Name: .... Maths Teacher: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 2

TRIAL HSC

2016

*Time allowed: 3 hours **plus** 5 minutes reading time*

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Marks for each question are indicated on the question.
- Board - Approved calculators may be used
- In Questions 11- 16, show relevant mathematical reasoning and/or calculations.
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black pen
- All answers are to be in the writing booklet provided
- A BOSTES reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Total marks – 100

Section 1 Multiple Choice

10 Marks

- Attempt Questions 1-10
- Allow 15 minutes for this section

Section II

90 Marks

- Attempt Questions 11-16
- Allow 2 hours 45 minutes for this section

Section I

10 marks

Attempt Questions 1- 10

Allow about 15 minutes for this section

Use the multiple- choice answer sheet located in your answer booklet for Questions 1 -10

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1. Which conic has eccentricity $\dfrac{\sqrt{3}}{3}$?

(A) $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$

(B) $\dfrac{x^2}{3^2}+\dfrac{y^2}{2^2}=1$

(C) $\dfrac{x^2}{3}-\dfrac{y^2}{2}=1$

(D) $\dfrac{x^2}{3^2}-\dfrac{y^2}{2^2}=1$

2. What value of z satisfies; $z^2=20i-21$?

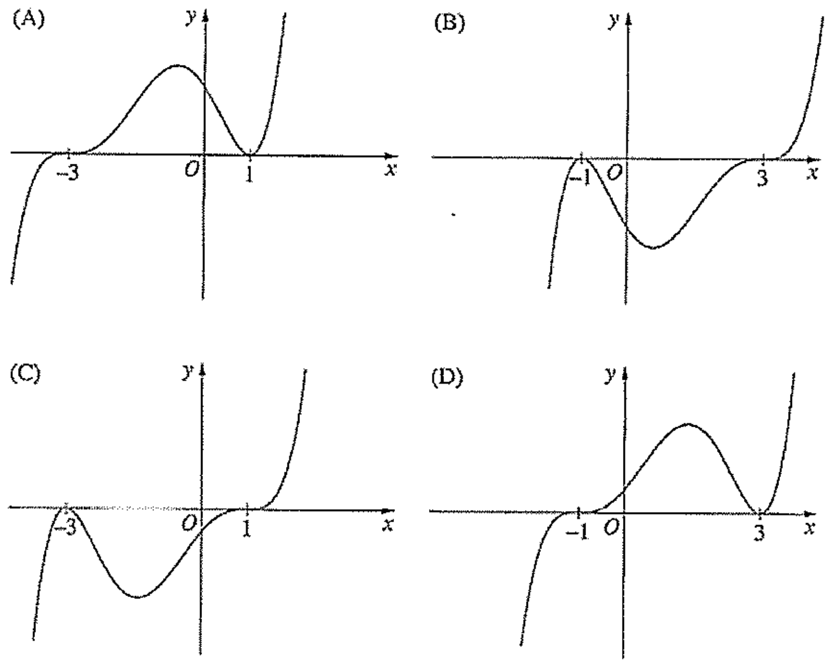
(A) $-2+5i$

(B) $2-5i$

(C) $2+5i$

(D) $5-2i$

3. Which graph represents the curve, $y=(x+3)^2(x-1)^3$?



4. The polynomial $2x^4-17x^3+45x^2-27x-27$ has a triple root at $x=\alpha$.

What is the value of α ?

(A) $-\frac{1}{2}$

(B) $\frac{1}{2}$

(C) -3

(D) 3

5. If $z_1=1+2i$ and $z_2=3-i$ then $z_1\div\overline{z_2}$ is,

(A) $\frac{1}{2}-\frac{1}{2}i$

(B) $\frac{1}{2}+\frac{1}{2}i$

(C) $4+3i$

(D) $\frac{5}{8}+\frac{5}{8}i$

6. Which expression is equal to, $\int\frac{x^2}{\cos^2 x}dx$?

(A) $2x\tan x-2\int\tan xdx$

(B) $\frac{1}{3}(x^3\sec^2 x-\int x^3\tan xdx)$

(C) $x^2\tan\ x-2\int x\tan xdx$

(D) $x^2\tan x-2\int x\sec^2 xdx$

7. What is the natural domain of the function $f(x)=\frac{1}{2}(x\sqrt{x^2-1}-ln\left(x+\sqrt{x^2-1}\right))$?

(A) $x\leq-1$ or $x\geq1$

(B) $-1\leq x\leq1$

(C) $x\geq1$

(D) $x\leq-1$

8. If α, β, δ are the roots of $x^3 + x - 1 = 0$, then an equation with roots

$$\frac{(\alpha+1)}{2}, \frac{(\beta+1)}{2}, \frac{(\delta+1)}{2}$$ is?

(A) $x^3 - 3x^2 + 4x - 3 = 0$

(B) $x^3 + 3x^2 + 4x + 1 = 0$

(C) $x^3 - 6x^2 + 16x - 24 = 0$

(D) $8x^3 - 12x^2 + 8x - 3 = 0$

9. The complex number Z satisfies $|Z+2| = 1$

What is the smallest positive value of the $arg(z)$ on the Argand diagram?

(A) $\frac{\pi}{3}$

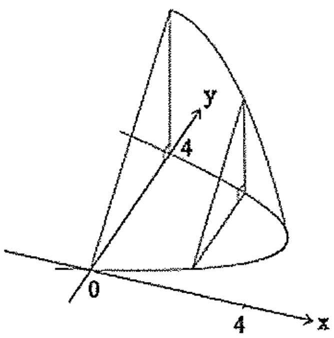
(B) $\frac{5\pi}{6}$

(C) $\frac{2\pi}{3}$

(D) $\frac{\pi}{6}$

10. The base of a solid is the region bounded by the parabola $x = 4y - y^2$ and the y axis.

Vertical cross sections are right angled isosceles triangles perpendicular to the $x-axis$ as shown.



Which integral represents the volume of this solid?

(A) $\int_0^4 2\sqrt{4-x}\,dx$

(B) $\int_0^4 \pi(4-x)\,dx$

(C) $\int_0^4 (8-2x)\,dx$

(D) $\int_0^4 (16-4x)\,dx$

Question 11 **(15 marks)**

(a) Express $\frac{18+4i}{3-i}$ in the form, $x+iy$, where x and y are real. **2**

(b) Consider the complex numbers $z=-1+\sqrt{3}i$ and $w=\sqrt{2}\left(cos\left(\frac{-\pi}{4}\right)+isin\left(\frac{-\pi}{4}\right)\right)$

(I) Evaluate $|z|$ **1**

(II) Evaluate $arg(z)$ **1**

(III) Find the argument of $\frac{w}{z}$ **2**

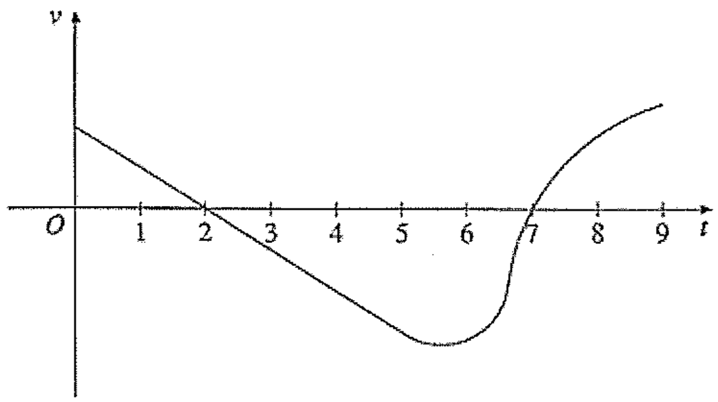
(c) (i) Find A, B and C such that **3**

$$\frac{1}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

(ii) Hence, or otherwise, find;

$$\int\frac{dx}{x(x^2+4)}$$ **2**

(d)



A particle moves along the $x-$ axis. At time, $t=0$, the particle is at $x=0$.

Its velocity v at time t is shown on the graph above.

Copy or trace this graph onto your answer page.

(i) At what time is the acceleration the greatest? Explain your answer. **1**

(ii) At what time does the particle first return to $x=0$? Explain your answer. **1**

(iii) Sketch the displacement time graph for the particle in the interval, $0\leq t\leq 9$. **2**

Question 12 **(15 marks)** **START THIS QUESTION ON A NEW PAGE.**

(a) Find $\int x\sqrt{x+1}\,dx$ **2**

(b) Evaluate

(i) $$\int_0^{\frac{\pi}{4}} \sin x \cos 2x \mathrm{d}x$$ **2**

(ii) $$\int_1^e \frac{\ln x}{x^2}\,dx$$ **2**

(c) Find the equation of the normal to the curve, $3x^2y^3 + 4xy^2 = 6 + y$ at the point (1,1). **4**

(d)

(i) Prove that,

$$\cos(A-B)x - \cos(A+B)x = 2\sin Ax \sin Bx$$ **1**

(ii) Using the above result, express the equation $\sin 3x \sin x = 2\cos 2x + 1$,

as a quadratic equation in terms of $\cos 2x$ **2**

(iii) Hence, solve, $\sin 3x \sin x = 2\cos 2x + 1$ for $0 \le x \le 2\pi$ **2**

Question 13 **(15 marks)** **START THIS QUESTION ON A NEW PAGE.**

(a) The function $y = f(x)$ is defined by the equation;

$$f(x) = \frac{x(x-4)}{4}$$

Without the use of calculus, draw sketches of each of the following, clearly labelling any Intercepts, asymptotes and turning points.

(i) $y = f(x)$ **1**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \dfrac{x|x-4|}{4}$ **2**

(iv) $y = \tan^{-1} f(x)$ **2**

(v) $y = e^{f(x)}$ **2**

(b) Sketch the locus of z satisfying

(i) $Re(z) = |z|$ **2**

(ii) $Im(z) \geq 2$ and $|z-1| \leq 2$ **2**

(c) Write down the domain and range of $y = 2\sin^{-1}\sqrt{1-x^2}$ **2**

Question 14 **(15 marks)** **START THIS QUESTION ON A NEW PAGE.**

(a) Use the substitution $t=\tan\frac{x}{2}$ to find **4**

$$\int_0^{\pi/2} \frac{1}{5+4\cos x+3\sin x}\,dx$$

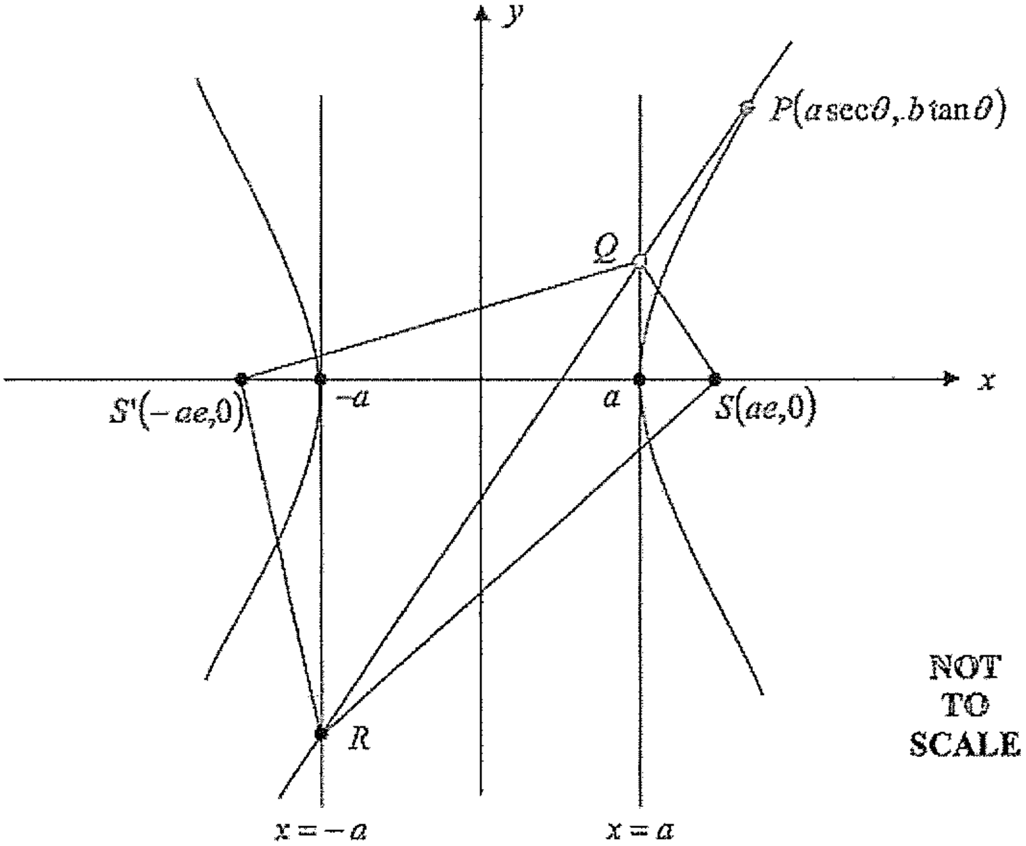
(b) The area enclosed by the curves $y=\sqrt{x}$ and $y=x^2$ is rotated about the $y-$ axis.

Use the method of ***cylindrical shells*** to find the volume of the solid formed. **4**

Question 14 continues on the next page….

Question 14 continued....

(c)



$P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

The tangent at P meets the line $x=-a$ and $x=a$ at R and Q respectively.

(i) Show that the equation of the tangent is given by $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$. **2**

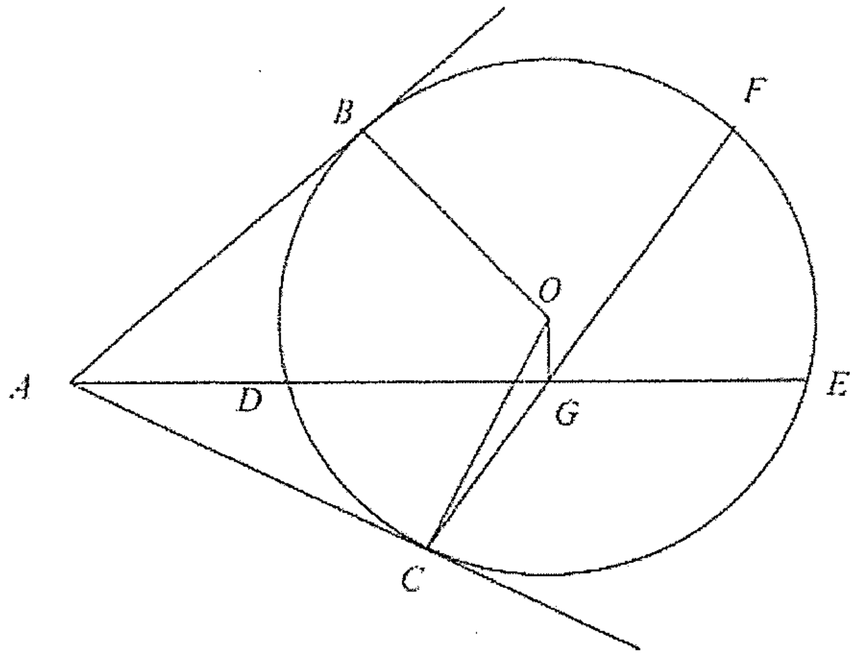
(ii) Find the coordinates of Q and R. **1**

(iii) Show that QR subtends a right angle at the focus $S\left(ae,0\right)$. **2**

(iv) Deduce that Q,S,R,S' are concyclic. **2**

Question 15 (15 marks) **START THIS QUESTION ON A NEW PAGE.**

(a) In the diagram, AB and AC are tangents from A to the circle with centre O , meeting the circle at B and C respectively. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.



(i) Copy the diagram, using about one third of the page, into your answer booklet and prove that $ABOC$ and $AOGC$ are cyclic quadrilaterals **3**

(ii) Explain why $\angle OGF = \angle OAC$. **1**

(iii) Prove that $BF \parallel AE$ **3**

(b)

(i) Let $I_n = \int_0^1 x^n \sqrt{1-x^3}\,dx$ for $n \geq 2$.

Show that: $I_n = \dfrac{2n-4}{2n+5}.I_{n-3}$ for $n \geq 5$ **3**

(ii) Hence find I_8 **2**

(c) A sequence of numbers is given by $T_1 = 6$ $T_2 = 27$ and $T_n = 6T_{n-1} - 9T_{n-2}$ for $n \geq 3$.

Prove by Mathematical Induction that:

$$T_n = (n+1) \times 3^n \text{ for } n \geq 1$$ **3**

Question 16 (15 marks) **START THIS QUESTION ON A NEW PAGE.**

(a) Show that the minimum value of $ae^{mx} + be^{-mx}$ is $2\sqrt{ab}$

if a, b and m are all positive constants. **4**

(b) A particle of mass 1 kilogram is projected upwards under gravity (g) with a speed of $2k$ in a medium in which resistance to motion is $\frac{g}{k^2}$ times the square of the speed, where k is a positive constant.

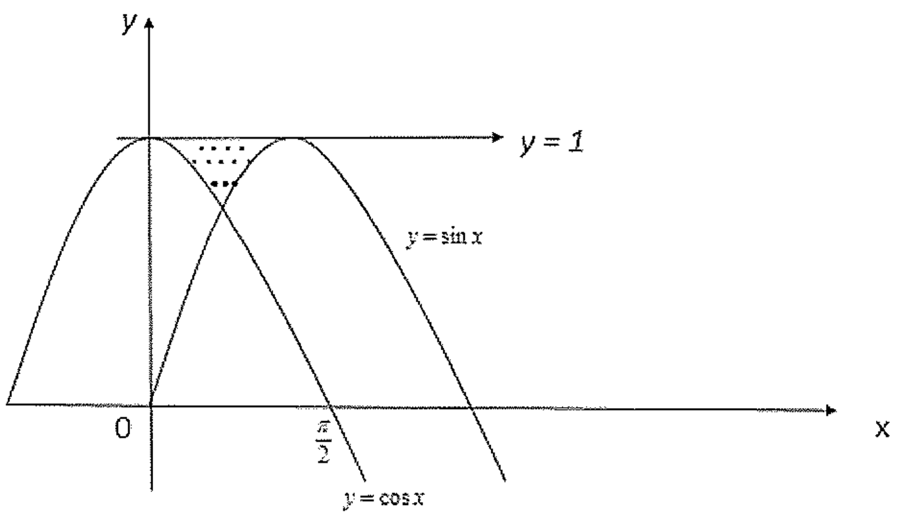
(i) Show that the maximum height (H) reached by the particle is

$$H = \frac{k^2}{2g}\ln 5$$ **3**

(ii) Show that the speed with which the particle returns to its starting point is given by $V = \frac{2k}{\sqrt{5}}$ **4**

(c) The shaded region in the diagram is bounded by the curves $y = \sin x$, $y = \cos x$ and the line $y = 1$.

This region is rotated around the $y-$ axis.



Calculate the volume of the solid formed, using the process of ***Volume by Slicing***. **4**

END OF ASSESSMENT TASK

STHS - Ext 2 Trial - Suggestion Solution

Section 1

1. A 2. C 3. C 4. D 5. B

6. C * (all given) 7. C 8. D 9. B 10. C

Section 2

Question 11

a) $\frac{18+4i}{3-i} \times \frac{3+i}{3+i}$

$= \frac{54+18i+12i-4}{10}$

$= \frac{50+30i}{10}$

$= 5+3i$

b) $\omega = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ $z = -1+\sqrt{3}i$

i) $|z| = \sqrt{1+3}$

$= 2$

ii) $\arg(z) = \frac{2\pi}{3}$

iii) $\arg\left(\frac{\omega}{z}\right) = \arg(\omega) - \arg(z)$

$= -\frac{\pi}{4} - \frac{2\pi}{3}$

$= -\frac{11\pi}{12}$.

c) $\frac{1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$\therefore 1 = A(x^2+4) + x(Bx+c)$

let $x = 0$

$1 = A(4) \longrightarrow A = \frac{1}{4}$

equating: $0 = A + B$

$B = -\frac{1}{4}$

$0 = c$

ie $A = \frac{1}{4}$ $B = \frac{-1}{4}$ $C = 0$

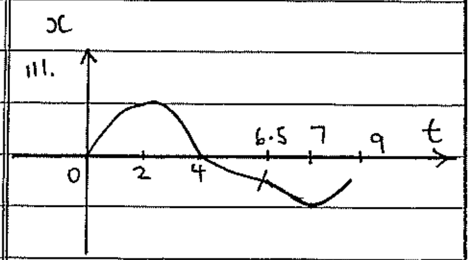
ii. Now $\int \frac{dx}{x(x^2+4)} = \int \frac{1}{4x} - \frac{\frac{1}{4}x}{x^2+4}\,dx$

$= \frac{1}{4}\ln x - \frac{1}{8}\ln(x^2+4) + C$

d) i. $t = 6.5$ (point of inflection on vel. curve is greatest acc)

ii. When the area above the t-axis equals area below $\therefore$ at $t = 4$

iii.



Question 12

a) $\int x\sqrt{x+1}\,dx$

one method:

let $u = x+1$

$\frac{du}{dx} = 1$ $\therefore du = dx$

$= \int (u-1)\sqrt{u}\,du$

$= \int u\sqrt{u} - \sqrt{u}\,du$

$= \int u^{3/2} - u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

ii. $\int_1^e \frac{\ln x}{x^2}\,dx$

$= \int_1^e x^{-2}\ln x\,dx$

$= \ln x \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x}\cdot\frac{-1}{x}\,dx$

$= -\left.\frac{\ln x}{x}\right]_1^e + \int_1^e x^{-2}\,dx$

$= -\left[\frac{1}{e} - 0\right] + \left[-\frac{1}{x}\right]_1^e$

$= -\frac{1}{e} + \left[-\frac{1}{e} - (-1)\right]$

$= 1 - \frac{2}{e}$

b) i. $\int_0^{\pi/4} \sin x \cos 2x\,dx$

$= \int_0^{\pi/4} \sin x\,(2\cos^2 x - 1)\,dx$

$= \int_0^{\pi/4} 2\sin x(\cos x)^2 - \sin x\,dx$

$= \left[\frac{-2}{3}\cos^3 x + \cos x\right]_0^{\pi/4}$

$= \frac{-2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{\sqrt{2}} - \left[\frac{-2}{3}(1)^3 + 1\right]$

$= \frac{2}{3\sqrt{2}} - \frac{1}{3}$

c) $3x^2y^3 + 4xy^2 = 6 + y$ @$(1,1)$

$3x^2\left[3y^2\frac{dy}{dx}\right] + y^3.6x + 4x\,2y\frac{dy}{dx} + 4y^2 = \frac{dy}{dx}$

$6xy^3 + 4x^2 = \frac{dy}{dx}(1 - 9x^2y^2 - 8xy)$

at $(1,1)$

$\frac{dy}{dx} = \frac{10}{-16}$

$M_T = -\frac{5}{8}$ $\therefore M_N = \frac{8}{5}$

$y - 1 = \frac{8}{5}(x-1)$

$5y - 5 = 8x - 8$

$8x - 5y - 3 = 0$

Question 12 - con't.

d)

i. Prove that

$\cos(A-B)x - \cos(A+B)x = 2\sin Ax \sin Bx$

LHS $= \cos Ax \cos Bx + \sin Ax \sin Bx - [\cos Ax \cos Bx - \sin Ax \sin Bx]$

$= 2\sin Ax \sin Bx$

$=$ RHS.

ii. $\sin 3x \sin x = 2\cos 2x + 1$

$A=3$
$B=1$

$\therefore \left[\cos(3-1)x - \cos(3+1)x\right] \div 2 = \cos 2x + 1$

$\cos 2x - \cos 4x = 2\cos 2x + 2$

$\cos 2x - [2\cos^2 2x - 1] = 2\cos 2x + 2$

$\cos 2x - 2\cos^2 2x + 1 = 2\cos 2x + 2$

$2\cos^2 2x + 3\cos 2x + 1 = 0$

iii. hence,

$(2\cos 2x + 1)(\cos 2x + 1) = 0$

$\cos 2x = -1/2$ or $\cos 2x = -1$

$2x = 2\pi/3, 4\pi/3, 8\pi/3, 10\pi/3$ or $2x = \pi, 3\pi$

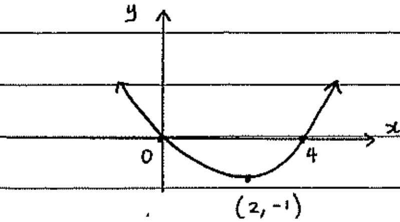
$x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$ $\quad x = \pi/2, 3\pi/2$

③

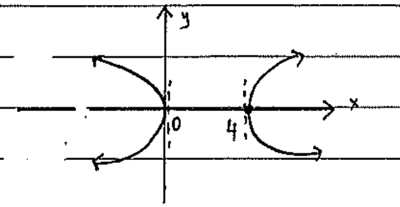
Question 13

a) $f(x) = \dfrac{x(x-4)}{4}$

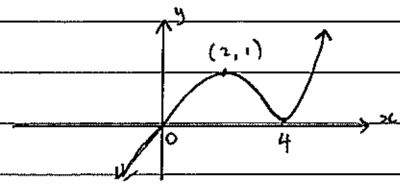
i.



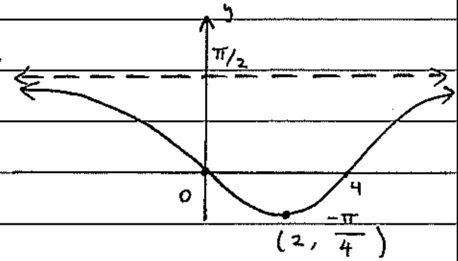
ii.



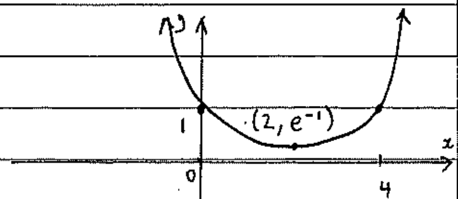
iii.



iv.



v.



b) $R(z) = |z|$

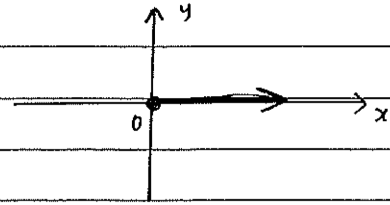
let $z = x + iy$

$x = \sqrt{x^2 + y^2}$ $\quad x \geq 0$

$x^2 = x^2 + y^2$

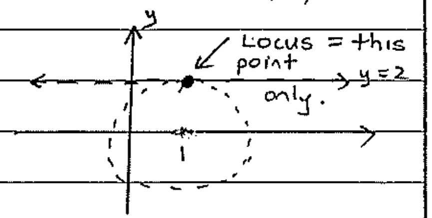
$y^2 = 0$

$y = 0$ but $x \geq 0$



ii. $Im(z) \geq 2$ $\quad |z-1| \leq 2$

$\therefore y \geq 2$ $\quad$ circle centre $(1, 0)$



c) $y = 2\sin^{-1}\sqrt{1-x^2}$

$y/2 = \sin^{-1}\sqrt{1-x^2}$

D: $-1 \leq x \leq 1$

R: $0 \leq y \leq \pi$

④

Question 14.

a)

$x = \pi/2 \quad t = \tan^{\pi}/4 = 1$

$x = 0 \quad t = \tan 0 = 0$

$\therefore \int_0^1 \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2\,dt}{t^2+1}$

$= \int_0^1 \frac{2\,dt}{5(1+t^2)+4(1-t^2)+6t}$

$= \int_0^1 \frac{2}{t^2+6t+9}\,dt$

$= 2\int_0^1 \frac{1}{(t+3)^2}\,dt$

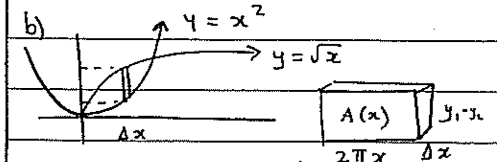
$= 2\int_0^1 (t+3)^{-2}\,dt$

$= \left.\frac{-2}{t+3}\right]_0^1$

$= \left(^{-2}/_4\right) - \left(^{-2}/_3\right)$

$= 1/6$

b)



$\Delta V = 2\pi x(\sqrt{x} - x^2)\,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_0^1 2\pi x(\sqrt{x} - x^2)\,\Delta x$

$= 2\pi \int_0^1 x^{3/2} - x^3\,dx$

$= \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 \times 2\pi$

$= 2\pi\left[2/5 - 1/4 - 0\right] = \frac{3\pi}{10}u^3$

c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$2x/a^2 - 2y/b^2 \cdot \frac{dy}{dx} = 0$

$dy/dx = \frac{x}{y}\,\frac{b^2}{a^2}$

$P(a\sec\theta, b\tan\theta)$

$m_T = b/a\sin\theta$

Now

$y - \frac{b\sin\theta}{\cos\theta} = \frac{b}{a\sin\theta}\left(x - \frac{a}{\cos\theta}\right)$

$a\sin\theta\, y - \frac{ab\sin^2\theta}{\cos\theta} = bx - \frac{ab}{\cos\theta}$

$(\div ab)$

$\frac{\sin\theta\, y}{b} - \frac{x}{a} = \frac{\sin^2\theta - 1}{\cos\theta}$

$\div -\cos\theta$

$-\tan\theta\,\frac{y}{b} + \frac{x}{a\cos\theta} = 1$

or

$\frac{x\sec\theta}{a} - \frac{\tan\theta}{b}y = 1$

ii. at Q $x = a$

$\frac{a\sec\theta}{a} - \frac{\tan\theta}{b}y = 1$

$1/\cos\theta - y/b\,\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$

$1 - y/b\sin\theta = \cos\theta$

$y = \frac{b(1-\cos\theta)}{\sin\theta}$

OR $y = b\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$Q\left[a, \frac{b(1-\cos\theta)}{\sin\theta}\right]$

(5)

at $R = -a$

$\frac{-a\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

$-1 - \frac{y}{b}\sin\theta = \cos\theta$

$y = \frac{-b - b\cos\theta}{\sin\theta}$

or $y = \frac{-b(1+\sec\theta)}{\tan\theta}$

iii) $S(ae, 0)$

$m_{SQ} = \frac{0 - \frac{b(1-\cos\theta)}{\sin\theta}}{ae - a}$

$= \frac{\frac{-b(1-\cos\theta)}{\sin\theta}}{a(e-1)}$

$m_{RS} = \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{a(e+1)}$

Now $m_{SQ} \times m_{RS}$

$= \frac{\frac{-b(1-\cos\theta)}{\sin\theta}}{a(e-1)} \times \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{a(e+1)}$

$= \frac{\frac{-b^2\cancel{(1-\cos^2\theta)}}{\cancel{\sin^2\theta}}}{a^2(e^2-1)}$

$= \frac{-b^2}{a^2(e^2-1)}$ but $a^2(e^2-1) = b^2$

$= -1 \therefore \angle QSR = 90°$

iv) $m_{QS'} = \frac{b(1-\cos\theta)}{a(e-1)\sin\theta}$

$m_{RS'} = \frac{b(1+\cos\theta)}{-a(e+1)\sin\theta}$

$\therefore m_{QS'} \times m_{RS'}$

$= \frac{b^2(1-\cos^2\theta)}{-a^2(e^2-1)\sin^2\theta}$

$= \frac{b^2}{-b^2}$

$= -1$

$\therefore \angle QS'R = 90°$

and $\angle QSR + \angle QS'R = 180°$

making $QSRS'$ a cyclic quad. as opposite angles are supplementary.

(6)

Question 15.

Join AO, BF BC

i. $\angle ABO = \angle OCA = 90^\circ$

(radii to tangent at point of contact is 90°)

$\therefore$ opposite angles in ABOC are supplementary and ABOC is a cyclic quadrilateral.

Now, $\angle ABO = 90^\circ$

(AO is a diameter OR Line from midpt to centre is perpendicular)

$\angle OGA = \angle OCA$ (angles at circumference of circle OGCA)

$= 90^\circ$

$\therefore$ AOGC is a cyclic quad as opposite angles are supplementary.

ii. $\angle OGF = \angle OAC$

exterior angle of a cyclic quadrilateral equals opposite interior angle (AOGC).

iii. let $\angle OGF = \angle OAC = \alpha$

$\therefore \angle FGE = 90^\circ - \alpha$ (straight line)

and

$\angle OBC = \angle OAC$ (angles in the same segment of ABOC)

$= \alpha$

as $BO = OC$ radii

$\angle OCB = \angle CBO$ (equal angles opposite equal sides)

$= \alpha$

Now in ΔBOC

$\angle BOC = 180 - 2\alpha$ (angle sum)

$\therefore \angle BFC = 90 - \alpha$

(angle at the circumference is half the angle at the centre on arc BC)

$\therefore \angle BFC = \angle FGE$ $(90 - \alpha)$

and the alternate angles are equal

$\therefore BF \parallel AE$

Question 15 con't

$I_n = \int_0^1 x^n \sqrt{1-x^3}\, dx \quad n \geq 2$

$= \int_0^1 x^{n-2} \cdot x^2 \sqrt{1-x^3}\, dx$

$= \left[x^{n-2} (1-x^3)^{3/2} \cdot \frac{-2}{9} \right]_0^1 - \int (n-2) x^{n-3} \cdot \frac{-2}{9} (1-x^3) \sqrt{1-x^3}\, dx$

$I_n = 0 + \frac{2(n-2)}{9} \int_0^1 x^{n-3} \sqrt{1-x^3}\, dx - \frac{2(n-2)}{9} \int_0^1 x^n \sqrt{1-x^3}\, dx$

$I_n \left[1 + \frac{2(n-2)}{9} \right] = \frac{2(n-2)}{9} \int_0^1 x^{n-3} \sqrt{1-x^3}\, dx$

$I_n \left[\frac{9 + 2n - 4}{9} \right] = \frac{2n-4}{9} I_{n-3}$

$I_n = \frac{2n-4}{2n+5} I_{n-3}$

ii)

$I_8 = \left(\frac{16-4}{16+5} \right) I_5$

$= \frac{12}{21} \left[\frac{(10-4)}{10+5} I_2 \right]$

$= \frac{12}{21} \cdot \frac{6}{15} \cdot \frac{2}{9}$

$= \frac{16}{315}$

Now $I_2 = \int_0^1 x^2 \sqrt{1-x^3}\, dx$

$= \frac{(1-x^3)^{3/2}}{3/2 \cdot -3}$

$= \left[\frac{-2}{9} (1-x^3)^{3/2} \right]_0^1$

$= \frac{-2}{9} \left[0 - 1^{3/2} \right]$

$= \frac{2}{9}$

Question 15 con't

c)

$T_1 = 6 \quad T_2 = 27$

$T_n = 6T_{n-1} - 9T_{n-2} \quad n \geq 3$

$T_n = (n+1)3^n$ for $n \geq 1$

Test $n = 1$

$T_1 = (1+1) \times 3^1$

$= 2 \times 3$

$= 6$ which is given

$\therefore$ True for $n = 1$

Assume true for $n = K$

ie $T_K = (K+1)3^K$

where

$T_K = 6T_{K-1} - 9T_{K-2}$

Prove true for $n = K+1$

aim to prove

$T_{K+1} = 6T_K - 9_{K-1} = (K+1+1).3^{K+1}$

$T_{K+1} = 6T_K - 9_{K-1}$

$= 6(K+1).3^K - 9\left[K(3^{K-1})\right]$ By assumption

$= 2(K+1) \times 3 \times 3^K - 3^2.K.3^{K-1}$

$= 2(K+1).3^{K+1} - K3^{K+1}$

$= (2K+2-K)3^{K+1}$

$= (K+2).3^{K+1}$

$= (K+1+1).3^{K+1}$

as required

(statement Required).

9

Question 16

let $P = ae^{mx} + be^{-mx}$

$\frac{dP}{dx} = mae^{mx} - mbe^{-mx} = 0$

$ae^{mx} = be^{-mx}$

$ae^{mx} = \frac{b}{e^{mx}}$

$e^{2mx} = b/a$

$2mx = \ln(b/a)$

$x = \frac{1}{2m}\ln\left(\frac{b}{a}\right)$

test

$d^2P/dx^2 = m^2ae^{mx} + m^2be^{-mx}$

at $x = \frac{1}{2m}\ln(b/a)$

$\frac{d^2P}{dx^2} > 0$ as $e^{-m}x > 0$

$e^{mx} > 0$

and $a, b, m > 0$

$\therefore$ min value is when

$x = \frac{1}{2m}\ln(b/a)$

$P = ae^{m(\frac{1}{2m}\ln b/a)} + be^{-m(\frac{1}{2m}\ln b/a)}$

$= ae^{\frac{1}{2}\ln b/a} + be^{-\frac{1}{2}\ln b/a}$

$= ae^{\ln\sqrt{b/a}} + be^{\ln\sqrt{a/b}}$

$= a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}}$

$= \sqrt{\frac{a^2b}{a}} + \sqrt{\frac{b^2a}{b}}$

$= \sqrt{ab} + \sqrt{ba} = 2\sqrt{ab}.$

b) +ve $\uparrow$ $\downarrow g$ $\downarrow R$ $\quad m = 1kg$

1. $m\ddot{x} = -mg - \frac{g}{K^2}v^2 \quad m = 1$

$\therefore \ddot{x} = -g - \frac{g}{K^2}v^2$

$v\frac{dv}{dx} = -g\left(\frac{K^2+v^2}{K^2}\right)$

$\frac{dv}{dx} = -g\left(\frac{K^2+v^2}{vK^2}\right)$

$\frac{dx}{dv} = \frac{-1}{g}\frac{vK^2}{K^2+v^2}$

$x = -\frac{K^2}{g}\int\frac{v}{K^2+v^2}\,dv$

$x = \frac{-K^2}{g}.\frac{1}{2}\ln(K^2+v^2) + C_1$

$x = 0$
$v = 2k$

$\therefore C_1 = K^2/2g \ln(5k^2)$

$\therefore x = \frac{-K^2}{2g}\ln(K^2+v^2) + \frac{K^2}{2g}\ln(5k^2)$

max height $v = 0$

$x = -K^2/2g\ln K^2 + K^2/2g\ln(5k^2)$

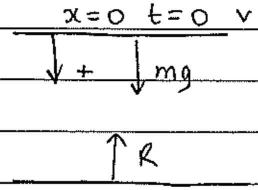
$= K^2/2g\ln\left[5k^2/K^2\right]$

$= \frac{K^2}{2g}\ln 5.$

10

Question 16 con't

b) $x=0 \quad t=0 \quad v=0$



$(m=1)$

$\ddot{x} = g - R$

$\ddot{x} = g - \frac{g}{k^2} v^2$

$v\frac{dv}{dx} = g - \frac{gv^2}{k^2}$

$\frac{dv}{dx} = \frac{g}{v} - \frac{gv}{k^2}$

$= \frac{gk^2 - gv^2}{vk^2}$

$\frac{dx}{dv} = \frac{vk^2}{gk^2 - gv^2}$

$x = \int \frac{vk^2}{gk^2 - gv^2}\, dv$

$x = \frac{k^2}{g} \times \frac{-1}{2} \ln(k^2 - v^2) + C_2$

$x=0$, $v=0$

$C_2 = \frac{k^2}{2g} \ln k^2$

$x = \frac{-k^2}{2g} \ln(k^2 - v^2) + \frac{k^2}{2g} \ln k^2$

Now $x = \frac{k^2}{2g} \ln 5$

$\frac{k^2}{2g} \ln 5 = \frac{-k^2}{2g} \ln(k^2 - v^2) + \frac{k^2}{2g} \ln(k^2)$

$\ln 5 = -\ln(k^2 - v^2) + \ln k^2$

$\ln 5 = \ln\left(\frac{k^2}{k^2 - v^2}\right)$

$5(k^2 - v^2) = k^2$

$-5v^2 = -4k^2$

$v^2 = \frac{4k^2}{5}$

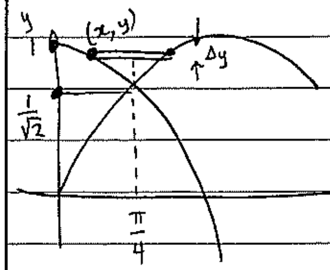
$\therefore v = \sqrt{\frac{4k^2}{5}}$

$v > 0$

$v = \frac{2k}{\sqrt{5}}.$

11

Question 16 con't

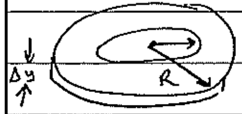


x lies of

$y = \cos x$

$\therefore x = \cos^{-1} y$

(I used symmetry can do $\sin^{-1} y$ & $\cos^{-1} y$)



$r = x = \cos^{-1} y$

$R = \frac{\pi}{2} - x = \frac{\pi}{2} - \cos^{-1} y$

$\Delta V = \pi (R^2 - r^2) \Delta y$

$= \pi \left[\frac{\pi}{2} - \cos^{-1} y - \cos^{-1} y\right]\left[\frac{\pi}{2} - \cos^{-1} y + \cos^{-1} y\right] \Delta y$

$= \pi \left[\frac{\pi}{2} - 2\cos^{-1} y\right]\left[\frac{\pi}{2}\right] \Delta y$

$= \frac{\pi^2}{2}\left[\frac{\pi}{2} - 2\cos^{-1} y\right] \Delta y$

Total volume

$= \lim_{\Delta y \to 0} \sum_{1/\sqrt{2}}^{1} \frac{\pi^2}{2}\left[\frac{\pi}{2} - 2\cos^{-1} y\right] \Delta y$

$= \frac{\pi^2}{2} \int_{1/\sqrt{2}}^{1} \frac{\pi}{2} - 2\cos^{-1} y \, dy$

$= \frac{\pi^2}{2}\left[\frac{\pi}{2} y - \left[2y\cos^{-1} y - \int 2y \cdot \frac{-1}{\sqrt{1-y^2}} dy\right]\right]$

$= \frac{\pi^2}{2}\left[\frac{\pi}{2} y - 2y\cos^{-1} y + 2\sqrt{1-y^2}\right]_{1/\sqrt{2}}^{1}$

$V = \frac{\pi^2}{2}\left[\frac{\pi}{2} - 2(0) + 0 - \left(\frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}}\cdot\frac{\pi}{4} + \frac{2}{\sqrt{2}}\right)\right]$

$= \frac{\pi^2}{2}\left[\frac{\pi}{2} - \left(\frac{\pi}{2\sqrt{2}} - \frac{\pi}{2\sqrt{2}} + \frac{2}{\sqrt{2}}\right)\right]$

$= \frac{\pi^2}{2}\left[\frac{\pi}{2} - \sqrt{2}\right] u^3.$

other solutions such as

$\frac{\pi^2}{2}\int_{1/\sqrt{2}}^{1} 2\sin^{-1} y - \frac{\pi}{2}\, dy$

can be used.

☺

12