Sydney Technical High School

Name: ....

Maths Class: ....

Year 12

MATHEMATICS EXTENSION 1

HSC COURSE

ASSESSMENT 3

JUNE, 2017

Time allowed: 90 minutes

General Instructions:

- Write using black or blue pen
- In Questions 6–11, show relevant mathematical reasoning and/ or calculations
- Approved calculators may be used
- Full marks may not be awarded for careless work or illegible writing
- **Begin each question on a new page**
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the back of this paper

Total Marks 65

Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-11
60 Marks

Section 1
Multiple Choice (5 marks)
Use the multiple choice answer sheet for Question 1-5

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1. The table below shows the values of a function f(x) for five values of x.

x	2	2.25	2.5	2.75	3
f(x)	3	4	-1	3	7

What value is an estimate for $\int_2^3 f(x)\,dx$ using Simpson's Rule with these five values?

(A) 3

(B) 4

(C) 5

(D) 6

2. The curve $y = 2x^{\frac{1}{3}}$ is reflected in the line $y = x$.
What is the equation of the reflected curve?

(A) $y = \dfrac{x^3}{16}$

(B) $y = \dfrac{x^3}{8}$

(C) $y = \dfrac{x^3}{4}$

(D) $y = \dfrac{x^3}{2}$

3. Which of the following is equal to $\log_{\frac{1}{a}} x$?

(A) $-\log_a x$

(B) $\dfrac{-1}{\log_a x}$

(C) $\dfrac{1}{\log_a x}$

(D) $\log_a x$

4. What is the domain and range of $y = \sin^{-1}\left(\frac{x}{3}\right)$?

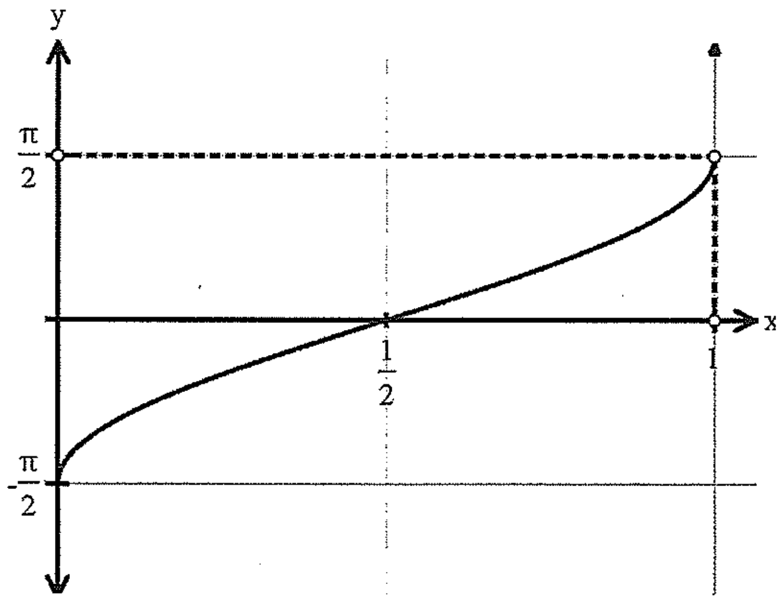
(A) D: $-3 \leq x \leq 3$ R: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(B) D: $-\frac{2}{3} \leq x \leq \frac{2}{3}$ R: $0 \leq y \leq \pi$

(C) D: $-\frac{1}{3} \leq x \leq \frac{1}{3}$ R: $-\pi \leq y \leq \pi$

(D) D: $-1 \leq x \leq 1$ R: $-\pi \leq y \leq \pi$

5. The diagram shows the graph of a function.



(A) $y = -\cos^{-1}(2x-1)$

(B) $y = \sin^{-1}(2x-1)$

(C) $y = \sin^{-1}(x-1)$

(D) $y = -\cos^{-1}(x-1)$

Section II

Total Marks (60)
Attempt Questions 6 – 11.

Answer each question in your writing booklet.
In Questions 6-11, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (10 Marks)

(a) Differentiate with respect to x

i) $e^{\tan 2x}$ **1**

ii) $y = \tan^{-1}\left(\dfrac{2}{x}\right)$ **2**

(b) i) Sketch the graph of the function $f(x) = e^x - 4$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. **2**

ii) On the same diagram sketch the graph of the function $y = f^{-1}(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. **2**

iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **2**

iv) Explain why the coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$ **1**

End of Question 6

Question 7 **(10 Marks)** Use a Separate Sheet of paper

(a) Find $\int e^x(e^x + 1)\ dx$ **2**

(b) Find $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{3}\right)^2$ and hence find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{\tan^{-1}\frac{x}{3}}{x^2+9}\ dx$ **3**

(c) Use the Trapezoidal Rule with 5 function values to obtain an estimate for: **2**

$$\int_{-2}^{6} \log_e \sqrt{x+3}\ dx$$

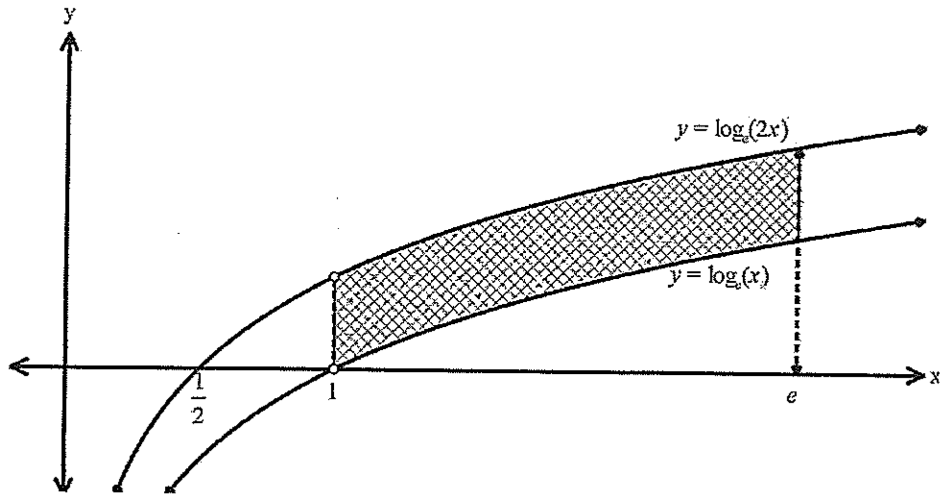
Simplify your answer as much as possible.

(d) Find the exact value of $\sin\left[\cos^{-1}\left(\dfrac{2}{3}\right) + \tan^{-1}\left(-\dfrac{3}{4}\right)\right]$ **3**

End of Question 7

Question 8 **(10 Marks)** Use a Separate Sheet of paper

(a) The curves of $y = \log x$ and $y = \log_e 2x$ area drawn below.



Find the shaded area between the curves $y = \log_e(2x)$ and $y = \log_e x$ and the lines x=1 and $x = e$. **4**

(b) Find the general solutions of the equation $\sin 2\theta = \sin^2\theta$ **3**

(c) Use the substitution $x = u^2 - 1$,, $u \geq 0$, to evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$ **3**

End of Question 8

Question 9 **(10 Marks)** Use a Separate Sheet of paper

(a) i) Show that $e^{1 - \ln 2} = \frac{e}{2}$ **1**

ii) Find the equation of the tangent to the curve $y = e^{1 - 4x}$ at $x = \frac{\ln 2}{4}$ **3**

(b) Prove that $\frac{d}{dx}\left(\log_e \frac{x^2 + 1}{\sqrt{x}}\right)$ may be written as $\frac{3x^2 - 1}{2x(1 + x^2)}$ **3**

(c) The region bounded by the curve $y = \cos^{-1} x$ and the y axis between $y = \frac{\pi}{12}$ and $y = \frac{\pi}{4}$ is rotated through one complete revolution about the y axis. Find the exact volume of the solid formed. **3**

End of Question 9

Question 10 **(10 Marks)** Use a Separate Sheet of paper

(a) i) Show that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ **2**

ii) Hence show that $\tan 15° + \cot 15° = 4$ **2**

(b) Consider the functions $y = -\cos^{-1}x$ and $y = 2\tan^{-1}(x-1)$.

i) Show that the graphs of these functions intersect on the y-axis. **2**

ii) Show that the graphs have a common tangent at this point of intersection. **2**

(c) Find $\int \sin^2 3x \; dx$ **2**

End of Question 10

Question 11 (10 Marks) Use a Separate Sheet of paper

(a) Find the value of $\lim\limits_{x \to 0} \dfrac{3x \cos 4x}{\sin 2x}$ **2**

(b) Consider the function $f(x) = 2\tan x$ where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

i) Sketch the graph of $y = f(x)$.

ii) Find the inverse function of $f^{-1}(x)$ and state its domain. **2**

iii) Hence or otherwise, find the area of the region bounded by the curve of $y = f^{-1}(x)$ the x-axis and the lines $x = 0$ and $x = 2$. Express your answer as an exact value. **3**

(c) Evaluate $\displaystyle\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$ **3**

End of Examination

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Sydney Technical High School
Extension One - Assessment 3
June 2017

ultiple choice

$\int_2^3 f(x)\,dx$.

$\approx \frac{0.25}{3}\left(3+4\times4+2\times(-1)+4\times3+7\right)$

$= 3$ A

$y = 2x^{\frac{1}{3}}$
$x = 2y^{\frac{1}{3}}$
$\frac{x}{2} = y^{\frac{1}{3}}$
$\frac{x^3}{8} = y$ B

$y = \log_{\frac{1}{a}} x$
$x = \left(\frac{1}{a}\right)^y$
$x = a^{-y}$
$-y = \log_a x$

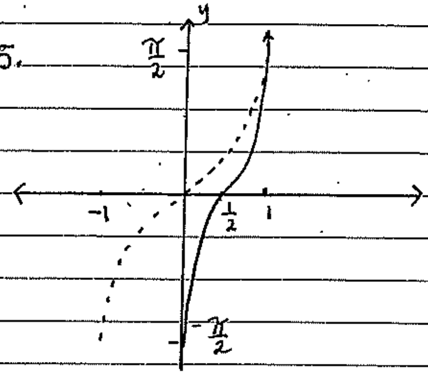
$\log_{\frac{1}{a}} x = -\log_a x$ A

Domain: $-1 \le \frac{x}{3} \le 1$
ie $-3 \le x \le 3$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

A

5.



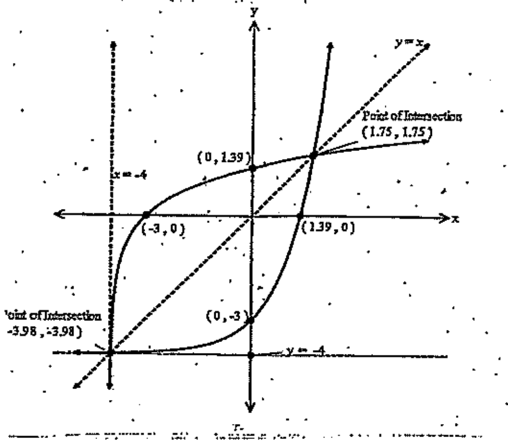
B

Question 6.

a i) $y = e^{\tan 2x}$
$\frac{dy}{dx} = 2\sec^2 2x\, e^{\tan 2x}$

ii) $y = \tan^{-1}\left(\frac{2}{x}\right)$ or $y = \tan^{-1}(2x^{-1})$
$\frac{dy}{dx} = \frac{-2x^{-2}}{1+\frac{4}{x^2}}$

$= \frac{-2}{x^2+4}$



b iii) $x = e^y - 4$
$x + 4 = e^y$
$\ln(x+4) = y$

b iv) The two curves intersect on the line $y = x$

$\therefore x = e^x - 4$
$e^x - x - 4 = 0$

Question 7

) $\int e^x(e^x+1)\,dx = \int e^{2x} + e^x\,dx$
$= \frac{1}{2}e^{2x} + e^x + c$

b) $y = \left(\tan^{-1}\frac{x}{3}\right)^2$

$\frac{dy}{dx} = 2\left(\tan^{-1}\frac{x}{3}\right) \times \frac{3}{3^2+x^2}$

$= \frac{6\tan^{-1}\left(\frac{x}{3}\right)}{9+x^2}$

$\int_0^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{x}{3}\right)}{9+x^2}\,dx = \frac{1}{6}\left[\tan^{-1}\left(\frac{x}{3}\right)^2\right]$

$= \frac{1}{6}\left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)^2 - \tan^{-1}(0)^2\right]$

$= \frac{1}{6}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)^2\right]$

$= \frac{1}{6} \times \left(\frac{\pi}{6}\right)^2$

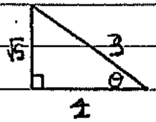
$= \frac{\pi^2}{216}$

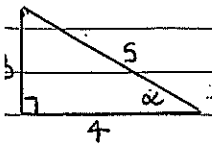
c) $A \approx \left[\frac{6--2}{8}\left(0 + 2(\ln 3 + \ln 5 + \ln 7) + \ln 3\right)\right]$

$\approx \ln 3 + \ln 5 + \ln 7 + \ln 3$
$\approx 2\ln 3 + \ln 5 + \ln 7$
$\approx \ln(9 \times 5 \times 7)$
$\approx \ln(315)$ unit^2
OR
5.79 unit^2 (2dp)

1) $\sin\left[\cos^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(-\frac{3}{4}\right)\right]$

let $\theta = \cos^{-1}\left(\frac{2}{3}\right)$





let $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$\sin\left[\cos^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right]$

$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$

$\sin(\theta - \alpha) = \frac{\sqrt{5}}{3} \times \frac{4}{5} - \frac{2}{3} \times \frac{3}{5}$

$= \frac{4\sqrt{5}}{15} - \frac{2}{5}$

$= \frac{4\sqrt{5} - 6}{15}$

Question 8

) Area $= \int_1^e \ln(2x) - \ln x \, dx$

$= \int_1^e \ln 2 + \ln x - \ln x \, dx$

$= \int_1^e \ln 2 \, dx$

$= \left[x \ln 2\right]_1^e$

$= e\ln 2 - \ln 2$

$= \ln 2 (e - 1)$

or

$= 1.19$ units^2

b) $\sin 2\theta = \sin^2\theta$

$2\sin\theta\cos\theta = \sin^2\theta$

$2\sin\theta\cos\theta - \sin^2\theta = 0$

$\sin\theta(2\cos\theta - \sin\theta) = 0$

$\sin\theta = 0$

$\theta = n\pi, \quad n \in \mathbb{Z}$

or

$2\cos\theta - \sin\theta = 0$

$\tan\theta = 2$

$\theta = n\pi + \tan^{-1} 2$

$n \in \mathbb{Z}$

c) $x = u^2 - 1 \quad u \geq 0$

$dx = 2u \, du$

$x = 0 \rightarrow u = 1$

$x = 3 \rightarrow u = 2$

$\int_0^3 \frac{x}{\sqrt{x+1}} \, dx = \int_1^2 \frac{u^2 - 1}{u} \times 2u \, du$

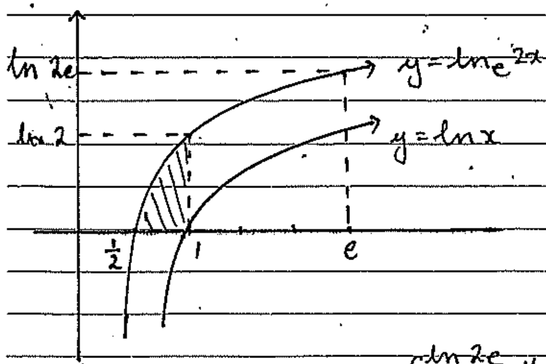
$= 2\int_1^2 u^2 - 1 \, du$

$= 2\left[\frac{u^3}{3} - u\right]_1^2$

$= 2\left[\left(\frac{8}{3} - 2\right) - \left(2 - 1\right)\right]$

$= \frac{8}{3}$

Question 8 (alternate solution)



$I_1 = e \times \ln 2e - \int_0^{\ln 2e} \frac{e^y}{2} \, dy$

$I_1 = e\ln 2e - \frac{1}{2}\left[e^{\ln 2e} - e^0\right]$

$I_1 = 2.384169$

$I_2 = e - \int_0^1 e^y \, dy$

$I_2 = e - \left[e^y\right]_0^1$

$I_2 = e - \left[e - e^0\right]$

$I_2 = 1$

$1 \times \ln 2 - \left[\frac{e^y}{2}\right]_0^{\ln 2}$

$= \ln 2 - \left[\frac{e^{\ln 2}}{2} - \frac{e^0}{2}\right]$

$= \ln 2 - \left[1 - \frac{1}{2}\right]$

$= 0.193147$

$= I_2 - I_1$

$= 2.384169 - 1$

$= 1.384169$

$\therefore$ Required Area

$= 1.384169 - 0.193147$

$= 1.191$

Question 9

i) $e^{1-\ln 2} = e \cdot e^{-\ln 2}$

$= e \cdot \ln^{\frac{1}{2}}$

$= e \times \frac{1}{2}$

$= \frac{e}{2}$

ii) $\frac{dy}{dx} = -4e^{1-4x}$

at $x = \frac{\ln 2}{4}$ $\quad y = \frac{e}{2}$

$m = -4e^{1-4\left(\frac{\ln 2}{4}\right)}$

$= -4e^{1-\ln 2}$

$= -4 \times \frac{e}{2}$

$= -2e$

Equation of tangent

$y - \frac{e}{2} = -2e\left(x - \frac{\ln 2}{4}\right)$

$y - \frac{e}{2} = -2ex + \frac{\ln 2}{2}$

$y = -2ex + \frac{\ln 2 + e}{2}$

) $y = \log(x^2+1) - \frac{1}{2}\log_e x$

$\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{1}{2x}$

$= \frac{4x^2 - 1 - x^2}{2x(x^2+1)}$

$= \frac{3x^2-1}{2x(1+x^2)}$

c) $V = \pi\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \cos^2 y \, dy$

$= \frac{\pi}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1+\cos 2y)\, dy$

$= \frac{\pi}{2}\left[y + \frac{1}{2}\sin 2y\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} - \frac{\pi}{12}\right) + \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right)\right]$

$= \frac{\pi}{2}\left(\frac{\pi}{6} + \frac{1}{4}\right)u^3$

$= \left(\frac{\pi^2}{12} + \frac{\pi}{8}\right)$ units^3

Question 10

a i) $\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{1+2\cos^2 x - 1}$

$= \frac{2\sin x\cos x}{2\cos^2 x}$

$= \tan x$

a ii) $\tan 15^\circ = \frac{\sin(2\times 15^\circ)}{1+\cos(2\times 15^\circ)}$

$= \frac{\sin 30^\circ}{1+\cos 30^\circ}$

$= \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$

$= \frac{1}{2+\sqrt{3}}$

$\cot 15^\circ = 2+\sqrt{3}$

$\tan 15^\circ + \cot 15^\circ$

$= \frac{1}{2+\sqrt{3}} + 2+\sqrt{3}$

$= \frac{1+(2+\sqrt{3})^2}{2+\sqrt{3}}$

$= \frac{1+4+4\sqrt{3}+3}{2+\sqrt{3}}$

$= \frac{8+4\sqrt{3}}{2+\sqrt{3}}$

$= 4$

b) i) $y = -\cos^{-1} x$

$y = 2\tan^{-1}(x-1)$

When $x=0$ $\quad y = -\cos^{-1} 0$

$y = -\frac{\pi}{2}$

$y = 2\tan^{-1}(-1)$

$y = 2\left(-\frac{\pi}{4}\right)$

$y = -\frac{\pi}{2}$

∴ The curves intersect at the same point $\left(0, -\frac{\pi}{2}\right)$ on the y-axis

ii) $y = -\cos^{-1} x$

$\frac{dy}{dx} = -\left(\frac{-1}{\sqrt{1-x^2}}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

When $x=0$ $\quad \frac{dy}{dx} = \frac{1}{\sqrt{1-0}} = 1$

For $y = 2\tan^{-1}(x-1)$

$\frac{dy}{dx} = \frac{2}{(x-1)^2+1} \times 1$

When $x=0$ $\quad \frac{dy}{dx} = \frac{2}{(0-1)^2+1} = 1$

∴ the tangents at $\left(0, -\frac{\pi}{2}\right)$ have the same gradient. Hence the curves have a common tangent at this point.

c) $\int \sin^2 3x \, dx$

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1-\cos 2A}{2}$

$\int \sin^2 3x\, dx = \frac{1}{2}\int 1 - \cos 6x \, dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right] + c$

$= \frac{1}{2}x - \frac{1}{12}\sin 6x + c$

Question 11

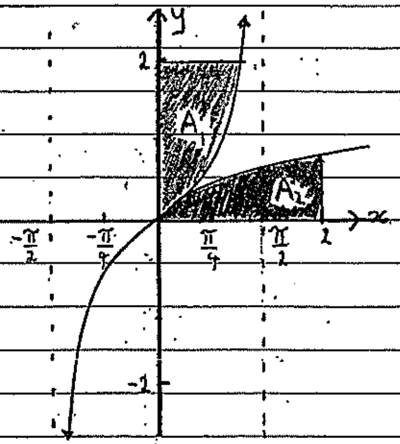
a) $\lim_{x \to 0} \frac{3x}{\sin 2x} \times \cos 4x$

$= \lim_{x \to 0} \frac{2x}{\sin 2x} \times \frac{3}{2} \times \lim_{x \to 0} \cos 4x$

$= 1 \times \frac{3}{2} \times 1$

$= \frac{3}{2}$

b) i)



ii) $x = 2\tan y$

$\tan y = \frac{x}{2}$

$f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$

Domain of $f^{-1}(x)$ is all real numbers

A_2 = Area of rectangle − area bounded by $y = 2\tan x$ the x-axis $x = 0$ and $x = \frac{\pi}{4}$

$A_2 = \frac{\pi}{4} \times 2 - \int_0^{\frac{\pi}{4}} 2\tan x \, dx$

$A_2 = \frac{\pi}{2} - 2\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$

$= \frac{\pi}{2} + 2\left[\log_e(\cos x)\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2} + 2\left[\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e(1)\right]$

$= \frac{\pi}{2} + 2\left[\log_e\left(2^{-\frac{1}{2}}\right)\right]$

$= \frac{\pi}{2} - \log_e 2$

c) $\int_0^{\frac{2}{5}} \frac{dx}{\sqrt{16 - 25x^2}} = \int_0^{\frac{2}{5}} \frac{dx}{5\sqrt{\frac{16}{25} - x^2}}$

$= \frac{1}{5}\left[\sin^{-1}\left(\frac{x}{\frac{4}{5}}\right)\right]_0^{\frac{2}{5}}$

$= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$

$= \frac{1}{5} \times \frac{\pi}{6}$

$= \frac{\pi}{30}$