Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

Assessment 1

March 2010

TIME ALLOWED: **70 minutes**

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- START ALL QUESTIONS ON A NEW PAGE
- Approved calculators may be used.

(FOR MARKERS USE ONLY)

1	2	3	TOTAL
/17	/17	/16	/50

QUESTION 1: (17 Marks)

Marks

5 (a) If $z = 1 - \sqrt{3}\,i$, find

(i) $\bar{z}$ (ii) $|z|$ (iii) $arg\ z$ (iv) $arg\ iz$ (v) $\frac{1}{z}$ (in simplest form)

(b) Given the ellipse $9x^2 + 16y^2 = 144$, find

1 (i) the length of the major axis

1 (ii) the eccentricity

1 (iii) the co-ordinates of the foci

1 (iv) the equations of the directrices

1 (v) the slope of the tangent at the point P $(3, \frac{3\sqrt{7}}{4})$

1 (vi) the equation of the normal at P $(3, \frac{3\sqrt{7}}{4})$
(DO NOT SIMPLIFY THIS)

2 (c) (i) Sketch the region where the inequalities

$$|z - 2| \leq |z - 2i| \quad \text{and} \quad |z - 1 - 2i| \leq 1$$

hold simultaneously.

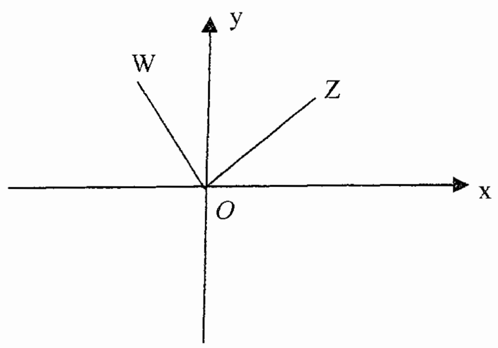
4 (ii) P is a point on the boundary of the region in part (i) above, and is represented by the complex number z, where $arg\, z = \frac{\pi}{4}$.

Find the 2 possibilities for z (in the form $a+ib$).

QUESTION 2: (17 Marks)

Marks

(a) The point Z, represents the complex number $z = 2 + 3i$



The line OZ is rotated anticlockwise by $\frac{\pi}{2}$ radians to form the line OW.

2 (i) Find the complex number w, represented by the point W.

2 (ii) Give the exact value of $\arg(\frac{z}{w})$

3 (b) For any point Z, representing the complex number z, you are given that

$$\arg(z - 1) - \arg(z - i) = 0$$

On an Argand Diagram, draw the locus of the point Z.

5 (c) (i) Prove De Moivre's Theorem by the process of Mathematical Induction. [NOTE: De Moivre's Theorem states that $(r cis\ \theta)^n = r^n cis\ n\theta$]

2 (ii) Express both $1 + i$ and $1 - i$ in the form $r cis\ \theta$

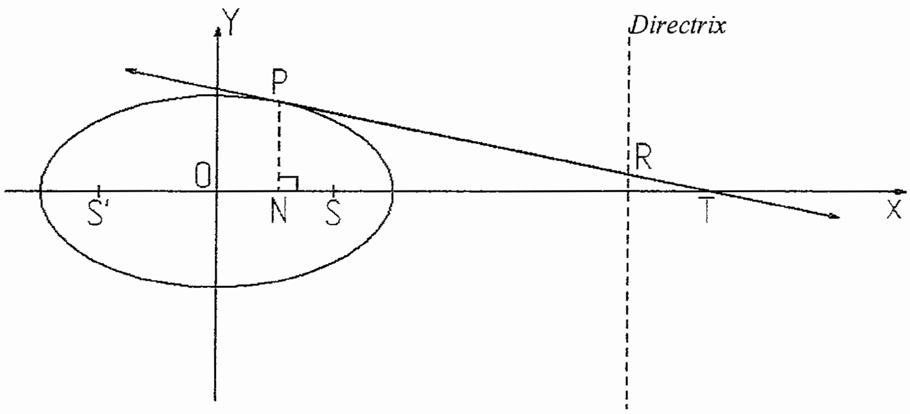
3 (iii) Using De Moivre's Theorem, or otherwise, and your answers to part (b) above, find, as a whole number, the value of

$$(1 + i)^9 + (1 - i)^9$$

QUESTION 3: (16 Marks)

Marks

(a) P (x_1, y_1) is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The tangent at P cuts the major axis of the ellipse at T and the Directrix at R, while N is the foot of the perpendicular from P to the x-axis.

O is the centre of the ellipse, while S and S' are the foci.

4 (i) Show that the equation of the tangent at P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
(Show all working)

1 (ii) Find the co-ordinates of the point T.

1 (iii) Show that ON.OT = a^2

QUESTION 3 continues overleaf....

.......QUESTION 3 continued

(b) You are given the curve $y^2 = x^2(4 - x^2)$,

1 (i) Find the points where this curve cuts the x-axis.

3 (ii) Use implicit differentiation, or otherwise, to show that $\frac{dy}{dx} = \frac{4x - 2x^3}{y}$

(iii) By taking the positive square root of the curve only, the curve becomes

$$y = x\sqrt{(4 - x^2)}$$

1 Show that, in this instance, $\frac{dy}{dx} = \frac{2(2 - x^2)}{\sqrt{4 - x^2}}$

3 (iv) Hence find the co-ordinates of the turning points on the new curve $y = x\sqrt{(4 - x^2)}$ and identify their nature.

DO NOT ATTEMPT TO FIND THE SECOND DERIVATIVE

2 (v) Hence neatly sketch the original curve $y^2 = x^2(4 - x^2)$, showing all features found in the parts above.

SOLUTIONS

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QUESTION 1:

(a) $z = 1 - \sqrt{3}i$

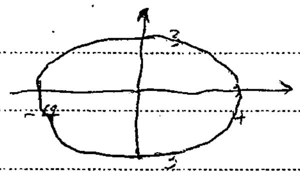
(i) $\bar{z} = 1 + \sqrt{3}i$ (ii) $|z| = \sqrt{1+3}$ $= 2$

(iii) $\arg z = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$ $= -\pi/3$ (iv) $-\pi/3 + \pi/2$ $= \pi/6$

(v) $\frac{1}{z} = \frac{1+\sqrt{3}i}{4}$

1 MARK EACH

(b) $x^2/16 + y^2/9 = 1$



(i) MAJOR AXIS is 8

(1) MARK each part

(ii) $b^2 = a^2(1-e^2)$

$9 = 16(1-e^2)$

$e^2 = 1 - 9/16 \Rightarrow e = \sqrt{7}/4$

(iii) Foci are $(\pm\sqrt{7}, 0)$

(iv) Directrices are $x = \pm\frac{16\sqrt{7}}{7}$ (OR $16/\sqrt{7}$)

(v) $2x/16 + 2y/9 \frac{dy}{dx} = 0$

$\therefore \; dy/dx = -\frac{2x}{16} \times \frac{9}{2y}$

$= -9x/16y$

At $x = 3$, $y = \frac{3\sqrt{7}}{4}$, $m_T = -\frac{9\sqrt{7}}{28}$

(vi) $m_N = 28/9\sqrt{7}$

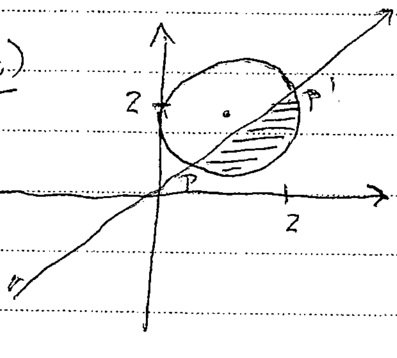
$y - 3\sqrt{7}/4 = 28/9\sqrt{7}\,(x-3)$

$9y\sqrt{7} - \frac{27}{4} \times 7 = 28x - 84$

$112x - 36y\sqrt{7} - 147 = 0$

any of these forms for a mark

(c)



(1) MARK for circle

(1) MARK for the line

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(ii) $y = x \quad (1)$ $(x-1)^2 + (y-2)^2 = 1 \quad (2)$	(1) for these 2 equations
$x^2 - 2x + 1 + x^2 - 4x + 4 = 1$ $\therefore \quad 2x^2 - 6x + 4 = 0$ $x^2 - 3x + 2 = 0$ $(x-2)(x-1) = 0$ $x = 2$ OR $x = 1$ $y = 2$ OR $y = 1$	(1) mark for solving simultaneously
$\therefore$ P can be $(2,2)$ OR $(1,1)$ and z can be $2+2i$ OR $1+i$	(1) mark each answer $= 2$

SOLUTIONS

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MARKS and COMMENTS

QUESTION 2:

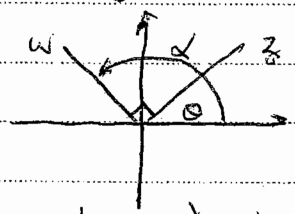
(a) (i) $w = iz$

$= \begin{cases} 2i - 3 \\ -3 + 2i \end{cases}$

1 for this
1 for either
2

(ii) $\arg\left(\frac{z}{w}\right) = \arg \overset{\theta}{z} - \arg \overset{\alpha}{w}$

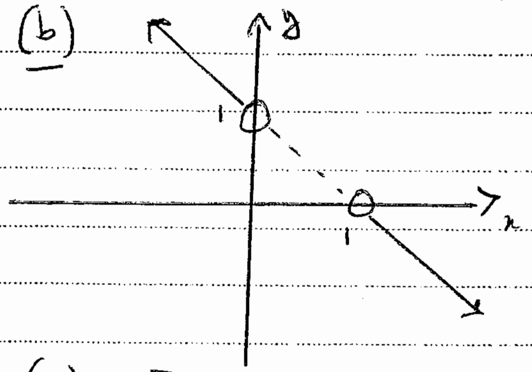
$= -\frac{\pi}{2}$



2 MARKS

OR by rationalising the fraction and finding the answer was purely negative complex

(b)



1 MARK for the line
1 " for gap in middle
1 " for open circles

(c) For $n = 1$, $r \operatorname{cis} \theta = r \operatorname{cis} \theta$ TRIVIAL

(i) For $n = 2$ $(r \operatorname{cis} \theta)^2 = r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$\therefore$ true for $n = 1, 2$

① for testing 1,2

Assume the formula is true for $n = k$

$\therefore (r \operatorname{cis} \theta)^k = r^k \operatorname{cis} k\theta$

For $n = k+1$

$(r \operatorname{cis} \theta)^{k+1} = (r \operatorname{cis} \theta)^k (r \operatorname{cis} \theta)$ ← ①

$= r^k \operatorname{cis} k\theta \cdot r \operatorname{cis} \theta$ ①

$= r^{k+1}[\cos k\theta \cos\theta - \sin\theta \sin k\theta + i(\cos\theta \sin k\theta + \cos k\theta \sin\theta$

$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$ ← ①

$= r^{k+1} \operatorname{cis}(k+1)\theta$

$\therefore$ If the formula is true for $n = k$, it is true for $n = k+1$

BUT it is true for $n = 2$, so it is true for $n = 3$

etc.

① for acceptable conclusion

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QUESTION 2 CONT...

(ii) $1+i = \sqrt{2}\,\text{cis}\,\pi/4$

$1-i = \sqrt{2}\,\text{cis}\,(-\pi/4)$ OR $\sqrt{2}\,\text{cis}\,(7\pi/4)$ } 1 each part

(iii) $(1+i)^9 = (\sqrt{2})^9\,\text{cis}\,9\pi/4$

$(1-i)^9 = (\sqrt{2})^9\,\text{cis}\,(-9\pi/4)$ } 1

$\therefore (1+i)^9 + (1-i)^9 = (\sqrt{2})^9\left[\cos\pi/4 + i\sin\pi/4 + \cos(-\pi/4) + i\sin(-\pi/4)\right]$

$= (\sqrt{2})^9\left[2\cos\pi/4\right]$ ← 1

$= 16\sqrt{2} \times 2/\sqrt{2}$

$= 32$ ← 1

SOLUTIONS

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QUESTION 3:

(a) (i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$= -\frac{xb^2}{a^2y}$$ ① for this

At (x_1, y_1) $m_T = -\frac{x_1 b^2}{y_1 a^2}$ ① " "

Tangent is

$$y - y_1 = -\frac{x_1 b^2}{y_1 a^2}(x - x_1)$$

$$yy_1a^2 - y_1^2a^2 = -xx_1b^2 + x_1^2b^2$$ ①

$$\therefore yy_1a^2 + xx_1b^2 = x_1^2b^2 + y_1^2a^2$$

DIVIDE by a^2b^2

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$ ①

$$= 1$$

(ii) T is $\left(\frac{a^2}{x_1}, 0\right)$ 1 MARK

(iii) $ON.OT = x_1 . \frac{a^2}{x_1}$ 1 MARK

$$= a^2$$

(b) (i) $x = 2$ or $x = -2$ 1 MARK

(ii) $2y\frac{dy}{dx} = (4 - x^2)2x + x^2(-2x)$ 2 MARKS

$$= 8x - 4x^3$$

$$\frac{dy}{dx} = \frac{8x - 4x^3}{2y}$$ 1 for simplifying

$$= \frac{4x - 2x^3}{y}$$

(iii) Using $y = x\sqrt{4 - x^2}$

$$\frac{dy}{dx} = \frac{4x - 2x^3}{x\sqrt{4 - x^2}}$$ 1 MARK

$$= \frac{4 - 2x^2}{\sqrt{4 - x^2}}$$

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Q 3 CONT....

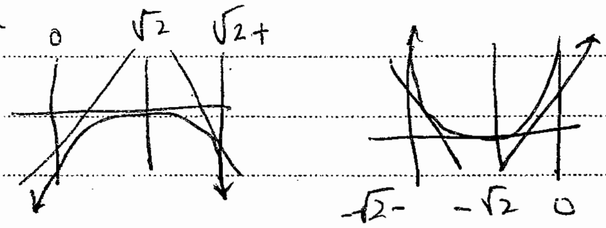
(iv) T.P.'s at $\frac{dy}{dx} = 0$

$\therefore 2(2 - x^2) = 0$

$\therefore x = \sqrt{2}, y = 2$ OR $x = -\sqrt{2}, y = -2$

① MARK for each part = 2.

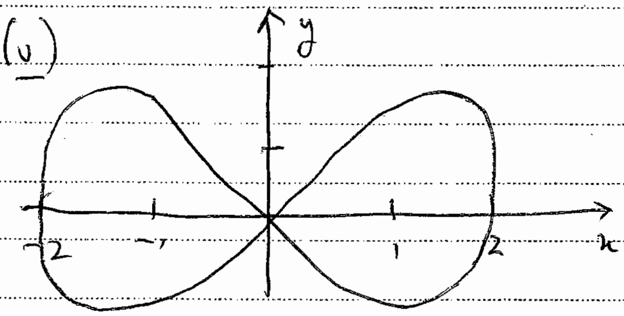
TESTING $\frac{dy}{dx}$



① MARK.

$\therefore$ MAX at $(\sqrt{2}, 2)$ MIN $(-\sqrt{2}, -2)$

(v)



② MARKS

Only ① if only half a graph.