrac{x}{8}$

$\therefore$ at $P(8p, 4p^2)$ $m_T = p$

$\therefore$ eqn tang $y - 4p^2 = p(x - 8p)$

$y = px - 4p^2$

ii) sub $T(-2, -2)$ into tangent

$-2 = -2p - 4p^2$

$4p^2 + 2p - 2 = 0$

$(2p - 1)(p + 1) = 0$ $\therefore p = \frac{1}{2}, -1$

$\therefore$ co-ord $P(4, 1)$ $p = 1/2$

co-ord $Z(-8, 4)$ $p = -1$

iii) $m_{RQ} = \dfrac{4r^2 - 4q^2}{8r - 8q} = \dfrac{4(r-q)(r+q)}{8(r-q)}$

$= \dfrac{r+q}{2}$

$\therefore$ eqn RQ: $y - 4r^2 = \dfrac{r+q}{2}(x - 8r)$

$y - 4r^2 = \left(\dfrac{r+q}{2}\right)x - \dfrac{8r(r+q)}{2}$

$y - \cancel{4r^2} = \left(\dfrac{r+q}{2}\right)x - \cancel{4r^2} - 4rq$

$y = \left(\dfrac{r+q}{2}\right)x - 4rq$ *

iv) If RQ is a focal chord passes through $(0,4)$ sub into *

$4 = -4rq$

$\therefore rq = -1$

b) i) use sine rule in ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a \sin B = b \sin A \quad \text{—— ①}$$

ii) In ΔACD $\cos A = \frac{AD}{b}$

$AD = b \cos A$

In ΔDBC $\cos B = \frac{DB}{a}$

$DB = a \cos B$

since $c = AD + DB$

$\therefore c = b \cos A + a \cos B$

iii)

$$c^2 = (b \cos A + a \cos B)^2$$

$$c^2 = b^2 \cos^2 A + 2ab \cos A \cos B + a^2 \cos^2 B$$

$$4ab \cos A \cos B = b^2 \cos^2 A + 2ab \cos A \cos B + a^2 \cos^2 B$$

$$0 = b^2 \cos^2 A - 2ab \cos A \cos B + a^2 \cos^2 B$$

$$0 = (b \cos A - a \cos B)^2$$

$$b \cos A = a \cos B \quad \text{—— ②}$$

$\div$ ① by ②

$$\frac{a \sin B}{a \cos B} = \frac{b \sin A}{b \cos A}$$

$$\tan B = \tan A$$

$$\therefore A = B$$

$\therefore$ from ① or ②

$$a \cancel{\sin B} = b \cancel{\sin B}$$

$$\therefore a = b$$