

Your Name: Teacher:

SYDNEY TECHNICAL HIGH SCHOOL
YEAR 12 EXTENSION 2 MATHEMATICS
ASSESSMENT TASK

MARCH 2007

Time allowed: 70 mins

Instructions

- * Attempt all questions.
- * Answers to be written on the paper provided.
- * Start each question on a new page.
- * Marks may not be awarded for careless or badly arranged working.
- * Indicated marks are a guide and may be changed slightly if necessary during the marking process.
- * **These questions must be handed in attached to the top of your solutions.**

Q1	Q2	Q3	TOTAL
/17	/16	/17	/50

QUESTION 1

- (a) Suppose that $z = 2 - i$. 2

Express the following in the form $x + iy$ where x and y are real numbers:

i) $\overline{(iz)}$

ii) $\frac{1}{z}$

- (b) i) Express $-1 - i\sqrt{3}$ in modulus argument form. 4

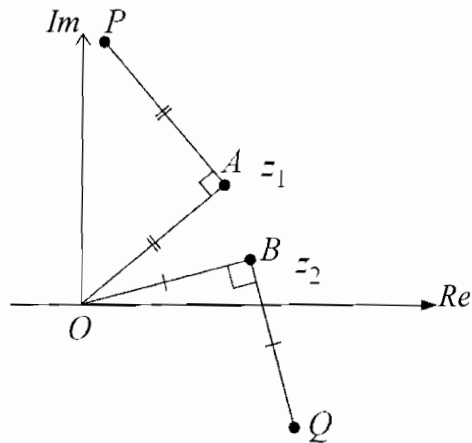
ii) Hence evaluate $(-1 - i\sqrt{3})^3$

- (c) i) Simplify $(-2i)^3$. 4

ii) Hence or otherwise find all complex numbers, z , such that $z^3 = 8i$. Express your answers in the form $x + iy$.

- (d) Sketch the region where the inequalities $|z - 3 + i| \leq 5$ and $|z + 1| \leq |z - 1|$ both hold. 3

- (e) 4



The points A and B in the complex plane correspond to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

- (i) Explain why P corresponds to the complex number $(1 + i)z_1$.
- (ii) Find a similar expression for the complex number corresponding to Q .
- (iii) Let M be the midpoint of PQ . Give an expression in terms of z_1 and z_2 for the complex number that corresponds to M .

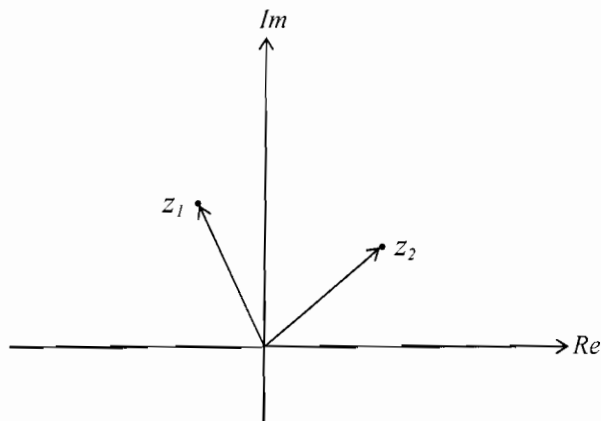
QUESTION 2 (Begin on a new page)

(a) If $1 - 3i$ is a root of the equation $2z^3 - 3z^2 + 18z + 10 = 0$ **2**

(i) Explain why $1 + 3i$ is also a root.

(ii) Find all roots of the equation.

(b) z_1 and z_2 are complex numbers such that $\frac{z_1 + z_2}{z_1 - z_2} = 2i$. **6**



(i) Copy the diagram onto your answer page.

On the diagram show the vectors $z_1 + z_2$ and $z_1 - z_2$.

(ii) Show that $|z_1| = |z_2|$.

(iii) If the angle between the vectors z_1 and z_2 is α , show that $\tan \frac{\alpha}{2} = \frac{1}{2}$.

(c) (i) Solve $z^5 = -1$ over the complex field showing the roots on an Argand diagram. **2**

(ii) If ω is the complex root of $z^5 = -1$ with the smallest positive argument, **3**
show that the other complex roots are $-\omega^2$, ω^3 and $-\omega^4$.

(iii) Using ω as in part (ii), simplify $(1 - \omega + \omega^2 - \omega^3)^8$. **3**

QUESTION 3 (Begin on a new page)

- (a) Sketch the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. State the eccentricity and show clearly the coordinates of the foci and the equations of the directrices. **4**
- (b) The point C (x_0, y_0) lies outside the ellipse in (a). Give a full geometrical description of the locus of the point C if the chord of contact from C always passes through the point (2, 2). **3**
- (c) Prove that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P (x_1, y_1) is $\frac{x_1 y}{a^2} - \frac{x y_1}{b^2} = \frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{b^2}$. **4**
- (d) The tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P (x_1, y_1) cuts the y axis at A while the normal at P cuts the y axis at B. If S is a focus of the ellipse, show that $\angle ASB = 90^\circ$. (You do *not* have to derive the equation of the tangent.) **5**
- (e) Deduce the geometrical relationship between the points A, P, S and B in part (d). **1**

End of Paper

QUESTION 1

a) i) $\overline{iz} = \overline{2i - i^2}$
 $= 1 - 2i$ (1)

ii) $\frac{1}{z} = \frac{1}{2-i} \times \frac{2+i}{2+i}$
 $= \frac{2+i}{5}$ (1)

b) i) $-1 - i\sqrt{3}$

Modulus = 2

$\arg(-1 - i\sqrt{3}) = -\frac{2\pi}{3}$

$\therefore -1 - i\sqrt{3} = 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$ (2)

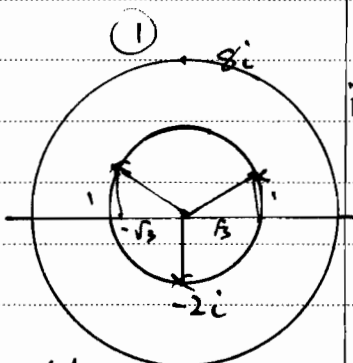
(ii) $(-1 - i\sqrt{3})^3 = \left(2 \operatorname{cis} -\frac{2\pi}{3}\right)^3$
 $= 8 \operatorname{cis} -2\pi$
 $= 8 \operatorname{cis} 0$
 $= 8$ (2)

c) i) $(-2i)^3 = -8i^3$
 $= 8i$ (1)

ii) $z^3 = 8i$

Roots are

$-2i, \sqrt{3}+i, -\sqrt{3}+i$

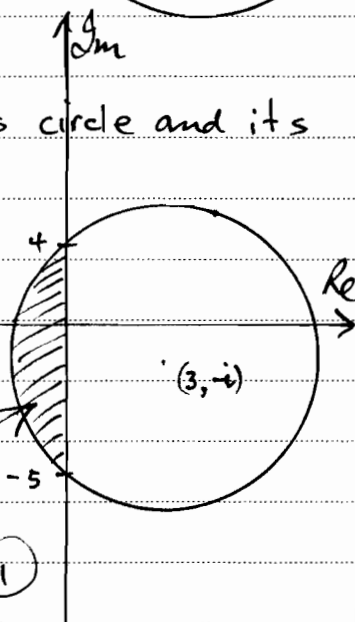


d)

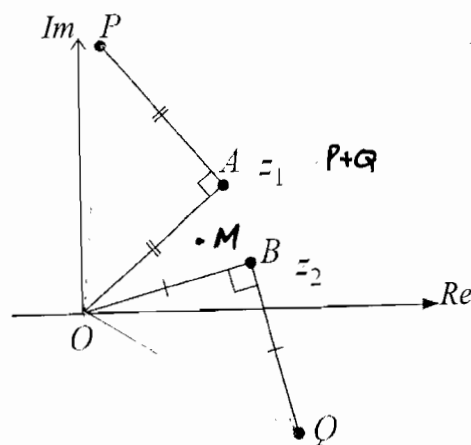
$|z - (3-i)| \leq 5$ is circle and its interior. (1)

$|z+1| \leq |z-1|$ is everything on and to the left of the imaginary axis (1)

Required region. (1)



(e)



i) \vec{AP} represents the complex number iz , and P represents the complex number obtained by adding \vec{OA} and \vec{AP} (1)

ie $z_1 + iz_1 = (1+i)z_1$

ii) Q represents the complex number $z_2 - iz_2$

ie $(1-i)z_2$ (1)

iii) M is midpoint of vector to $P+Q$ ie $\frac{P+Q}{2}$

$\therefore M: \left(\frac{z_1+z_2}{2} + i \frac{z_1-z_2}{2} \right)$ (2)

QUESTION 2

a) i) Since coefficients are real, roots occur in conjugate pairs. OR

$1+3i$ could be shown to be a root by substitution. (1)

ii) Known factors of $2z^3 - 3z^2 + 18z + 10 = 0$ are $(z - 1 - 3i)(z - 1 + 3i)$

ie $(z^2 - 2z + 10)$ is a factor

\therefore Third factor is $(2z+1)$

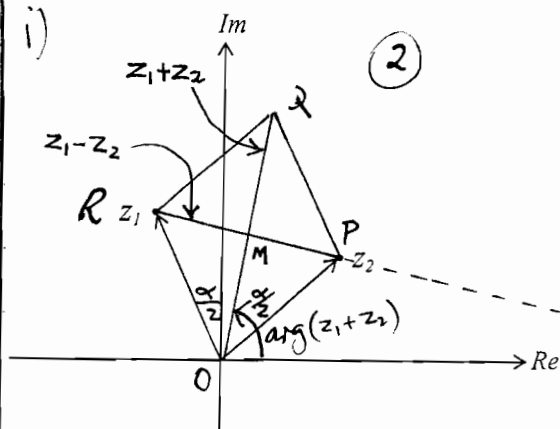
\therefore Final root is $-\frac{1}{2}$ (1)

QUESTION 2 (cont.)

(b)

z_1 and z_2 are complex numbers such that $\frac{z_1 + z_2}{z_1 - z_2} = 2i$

i)



ii) Since $\frac{z_1 + z_2}{z_1 - z_2} = 2i$

$\arg(z_1 + z_2) - \arg(z_1 - z_2) = \arg 2i = \frac{\pi}{2}$
ie $\angle OMP = 90^\circ$

$\therefore OPQR$ is a rhombus

$\therefore |z_1| = |z_2|$ (all sides equal)

iii) Since $\frac{z_1 + z_2}{z_1 - z_2} = 2i$,

$$\left| \frac{z_1 + z_2}{z_1 - z_2} \right| = |2i| = 2$$

ie $|z_1 + z_2| = 2|z_1 - z_2|$

ie. $OQ = 2 RP$

$\therefore MP = \frac{1}{2} OM$

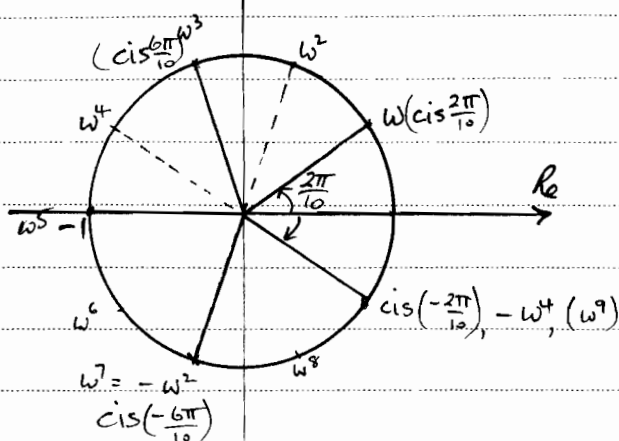
$\therefore \tan \frac{\alpha}{2} = \frac{MP}{OM} = \frac{1}{2}$

$$(c) i) z^5 = -1$$

$$\therefore z = \cos \frac{2\pi}{10}, \cos \frac{6\pi}{10}, -1, \cos \frac{-2\pi}{10},$$

$$\cos \left(-\frac{6\pi}{10} \right).$$

(2)



(ii) If $\cos \frac{2\pi}{10}$ is w then $\cos \frac{6\pi}{10}$ becomes w^3 and the roots $\cos \frac{-2\pi}{10}$ and $\cos \frac{-6\pi}{10}$ can be seen to be the opposites of w^4 and w^2 resp. \therefore Roots may be designated as $w, w^3, -1, -w^2, -w^4$.

(iii) Now $z^5 + 1 = 0$ and since coefficient of z^4 is zero the sum of the roots is zero.

$$\text{ie } w + w^3 - 1 - w^2 - w^4 = 0$$

$$\therefore -w^4 = 1 - w - w^3 + w^2$$

$$\therefore (1 - w - w^3 + w^2)^8 = (-w^4)^8$$

$$= w^{32}$$

$$= (w^5)^6 \times w^2$$

(3)

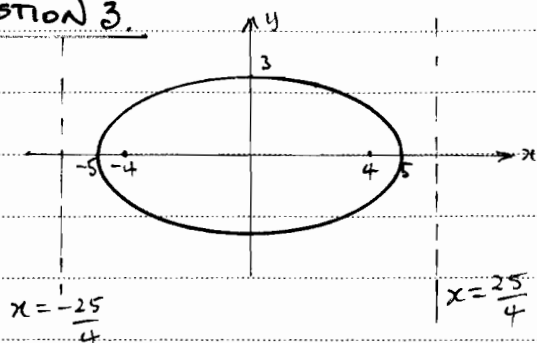
$$= 1 \times w^2$$

$$= \cos \frac{4\pi}{10}$$

$$\text{or } \cos \frac{2\pi}{5}$$

QUESTION 3.

(a)



$$a=5, b=3$$

$$a^2c^2 = 25 - 9 = 16 \quad (\rightarrow ac = 4)$$

$$\therefore e = \frac{4}{5} \quad (4)$$

(b) Chord of contact: $\frac{xx_0}{25} + \frac{yy_0}{9} = 1$

This always contains the point (2,2)

$$\therefore \frac{2x_0}{25} + \frac{2y_0}{9} = 1$$

ie (x_0, y_0) lies on the st.

$$\text{line } \frac{2x}{25} + \frac{2y}{9} = 1 \quad (3)$$

$$(c) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{d}{dx} \frac{x^2}{a^2} + \frac{d}{dx} \frac{y^2}{b^2} = \frac{d}{dx} 1$$

$$\text{ie } \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{xb^2}{ya^2} \quad \checkmark$$

\therefore at (x_1, y_1) the slope of the normal is $\frac{y_1 a^2}{x_1 b^2} \quad \checkmark$

$$\therefore \text{Eqn of Normal is } y - y_1 = \frac{y_1 a^2}{x_1 b^2} (x - x_1) \quad \checkmark$$

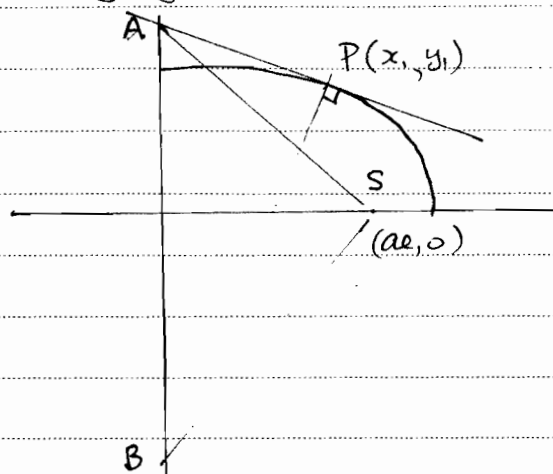
$$\text{ie } x_1 y_1 b^2 - x_1 y_1 b^2 = x y_1 a^2 - x_1 y_1 a^2$$

$$\text{ie } \frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{a^2} = \frac{x y_1}{b^2} - \frac{x_1 y_1}{b^2} \quad \checkmark \text{ on}$$

dividing through by $a^2 b^2$

Hence required form follows on rearranging.

(d)



$$\text{Tangent at P } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

cuts at A when $x=0$

$$\text{ie } y = \frac{b^2}{y_1} \quad (1)$$

$$\text{Normal at P, } \frac{x_1 y_1}{a^2} - \frac{x y_1}{b^2} = \frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{b^2}$$

cuts at B when $x=0$

$$\text{ie } y = \frac{a^2}{x_1} \left(\frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{b^2} \right)$$

$$= y_1 \left(1 - \frac{a^2}{b^2} \right) \quad (1)$$

$$\text{Slope AS} = \frac{b^2/y_1}{-ae} \quad (1)$$

$$\text{Slope BS} = \frac{y_1 (1 - \frac{a^2}{b^2})}{-ae} \quad (1)$$

The product of these is $\frac{b^2 - a^2}{a^2 e^2}$.

$$\text{But } a^2 c^2 = a^2 - b^2$$

$$\therefore \text{Product of Slopes is } \frac{b^2 - a^2}{a^2 - b^2} = -1 \quad (1)$$

$$\therefore AS \perp BS$$

(e) APSB is a cyclic quadrilateral with AB as diameter of the