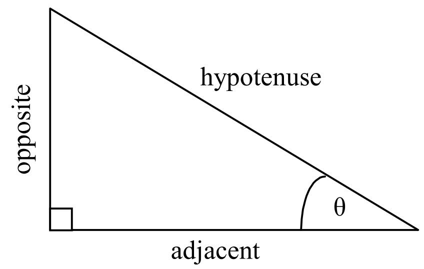
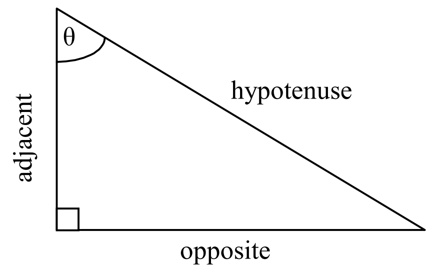
Trigonometry

- Trigonometric Ratios
- Exact Values & Triangles
- Trigonometric Identities
- ASTC Rule
- Trigonometric Graphs
- Sine & Cosine Rules
- Area of a Triangle
- Trigonometric Equations
- Sums and Differences of angles
- Double Angles
- Triple Angles
- Half Angles
- T – formula
- Subsidiary Angle formula
- General Solutions of Trigonometric Equations
- Radians
- Arcs, Sectors, Segments
- Trigonometric Limits
- Differentiation of Trigonometric Functions
- Integration of Trigonometric Functions
- Integration of $\sin^2 x$ and $\cos^2 x$
- INVERSE TRIGNOMETRY
  - Inverse Sin – Graph, Domain, Range, Properties
  - Inverse Cos – Graph, Domain, Range, Properties
  - Inverse Tan – Graph, Domain, Range, Properties
  - Differentiation of Inverse Trigonometric Functions
  - Integration of Inverse Trigonometric Functions

Trigonometric Ratios





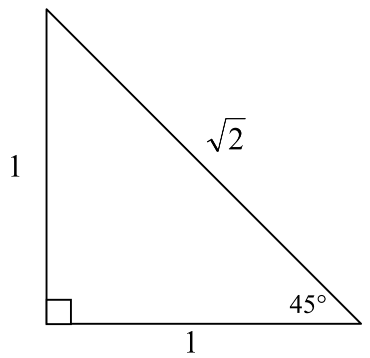
Sine	$\sin\theta$		$= \dfrac{opposite}{hypotenuse}$
Cosine	$\cos\theta$		$= \dfrac{adjacent}{hypotenuse}$
Tangent	$\tan\theta$		$= \dfrac{opposite}{adjacent}$
Cosecant	$\operatorname{cosec}\theta$	$= \dfrac{1}{\sin\theta}$	$= \dfrac{hypotenuse}{opposite}$
Secant	$\sec\theta$	$= \dfrac{1}{\cos\theta}$	$= \dfrac{hypotenuse}{adjacent}$
Cotangent	$\cot\theta$	$= \dfrac{1}{\tan\theta}$	$= \dfrac{adjacent}{opposite}$

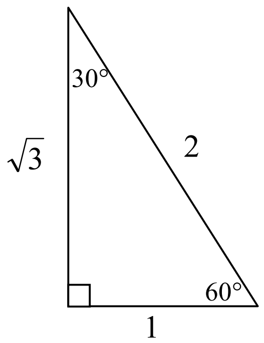
$\sin\theta$	$= \cos(90° - \theta)$
$\cos\theta$	$= \sin(90° - \theta)$
$\tan\theta$	$= \cot(90° - \theta)$
$\operatorname{cosec}\theta$	$= \sec(90° - \theta)$
$\sec\theta$	$= \cos ec(90° - \theta)$
$\cot\theta$	$= \tan(90° - \theta)$

60 seconds = 1 minute   60’’ = 1’
60 minutes = 1 degree   60’ = 1°

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Exact Values & Triangles



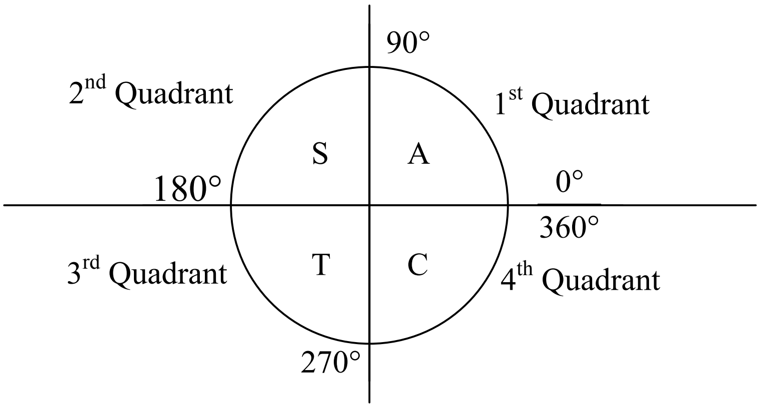


	0°	30°	60°	45°	90°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	–1
tan	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	—	0
cos ec	—	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	1	—
sec	1	$\frac{2}{\sqrt{3}}$	2	$\sqrt{2}$	—	–1
cot	—	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	1	0	—

Trigonometric Identities

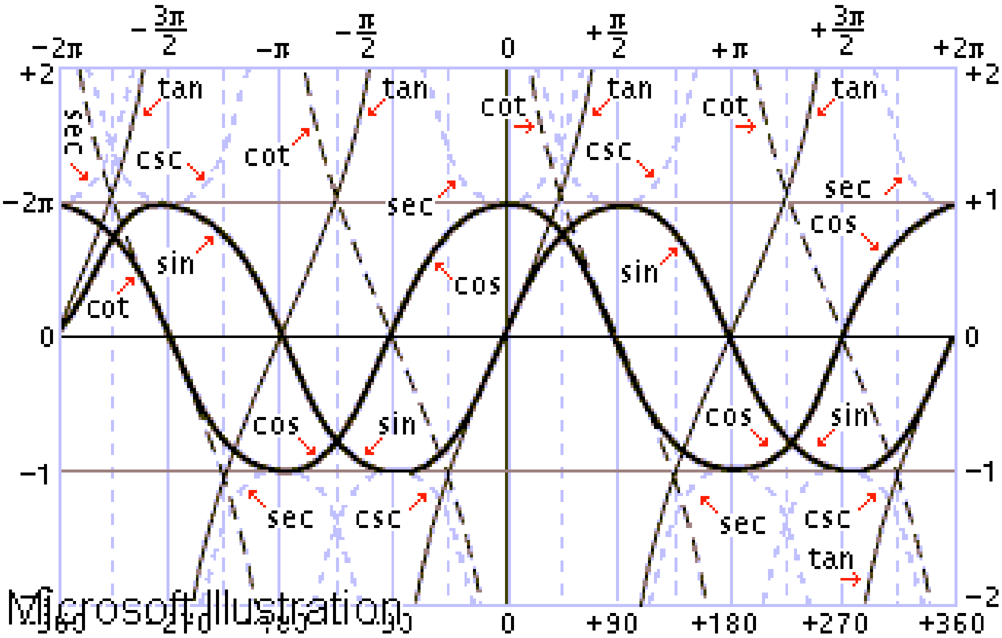
$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\cos^2\theta &= 1 - \sin^2\theta \\
\sin^2\theta &= 1 - \cos^2\theta \\[1em]
1 + \cot^2\theta &= \operatorname{cosec}^2\theta \\
\cot^2\theta &= \operatorname{cosec}^2\theta - 1 \\
1 &= \operatorname{cosec}^2\theta - \cot^2\theta \\[1em]
\tan^2\theta + 1 &= \sec^2\theta \\
\tan^2\theta &= \sec^2\theta - 1 \\
1 &= \sec^2\theta - \tan^2\theta
\end{aligned}$$

ASTC Rule



<u>First Quadrant:</u> **A**ll positive		
$\sin\theta$	$\sin\theta$	+
$\cos\theta$	$\cos\theta$	+
$\tan\theta$	$\tan\theta$	+
<u>Second Quadrant:</u> **S**ine positive		
$\sin(180°-\theta)$	$\sin\theta$	+
$\cos(180°-\theta)$	$-\cos\theta$	–
$\tan(180°-\theta)$	$-\tan\theta$	–
<u>Third Quadrant:</u> **T**angent positive		
$\sin(180°+\theta)$	$-\sin\theta$	–
$\cos(180°+\theta)$	$-\cos\theta$	–
$\tan(180°+\theta)$	$\tan\theta$	+
<u>Fourth Quadrant:</u> **C**osine positive		
$\sin(360°-\theta)$	$-\sin\theta$	–
$\cos(360°-\theta)$	$\cos\theta$	+
$\tan(360°-\theta)$	$-\tan\theta$	–

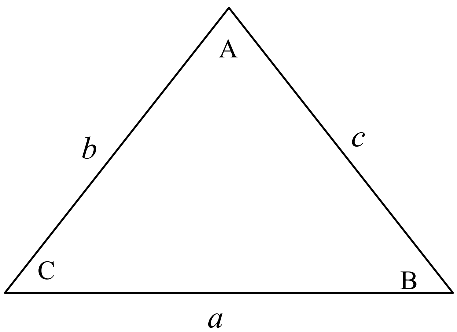
Trigonometric Graphs



Sine & Cosine Rules

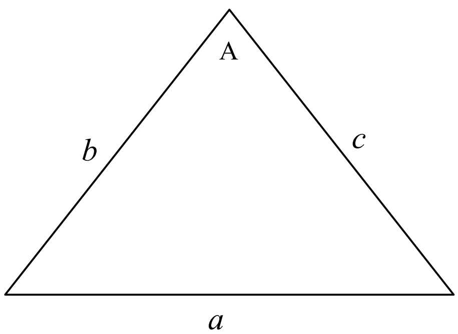
Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{OR} \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



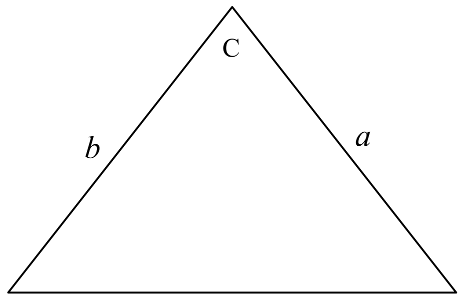
Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Area of a Triangle

$A = \frac{1}{2}ab\sin C$	➢ C is the angle ➢ a & b are the two adjacent sides



Trigonometric Equations

- Check the domain eg. $0° \le \theta \le 360°$
- Check degrees ($0° \le \theta \le 360°$) or radians ($0 \le \theta \le 2\pi$)
- If double angle, go 2 revolutions
- If triple angle, go 3 revolutions, etc…
- If half angles, go half or one revolution (safe side)

Example 1

Solve $\sin\theta = \frac{1}{2}$ for $0° \le \theta \le 360°$

$$\begin{aligned} \sin\theta &= \tfrac{1}{2} \\ \theta &= 30°, 150° \end{aligned}$$

Example 2

Solve $\cos 2\theta = \frac{1}{2}$ for $0° \le \theta \le 360°$

$$\begin{aligned} \cos 2\theta &= \tfrac{1}{2} \\ 2\theta &= 60°, 300°, 420°, 660° \\ \theta &= 30°, 150°, 210°, 330° \end{aligned}$$

Example 3

Solve $\tan\frac{\theta}{2} = 1$ for $0° \le \theta \le 360°$

$$\begin{aligned} \tan\tfrac{\theta}{2} &= 1 \\ \tfrac{\theta}{2} &= 45°, 225° \\ \theta &= 90° \end{aligned}$$

Example 4

$\sin 2\theta + \cos\theta = 0$

$$\begin{aligned} 2\sin\theta\cos\theta + \cos\theta &= 0 \\ \cos\theta(2\sin\theta + 1) &= 0 \end{aligned}$$

$$\begin{aligned} \cos\theta &= 0 \\ \theta &= 90°, 270° \end{aligned} \qquad \begin{aligned} \sin\theta &= -\tfrac{1}{2} \\ \theta &= 210°, 330° \end{aligned}$$

Example 5

$3\sin\theta - \cos 2\theta = -2$

$$\begin{aligned} 3\sin\theta - (1 - 2\sin^2\theta) &= -2 \\ 2\sin^2\theta + 3\sin\theta + 1 &= 0 \\ (2\sin\theta + 1)(\sin\theta + 1) &= 0 \end{aligned}$$

$$\begin{aligned} \sin\theta &= -\tfrac{1}{2} \\ \theta &= 210°, 330° \end{aligned} \qquad \begin{aligned} \sin\theta &= -1 \\ \theta &= 270° \end{aligned}$$

Sums and Differences of angles

$\sin(\alpha+\beta)$	$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta)$	$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha+\beta)$	$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta)$	$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\tan(\alpha+\beta)$	$= \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta)$	$= \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Double Angles

$\sin 2\theta$	$= 2\sin\theta\cos\theta$
$\cos 2\theta$	$= \cos^2\theta - \sin^2\theta$ $= 1-2\sin^2\theta$ $= 2\cos^2\theta - 1$
$\tan 2\theta$	$= \dfrac{2\tan\theta}{1-2\tan^2\theta}$
$\sin^2\theta$	$= \tfrac{1}{2}(1-\cos 2\theta)$
$\cos^2\theta$	$= \tfrac{1}{2}(1+\cos 2\theta)$

Triple Angles

$\sin 3\theta$	$= 3\sin\theta - 4\sin^3\theta$
$\cos 3\theta$	$= 4\cos^3\theta - 3\cos\theta$
$\tan 3\theta$	$= \dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

Half Angles

$\sin\theta$	$= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$
$\cos\theta$	$= \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}$ $= 1-2\sin^2\tfrac{\theta}{2}$ $= 2\cos^2\tfrac{\theta}{2} - 1$
$\tan\theta$	$= \dfrac{2\tan\tfrac{\theta}{2}}{1-2\tan^2\tfrac{\theta}{2}}$

Deriving the Triple Angles

$\sin 3\theta$	$= \sin(2\theta + \theta)$	
	$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$	Normal double angle
	$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$	Expand double angle
	$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$	Multiply
	$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$	Change $\sin^2\theta + \cos^2\theta = 1$
	$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$	Simplify
	$= 3\sin\theta - 4\sin^3\theta$	
$\cos 3\theta$	$= \cos(2\theta + \theta)$	
	$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$	
	$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$	
	$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$	
	$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$	
	$= 2\cos^2\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$	
	$= 4\cos^3\theta - 3\cos\theta$	
$\tan 3\theta$	$= \tan(2\theta + \theta)$	
	$= \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$	
	$= \dfrac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta\tan\theta}{1-\tan^2\theta}}$	
	$= \dfrac{\frac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}$	
	$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	

T – Formulae

Let t = $\tan\frac{\theta}{2}$

$\sin\theta = \dfrac{2t}{1+t^2}$	
$\cos\theta = \dfrac{1-t^2}{1+t^2}$	
$\tan\theta = \dfrac{2t}{1-t^2}$	

$\sin\theta$	$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$	Using half angles
	$= \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}$	Divide by "1" $\sin^2\theta+\cos^2\theta=1$
	$= \dfrac{\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}$	Divide top and bottom by $\cos^2\theta$
	$= \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$	cos ' cancel; $\frac{\sin}{\cos}$ becomes tan
	$= \dfrac{2t}{1+t^2}$	

$\cos\theta = \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
$= \dfrac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}$	$= \dfrac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}$
$= \dfrac{\dfrac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}$	$= \dfrac{2t}{1-t^2}$
$= \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$	
$= \dfrac{1-t^2}{1+t^2}$	

Subsidiary Angle Formula

$$\begin{aligned} a\sin x + b\cos x &= R(\sin x\cos x + \cos x\sin x) \\ &= R\sin x\cos x + R\cos x\sin x \end{aligned}$$

$$\begin{aligned} a &= R\cos x & \therefore a^2 &= R^2\cos^2 x \\ b &= R\sin x & \therefore b^2 &= R^2\sin^2 x \end{aligned}$$

$$\sin^2 x + \cos^2 x = 1 \quad = \frac{a^2+b^2}{R^2}$$

$R = \sqrt{a^2+b^2}$		$\tan\alpha = \frac{b}{a}$
$a\sin x + b\cos x$	= C	$R\sin(x+\alpha)$
$a\sin x - b\cos x$	= C	$R\sin(x-\alpha)$
$a\cos x + b\sin x$	= C	$R\cos(x+\alpha)$
$a\cos x - b\sin x$	= C	$R\cos(x-\alpha)$

Example 1

Find x. $\sqrt{3}\sin x - \cos x = 1$

$$\begin{aligned} \text{R} &= \sqrt{\sqrt{3}^2 + 1^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \tan\alpha &= \frac{1}{\sqrt{3}} \\ \alpha &= 30^\circ \end{aligned}$$

$$\begin{aligned} 2\sin(x-30) &= 1 \\ \sin(x-30) &= \tfrac{1}{2} \\ x-30 &= 30^\circ,\ 150^\circ \\ x &= 60^\circ,\ 180^\circ \end{aligned}$$

General Solutions of Trigonometric Equations

$\sin\theta = \sin\alpha$	Then $\theta = n\pi + (-1)^n \alpha$
$\cos\theta = \cos\alpha$	Then $\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	Then $\theta = n\pi + \alpha$

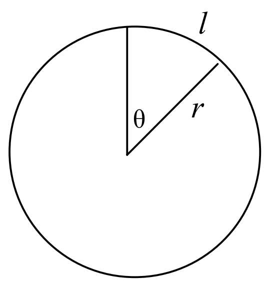
Radians

$\pi^c = 180°$
$1° = \dfrac{\pi^c}{180}$

Arcs, Sectors, Segments

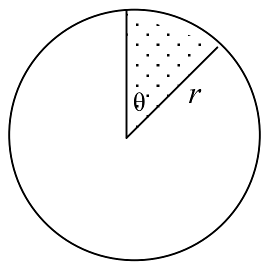
Arc Length

$$l = r\theta$$



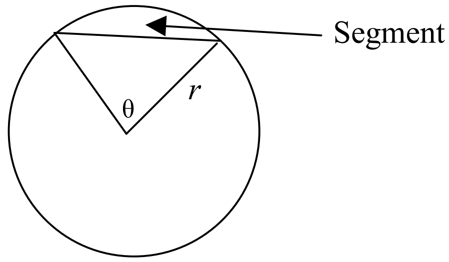
Area of Sector

$$A = \tfrac{1}{2} r^2 \theta$$



Area of Segment

$$A = \tfrac{1}{2} r^2 (\theta - \sin\theta)$$



Trigonometric Limits

$\lim_{x\to 0}\frac{\sin x}{x}$	$=\lim_{x\to 0}\frac{\tan x}{x}$	$=\lim_{x\to 0}\cos x$	$=1$

Differentiation of Trigonometric Functions

$\frac{d}{dx}(\sin x)$	$=\cos x$
$\frac{d}{dx}[\sin f(x)]$	$=f'(x)\cos f(x)$
$\frac{d}{dx}(\sin(ax+b))$	$=a\cos(ax+b)$
$\frac{d}{dx}(\cos x)$	$=-\sin x$
$\frac{d}{dx}[\cos f(x)]$	$=-f'(x)\sin f(x)$
$\frac{d}{dx}(\cos(ax+b))$	$=-a\sin(ax+b)$
$\frac{d}{dx}(\tan x)$	$=\sec^2 x$
$\frac{d}{dx}[\tan f(x)]$	$=f'(x)\sec^2 f(x)$
$\frac{d}{dx}(\tan(ax+b))$	$=a\sec^2(ax+b)$
$\frac{d}{dx}\sec x$	$=\sec x.\tan x$
$\frac{d}{dx}\cos ecx$	$=-\cot x.\cos ecx$
$\frac{d}{dx}\cot x$	$=-\cos ec^2 x$

Integration of Trigonometric Functions

$\int \cos ax \ \mathrm{dx}$	$= \frac{1}{a}\sin ax + c$
$\int \sin ax \ \mathrm{dx}$	$= -\frac{1}{a}\cos ax + c$
$\int \sec^2 ax \ \mathrm{dx}$	$= \frac{1}{a}\tan ax + c$
$\int \frac{1}{\sqrt{a^2 - x^2}} \ \mathrm{dx}$	$= \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int -\frac{1}{\sqrt{a^2 - x^2}} \ \mathrm{dx}$	$= \cos^{-1}\left(\frac{x}{a}\right) + c$ OR $-\sin^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{1}{a^2 + x^2} \ \mathrm{dx}$	$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int \operatorname{cosec}^2 ax \ \mathrm{dx}$	$= -\frac{1}{a}\cot ax + c$
$\int \sec ax . \tan ax \ \mathrm{dx}$	$= \frac{1}{a}\sec ax + c$
$\int \operatorname{cosec} ax . \cot ax \ \mathrm{dx}$	$= -\frac{1}{a}\operatorname{cosec} ax + c$

Integration of $\sin^2 x$ and $\cos^2 x$

$$\begin{aligned}
\cos 2x &= 2\cos^2 x - 1\\
\cos 2x + 1 &= 2\cos^2 x\\
\tfrac{1}{2}(\cos 2x + 1) &= \cos^2 x\\
\int \cos^2 x \ \mathrm{dx} &= \tfrac{1}{2}\int (\cos 2x + 1) \ \mathrm{dx}\\
&= \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2x + x\right) + C\\
&= \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + C
\end{aligned}$$

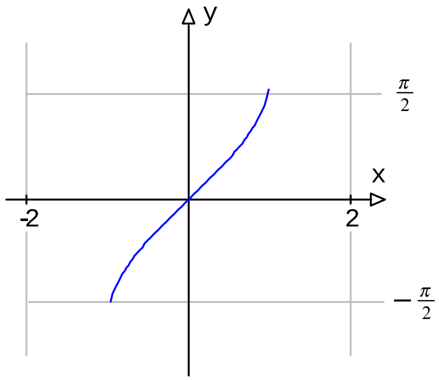
$$\int \cos^2 x \ \mathrm{dx} = \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + C$$

$$\begin{aligned}
\cos 2x &= 1 - \sin^2 x\\
2\sin^2 x &= 1 - \cos 2x\\
\sin^2 x &= \tfrac{1}{2}(1 - \cos 2x)\\
\int \sin^2 x \ \mathrm{dx} &= \tfrac{1}{2}\int (1 - \cos 2x) \ \mathrm{dx}\\
&= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + C\\
&= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C
\end{aligned}$$

$$\int \sin^2 x \ \mathrm{dx} = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$$

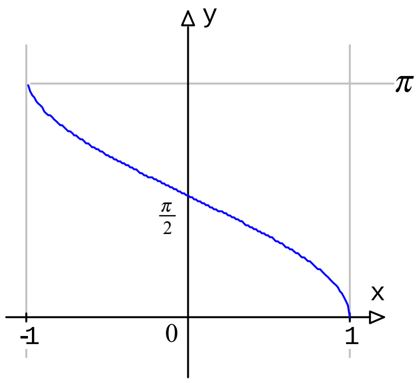
INVERSE TRIGNOMETRY

Inverse Sin – Graph, Domain, Range, Properties



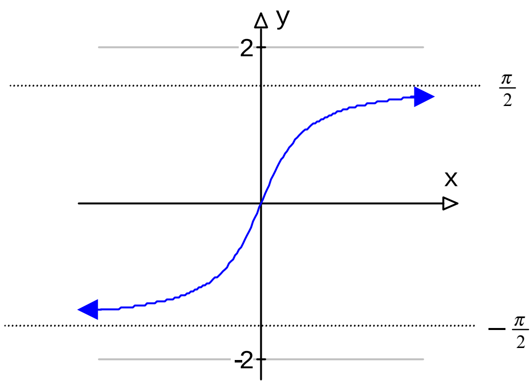
Domain	$-1 \le x \le 1$
Range	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
Property	$\sin^{-1}(-x) = -\sin^{-1} x$

Inverse Cos – Graph, Domain, Range, Properties



Domain	$-1 \le x \le 1$
Range	$0 \le y \le \pi$
Property	$\cos^{-1}(-x) = \pi - \cos^{-1} x$

Inverse Tan – Graph, Domain, Range, Properties



Domain	All real x
Range	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
Property	$\tan^{-1}(-x) = -\tan^{-1} x$

Differentiation of Inverse Trigonometric Functions

$\frac{d}{dx}\left(\sin^{-1} x\right)$	$= \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\left(\sin^{-1} \frac{x}{a}\right)$	$= \frac{1}{\sqrt{a^2-x^2}}$
$\frac{d}{dx}\left(\sin^{-1} f(x)\right)$	$= \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$\frac{d}{dx}\left(\cos^{-1} x\right)$	$= -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\left(\cos^{-1} \frac{x}{a}\right)$	$= -\frac{1}{\sqrt{a^2-x^2}}$
$\frac{d}{dx}\left(\cos^{-1} f(x)\right)$	$= -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$\frac{d}{dx}\left(\tan^{-1} x\right)$	$= \frac{1}{1+x^2}$
$\frac{d}{dx}\left(\tan^{-1} \frac{x}{a}\right)$	$= \frac{a}{a^2+x^2}$
$\frac{d}{dx}\left(\tan^{-1} f(x)\right)$	$= \frac{f'(x)}{a+[f(x)]^2}$

Integration of Inverse Trigonometric Functions

$\int \frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{dx}$	$= \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int -\frac{1}{\sqrt{a^2 - x^2}}\ \mathrm{dx}$	$= \cos^{-1}\left(\frac{x}{a}\right) + c$ OR $-\sin^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{1}{a^2 + x^2}\ \mathrm{dx}$	$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$