Parabola, Locus – Parametric Representation

Parabola

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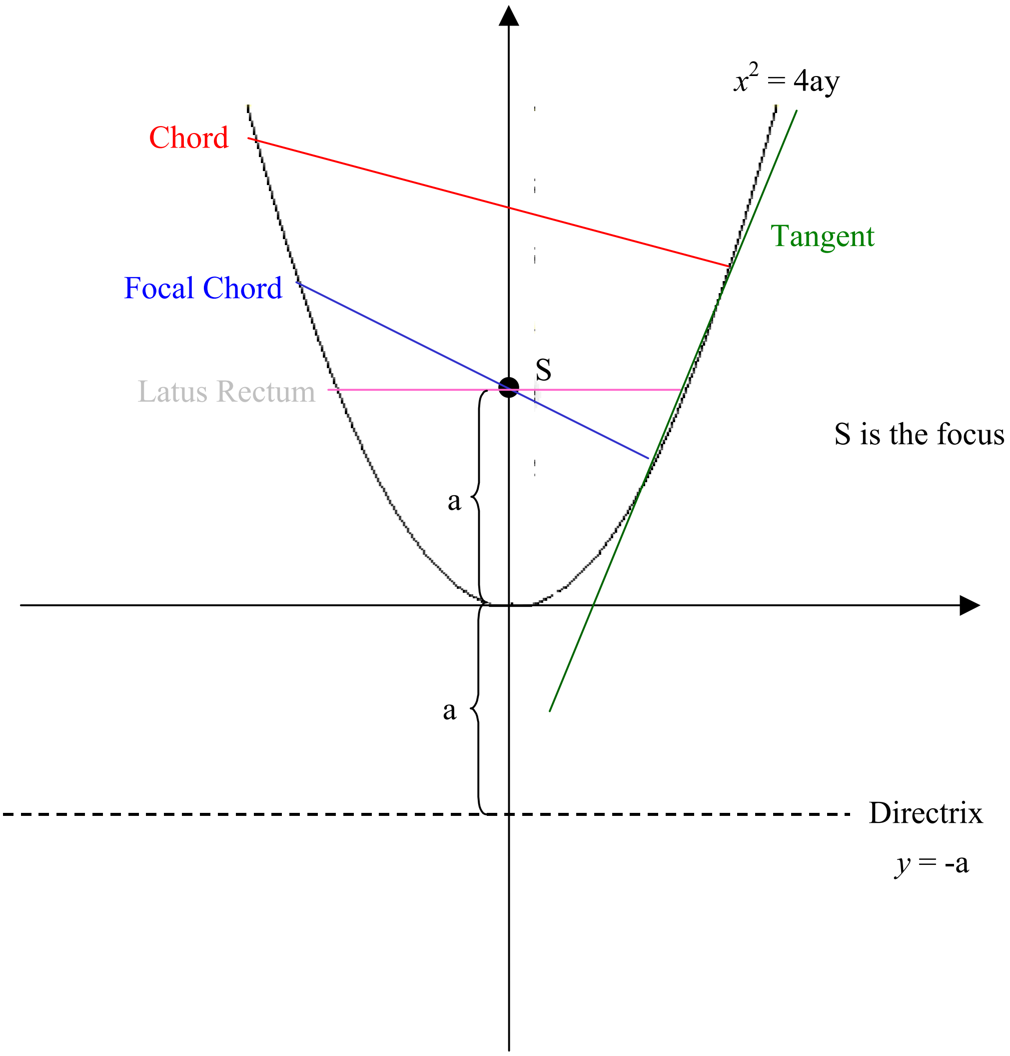
Parts of a Parabola & Definition

The locus of a parabola is a set of points equidistant from one fixed point and one fixed line.



$x^2 = 4ay$		$(x-h)^2 = 4a(y-k)$	
Focus	(0, a)	Focus	(h, k + a)
Directrix	y = -a	Directrix	y = k – a
Vertex	(0, 0)	Vertex	(h, k)

Parabola Summary

Dummy Variables	P(2ap,ap^2) Q(2aq,aq^2)
Gradient of Chord	$\dfrac{p+q}{2}$
Equation of Chord	$y-\frac{1}{2}(p+q)x+apq=0$ $y=\frac{1}{2}(p+q)x-apq$
The Focal Chord Length	$l=4a$
Gradient of Tangent	$\dfrac{x}{2a}$
Equation of Tangent	$y-px+ap^2=0$ $y=px-ap^2$
Intersection of two Tangents	$T(a(p+q),apq)$
Intersection of Tangents of Focal Chord	Tangents of the focal chord intersect at right angles on the directrix
Gradient of Normal	$-\dfrac{1}{p}$
Equation of Normal	$x+py-ap^3-2ap=0$ $x+py=ap^3+2ap$
Intersection of two Normals	$N(-apq(p+q),a(p^2+pq+q^2+2))$

Introduction of parameters (dummy variables)

In 3U, we want to use parameters (dummy variables). We use p & q, not x and y.

$(2ap, ap^2)$ & $(2aq, aq^2)$ represents points on the locus of a parabola

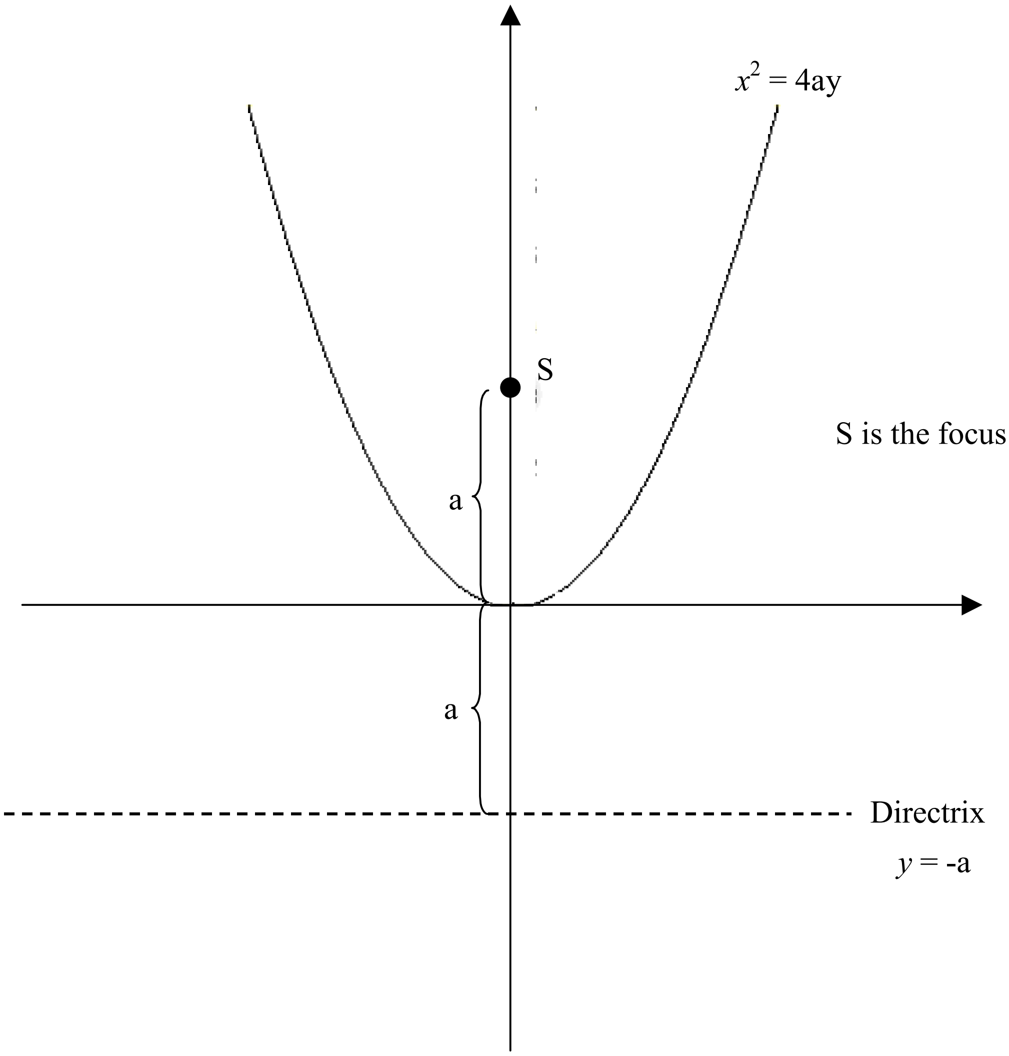
Let $x = 2ap$, $y = ap^2$

Eliminate "p"

$p = \frac{x}{2a}$

Sub in $y = ap^2$

$$\begin{aligned} y &= a\left(\frac{x^2}{4a^2}\right) \\ &= \frac{x^2}{4a} \\ \therefore x^2 &= 4ay \end{aligned}$$



CHORDS

Gradient of a Chord

$$
\begin{aligned}
m &= \frac{aq^2 - ap^2}{2aq - 2ap} \\
&= \frac{a(q-p)(q+p)}{2a(q-p)} \\
&= \frac{p+q}{2}
\end{aligned}
$$

Equation of a Chord (using the 2 point formula)

$$
\begin{aligned}
y - ap^2 &= \frac{p+q}{2}(x - 2ap) \\
2y - 2ap^2 &= (p+q)x - 2ap^2 - 2apq \\
0 &= y - \tfrac{1}{2}(p+q)x + apq \\
y &= \tfrac{1}{2}(p+q)x - apq
\end{aligned}
$$

If PQ is a Focal Chord, then pq = -1

For PQ to be a focal chord, it passes through (0, a)
Sub (0, a) into the Equation of a Chord

$$
\begin{aligned}
0 &= y - \tfrac{1}{2}(p+q)x + apq \\
0 &= a - \tfrac{1}{2}(p+q).0 + apq \\
0 &= a + apq \\
-a &= apq \\
-1 &= pq
\end{aligned}
$$

The Latus Rectum (Special Case Focal Chord) passes through (0,a)

x = 0; y = a

$pq = -1$

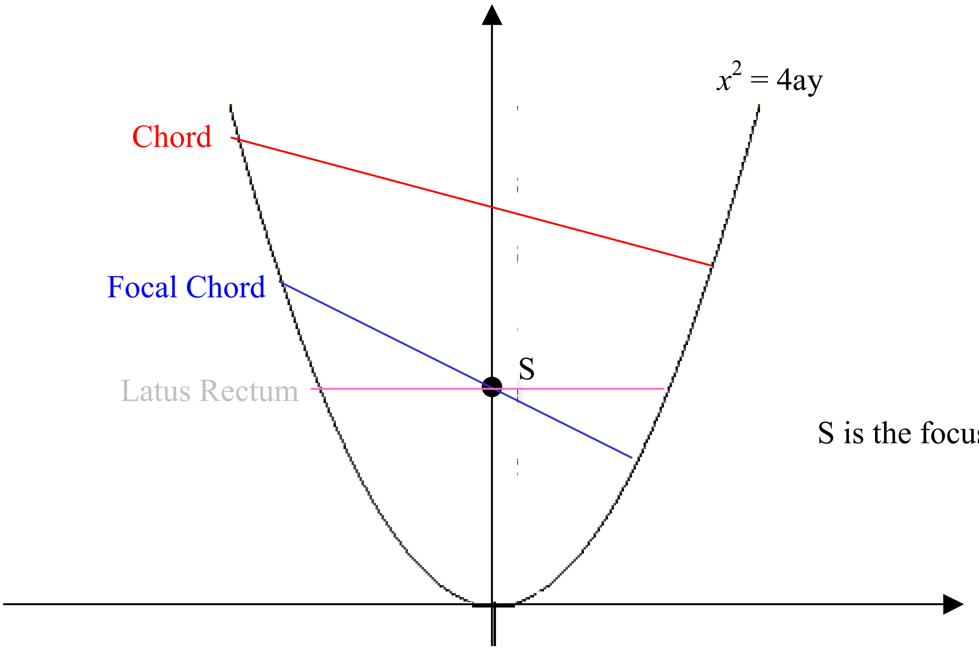
$y - \frac{1}{2}(p+q)x + apq$

Sub y = a in x

$x^2 = yaa$

$x = \pm 2a$

$\therefore l = 4a$



S is the focus

TANGENTS

Gradient of a Tangent

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$y` = \frac{x}{2a}$$

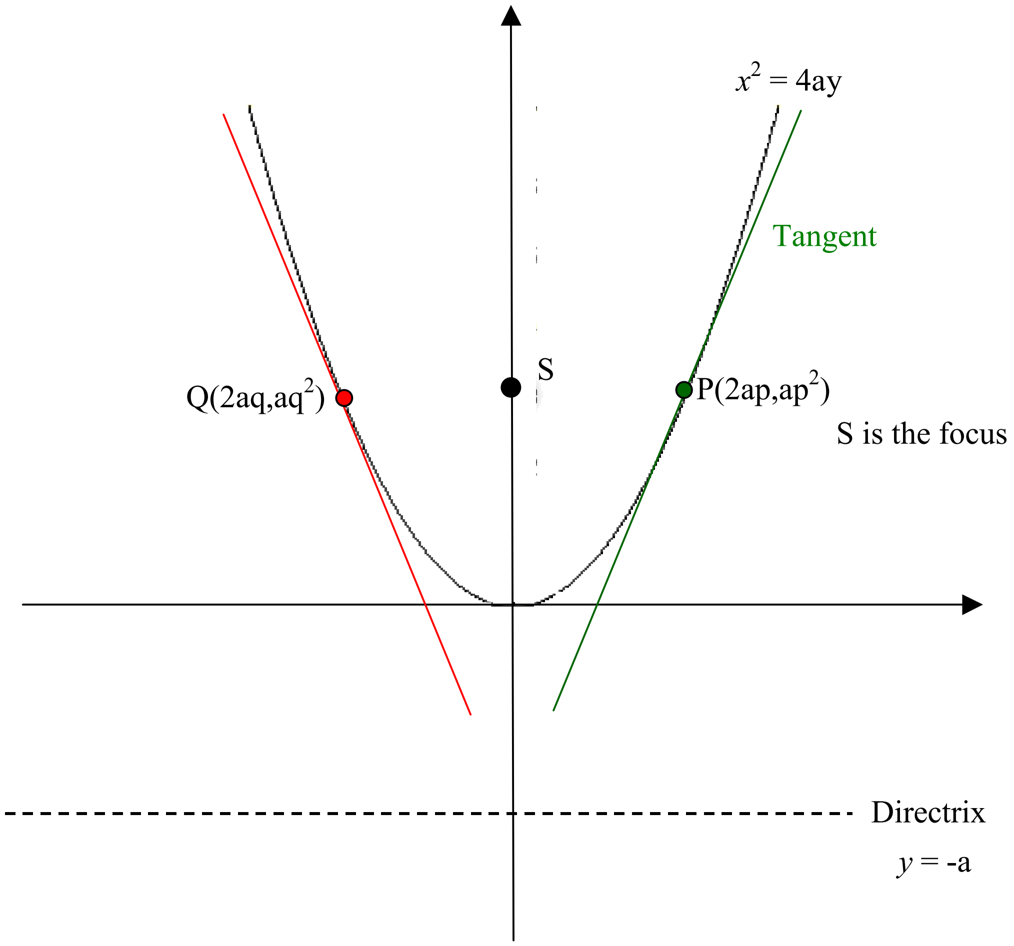
Equation of a Tangent at P(2ap,ap^2)

At point P The gradient is:

$$= \frac{2ap}{2a}$$
$$= p$$

Equation using 2 point formula:

$$y - ap^2 = p(x - 2ap)$$
$$y - ap^2 = px - 2ap^2$$
$$0 = y - px + ap^2$$
$$y = px - ap^2$$



Intersection of two Tangents

P $\quad y - px + ap^2 \quad$ Equation 1

Q $\quad y - qx + aq^2 \quad$ Equation 2

(Equation 1) – (Equation 2)

$$\begin{aligned}
-px + qx + ap^2 - aq^2 &= 0 \\
a\left(p^2 - q^2\right) &= px - qx \\
\frac{a(p-q)(p+q)}{(p-q)} &= \mathrm{x} \\
\mathrm{x} &= \mathrm{a(p + q)}
\end{aligned}$$

Sub into Equation 1

$$\begin{aligned}
y - pa(p+q) + ap^2 &= 0 \\
y - ap^2 - apq + ap^2 &= 0 \\
y &= \mathrm{apq}
\end{aligned}$$

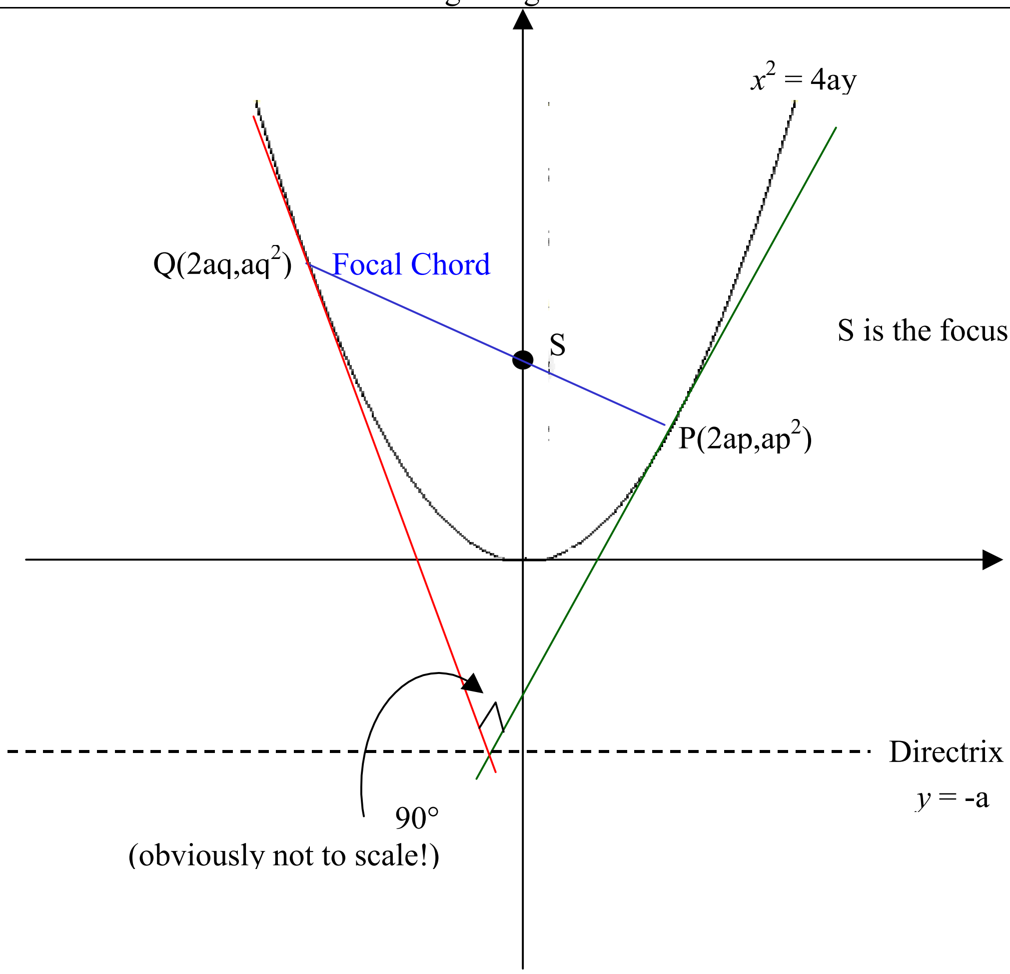
$$T(a(p+q), apq)$$

Intersection of Tangents of the Focal Chord

Since pq = -1

$$\begin{aligned}
\mathrm{apq} &= \text{-a} \\
\therefore y &= \text{-a}
\end{aligned}$$

Tangents of the focal chord intersect at right angles on the directrix



What is the condition for a line $y = mx + c$, so that it becomes a tangent to the parabola $x^2 = 4ay$?

Solve Simultaneous Equations

$$x^2 = 4ay$$
$$y = mx + c$$

$$\begin{aligned} x^2 &= 4a(mx + c) \\ &= 4amx + 4ac \\ 0 &= x^2 - 4amx - 4ac \end{aligned}$$

A Quadratic Equation

We want 1 point of intersection, roots must be equal so $\Delta = 0$

$$\begin{aligned} 0 &= (4am)^2 - 4(1)(-4ac) \\ &= 16a^2m^2 + 16ac \\ &= am^2 + c \\ c &= -am^2 \end{aligned}$$

If…

c	$> -am^2$	No intersection
c	$< -am^2$	2 points of intersection

NORMALS

Gradient of a Normal

If $m_T = p$

$m_N = -\frac{1}{p}$ MN = -1

Equation of a Normal at P(2ap,ap²)

$$\begin{aligned}
y - ap^2 &= \tfrac{1}{-p}(x - 2ap) \\
py - ap^3 &= -x + 2ap \\
x + py &= ap^3 + 2ap \\
0 &= x + py - ap^3 - 2ap
\end{aligned}$$

Intersection of two Normals

P $x + py = ap^3 + 2ap$ Equation 1

Q $x + qy = aq^3 + 2aq$ Equation 2

(Equation 1) – (Equation 2)

$$\begin{aligned}
py - qy &= ap^3 - aq^3 + 2ap - 2aq \\
y(p - q) &= a(p^3 - q^3) + 2a(p - q) \\
y &= a(p^2 + pq + q^2) + 2a \\
&= a(p^2 + pq + q^2 + 2)
\end{aligned}$$

Sub into Equation 1

$$\begin{aligned}
x + pa(p^2 + pq + q^2 + 2) &= ap^3 + 2ap \\
x + ap^3 + ap^2q + apq^3 + 2ap &= ap^3 + 2ap \\
x &= -apq(p + q)
\end{aligned}$$

$$N(-apq(p+q), a(p^2 + pq + q^2 + 2))$$

Reflection Property

We want to prove <MPR = <SPT = <STP

$SP = PN$ Definition of a parabola

$PN = a + ap^2$

$SP = a + ap^2$

Find where Tangent at P cuts the x axis

$y - px + ap^2 = 0$

At T, x = 0

$y - 0 + ap^2 = 0$

$y = -ap^2$

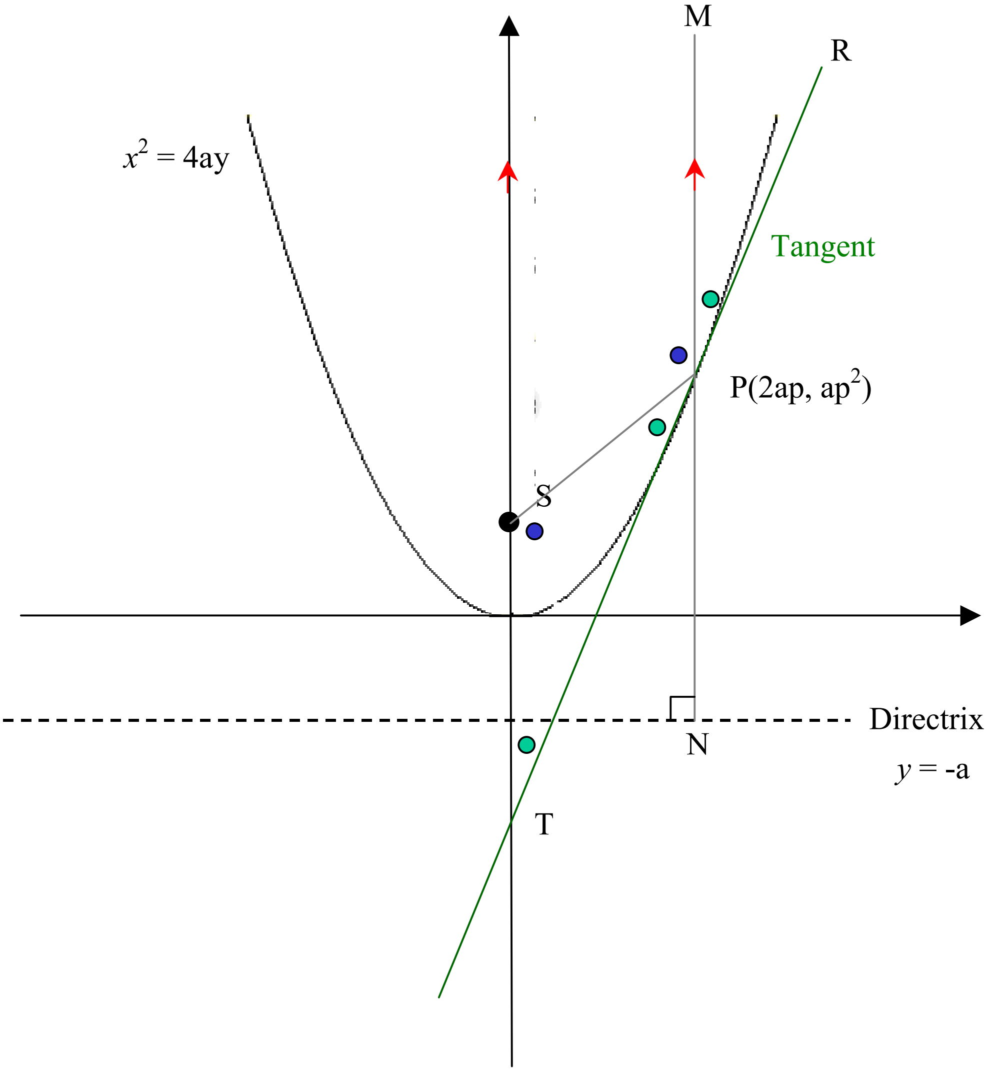
$T(0, -ap^2)$

$ST = a + ap^2$

$ST = SP$

$\therefore \Delta TSP$ is isosceles

<STP = <SPT	➢ Base angles of an isosceles triangle
<TSP = <SPM	➢ Alternate angles in parallel lines
<SPT = <MPR	➢ Angles in a straight line add to 180°, angle sum of a triangle.



Chord of Contact from (x_o , y_o) to $x^2 = 4ay$

Tangents P and Q are drawn from an external point (x_o , y_o). PQ is the resultant Chord.

PQ $\quad y-\frac{1}{2}(p+q)x+apq=0$

T $\quad [a(p+q),apq]$

So

$x_0 = a(p+q)$ $\qquad$ $y_0 = apq$

$p+q=\dfrac{x_0}{a}$ $\qquad$ $pq=\dfrac{y_0}{a}$

$$\begin{aligned} 0 &= y-\frac{1}{2}\left(\frac{x_0}{a}\right)x+a\left(\frac{y_0}{a}\right) \\ &= 2ay - xx_0 + 2ay_0 \end{aligned}$$

$$xx_o = 2a(y+y_0)$$