

Name : _____

Teacher/ Class : _____

SYDNEY TECHNICAL HIGH SCHOOL



HSC ASSESSMENT TASK 1

DECEMBER 2007

MATHEMATICS - EXTENSION 1

Time Allowed : 70 minutes

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start **each** question on a new **page**.
- Diagrams unless otherwise stated are not to scale.

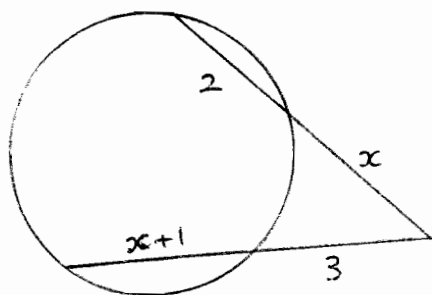
Q1	Q2	Q3	Q4	Q5	TOTAL
/10	/10	/10	/10	/10	/50

Question 1

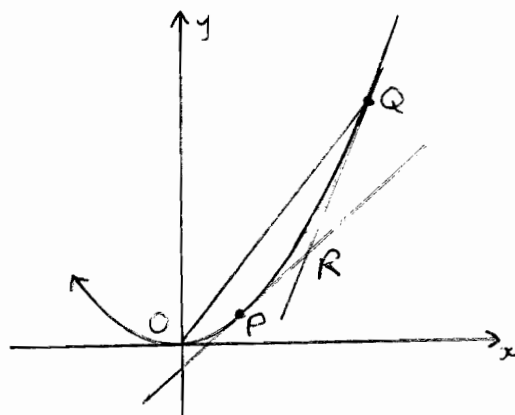
- a) Given the parabola described by $x = 6t, y = -3t^2$
- Write the cartesian equation of this parabola 1
 - Find the coordinates of the focus and the equation of the directrix 2
 - Find the length of the latus rectum 1
- b) The sum to n terms of a certain series is given by $S_n = \frac{n}{n+1}$
- Find the first and second terms of the sequence 2
 - Find the sum of the 9th, 10th and 11th terms 2
 - Find a simplified expression for the n th term, T_n 2

Question 2

- a) Find the value of x 2



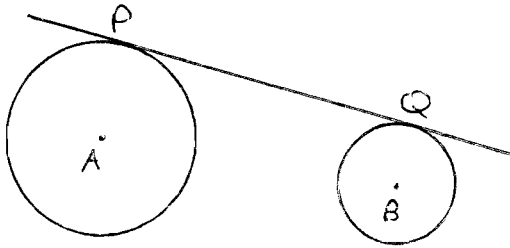
- b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.
 OQ is a chord passing through the vertex.
 Tangents at P and Q intersect at R .



- Derive the equation of the tangent at P . 2
- Find the coordinates of R 2
- Find the coordinates of the point where the tangent at P intersects the directrix 1
- Chord OQ is parallel to the tangent at P . Find a condition linking p and q 1
- Show that, as P moves on its parabola, R moves on another parabola $x^2 = \frac{9}{2}ay$ 2
 (You may use the result from part iv)

Question 3

(a)



Two circles with centres A and B have radii 14 cm and 7 cm respectively. The distance between the centres is 25 cm.

1

Find the length of the common tangent PQ.

(b) Find the number that must be added to each of 1, 3, 4 in order to form a geometric sequence.

2

(c) Find the sum of all the integers from 501 to 1600 (inclusive), but excluding all multiples of 8.

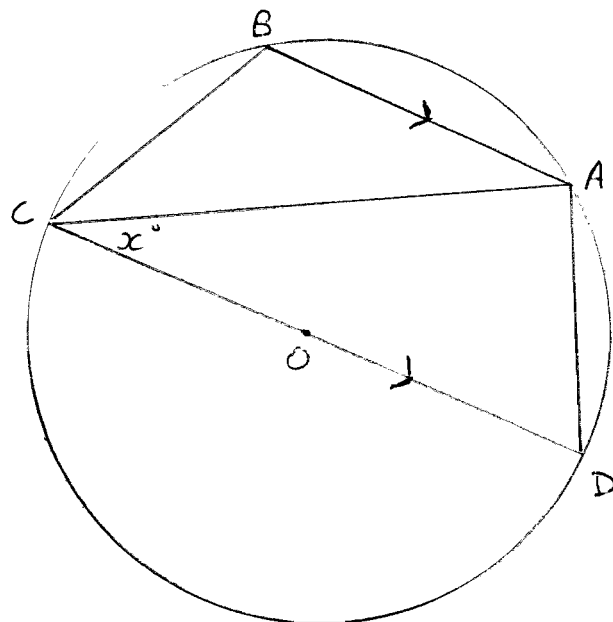
4

(d) Points A, B, C, D are points on the circle, centre O.

$CD \parallel BA$ and $\angle ACD = x^\circ$

Find an expression for $\angle BCD$

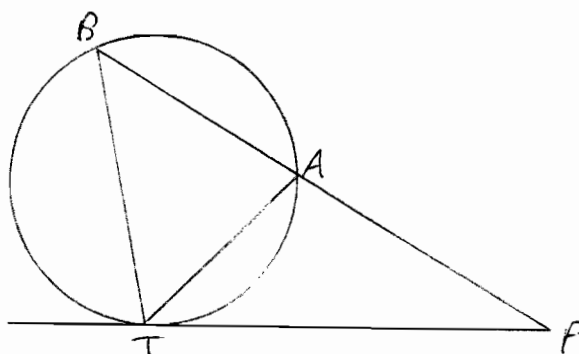
in terms of x , giving reasons.



3

Question 4

- (a) TP is a tangent to the circle and BP is a secant.
BP cuts the circle at A.



- i) Prove that triangles TAP and BTP are similar. 3
 - ii) Hence show that $PT^2 = PA.PB$ 1
- (b) The curve $y = ax^2 + bx$ has a turning point at $(1, -2)$. Find the values of a and b . 3
- (c) Sketch a curve $y = f(x)$ satisfying the following conditions: 3
- $f'(1) = f'(2) = f'(3) = 0$
- $f'(x) < 0$ for $x < 1$, for $1 < x < 2$, for $x > 3$
- $f'(x) > 0$ for $2 < x < 3$.

Question 5

- (a) Use mathematical induction to prove that $8^n - 5^n$ is divisible by 3 for all positive integers n . 4
- (b) A man borrows \$40,000 for a new car. The loan is to be repaid over 5 years by equal monthly instalments of \$R per month. Interest of 12% p.a is calculated on the balance owing.
- Let A_n be the amount owing after n months.
- (i) Write an expression for A_1 and show that $A_2 = 40\,000(1.01)^2 - 1.01R - R$ 1
 - (ii) Find the amount of each monthly repayment to the nearest cent. 3
 - (iii) Suppose instead that the man repays \$1200 every month. How many months will it take to repay the loan? 2

Solutions

$$\begin{aligned} \textcircled{1} \text{ a) } x = \frac{x}{6} &\Rightarrow y = -3\left(\frac{x}{6}\right)^2 \\ \text{i) } &= -3 \times \frac{x^2}{36} \\ &= -\frac{x^2}{12} \end{aligned}$$

$$\therefore x^2 = -12y$$

$$\text{ii) } F(0, -3)$$

$$\text{directrix } y = 3$$

$$\text{iii) } LR = 12 \text{ units}$$

$$\text{b) i) } T_1 = S_1 = \frac{1}{2}$$

$$\begin{aligned} T_2 &= S_2 - S_1 \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \text{ii) } T_9 + T_{10} + T_{11} &= S_{11} - S_8 \\ &= \frac{11}{12} - \frac{8}{9} \\ &= \frac{1}{36} \end{aligned}$$

$$\begin{aligned} \text{iii) } T_n &= S_n - S_{n-1} \\ &= \frac{n}{n+1} - \frac{n-1}{n} \\ &= \frac{n^2 - (n-1)(n+1)}{n(n+1)} \\ &= \frac{1}{n(n+1)} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \text{ a) } x(x+2) &= 3(x+4) \\ x^2 + 2x - 3x - 12 &= 0 \\ x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \\ \therefore x &= 4 \text{ only} \end{aligned}$$

$$\begin{aligned} \text{b) i) } y &= \frac{x^2}{4a} \Rightarrow \frac{dy}{dx} = \frac{x}{2a} \\ \text{When } x &= 2ap, \frac{dy}{dx} = \frac{2ap}{2a} \\ &= p \end{aligned}$$

\therefore eqn tangent at P:

$$\begin{aligned} y - ap^2 &= p(x - 2ap) \\ &= px - 2ap^2 \\ \therefore y &= px - ap^2 \end{aligned}$$

$$\begin{aligned} \text{iii) } y &= -a \Rightarrow -a = px - ap^2 \\ \therefore px &= ap^2 - a \\ \therefore x &= \frac{ap^2 - a}{p} \\ \therefore \text{point is } &\left(\frac{ap^2 - a}{p}, -a\right) \end{aligned}$$

$$\begin{aligned} \text{ii) } y &= px - ap^2, y = qx - aq^2 \\ \text{At R: } &px - ap^2 = qx - aq^2 \\ px - qx &= ap^2 - aq^2 \\ \therefore x &= \frac{a(p^2 - q^2)}{p - q} \end{aligned}$$

$$\begin{aligned} \text{Subst. } \Rightarrow y &= p \times \frac{a(p^2 - q^2)}{p - q} - ap^2 \\ &= \frac{ap^3 - apq^2}{p - q} - ap^2 \\ &= \frac{ap^3 - apq^2 - ap^2(p - q)}{p - q} \\ &= \frac{ap^3 - apq^2 - ap^3 + ap^2q}{p - q} \\ &= \frac{-apq^2 + ap^2q}{p - q} \\ &= \frac{apq(p - q)}{p - q} \\ &= apq \end{aligned}$$

iv) $M_{OQ} = p$

$$\therefore \frac{aq^2 - 0}{2aq - 0} = p$$

$$\therefore \frac{q}{2} = p \text{ or } q = 2p$$

v) For R: $x = a(p+q)$
 $y = apq$

Subst. $p = \frac{q}{2}$:

$$\therefore x = a\left(\frac{q}{2} + q\right)$$

$$= \frac{3aq}{2} \quad (1)$$

$$\therefore y = a \frac{q}{2} q$$

$$= \frac{aq^2}{2} \quad (2)$$

From (1): $q = \frac{2x}{3a}$

Sub in (2): $y = \frac{a}{2} \left(\frac{2x}{3a}\right)^2$

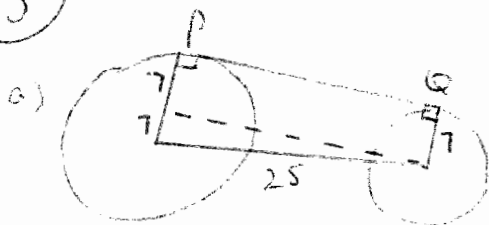
$$= \frac{a}{2} \times \frac{4x^2}{9a^2}$$

$$= \frac{2x^2}{9a}$$

$$\therefore 9ay = 2x^2$$

$$\therefore x^2 = \frac{9}{2} ay$$

(3)



$$PQ^2 = 25^2 - 7^2$$

b)

$1+x, 3+x, 4+x$ forms a G.P.

$$\text{if } \frac{3+x}{1+x} = \frac{4+x}{3+x}$$

$$\therefore x^2 + 6x + 9 = x^2 + 5x + 4$$

$$\therefore x = -5 \text{ (i.e. } -4, -2, -1)$$

c) $\text{Sum} = (501 + 502 + \dots + 1600)$
 $- (504 + 512 + \dots + 1600)$

$$= \frac{1100}{2} (501 + 1600) - \frac{138}{2} (504 + 1600)$$

$$= 1155550 - 145176$$

$$= 1010374$$

d) $\angle CAB = x^\circ$ (alternate angles, $BA \parallel CD$)
 $\angle CAD = 90^\circ$ (angle in semicircle)

$$\therefore \angle BAD = 90^\circ + x^\circ$$

$$\therefore \angle BCD = 180 - (90 + x)^\circ \text{ (supplementary opposite angles, cyclic quad.)}$$

$$= (90 - x)^\circ$$

(4) a)

i) LP is common

$$\angle ATP = \angle TBP \text{ (angle in alternate segment rule)}$$

$$\therefore \triangle TAP \parallel \triangle TBP \text{ (equiangular)}$$

ii) $\frac{PT}{PB} = \frac{PA}{PT}$ (equal ratio of sides in similar triangles)

$$PT^2 = PA \cdot PB$$

$$b) \frac{dy}{dx} = 2ax + b$$

$$\text{Sub } \frac{dy}{dx} = 0, x = 1:$$

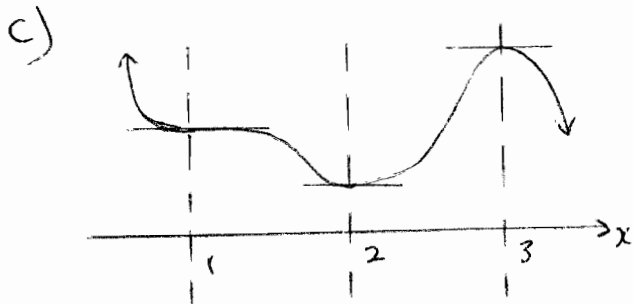
$$\Rightarrow 0 = 2a + b - \textcircled{1}$$

$$\text{Sub } x = 1, y = -2:$$

$$\Rightarrow -2 = a + b - \textcircled{2}$$

$$\textcircled{1} - \textcircled{2}: \boxed{a = 2}$$

$$\text{Sub in } \textcircled{1}: \boxed{b = -4}$$



⑤ a) Step 1 Prove true for $n=1$
 $8^1 - 5^1 = 3$ (divisible by 3)

Step 2 Assume true for $n=k$
 i.e. assume $8^k - 5^k = 3P$
 (P is integral)

Step 3 Prove true for $n=k+1$
 i.e. prove $8^{k+1} - 5^{k+1} = 3Q$
 (Q is integral)

$$\begin{aligned} \text{Now, } 8^{k+1} - 5^{k+1} &= 8 \times 8^k - 5 \times 5^k \\ &= 8(8^k - 5^k) + 3 \times 5^k \\ &= 8 \times 3P + 3 \times 5^k \end{aligned}$$

Step 4 The result is true for $n=1$.
 From step 3, it must also be true for
 $n=1+1=2$, then $n=2+1=3$ and so on
 for all positive, integral n .

$$b) i) A_1 = 40000 \times 1.01 - R$$

$$\begin{aligned} A_2 &= A_1 \times 1.01 - R \\ &= 40000 \times 1.01^2 - 1.01R - R \end{aligned}$$

$$ii) A_{60} = 40000 \times 1.01^{60} - 1.01^{59}R - \dots - R$$

$$\text{But } A_{60} = 0,$$

$$\therefore 40000 \times 1.01^{60} - (1.01^{59}R + \dots + R) = 0$$

$$\therefore 40000 \times 1.01^{60} = \frac{R(1.01^{60} - 1)}{0.01}$$

$$\therefore R = 40000 \times 1.01^{60} \times \frac{0.01}{1.01^{60} - 1}$$

$$= \$889.78 \text{ per month.}$$

iii)

$$1200 = 40000 \times 1.01^n \times \frac{0.01}{1.01^n - 1}$$

$$= \frac{400 \times 1.01^n}{1.01^n - 1}$$

$$\therefore 1200 \times 1.01^n - 1200 = 400 \times 1.01^n$$

$$800 \times 1.01^n = 1200$$

$$1.01^n = 1.5$$

$$(n > 40)$$

\therefore 41 months needed to
 repay the loan