Binomial Theorem & Binomial Probability

- Pascal's Triangle
- General Expansion
- Special Expansion
- Finding Terms
- Equidistant Coefficients
- Pascal's Relation
- Pairing Off Method
- Sums of Coefficients
- Greatest Coefficient
- Extra stuff

- Success and Failure

Pascal's Triangle

						1							1	2^0
					1		1						2	2^1
				1		2		1					4	2^2
			1		3		3		1				8	2^3
		1		4		6		4		1			16	2^4
	1		5		10		10		5		1		32	2^5
1		6		15		20		15		6		1	64	2^6

2^{n-1} n = line/ row

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Where T_{r+1} term is important to identify a particular term
$T_n = T_{r+1}$

General Expansion $(a+x)^n$

$$(a+x)^n = \underbrace{{}^nC_0a^nx^0}_{1^{st}} + {}^nC_1a^{n-1}x^1 + \ldots + \underbrace{{}^nC_ra^{n-r}x^r}_{\text{Term } r+1} + \ldots + \underbrace{{}^nC_na^0x^n}_{\text{last}}$$

Special Expansion $(1+x)^n$

$$(1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$

Finding Terms

Example 1

Find Term 4

$$(1+x)^7 = {}^7C_0x^0 + {}^7C_1x^1 + {}^7C_2x^2 + {}^7C_3x^3 + {}^7C_4x^4 + {}^7C_5x^5 + {}^7C_6x^6 + {}^7C_7x^7$$
$$= x^0 + 7x^1 + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$$

$$\begin{aligned} {}^7C_4 &= \frac{7!}{4!(7-4)!} \\ &= \frac{7!}{4!.3!} \\ &= \frac{5040}{144} \\ &= 35 \end{aligned}$$

Example 2

Find the term independent of x in $\left(3x^2 - \frac{1}{2x}\right)^3$

$$\begin{aligned} T_{r+1} = {}^nC_r a^{n-r} x^r &= {}^3C_r\left(3x^2\right)^{3-r}\left(-\tfrac{1}{2x}\right)^r \\ &= x^{6-2r}.x^r = x^0 \\ & 6 - 3r = 0 \\ & \therefore r = 2 \end{aligned}$$

Equidistant Coefficients

Equidistant coefficients are equal $\quad {}^nC_r = {}^nC_{n-r}$

Proof

$$\text{LHS} = {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{r!(n-r)!}$$

$$\text{RHS} = {}^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!}$$

Pascal's Relation

${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$

			1					
		1		1				
	1		2		1			nth row
1		3		3		1		$n+1$ row

Example

${}^{n}C_{r-1} \quad + {}^{n}C_{r} \quad = {}^{n+1}C_{r}$

$2 \quad + 1 \quad = 3$

Proof 1

LHS $= {}^{n}C_{r} + {}^{n}C_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!(n-r+1) + r.n!}{r!(n-r+1)!}$$

$$>>> \frac{(n-r+1)!}{(n-r)!}$$

$$>>> \frac{1.2.3...(n-r)(n-r+1)}{1.2.3...(n-r)}$$

$$>>> (n-r+1)$$

$$= \frac{n!(n-r+1+r)}{r!(n-r+1)!}$$

$$= \frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!}$$

= RHS

RHS $= {}^{n+1}C_{r}$

$$= \frac{(n+1)!}{r!(n-r+1)!}$$

Proof 2

Expand $(1+x)^{n+1}$ in two ways. Compare the coefficients of x^r.

1st

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n.(1+x)\\ &= (1+x)\left({}^nC_0x^0 + {}^nC_1x^1 + \ldots + {}^nC_{r-1}x^{r-1} + {}^nC_rx^r + \ldots + {}^nC_nx^n\right)\end{aligned}$$

Coefficients of x^r

$$= \left[1\times {}^nC_r + {}^nC_{r-1}\right]x^r$$
$$= \left[{}^nC_r + {}^nC_{r-1}\right]x^r$$

2nd

$$(1+x)^{n+1} = {}^{n+1}C_0x^0 + {}^{n+1}C_1x^1 + \ldots + {}^{n+1}C_rx^r + \ldots + {}^{n+1}C_{n+1}x^{n+1}$$

Coefficients of x^r

$$= {}^{n+1}C_r$$

$$\therefore {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

Pairing Off Method

Find the coefficient of x in the expansion of $\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$

**Do Not Expand

$$= \left[{}^4C_0x^4\left(\tfrac{1}{x}\right)^0 + {}^4C_1x^3\left(\tfrac{1}{x}\right)^1 + {}^4C_2x^2\left(\tfrac{1}{x}\right)^2 + {}^4C_3x^1\left(\tfrac{1}{x}\right)^3 + {}^4C_4x^0\left(\tfrac{1}{x}\right)^4\right]$$

$$\times$$

$$\left[{}^3C_0x^3\left(-\tfrac{1}{x}\right)^0 + {}^3C_1x^2\left(-\tfrac{1}{x}\right)^1 + {}^3C_2x^1\left(-\tfrac{1}{x}\right)^2 + {}^3C_3x^0\left(-\tfrac{1}{x}\right)^3\right]$$

Coefficients of x are:

$$= {}^4C_0.{}^3C_3 + {}^4C_1.{}^3C_2 + {}^4C_2.{}^3C_1 + {}^4C_3.{}^3C_0$$
$$= -1 + 12 - 18 + 4$$
$$= -3$$

Sums of Coefficients

$$\sum_{r=0}^{n} {}^nC_r = 2^n$$

OR

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

Greatest Coefficient

$(a+b)^n$

$$\begin{aligned}\frac{T_{r+1}}{T_r} &= \frac{{}^nC_r a^{n-r} b^r}{{}^nC_{r+1} a^{n-r+1} b^{r-1}} \\ &= \frac{b}{a} \cdot \frac{n!}{r!(n-r)!} \cdot \frac{(r-1)!(n-r+1)!}{n!} \\ &= \frac{b}{a} \cdot \frac{n-r+1}{r} \geq 1\end{aligned}$$

Example 1

Find the greatest coefficient in $(5+2x)^{12}$ when $x=3$

$$\begin{aligned} &= \frac{2x}{5} \cdot \frac{(12-r+1)}{r} \\ &= \frac{6}{5} \cdot \frac{(13-r)}{r} \\ &= \frac{78-6r}{5r} \geq 1 \end{aligned}$$

r is an integer

$$\begin{aligned} 78-6r &\geq 5r \\ 78 &\geq 11r \\ r &\leq \frac{78}{11} \\ r &= 7 \end{aligned}$$

$$\begin{aligned} T_{r+1} = T_8 &= {}^{12}C_7 a^{12-7} b^7 \\ &= 6.93 \text{ x } 10^{11} \end{aligned}$$

Example 2

Find the greatest coefficient in $(7+3x)^{12}$ when $x=2$

$$\begin{aligned} &= \frac{6}{7} \cdot \frac{(13-r)}{r} \geq 1 \\ 78-6r &\geq 7r \\ 78 &\geq 13r \\ r &= 7 \end{aligned}$$

$$\begin{aligned} &= {}^{12}C_6 7^6 6^6 \\ &= 5.07 \text{ x } 10^{12} \end{aligned}$$

Extra Examples

Example 1

Find the middle term of $(a-2b)^8$

There are 9 terms. Middle term is T_5.

$$\begin{aligned} {}^8C_4(a)^4(-2b)^4 &= 70a^4 16b^4 \\ &= 1120a^4b^4 \end{aligned}$$

Example 2

Fine the middle term of $\left(\frac{1}{x}+x^2\right)^{11}$

There are 12 terms. Middle term is T_6, T_7.

$${}^{11}C_5\left(\tfrac{1}{x}\right)^6\left(x^2\right)^5 = 462x^4$$

$${}^{11}C_6\left(\tfrac{1}{x}\right)^5\left(x^2\right)^6 = 462x^7$$

Example 3

$(1+x)^8(1+x)^8 = (1+x)^{16}$

Prove ${}^{16}C_3 = 2\left[{}^8C_0.{}^8C_3 + {}^8C_1.{}^8C_2\right]$

$$\text{RHS} = \left[{}^8C_0x^0 + {}^8C_1x^1 + \ldots + {}^8C_8x^8\right] \times \left[{}^8C_0x^0 + {}^8C_1x^1 + \ldots + {}^8C_8x^8\right]$$

Coefficients of x^3

$$\begin{aligned} &= {}^8C_0.{}^8C_3 + {}^8C_1.{}^8C_2 + {}^8C_2.{}^8C_1 + {}^8C_3.{}^8C_0 \\ &= 2\left[{}^8C_0.{}^8C_3 + {}^8C_1.{}^8C_2\right] \end{aligned}$$

$$\begin{aligned} \text{LHS} &= {}^{16}C_3x^3 \\ &= {}^{16}C_3 \end{aligned}$$

Prove ${}^{16}C_8 = \left({}^8C_0\right)^2 + \ldots + \left({}^8C_8\right)^2$

$$\begin{aligned} &= {}^8C_0.{}^8C_8 + {}^8C_1.{}^8C_7 + \ldots + {}^8C_0.{}^8C_8 \\ &= \left({}^8C_0\right)^2 + \left({}^8C_1\right)^2 + \ldots + \left({}^8C_8\right)^2 \\ &= \text{Proven} \end{aligned}$$

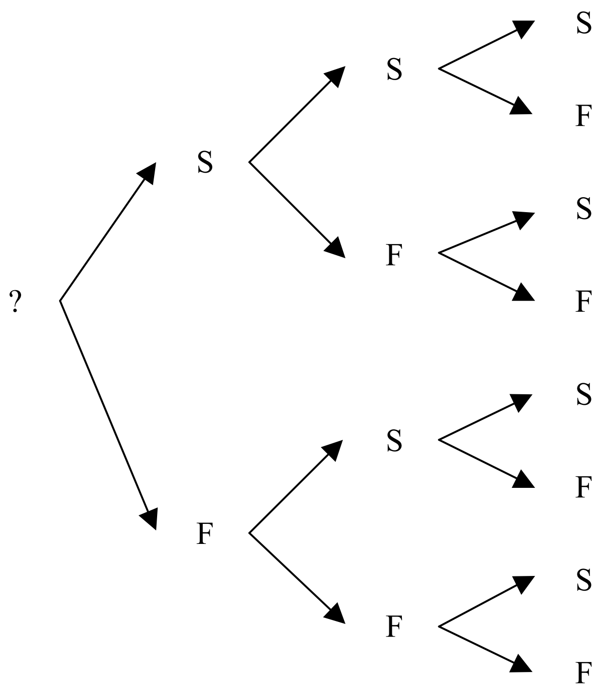
Success and Failure

$(q+p)^n = {}^nC_r q^{n-r} p^r$

Let p be the one you want "Success"
Let q be the one you don't want "Failure"
There are only 2 options – Binomial

Example 1

A die is tossed 3 times. "6" outcome



SSS	$\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0$	p^3q^0
SSF	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1$	p^2q^1
SFS	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1$	p^2q^1
SFF	$\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$	p^1q^2
FSS	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1$	p^2q^1
FSF	$\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$	p^1q^2
FFS	$\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$	p^1q^2
FFF	$\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3$	p^0q^3

Example 2

An archer has a record of hitting the target 3 out of 4 occasions. He tries 5 times. Find the probability a) Exactly 3 hits

b) Exactly 4 hits

c) Bull's eye only in 2nd round

d) 1 bull's eye

A $\quad {}^5C_3 q^2 p^3$ $= 10\times\left(\frac{1}{4}\right)^2\times\left(\frac{3}{4}\right)^3$ $= \dfrac{135}{512}$	B $\quad {}^5C_4 q^1 p^4 + {}^5C_5 q^0 p^5$ $= 5\times\left(\frac{1}{4}\right)^1\times\left(\frac{3}{4}\right)^4 + 1\times\left(\frac{3}{4}\right)^5$ $= \dfrac{405}{1024}+\dfrac{243}{1024}$ $= \dfrac{81}{128}$
C $\quad \dfrac{1}{4}\times\dfrac{3}{4}\times\dfrac{1}{4}\times\dfrac{1}{4}\times\dfrac{1}{4}$ $= \left(\dfrac{1}{4}\right)^2\left(\dfrac{3}{4}\right)$ $= \dfrac{3}{1024}$	D $\quad {}^5C_1 q^4 p^1$ $= 5\times\left(\frac{1}{4}\right)^4\times\left(\frac{3}{4}\right)$ $= \dfrac{15}{1024}$

Example 3

For a certain species of bird, there is a 3 in 5 chance that a fledgling will survive the 1st month. From a breed of 10 chicks, find the probability that:

a) P (0 survive)

b) P (more than 1 survive)

c) P (3 survive)

A	${}^{10}C_0\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^{10}$	$= 1\times\left(\frac{2}{5}\right)^{10}$ $= \dfrac{1024}{9765625}$
B	$1-\left[P_0+P_1\right]$	$= 1-\left[{}^{10}C_0\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^{10}+{}^{10}C_1\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^9\right]$ $= 1-\dfrac{16384}{9762625}$ $= \dfrac{9748881}{9765625}$
C	${}^{10}C_3 q^7 p^3$	$= 120\times\left(\frac{2}{5}\right)^7\times\left(\frac{3}{5}\right)^3$ $= \dfrac{414720}{9762625}$ $= 0.0425$