Name;_______________________ Teacher: _______________________

SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2010

MATHEMATICS

Time Allowed: 3 hours plus 5 minutes reading time

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted (12 marks each).
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- A table of standard integrals is attached.

(for Markers Use Only)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	**Total**
										/120

Question 1: (12 marks) **Marks**

a) Expand and simplify $(\sqrt{2}-3)^2$ 1

b) Find $e^{-0.6}$ correct to three decimal places. 2

c) Find the compound interest earned on \$80 000 if invested for three years at a rate of 6% per annum compounding quarterly. 2

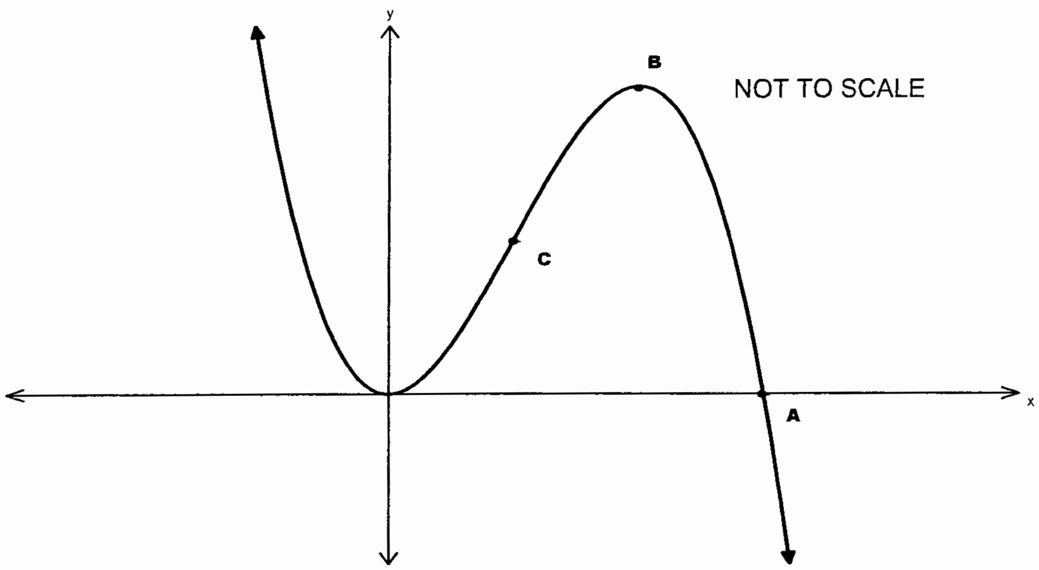
d) Solve the equation $4x^2 = x$ 2

e) Solve the equation $|4-x| = 2x$. 2

f) Sketch the parabola $x^2 = -4y + 8$ showing its focus and directrix. 3

Question 2: (12 marks) (Start a new page)

a)



The graph represents the function $y = 6x^2 - x^3$.

The point A is an x intercept.

The point B is a local maximum.

The point C is a point of inflexion.

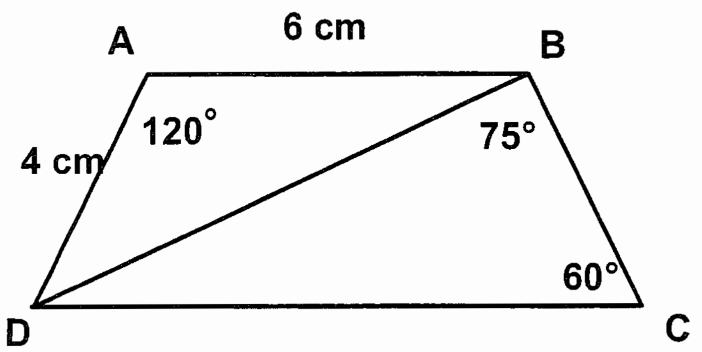
Find

(i) the coordinates of A 1

(ii) the coordinates of B 2

(iii) the coordinates of C 2

b)



(i) Find the length of BD as a simplified surd. 2

(ii) Find the length of BC correct to one decimal place. 3

(iii) Find the area of triangle ABD as a surd. 2

Question 3: (12 marks) (Start a new page)

a) The first term of an arithmetic sequence is 4 and the fifth term is four times the third term. Find the common difference. 2

b) Determine the derivates of:

(i) $(3x+7)^{14}$ 1

(ii) $\frac{2x}{x^2-1}$ 2

c) Find the **EXACT** values of the following definite integrals:

(i) $\int_0^1 e^{3x}\,dx$ 2

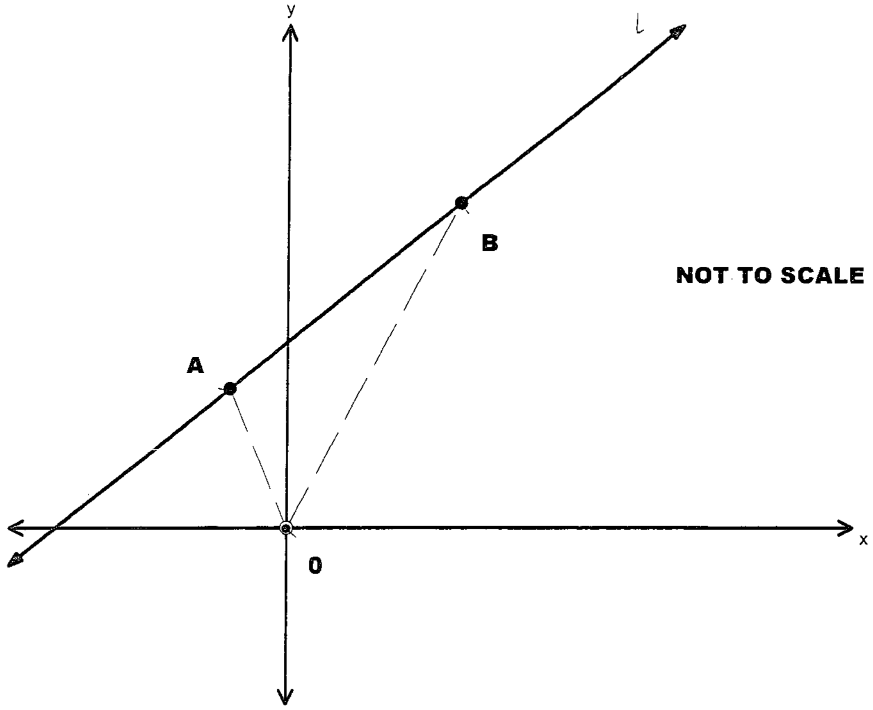
(ii) $\int_0^1 \frac{1}{1+x}\,dx$ 2

d) The sum of the first four terms of a geometric sequence is 30 and the limiting sum is 32. If the common ratio is negative, find the common ratio and the first term. 3

Question 4: (12 marks) (Start a new page)

a) Find the equation of the line (in general form) perpendicular to $2x-3y-6=0$ and intersecting it on the x axis. 3

b) The line l passes through A (-1,3) and B (3,7).



(i) Find the length of AB (in exact form) 1

(ii) Find the equation of the line l. 2

(iii) Show that the distance from O to the line l is $\frac{4}{\sqrt{2}}$ units. 2

(iv) Calculate the area of ΔAOB. 1

c) The points A(2,-6) and B(4,8) are at opposite ends of the diameter of a circle.

Find

(i) the centre of the circle. 1

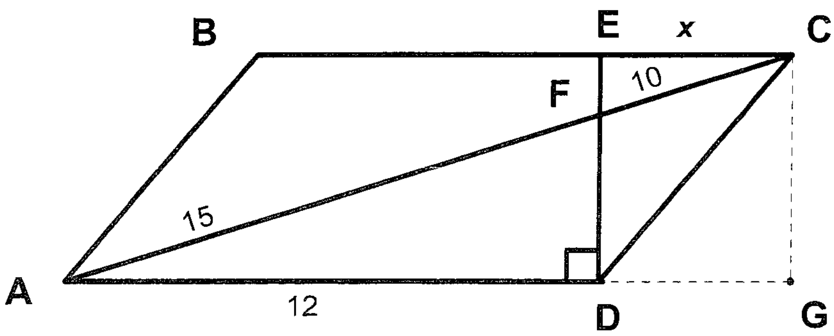
(ii) the radius of the circle. 1

(iii) the equation of the circle. 1

Question 5:

a) Given $f'(x) = 3x^2 - 4$, find $y = f(x)$ if the function passes through (3, 8) 2

b)



EC = x
AD = 12
FC = 10
AF = 15
ED||CG

NOT TO SCALE

ABCD is a parallelogram.

Copy the above diagram onto your writing paper

(i) Prove $\Delta EFC\ |||\ \Delta DFA$. 2

(ii) Find the value of x. (with a reason) 2

(iii) Find the length of CG. (with reasons) 2

c) A person wishes to invest \$A at the beginning of each month at a compound interest rate of 0.6% per month. How much does the person invest each month in order to have \$20 000 saved at the end of the first year? 4

Question 6: (12 marks) (Start a new page)

a) (i) Sketch $y = x^2 + 6$ and $y = 12 - x$ on the same axes. 3

Find the x coordinate of the points in intersection.

(ii) Find the area in the first quadrant bounded by the y axis, $y = x^2 + 6$ and $y = 12 - x$. 2

b) Use Simpson's Rule with 5 function values to estimate 3

$\int_0^4 \sqrt{5 + x^2}dx$ correct to 2 decimal places.

c) Find $log_{11} 57$ correct to 2 decimal places. 1

d) (i) Use logarithm laws to simplify $ln(\frac{\sqrt{x-1}}{x^2+1})$ 1

(ii) Hence find $\frac{d}{dx}(ln \frac{\sqrt{x-1}}{x^2+1})$ 2

Question 7: (12 marks) (Start a new page)

a) The size of a colony of insects is given by the equation

$P = 3000e^{kt}$

Where P is the population after t days.

(i) Write down the initial population. 1

(ii) If there are 3600 insects after one day, find the value of k, correct to 2 decimal places. 2

(iii) When will the colony double its initial population? (Answer correct to the nearest day). 2

(iv) What is the rate of growth of the population after 2 days? 2

b) Find the volume of the solid formed when the curve $y = \sqrt{x}$ is rotated about the x axis between $x = 1$ and $x = 5$. (Leave your answer in terms of π) 2

c) Consider the equation $2x^2 - (3 + k)x + 2 = 0$.

For what values of k does the equation have

(i) equal roots 2

(ii) different real roots 1

Question 8: (12 marks) (Start a new page)

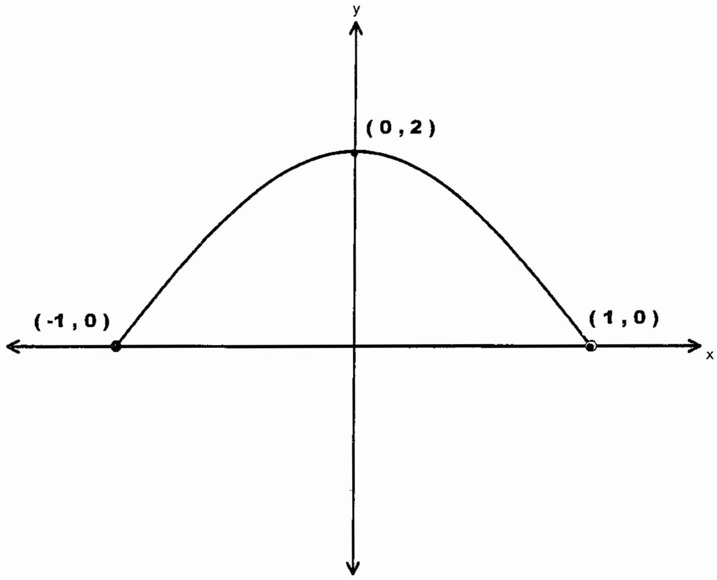
a) Differentiate

(i) $\sin(1 - 2x^3)$ 2

(ii) $tan3x$ 2

(iii) cos^2x 2

b)



An ornamental arch window 2 metres wide and 2 metres high is to be made in the shape of an arc of either a cosine curve or a parabola, as illustrated on axes above.

i) If the arch is made in the shape of the curve $y = 2\,cos\,\frac{\pi}{2}x$, 2

find the area of the window. (Your answer may be left in terms of π).

ii) If the arch is made the shape of an arc of a parabola, find :

α) the equation of the parabola 2

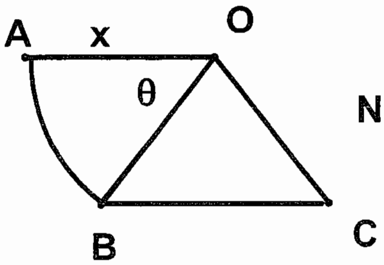
β) the area of the window 2

Question 9 (12 marks) (Start a new page)

a) (i) On the same diagram sketch the curve $y = sin\pi x$ and the line $y = x$, in the domain $-1 \leq x \leq 1$. 3

(ii) Hence find the number of solutions to the equation $sin\pi x - x = 0$ in the domain $-1 \leq x \leq 1$. 1

b)



NOT TO SCALE

The diagram shows a sector OAB of a circle, centre O and radius x metres. Arc AB subtends an angle θ radians at O. An equilateral triangle BCO adjoins the sector.

Write down expressions for:

(i) the perimeter of the figure ABCO. 1

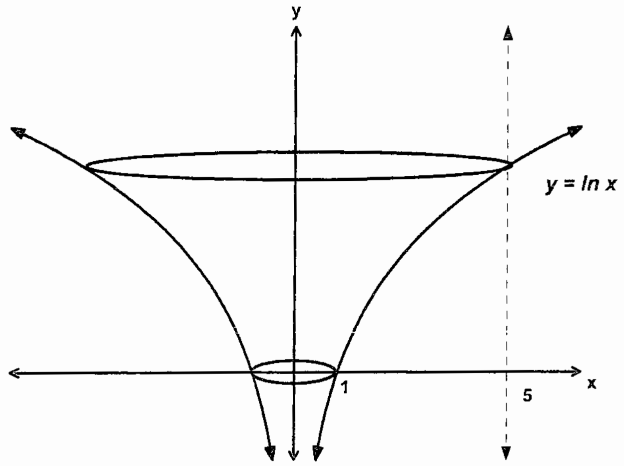
(ii) the area of the figure ABCO. 2

c) (i) Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ 2

(ii) Hence find the equation of the normal to $y = \sec x$ at the point where $x = \frac{\pi}{4}$ (leave your answer in exact form) 3

Question 10 (12 marks) (Start a new page)

a)



NOT TO SCALE

The interior of a bowl is shaped by rotating the arc of the curve $y = log_e x$ from $x = 1$ to $x = 5$ around the y axis. Calculate the capacity of the bowl in terms of π. 3

b) Given that $a^2 + b^2 = 7ab$, use this result to show that

(i) $(\frac{a+b}{3})^2 = ab$ 1

(ii) and hence using part i) write

$$\log\left(\frac{a+b}{3}\right) - \frac{1}{2}(loga + \log b)$$ 2

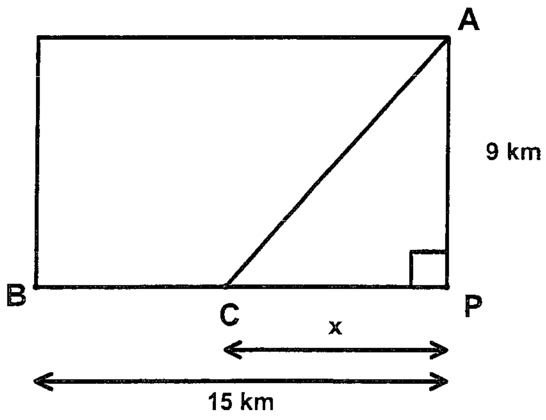
in simplest form.

c) The diagram shows a rectangular field measuring 9 km by 15 km. From A, a bike rider wishes to go to B.

Riding across the field from A to C, he can average 5km/hr.

Along the road BC, he can average 13 km/hr.

Let PC = x



(i) Show that the time he takes to go from A to C is $\frac{\sqrt{81+x^2}}{5}$ 1

(ii) Show that the total time he will take to go from A to C to B will be

$$T = \frac{\sqrt{81+x^2}}{5} + \frac{15-x}{13}$$ 1

(iii) Show that the shortest time for the journey will occur when he rides to a point $3\frac{3}{4}km$ from P. 4

QUESTION 1

i) $(\sqrt{2}-3)^2 = 2-6\sqrt{2}+9$

$= \underline{11-6\sqrt{2}}$

b) -0.6

$e \quad = \underline{0.549}$

c) $\text{Amt} = 80000\left(1+\frac{3}{200}\right)^{12}$

$\text{Amt} = \$95649.45$

$\therefore$ Interest $= \underline{\$15649.45}$

d) $4x^2 = x$

$4x^2 - x = 0$

$x(4x-1)=0$

$\therefore x = 0, \ 1/4$

e) $|4-x| = 2x$

$4-x = 2x \qquad 4-x = -2x$

$4 = 3x \qquad 4 = -x$

$x = 4/3 \qquad x = -4$

check solutions:

$\left|4-\frac{4}{3}\right| = 2\times\frac{4}{3} \qquad |4--4| = -8$

$2\,2/3 = 2\,2/3 \checkmark \qquad 8 \neq -8$

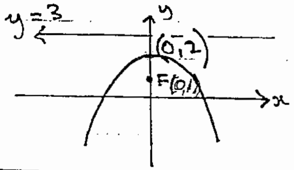
$\therefore$ $\underline{x = 4/3 \text{ only solution}}$

f) $x^2 = -4y+8$ vertex $(0,2)$

$x^2 = -4(y-2)$ Focal length $a=1$

Focus $(0,1)$

Directrix $y=3$



QUESTION 2

a) $y = 6x^2 - x^3$

i) $y = 0 \quad x^2(6-x)=0$

$\therefore x = 0, 6$

$\therefore A(6,0)$

ii) $\frac{dy}{dx} = 12x - 3x^2$

st pts $y'=0$

$3x(4-x)=0$

$x=0 \qquad x=4$

$\therefore \underline{B(4,32)}$

iii) $\frac{d^2y}{dx^2} = 12-6x$ pt inf $y''=0$

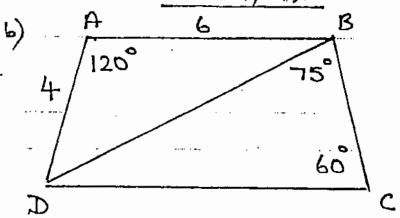
$12-6x=0$

$12=6x$

$x=2$

$\therefore \underline{C(2,16)}$

b)



i) $BD^2 = 6^2+4^2-2.4.6.\cos120°$

$= 36+16-48\times-\frac{1}{2}$

$= 52+24$

$= 76$

$\therefore BD = \sqrt{76}$ units

$\underline{BD = 2\sqrt{19}}$ simplified surd

ii) $\frac{BC}{\sin45°} = \frac{2\sqrt{19}}{\sin60°}$

$BC = \frac{2\sqrt{19}.\sin45°}{\sin60°}$

$\underline{BC = 7.1 \text{ units}}$

iii) $\triangle ABD = \frac{1}{2}.6.4\sin120°$

$= 12.\frac{\sqrt{3}}{2}$

$= \underline{6\sqrt{3}\,\text{unit}^2}$

QUESTION 3

a) AP: $a=4 \qquad T_5 = 4T_3$

$a+4d = 4(a+2d)$

$4+4d = 4(4+2d)$

$4+4d = 16+8d$

$-12 = 4d$

$\underline{d=-3}$

b) i) $\frac{d}{dx}(3x+7)^{14} = 14.3(3x+7)^{13}$

$= \underline{42(3x+7)^{13}}$

ii) $\frac{d}{dx}\left(\frac{2x}{x^2-1}\right) \Rightarrow$ quotient rule

$u=2x \qquad v=x^2-1$

$u'=2 \qquad v'=2x$

$\therefore \frac{d}{dx}\left(\frac{2x}{x^2-1}\right) = \frac{2(x^2-1)-2x.2x}{(x^2-1)^2}$

$= \frac{2x^2-2-4x^2}{(x^2-1)^2}$

$= \underline{\frac{-2x^2-2}{(x^2-1)^2}}$

c) i) $\int_0^1 e^{3x}\,dx = \left[\frac{e^{3x}}{3}\right]_0^1$

$= \frac{1}{3}\left[e^3 - e^0\right]$

$= \underline{\frac{1}{3}\left(e^3-1\right)}$

ii) $\int_0^1 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^1$

$= \ln 2 - \cancel{\ln 1}^{\,0}$

$= \underline{\ln 2}$

d) GP: $S_4 = 30 \qquad S_\infty = 32$

$30 = \frac{a(1-r^4)}{1-r}$ ① $\qquad 32 = \frac{a}{1-r}$ ②

① ÷ ② $\quad \frac{a(1-r^4)}{1-r} \div \frac{a}{1-r} = \frac{30}{32}$

$\frac{\cancel{a}(1-r^4)}{\cancel{1-r}} \times \frac{\cancel{1-r}}{\cancel{a}} = \frac{15}{16}$

$\frac{1-r^4}{1} = \frac{15}{16}$

$16(1-r^4) = 15$

$16-16r^4 = 15$

$1 = 16r^4$

$r^4 = \frac{1}{16}$

$\underline{r = -\frac{1}{2}}$ since negative

$\therefore$ from ② $\quad 32 = \frac{a}{1--\frac{1}{2}}$

$\underline{a = 48}$

QUESTION 4

a) grad of $2x-3y-6=0$

$3y = 2x-6$

$y = \frac{2x}{3} - 2$

$\therefore m_1 = \frac{2}{3} \quad \therefore$ perp $m_2 = -\frac{3}{2}$

$2x-3y-6=0$ cuts x axis at $(3,0)$

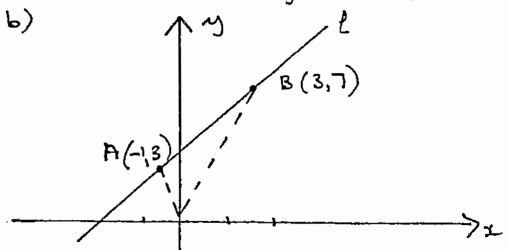
$\therefore$ required line $m_2 = -\frac{3}{2}$ thru $(3,0)$

$y-0 = -\frac{3}{2}(x-3)$

$2y = -3x+9$

$\underline{3x+2y-9=0}$

b)



i) $AB = \sqrt{(3--1)^2+(7-3)^2}$

$= \sqrt{16+16}$

$= \sqrt{32}$

$= \underline{4\sqrt{2} \text{ units}}$

ii) ℓ: $m_\ell = \frac{4}{4} = 1$

$y-7 = 1(x-3)$

$y-7 = x-3$

$\underline{y = x+4}$

$x-y-4=0$ general form

iii) $p = \left|\frac{0\times1+0\times-1+4}{\sqrt{1^2+1^2}}\right|$

$p = \left|\frac{+4}{\sqrt{2}}\right| \quad \therefore$ perp dist is $\underline{4/\sqrt{2} \text{ units}}$

iv) Area $\triangle AOB$

$= \frac{1}{2}.4\sqrt{2}.\frac{4}{\sqrt{2}}$

$= \underline{8 \text{ units}^2}$

c) i) Centre $(3,1)$

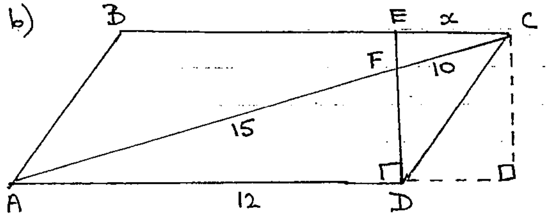
ii) $R = \sqrt{(3-4)^2 + (1-8)^2}$
$= \sqrt{1^2 + 7^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$ units

iii) $(x-3)^2 + (y-1)^2 = 50$

QUESTION 5

a) $f'(x) = 3x^2 - 4$
$f(x) = x^3 - 4x + c$
sub $(3,8)$ $\quad 8 = 27 - 12 + c$
$8 = 15 + c$
$c = -7$
$\therefore f(x) = x^3 - 4x - 7$

b)



i) In Δ's EFC, DFA
$E\hat{F}C = D\hat{F}A$ (vertically opposite)
$E\hat{C}F = F\hat{A}D$ (alternate BC || AD)
$\therefore \Delta EFC \,|||\, \Delta DFA$ (equiangular)

ii) $\frac{x}{12} = \frac{10}{15}$ (ratio of corsp. sides in simliar triangles)
$15x = 120$
$x = 8$

iii) pythag theroem in Δ AFD $\therefore$ FD = 9
" " in Δ ECF $\therefore$ EF = 6
$\therefore$ ED = 15
ED = CG = 15 (opp sides rectangle)

c) $20{,}000 = A(1.006)^{12} + A(1.006)^{11} + \cdots + A(1.006)$
$20{,}000 = A[\underbrace{1.006 + \cdots + 1.006^{12}}]$
AP: $a = 1.006$ $\quad r = 1.006$
$n = 12$

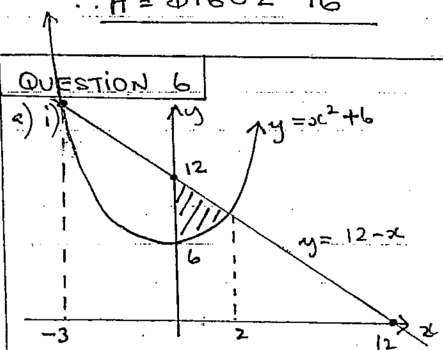
$20{,}000 = A\left[\frac{1.006(1.006^{12}-1)}{1.006-1}\right]$

$\frac{20{,}000(.006)}{1.006(1.006^{12}-1)} = A$

$\therefore A = \$1602.76$

QUESTION 6

a) i)



$x^2 + 6 = 12 - x$
$x^2 + x - 6 = 0$
$(x+3)(x-2) = 0$
$x = -3 \qquad x = 2$

ii) $A = \int_0^2 (12-x) - (x^2+6)\,dx$

$A = \int_0^2 (6 - x - x^2)\,dx$

$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$

$= \left(12 - 2 - \frac{8}{3}\right) - 0$

$= 7\frac{1}{3}$ unit^2

b)

x	0	1	2	3	4
y	$\sqrt{5}$	$\sqrt{6}$	3	$\sqrt{14}$	$\sqrt{21}$
	F	y_1	y_2	y_3	L

$\int_0^4 \sqrt{5+x^2}\,dx = \frac{1}{3}\left[\sqrt{5} + \sqrt{21} + 4(\sqrt{6} + \sqrt{14}) + 2.3\right]$

$= 12.53$

c) $\log_{11} 57 = \frac{\ln 57}{\ln 11}$
$= 1.69$

d) i) $\ln\left(\frac{\sqrt{x-1}}{x^2+1}\right) = \ln(\sqrt{x-1}) - \ln(x^2+1)$
$= \frac{1}{2}\ln(x-1) - \ln(x^2+1)$

ii) $\frac{d}{dx}\left(\frac{\sqrt{x-1}}{x^2+1}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x-1) - \ln(x^2+1)\right)$

$= \frac{1}{2}\frac{1}{(x-1)} - \frac{2x}{x^2+1}$

$= \frac{1}{2(x-1)} - \frac{2x}{x^2+1}$

QUESTION 7

a) $P = 3000e^{kt}$

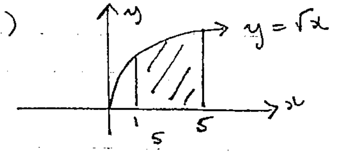
i) $t = 0 \quad P = 3000$

ii) $t = 1 \quad P = 3600$
$36\cancel{00} = 30\cancel{00}e^{k}$
$\frac{36}{30} = e^{k}$
$\ln\frac{6}{5} = k$
$k = 0.18$ (2 dec pl)

iii) $6\cancel{000} = 3\cancel{000}e^{.18t}$
$2 = e^{.18t}$
$\ln 2 = .18t$
$\therefore t = \frac{\ln 2}{.18}$
$t = 3.85 \quad \therefore t = 4$ days

iv) $\frac{dP}{dt} = k.3000e^{kt}$
$t = 2$
$\frac{dP}{dt} = .18 \times 3000 \times e^{.18\times 2}$
$\doteqdot 774$ insects / day

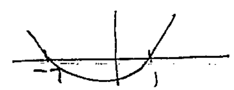
b)



$V_x = \pi\int_1^5 (\sqrt{x})^2\,dx$
$= \pi\int_1^5 x\,dx$
$= \pi\left[\frac{x^2}{2}\right]_1^5$
$= \frac{\pi}{2}[25 - 1]$
$= 12\pi$ unit^3

c) i) $\Delta = 0$
$(3+k)^2 - 4.2.2 = 0$
$9 + 6k + k^2 - 16 = 0$
$k^2 + 6k - 7 = 0$
$(k+7)(k-1) = 0$
$k = -7 \qquad k = 1$

ii) $\Delta > 0$



$k > 1, \quad k < -7$

QUESTION 8

a) i) $\frac{d}{dx}(\sin(1-2x^3)) =$
$-6x^2\cos(1-2x^3)$

ii) $\frac{d}{dx}(\tan 3x)$
$= 3\sec^2 3x$

iii) $\frac{d}{dx}(\cos x)^2$
$= -2.\sin x.\cos x$

b) i) $A = 2\int_0^1 2\cos\frac{\pi}{2}x\,dx$
$= 2\left[\frac{2.2}{\pi}\sin\frac{\pi x}{2}\right]_0^1$

$= \frac{8}{\pi}\left[\sin \frac{\pi}{2} x\right]_0^1$

$= \frac{8}{\pi}\left[\sin \frac{\pi}{2} - \cancel{\sin 0}^{\,0}\right]$

$= \frac{8}{\pi}$ unit^2

ii) α) use $(x-h)^2 = -4a(y-k)$

vertex $(0,2)$

$x^2 = -4a(y-2)$

sub pt $(1,0)$

$1 = -4a . -2$

$1 = 8a$

$a = 1/8$

eqn parab: $x^2 = -\frac{1}{2}(y-2)$

β) $2x^2 = -y + 2$

$y = 2 - 2x^2$

$A = 2\int_0^1 (2 - 2x^2)\,dx$

$= 2\left[2x - \frac{2x^3}{3}\right]_0^1$

$= 2\left[2 - \frac{2}{3}\right]$

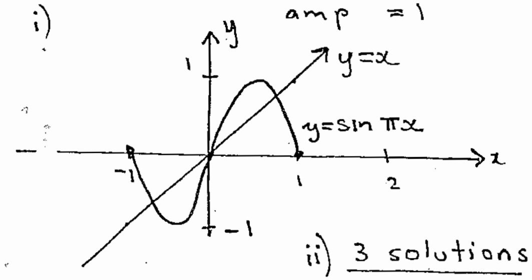
$= \frac{8}{3}$ unit^2

QUESTION 9

a) $y = \sin \pi x$ period $= \frac{2\pi}{\pi} = 2$

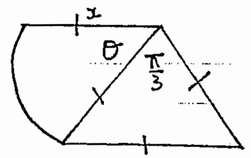
amp $= 1$

i)



ii) 3 solutions

b)



i) $P = 3x + x\theta$

ii) $A = \frac{1}{2}x^2\theta + \frac{1}{2}x^2 \sin\frac{\pi}{3}$

$= \frac{1}{2}x^2\theta + \frac{1}{2}x^2 \frac{\sqrt{3}}{2}$

$= \frac{1}{2}x^2\theta + \frac{\sqrt{3}}{4}x^2$

c) i) $\frac{d}{dx}(\sec x) = \frac{d}{dx}(\cos x)^{-1}$

$= -1 \cdot -\sin x\,(\cos x)^{-2}$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$

$= \tan x . \sec x$

ii) $m_T = \tan\frac{\pi}{4} . \sec\frac{\pi}{4}$

$m_T = 1 . \sqrt{2}$

$m_N = -\frac{1}{\sqrt{2}}$

at $x = \frac{\pi}{4}$ $y = \sec\frac{\pi}{4}$

$y = \sqrt{2}$

pt $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $m_N = -\frac{1}{\sqrt{2}}$

∴ normal $y - \sqrt{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$

$\sqrt{2}y - 2 = -x + \frac{\pi}{4}$

$x + \sqrt{2}y - 2 - \frac{\pi}{4} = 0$

QUESTION 10

a) $x = 1 \Rightarrow y = 0$ $x = 5 \Rightarrow y = \ln 5$

$V = \pi \int_0^{\ln 5} (e^y)^2\,dy$ | $y = \log_e x$, $e^y = x$

$V = \pi\left[\frac{e^{2y}}{2}\right]_0^{\ln 5}$

$V = \frac{\pi}{2}\left[e^{2\ln 5} - e^0\right]$

$= \frac{\pi}{2}\left[e^{\ln 25} - 1\right]$

$= \frac{24\pi}{2}$

$= 12\pi$ unit^3

b) i) $\left(\frac{a+b}{3}\right)^2 = \frac{a^2 + 2ab + b^2}{9}$

$= \frac{2ab + 7ab}{9}$

$= ab$

LH = RHS

ii) $\log\left(\frac{a+b}{3}\right)^2 = \log ab$

$2\log\left(\frac{a+b}{3}\right) = \log ab$

$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log ab$

$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$

$\therefore \log\left(\frac{a+b}{3}\right) - \frac{1}{2}\left[\log a + \log b\right] = 0$

c) i) $AC = \sqrt{x^2 + 81}$ pyth. th.

$D = S . T \therefore T = \frac{D}{S}$

$\therefore T_{AC} = \frac{\sqrt{x^2+81}}{5}$

ii) $T_{BC} = \frac{15 - x}{13}$

∴ Total Time

$T = \frac{\sqrt{x^2+81}}{5} + \frac{15-x}{13}$

iii) $\frac{dT}{dx} = \frac{1}{2} . \frac{2x}{5}(x^2+81)^{-\frac{1}{2}} - \frac{1}{13}$

$\frac{dT}{dx} = \frac{x}{5\sqrt{x^2+81}} - \frac{1}{13}$

st pt $T' = 0$

$\frac{x}{5\sqrt{x^2+81}} = \frac{1}{13}$

$13x = 5\sqrt{x^2+81}$

$169x^2 = 25(x^2+81)$

$169x^2 = 25x^2 + 2025$

$144x^2 = 2025$

$x^2 = \frac{2025}{144}$

$x = \pm\frac{45}{12}$

$x > 0$

test max/min for $x = 3.75$

x	3	$3\frac{3}{4}$	4
T'	−	0	+

∴ min Time if $x = 3.75$