Applications of Calculus to the Physical World

1. Rates of Change

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

2. Exponential Growth and Decay

$$\frac{dQ}{dt} = kQ$$

k = Constant
Q = Ae^{kt}

$$\frac{dN}{dt} = k(N - B)$$

N = $B + Ae^{kt}$
k = Constant
B = Constant

3. Motion in 2D

$$a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

4. Simple Harmonic Motion

$$\ddot{x} = -n^2 x$$

$$v^2 = n^2(a^2 - x^2)$$

$$T = \frac{2\pi}{n} = \frac{1}{F}$$

n Period
a Amplitude
T Period
F Frequency

5. Projectile Motion

$\ddot{x}$ = 0
$\dot{x}$ = $V\cos\theta$
x = $V\cos\theta t$

V = $\sqrt{\dot{x}^2 + \dot{y}^2}$

$\ddot{y}$ = $-g$ = -10
$\dot{y}$ = $-gt +$ $V\sin\theta$ = $-10t + V\sin\theta$
y = $-\dfrac{gt^2}{2} +$ $V\sin\theta t$ = $-5t^2 + V\sin\theta t$

Equation of Motion

$$= \frac{-gx^2(1+\tan^2\theta)}{2V^2} + x\tan\theta$$

Maximum Height

$$= \frac{V^2\sin^2\theta}{2g}$$

Time of Flight

$$= \frac{2V\sin\theta}{g}$$

Maximum Range

$$= \frac{V^2\sin 2\theta}{g}$$

Equations and Deriving Equations

Rates of Change

- What you want to find is on the LHS
- Get into form:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Exponential Growth and Decay

- Basic Form

$$\frac{dQ}{dt} = kQ$$

k = Constant
Q = Ae^{kt}

A = Constant – initial value
e =
k = Constant
t = Time

- Complex Form

$$\frac{dN}{dt} = k(N - B)$$

N = $B + Ae^{kt}$
k = Constant
B = Constant

Proof

$$\frac{dN}{dt} = k(N - B)$$ k, B are constants

Let u = N – B

$$\frac{du}{dt} = \frac{d}{dt}(N - B)$$

$$= \frac{dN}{dt} - \frac{dB}{dt}$$ B is a constant

$$= \frac{dN}{dt} - 0$$

$$= \frac{dN}{dt} = ku$$

This has solution u = Ae^{kt},
where A is a constant

But u = N – B
So N – B = Ae^{kt}
N = B + Ae^{kt}

N = B + Ae^{kt}

$$\frac{dN}{dt} = kAe^{kt}$$

= k(B + Ae^{kt} – B)
= k(N – B)

Motion in 2D

x	x	x	Displacement
v	$\dot{x}$	$\frac{dx}{dt}$	Velocity
a	$\ddot{x}$	$\frac{dv}{dt}$	Acceleration

At origin	$x = 0$
At rest	$\dot{x} = 0$
When velocity is constant	$a = 0$

Special Form: Expressing acceleration in terms of x, not t

$$a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

$$\text{Velocity} = \frac{dx}{dt}$$

$$\begin{aligned}\text{Acceleration} &= \frac{d}{dx}\left(\frac{dx}{dt}\right)\\ &= \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt}\\ &= \frac{dv}{dx}\times v\\ &= \frac{dv}{dx}\times\frac{d}{dv}\left(\tfrac{1}{2}v^2\right)\\ &= \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)\end{aligned}$$

SHM – Simple Harmonic Motion

Definition: $\ddot{x} = -n^2 x$

$v^2 = n^2(a^2 - x^2)$

$T = \dfrac{2\pi}{n} = \dfrac{1}{F}$

n	Period
a	Amplitude
T	Period
F	Frequency

Proof

$$\begin{aligned} x &= A\cos nt \\ \dot{x} &= -An\sin nt \\ \ddot{x} &= -An^2\cos nt \\ &= -n^2 A\cos nt \\ &= -n^2 x \end{aligned}$$

$$\ddot{x} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -n^2 x$$

$$\tfrac{1}{2}v^2 = \frac{-n^2x^2}{2} + C$$

At x = a, v = 0 $\quad a$ is the amplitude

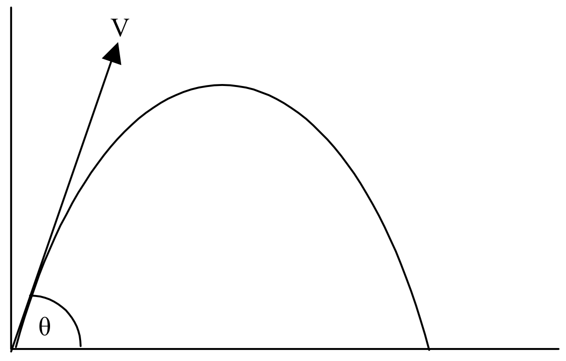
$$0 = \frac{-n^2x^2}{2} + C \qquad \therefore C = \frac{n^2a^2}{2}$$

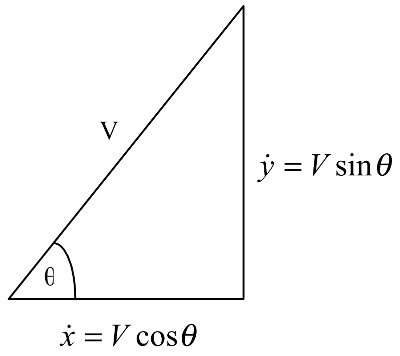
$$\begin{aligned} \tfrac{1}{2}v^2 &= \frac{-n^2a^2}{2} + \frac{n^2a^2}{2} \\ v^2 &= -n^2x^2 + n^2a^2 \\ &= n^2(a^2 - x^2) \\ \tfrac{dx}{dt} \quad v &= \pm n\sqrt{a^2 - x^2} \\ \frac{dt}{dx} &= \frac{1}{n\sqrt{a^2 - x^2}} \\ t &= \frac{1}{n}\int \frac{1}{a^2 - x^2} \\ &= \tfrac{1}{n}\cos^{-1}\left(\tfrac{x}{a}\right) + C \end{aligned}$$

At x = a, t = $\tfrac{1}{2}\cos^{-1}\left(\tfrac{x}{a}\right) + C \qquad \therefore C = 0$

$$\begin{aligned} nt &= \cos^{-1}\left(\tfrac{x}{a}\right) \\ \cos nt &= \tfrac{x}{a} \\ \therefore x &= a\cos nt \end{aligned}$$

Projectile Motion





$\ddot{x} = 0$

$\dot{x} = \mathrm{V}\cos\theta$

$x = \mathrm{V}\cos\theta t$

$\ddot{y} = -g$

$\dot{y} = -gt + \mathrm{V}\sin\theta$

$y = -\dfrac{gt^2}{2} + \mathrm{V}\sin\theta t$

At any time t, v the velocity of the particle can be given the equation $v = \sqrt{\dot{x}^2 + \dot{y}^2}$

MAX HEIGHT $\dot{y} = 0$

$0 = -gt + V\sin\theta$

$gt = V\sin\theta$

$t = \dfrac{V\sin\theta}{g}$ Time of Max Height

Sub into y equation

$y = \dfrac{-g}{2}\left(\dfrac{V\sin\theta}{g}\right)^2 + V\sin\theta\left(\dfrac{V\sin\theta}{g}\right)$

$y = -\dfrac{V^2\sin^2\theta}{2g} + \dfrac{V^2\sin^2\theta}{g}$

$y = \dfrac{V^2\sin^2\theta}{2g}$

MAX RANGE, TIME OF FLIGHT y = 0

$0 = -\dfrac{gt^2}{2} + V\sin\theta\, t$

$\dfrac{gt^2}{2} = V\sin\theta\, t$

$t = \dfrac{2V\sin\theta}{g}$ Time of Flight (Max Range)

Sub into x equation

$x = V\cos\theta\left(\dfrac{2V\sin\theta}{g}\right)$

$x = \dfrac{V^2\sin 2\theta}{g}$ **

$x = \dfrac{V^2}{g}$ only at 45°

** Max Angle (using double angles)

$\sin 2\theta = 1$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4} = 45°$

Cartesian Equation, Trajectory
$x = V\cos\theta\, t$ $y = \frac{-gt^2}{2} + V\sin\theta\, t$ $t = \frac{x}{V\cos\theta}$ Sub into y equation $y = \frac{-g}{2}\left(\frac{x}{V\cos\theta}\right)^2 + V\sin\theta\left(\frac{x}{V\cos\theta}\right)$ $y = \frac{-gx^2}{2V^2\cos^2\theta} + \frac{V\sin\theta x}{V\cos\theta}$ $y = \frac{-g\sec^2\theta x^2}{2V^2} + \tan\theta x$ $y = \frac{-gx^2(1+\tan^2\theta)}{2V^2} + \tan\theta x$ This is a quadratic in x and $\tan\theta$

Rates of Change

Problems Summary

<table>
<tr><td>Prisms, Spheres</td><td>

- $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$
- Sometimes with spheres, use 2 “dummy variables”

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<tr><td>Pyramids</td><td>

- Use similar triangles
- $V = \frac{1}{3}Ah$
- Try to eliminate unused variables

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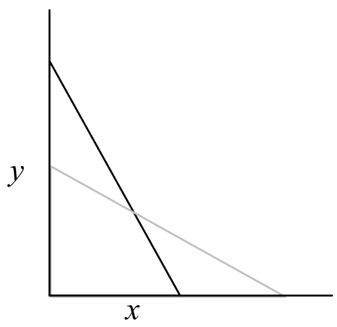
<tr><td>Ladder Problem type



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- Solve with Pythagoras Theorem
- Implicit differentiate with respect to time
- Sub in values

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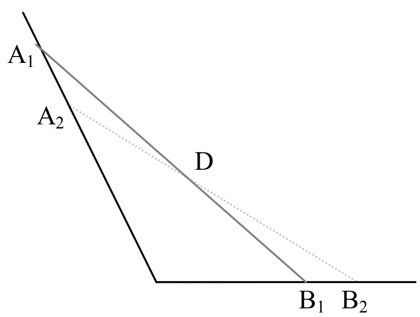
<tr><td>Boy Chase Girl type



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- Solve with Cos or Sin rules

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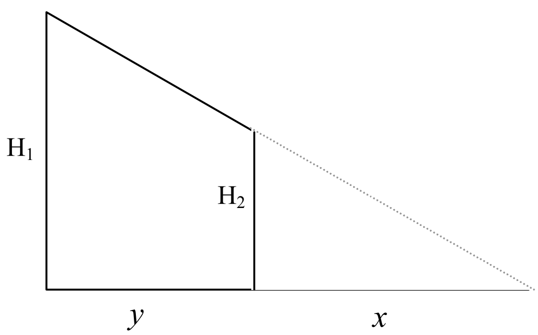
<tr><td>Shadow Problem type



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- Solve with similar triangles ratios of length
- Implicit differentiate with respect to time

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Rates of Change is the derivatives of functions with respect to time t.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

<u>Example 1</u>

The Surface Area of a sphere increases at 6cms^{-1}

Find a) The rate of change of radius when r = 5

b) The rate of change of volume when r = 5

A) $SA = 4\pi r^2$

Find : $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$ $\qquad \frac{dA}{dt} = 6 \qquad \frac{dA}{dr} = 8\pi r$

$$= 6 \times \frac{1}{8\pi r} \qquad \frac{dr}{dA} = \frac{1}{8\pi r}$$

$$= \frac{3}{4\pi r}$$

$$= \frac{3}{20\pi}\text{cms}^{-1}$$

B) $V = \frac{4}{3}\pi r^3$

Find: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\qquad \frac{dr}{dt} = \frac{3}{4\pi r} \qquad \frac{dV}{dr} = 4\pi r^2$

$$= \frac{3 \times 4\pi r^2}{4\pi r}$$

$$= 3r$$

$$= 15\text{cm}^3\text{s}^{-1}$$

Exponential Growth and Decay

Further Rates of Change

$$\frac{dN}{dt} = k(N - P) \qquad \text{k, P are constants}$$

$$N = P + Ae^{kt} \qquad \text{A is constant}$$

Example 1

In a certain town, the growth rate in population is given by

$$\frac{dN}{dt} = k(N - 125)$$

a) Show $N = 125 + Ae^{kt}$ is a solution of the differential equation
b) If the population is initially 25650, after 5 years it is 31100, find the population after 8 years
c) When will the population be 40000?

A) $N = 125 + Ae^{kt}$

$$\frac{dN}{dt} = Ake^{kt}$$
$$= k(Ae^{kt})$$
$$= k(125 + Ae^{kt} - 125)$$
$$= k(N - 125)$$

C) $N = 40000$

$$40000 = 125 + 25525e^{0.0587t}$$
$$1.6 = e^{0.0387t}$$
$$t = \frac{\ln 1.6}{0.0387}$$
$$\approx 11.5$$

B) $t = 0$, $N = 25650$

$$25650 = 125 + Ae^{kt}$$
$$A = 25525$$

$t = 5$, $N = 31100$

$$31100 = 125 + 25525e^{k5}$$
$$\frac{30975}{25525} = e^{k5}$$
$$\ln 1.2 = \ln e^{k5}$$
$$k = \frac{\ln 1.2}{5}$$
$$\approx 0.0387$$

<u>Example 2</u>
Newton's Law of Cooling

The rate of cooling is proportional to the excess of the temperature of the body over the surrounding medium of the room

Temperature of coffee	Room Temperature	In 10 mins, the temp. is
100°	25°	60°

a) What is the temperature of the coffee in 15 minutes?
b) What time will it reach 50°?

A)

$$\frac{dT}{dt} = k(T - P)$$

$T - P = Ae^{k0}$
$100 - 25 = Ae^{k0}$
$A = 75$

In this case, P is the room temperature
Finding k
$60 - 25 = 75Ae^{k10}$

$$\frac{35}{75} = e^{10k}$$

$$\ln\left(\tfrac{7}{15}\right) = 10k$$

$$k = \tfrac{1}{10}\ln\left(\tfrac{7}{15}\right)$$

So in 15 minutes time
$T = 25 + 17e^{k15}$
$= 48.9°$

B) $T = 50$
$50 - 25 = 75e^{kt}$

$$\tfrac{1}{3} = e^{kt}$$

$$\ln\left(\tfrac{1}{3}\right) = kt$$

$$t = \frac{\ln\left(\tfrac{1}{3}\right)}{k}$$

$= 14\text{min } 28\text{sec}$

Motion in 2D

$$a = \ddot{x} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

Example 1

A rocket is projected from Earth, when out of the Earth's atmosphere, has a retardation of $\frac{91000}{x^2}kms^{-2}$, where x km is the distance from the centre. 6400km = radius of Earth

a) If the rocket is moving at 4km^{-1} when it is 600km above the Earth's surface, find it's speed after a further 1000km.
b) Find also, the total distance traveled before first coming to rest.

A) v = 4kms^{-1}, x = 7000km

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -\frac{91000}{x^2}$$

$$\tfrac{1}{2}v^2 = \frac{91000}{x} + C$$

$$\tfrac{1}{2}(4)^2 = \frac{91000}{7000} + C$$

$$8 = 13 + C$$

$$C = -5$$

Finding v x = 8000km

$$\tfrac{1}{2}v^2 = \frac{91000}{7000} - 5$$

$$v^2 = 12.75$$

$$v = 3.75$$

B) v = 0 so v^2 = 0

$$0 = \frac{91000}{x^2} - 5$$

$$5x = 91000$$

$$x = 18200$$

18200 – 6400 = 1182km away form earth

Simple Harmonic Motion

$\ddot{x} = -n^2 x$

$v^2 = n^2(a^2 - x^2)$

$x = A\sin nt + A\cos nt$

Amplitude $= a$

Period $= \dfrac{2\pi}{n}$

Frequency $= \dfrac{1}{T} = \dfrac{n}{2\pi}$

Example 1

A particle moves along the x-axis according to the law x = 4sin3t

a) Show it is SHM.
b) Find when the particle is first at 2cm from positive 0. Find the velocity.
c) Find the greatest speed of the particle and the interval at which it moves.

A)

$x = 4\sin 3t$

$\dot{x} = 12\cos 3t$

$\ddot{x} = -36\sin 3t$

$= -9(4\sin 3t)$

$= -3^2 x$ is SHM

B)

$2 = 4\sin 3t$

$\frac{1}{2} = \sin 3t$

$3t = \frac{\pi}{6}$

$t = \frac{\pi}{18}$

$\dot{x} = 12\cos 3\left(\frac{\pi}{18}\right)$

$= 12\cos\frac{\pi}{6}$

$= 12.\frac{\sqrt{3}}{2}$

$= 6\sqrt{3}$

C)

$\dot{x} = 12\cos 3t$

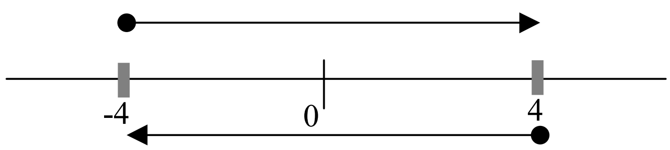
$\dot{x}$ is greatest when cos3t = 1

So the greatest velocity is 12cms^{-1}

At x = 0, v = 12

$$\begin{aligned} v^2 &= n^2(a^2 - x^2) \\ 12^2 &= 3^2(a^2 - 0^2) \\ 144 &= 9a^2 \\ a^2 &= 16 \\ a &= \pm 4 \end{aligned}$$

The interval at which it moves



Projectile Motion

$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = V\cos\theta$	$\dot{y} = -gt + V\sin\theta$
$x = V\cos\theta\, t$	$y = \dfrac{-gt}{2} + V\sin\theta\, t$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

Time of Flight $= \dfrac{V^2\sin^2\theta}{g}$

MAX Height $= \dfrac{2V\sin^2\theta}{2g}$

MAX Range $= \dfrac{V^2\sin 2\theta}{g}$

Cartesian Equation $= \dfrac{-g(1+\tan^2\theta)}{2V^2}x^2 + \tan\theta x$

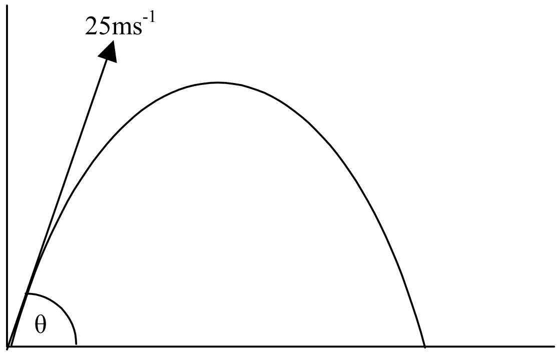
Types of projectile Questions:

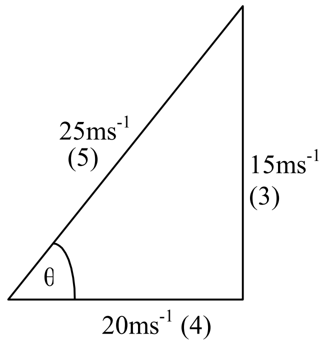
1) Level Ground
2) 2 Angles
3) Different Ground Levels
4) No $\dot{x}$, No $\dot{y}$

<u>Level Ground Question</u>

A ball is thrown with initial velocity $25ms^{-1}$ at angle $\tan^{-1}\left(\frac{3}{4}\right)$. Air resistance is neglected. Take 9 = $10ms^{-2}$. Find:

a) Maximum height
b) Time of flight and range
c) Velocity and direction after 0.5secs
d) Velocity and direction when it is 10m above the ground
e) Find the trajectory of the projectile





$$\ddot{x} = 0 \qquad \ddot{y} = -10$$
$$\dot{x} = 20 \qquad \dot{y} = -10t + 15$$
$$x = 20t \qquad y = -5t^2 + 15t$$

A) Max height when $\dot{y} = 0$

$$0 = -10t + 15$$
$$10t = 15$$
$$t = 1.5 \text{ secs}$$

Sub $t = 1.5$ into y

$$y = -5(1.5)^2 + 15(1.5)$$
$$= 11.25\text{m}$$

B) Time of flight and range when $y = 0$

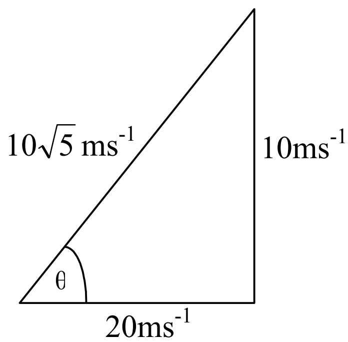
$$0 = -5t^2 + 15t$$
$$= -5t(t-3)$$
$$t = 0 \text{ or } 3 \qquad T = 3$$

Sub t = 3 into x

$$x = 20(3)$$
$$= 60$$

C) Velocity and Direction after 0.5 secs

$$\dot{x} = 20$$
$$\dot{y} = -10(0.5) + 15$$
$$= 10$$



Sub into $v = \sqrt{x^2 + y^2}$

$$v = \sqrt{20^2 + 10^2}$$
$$= 10\sqrt{5}$$

$$\tan^{-1}\left(\frac{10}{20}\right) = \theta$$
$$\theta = 26°\ 34'$$

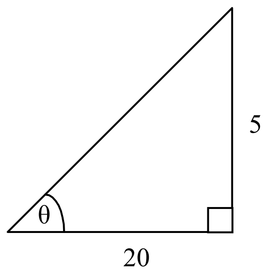
D) Velocity and Direction of projectile when it is 10m above the ground

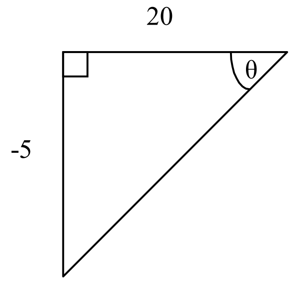
$$\begin{aligned} 10 &= -5t^2 + 15t \\ 2 &= -t^2 + 3t \\ 0 &= t^2 - 3t + 2 \\ &= (t-1)(t-2) \end{aligned}$$

Projectile is 10m above the ground at t = 1, 2

$$\begin{aligned} \dot{x} &= 20 \\ \dot{y} &= -10(1) + 15 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \dot{x} &= 20 \\ \dot{y} &= -10(2) + 15 \\ &= -5 \end{aligned}$$





$V = 5\sqrt{17}$ 14° 2'

$V = 5\sqrt{17}$ -14° 2'

E) Trajectory of projectile

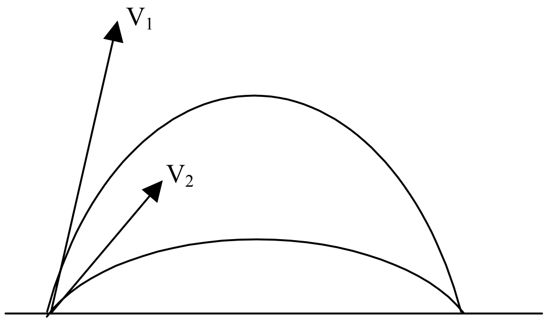
$$\begin{aligned} x &= 20t \\ t &= \frac{x}{20} \end{aligned}$$

Sub into y

$$\begin{aligned} y &= -5t^2 + 15t \\ &= -5\left(\frac{x^2}{400}\right) + 15\left(\frac{x}{20}\right) \\ &= -\frac{x^2}{80} + \frac{3x}{4} \\ &= \frac{x}{80}(60 - x) \end{aligned}$$

<u>2 Angles Question</u>

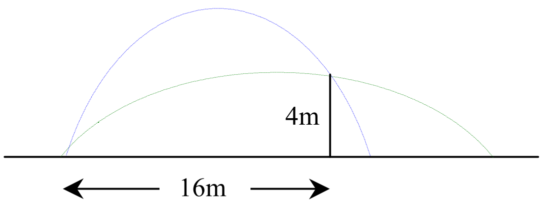
On level ground, projectiles with the same initial velocity may be projected at 2 different angles to reach a particular point x distance away. Except at 45° - the maximum angle.



Projectiles can be assumed as parabolic

A footballer kicks a ball at $16ms^{-1}$. The ball just passes over a wall 4m high when s/he is 16m away.

a) Show the angle is $5\tan^2\theta - 16\tan\theta + 9 = 0$
b) Find the two angles the ball may be kicked.



$$\begin{array}{llll} \ddot{x} & = 0 & \ddot{y} & = -10 \\ \dot{x} & = 16\cos\theta & \dot{y} & = -10t + 16\sin\theta \\ x & = 16\cos\theta t & y & = -5t^2 + 16\sin\theta t \end{array}$$

A)

$$\begin{aligned} 16 &= 16\cos\theta t \\ 1 &= \cos\theta t \\ t &= \sec\theta \end{aligned}$$

Sub t = secθ and y = 4 into y

$$\begin{aligned} 4 &= -5\sec^2\theta + 16\sin\theta.\sec\theta \\ &= -5(1 + \tan^2\theta) + 16\tan\theta \\ &= -5 - 5\tan^2\theta + 16\tan\theta \\ 0 &= 5\tan^2\theta - 16\tan\theta + 9 \end{aligned}$$

B) $0 = 5\tan^2\theta - 16\tan\theta + 9$

$$\begin{aligned} \tan\theta &= \frac{16 \pm \sqrt{16^2 - 4(5)(9)}}{2(5)} \\ &= \frac{16 \pm \sqrt{256 - 180}}{10} \\ &= \frac{16 \pm 2\sqrt{19}}{10} \\ &= \frac{8 \pm \sqrt{19}}{5} \end{aligned}$$

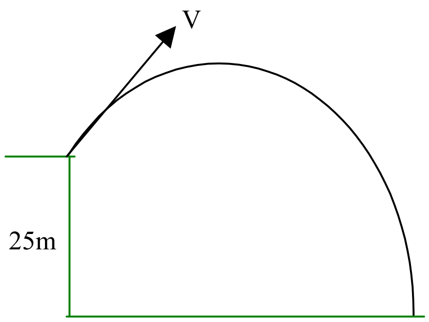
$\theta = 67° 58'$ and $36° 4'$

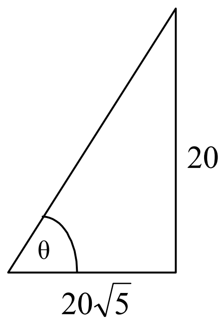
Different Ground Levels Question

A stone is thrown from the top of a 25m cliff. $\dot{x} = 20\sqrt{5}$, $\dot{y} = 20$. Find:

a) The distance away from the base of the cliff when it hits the sea

b) Maximum height above sea level





$\ddot{x} = 0$

$\dot{x} = 20\sqrt{5}$

$x = 20\sqrt{5}\,t$

$\ddot{y} = -10$

$\dot{y} = -10t + 20$

$y = -5t^2 + 20t$

A) Where it hits the sea y = –25

$-25 = -5t^2 + 20t$

$0 = t^2 - 4t - 5$

$= (t - 5)(t + 1)$

$t = 5, -1$

Sub t = 5 in x

$x = 20\sqrt{5}\,(5)$

$= 100\sqrt{5}$ from cliff base

B) Max height above sea level $\dot{y} = 0$

$0 = -10t + 20$

$t = 2$

Sub t = 2 in y

$y = -5(2)^2 + 20(2)$

$= -20 + 40$

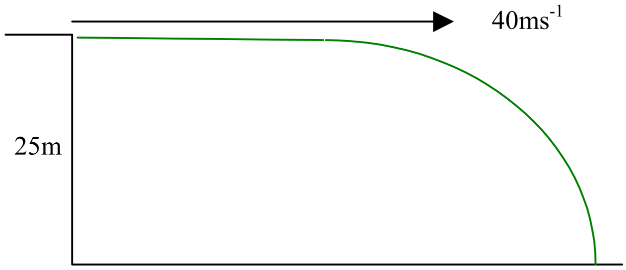
$= 20$

20 + 25 = 45m above sea level

No $\dot{x}$, No $\dot{y}$

A stone is thrown horizontally from the top of a 25m cliff with initial velocity of 40ms^{-1} Find:

a) Equation of motion
b) Where the stone hits the sea
c) The velocity of impact
d) If another stone is dropped from the same place, will it reach the sea at the same time?



$$\ddot{x} = 0 \qquad \ddot{y} = -10$$
$$\dot{x} = 40 \qquad \dot{y} = -10t$$
$$x = 40t \qquad y = -5t^2$$

A)

$$x = 40t$$
$$t = \frac{x}{40}$$

Sub $t = \frac{x}{40}$ in y

$$y = -5\left(\frac{x}{40}\right)^2$$
$$= \frac{-x^2}{320}$$

B)

$$-25 = -5t^2$$
$$5 = t^2$$
$$t = \pm\sqrt{5}$$

Sub t = $\sqrt{5}$ in x

$$x = 40\sqrt{5}$$

C)

$$\dot{x} = 40$$
$$\dot{y} = -10\sqrt{5}$$

$$v = \sqrt{40^2 + \left(-10\sqrt{5}\right)^2}$$
$$= 10\sqrt{21}$$

θ = 150° 48’ in the positive direction

D)

Yes, both will take $\sqrt{5}$ secs to reach the sea.