Sydney Technical High School

Name: ....

Maths Class: ....

Year 12
Mathematics Extension 2

HSC Course

Assessment 1

December, 2017

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

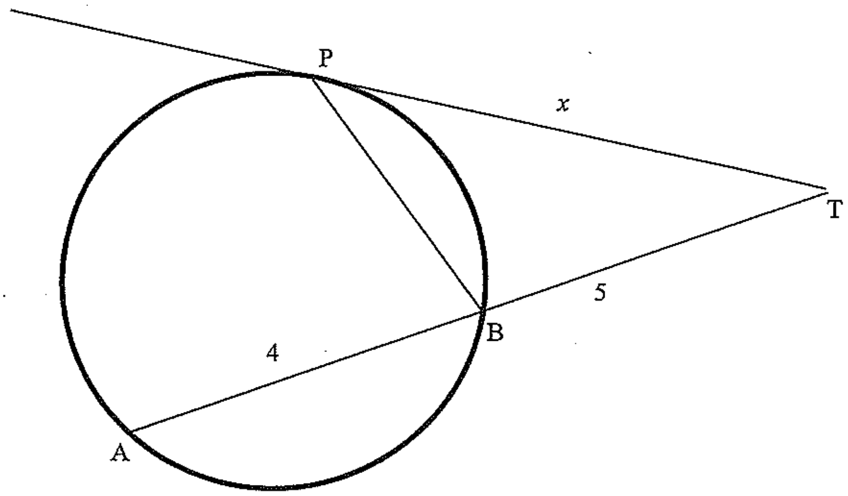
Section 1 Multiple Choice
Questions 1-5
5 Marks

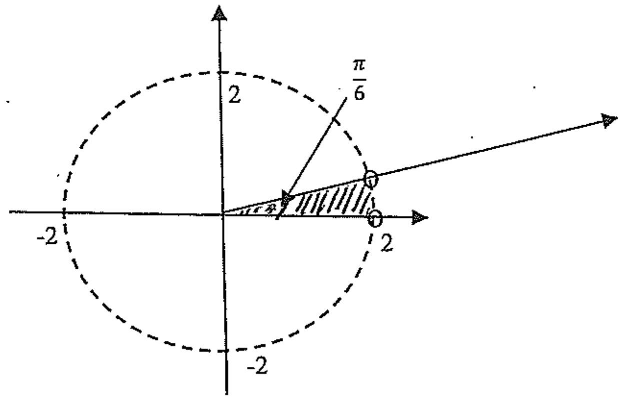
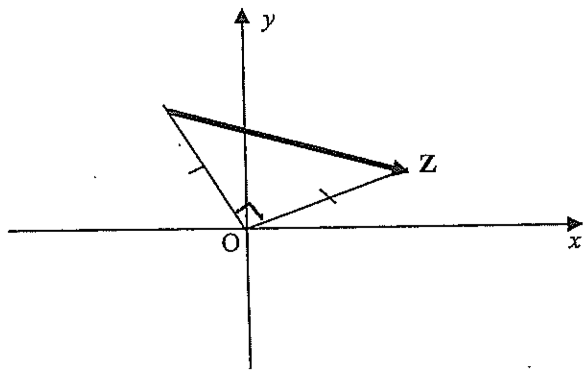
Section II Questions 6-9
52 Marks

SECTION I: MULTIPLE CHOICE

Use the Multiple Choice Answer sheet provided in your answer booklet.

All questions are worth 1 mark

1	The value of $\arg(z\bar{z})$ is A. 0 B $\frac{\pi}{2}$ C. π D. 2π
2	 In the diagram PT is tangent to the circle at P. The value of x is A. $2\sqrt{5}$ B. $3\sqrt{5}$ C. 20 D. 45
3	If k is a positive integer then $\displaystyle\sum_{k=1}^{n} k^3 - (k-1)^3 =$ A. $3n^3 - 3n + 1$ B. $n^3 - 1$ C. $1 - (n-1)^3$ D. n^3

4	The algebraic description of the following shaded area in the Complex Plane is:  A. $\lvert z\rvert < 2$ and $0 < \arg(z) < \frac{\pi}{6}$ B. $\lvert z\rvert \le 2$ and $0 \le \arg(z) \le \frac{\pi}{6}$ C $\lvert z\rvert < 2$ and $0 \le \arg(z) \le \frac{\pi}{6}$ D. $\lvert z\rvert \le 2$ and $0 < \arg(z) < \frac{\pi}{6}$
5	The point **Z** representing the complex number z is plotted as shown.  The vector in bold represents which complex number below? A. $z(1-i)$ B. $z(i-1)$ C. $z(z-i)$ D. $z(i-z)$

SECTION II

(START EACH QUESTION ON A NEW PAGE)

QUESTION 6: (13 Marks) **Marks**

(a) If $z = 5 + 2i$ find the value of: **4**

(i) $|z|$ (ii) $\bar{z}$ (iii) $z\bar{z}$ (iv) arg z (to the nearest minute)

(b) If $a = 3 - 4i$ and $b = 5 + 12i$ find the following, leaving your answer in the form $x + iy$: **2**

(i) ab (ii) $\dfrac{a}{b}$

(c) Find $\sqrt{7 - 24i}$ giving your answer in the form $a + ib$ **2**

(d) (i) Express $1 + i\sqrt{3}$ in mod-argument form. **1**

(ii) Hence find the value of $(1 + i\sqrt{3})^6$ in the form $a + ib$ **1**

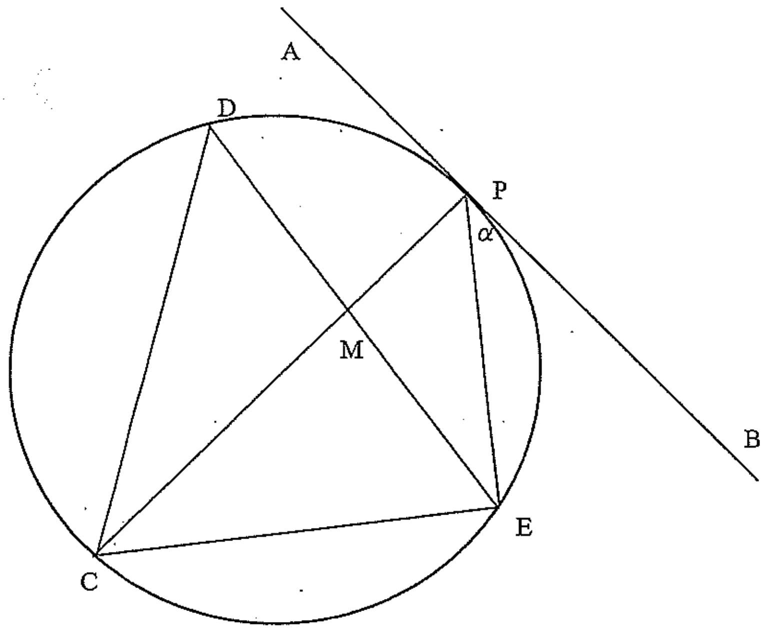
QUESTION 6 continues overleaf……

(e) In the diagram below, AB is a tangent to the circle touching it at P. **3**

C, D and E are points on the circumference of the circle, with PE bisecting $\angle BPC$.

DE and PC intersect at M

$\angle BPE = \alpha$



This diagram has been reproduced on page 4 of your answer booklet. Complete this question under that diagram.

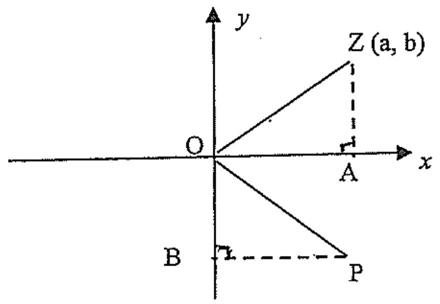
Prove that PE = CE.

QUESTION 7: (13 Marks) **Start a New Page** **Marks**

(a) Find the value of i^{2017} **1**

(b) (i) Prove that $\arg(i^3 z) = \arg z - \frac{\pi}{2}$ **1**

(ii) In the diagram below, Z represents the complex number $z = a + ib$ and P the complex number $i^3 z$. A and B are projections from Z and P respectively onto the co-ordinate axes. O is the origin.



Give the co-ordinates of the point P in terms of a and b. **1**

(c) Show that, for all positive integral values of m, **3**

$m + (m+2) + (m+4) + \ldots\ldots\ldots +3m$ is equal to four times the sum of the first m positive integers.

(DO NOT use Mathematical Induction)

QUESTION 7 continues overleaf….

(d) (i) Sketch the graph of the locus of z if $|z+2-i| = |z-2+i|$ **2**

(ii) Give both a geometric and an algebraic description of the locus as z varies **2**

(e) A, B, C and D are points on the Argand Diagram corresponding to the complex numbers α, β, γ and δ respectively, forming the quadrilateral ABCD.

(i) Describe the point given by the Complex Number $\frac{1}{2}(\alpha+\gamma)$ **1**

(ii) If $\alpha+\gamma=\beta+\delta$, prove that ABCD is a parallelogram. **2**

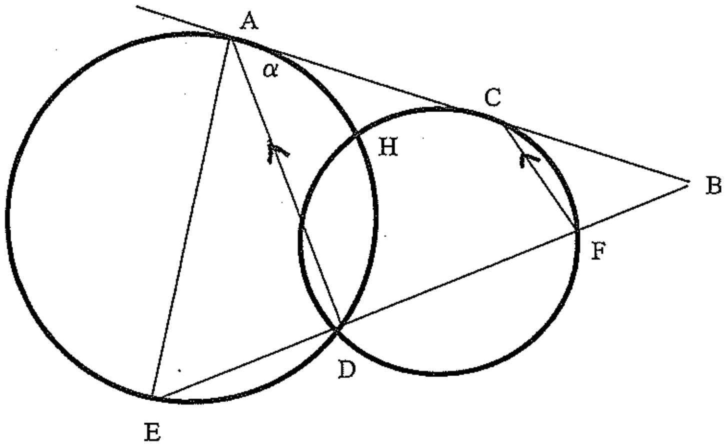
QUESTION 8: (13 Marks) Start a New Page

(a) Two circles of differing diameters intersect in D and H as shown

AB is the common tangent to both circles, touching them at A and C respectively.

The line BD cuts the smaller circle at F and produced it cuts the larger circle at E.

$AD \parallel CF$ and $\angle BAD = \alpha$



Copy or trace the diagram into your answer booklet

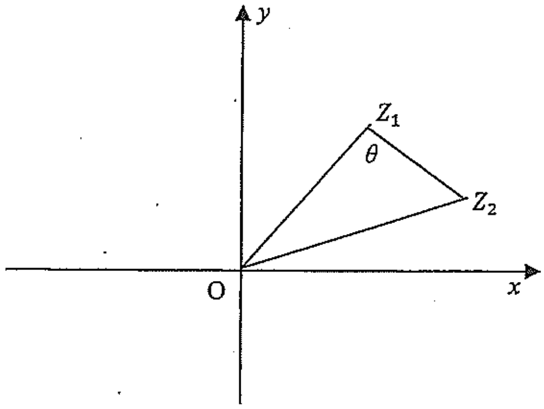
(i) Giving all reasons, prove that Δ BAD is similar to Δ BEA **2**

(ii) You can also assume that Δ BCF is similar to Δ BAD **3**

Deduce that BC.EA = EB.CF

Question 8 continues overleaf……

(b) The points Z_1 and Z_2 represent the complex numbers z_1 and z_2 as shown in the diagram below.



(i) If $\angle OZ_1Z_2 = \theta$, show that $arg\left(\frac{z_1 - z_2}{z_1}\right) = \theta$ **3**

(ii) If Z_2 is a fixed point and θ remains constant, briefly describe how Z_1 moves if it is NOT fixed. **1**

(c) (i) By use of a suitable diagram, or otherwise, prove the Triangle Inequality **1**

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

(ii) Prove that, for n≥ 2, **3**

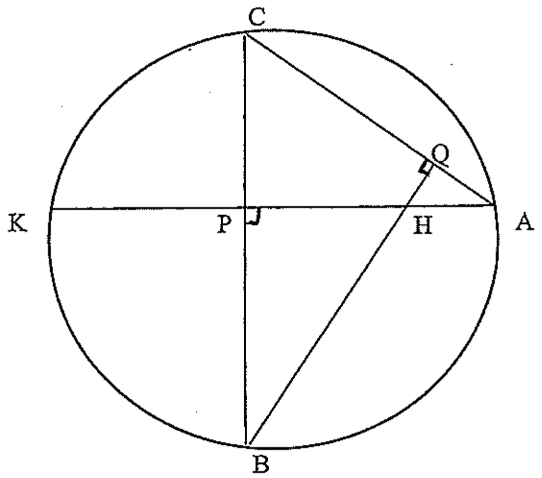
$$|z_1 + z_2 + \cdots \ldots \ldots + z_n| \leq |z_1| + |z_2| + \cdots \ldots \ldots + |z_n|$$

QUESTION 9: (13 Marks) Start a New Page **Marks**

(a) In the diagram below, ΔABC has been placed inside a circle so that A, B and C are on the circumference, as shown.

Two altitudes AP and BQ have been drawn in the triangle, intersecting at the point H. AP has been extended to the point K which is also on the circumference.

P is not necessarily the centre of the circle



Copy or trace the diagram into your answer booklet

(i) Prove that CPHQ is cyclic. **1**

(ii) Prove that HB = KB **3**

Question 9 continues overleaf….

(b) (i) Show that $(4k+3)\sqrt{k} < (4k+1)\sqrt{k+1}$ for all positive integers k. **2**

(ii) Prove, by Mathematical Induction, that **3**

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} < \frac{4n+3}{6}\sqrt{n} \qquad \text{for all integers } n > 0$$

(c) You are given that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ ***(Do not prove this)***

(i) By bracketing terms in pairs, or otherwise, show that the sum to 2n terms of the series **2**

$$1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \cdots + (4n-3)^2 - (4n-1)^2 \text{ is } -8n^2$$

(ii) Deduce the sum to $2n+1$ terms in its most simplified form. **2**

End of paper

EXTENSION 2 ASSESSMENT No 1

2017.

MULTIPLE CHOICE

1/ $\arg z\bar{z} = \arg z + \arg \bar{z}$
$= 0 \quad \therefore$ (A)

2/ $x^2 = 9 \times 5$
$x = 3\sqrt{5} \quad \therefore$ (B)

3/ $1^3 - 0^3 + 2^3 - 1^3 + \ldots + n^3 - (n-1)^3$
$= n^3 - 0 \quad \therefore$ (D)

4/ (C)

5/ (D)

SECTION 2

QUESTION 6: (a) (i) $\sqrt{29}$ (ii) $5-2i$ (iii) $\theta = 21°48'$ (iii) 29

(b) (i) $63+16i$ (ii) $-\frac{33}{169} - \frac{56}{169}i$

(c) $\sqrt{7-24i} = a+ib$

$a^2 - b^2 = 7$ and $2abi = -24i \Rightarrow b = -\frac{12}{a}$

$\therefore a^2 - \left(\frac{144}{a^2}\right) = 7$

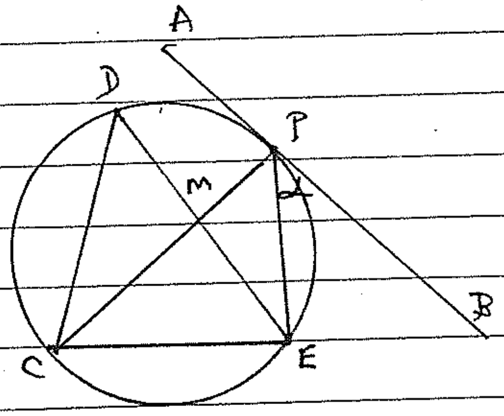
$\therefore a^4 - 7a^2 - 144 = 0$

$\therefore (a^2+9)(a^2-16) = 0$

$\therefore a = \pm 4 \quad b = \mp 3 \Rightarrow \sqrt{7-24i} = \pm(4-3i)$

(d) (i) $2\operatorname{cis}\frac{\pi}{3}$ (ii) $(1+i\sqrt{3})^6 = 64\operatorname{cis}2\pi$
$= 64$

(e)



$\angle BPE = \angle PCE$ (alternate segment theorem)

$\therefore \angle PCE = \alpha$

You are given that PC bisects $\angle CPB$ $\therefore \angle CPE = \alpha$

$\therefore \triangle CPE$ is isosceles

$\therefore PE = CE$ (Sides opposite equal angle are equal)

QUESTION 7:

(a) $i^{2017} = (i^4)^{504} i$
$= i$

(b) (i) $\arg(i^3 z) = \arg i^3 + \arg z$
$= -\pi/2 + \arg z$

(ii) P is $-i(a+ib)$ $= -i^2 b - ia$
$= b - ai$
$\therefore P$ is $(b, -a)$

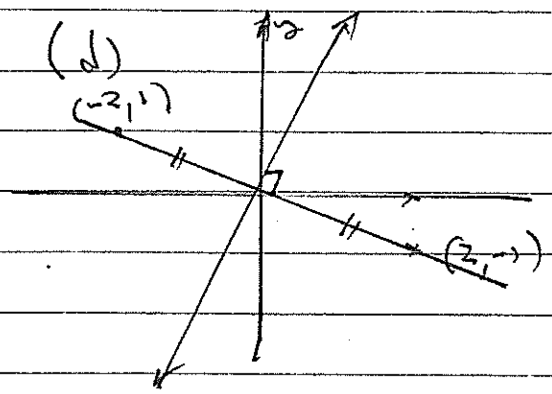
(c) Sum of first m positive integers is $S_m = \frac{m}{2}(m+1)$

$S_2 = \underbrace{m + (m+2) + \ldots + 3m}$

A.P. $a = m$
$n = m+1$

$\therefore S_2 = \frac{m+1}{2}[m + 3m]$
$= 2m(m+1)$
$= 4S_m$

(d)



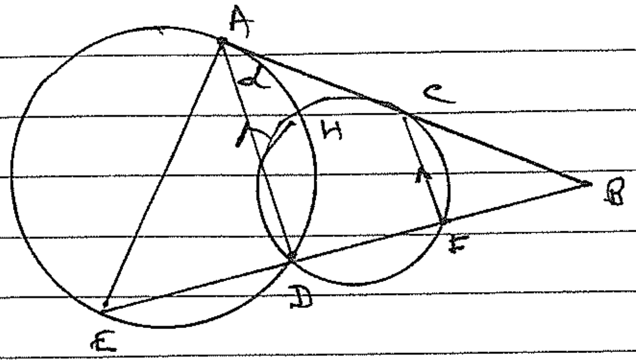
(ii) Geometric. Perpendicular bisector

Algebraic
$m = 2$
$\therefore y = 2x$.

(e) (i) The midpoint of AC
(ii) If $\alpha + \gamma = \beta + \delta$
then $\frac{\alpha+\gamma}{2} = \frac{\beta+\delta}{2}$
$\therefore$ midpoints of AC and BD are the same.
ie Diagonals bisect
$\therefore$ parallelogram

Question 8:

(a)



(i) In $\triangle BAD$ and $\triangle BEA$

$\angle BAD = \angle AEB = \alpha$ (alternate segment theorem)

$\angle B$ is common

$\therefore \triangle BAD \parallel\!\mid \triangle AEB$ (equiangular)

(ii) From part (i)

Since $\triangle BAD \parallel\!\mid \triangle BEA$

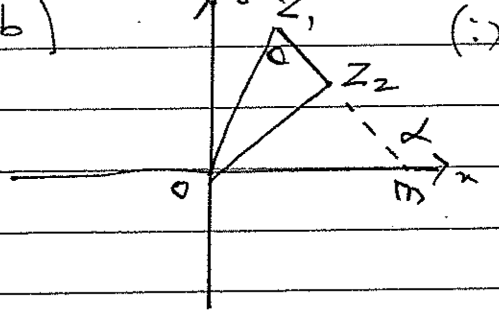
and $\triangle BAD \parallel\!\mid \triangle BCF$

$\therefore \triangle BEA \parallel\!\mid \triangle BCF$.

$\therefore \dfrac{BE}{EA} = \dfrac{BC}{CF}$ (corresponding sides in ratio in similar triangles)

$\therefore EB.CF = BC.EA$

(b)



(i) $\arg\left(\dfrac{z_1 - z_2}{z_1}\right) = \arg(z_1 - z_2) - \arg z_1$

let $\angle z_1 m x = \alpha = \arg(z_1 - z_2)$

Because it is the external angle of $\triangle OZ_1 Z_2$

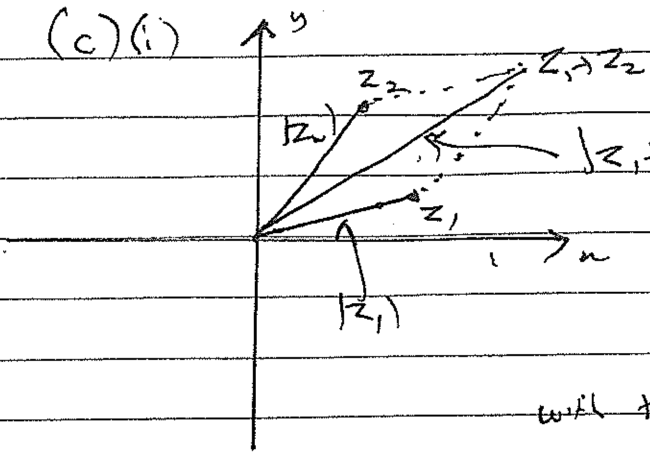
$\alpha = \theta + \arg z_1$

$\therefore \arg(z_1 - z_2) = \theta + \arg z_1$

$\therefore \arg\left(\dfrac{z_1 - z_2}{z_1}\right) = \theta$.

(ii) Z_1 moves in a circle, on the arc OZ_2 above the arc OZ_2 and does not include the points O or Z_2.

(c)(i)



In any triangle any side is less than the sum of the lengths of the other 2 sides.

ie $|z_1+z_2| \le |z_1|+|z_2|$

~~with the equality only if O, Z_1, Z_2 are~~ collinear.

(ii)

$$|z_1+z_2+\dots+z_n| \le |z_1+z_2+\dots+z_{n-1}| + |z_n|$$

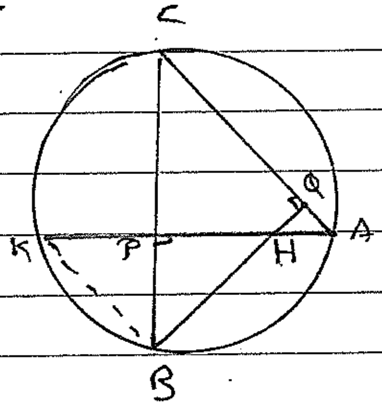
$$\le |z_1+z_2+\dots+z_{n-2}| + |z_{n-1}| + |z_n|$$

and so on until

$$\le |z_1|+|z_2|+\dots+|z_{n-1}|+|z_n|$$

QUESTION 9:

(c)



(i) In CPHQ, $\angle BPA = \angle CQB$

$\therefore$ CPHQ is cyclic (exterior angle is equal to the opposite angle)

(ii)

Join KB

If $\angle PCQ = \alpha$

then $\angle PHB = \alpha$ (external angle of cyclic quadrilateral)

and $\angle BKH = \angle PCQ = \alpha$

(angles at the circumference, standing on arc AB)

$\therefore \angle BCQ = \angle PHB$

$\therefore BK = HB$ (equal angles opposite equal sides in $\triangle KHB$)

(b) (i) $\left[(4k+3)\sqrt{k}\right]^2 = 16k^3+24k^2+9k$

$< 16k^3+24k^2+9k+1$

$= (4k+1)^2(k+1)$

$\therefore (4k+3)\sqrt{k} < (4k+1)\sqrt{k+1}$.

(ii) For $n=1$, $LHS = 1$

$RHS = 7/6\sqrt{1} = 7/6$

$> LHS$

$\therefore$ true for $n=1$

Assume it is true for $n=k$

$\therefore \sqrt{1}+\sqrt{2}+\ldots+\sqrt{k} < \frac{4k+3}{6}\sqrt{k}$.

For $n=k+1$

$\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}+\sqrt{k+1} < \frac{(4k+3)}{6}\sqrt{k}+(\sqrt{k+1})$

$< \frac{1}{6}(4k+1)\sqrt{k+1}+\sqrt{k+1}$ from part (i)

$= \frac{1}{6}\sqrt{k+1}\left[4k+1+6\right]$

$= \frac{1}{6}\sqrt{k+1}\,(4k+7)$

which is of the same form as for $n=k$.

$\therefore$ If the formula is true for $n=k$, it is true for $n=k+1$

But it is true for $n=1$

$\therefore$ " " " " $n=2$ and so on...

$\therefore$ ie true $\forall\, n \geq 1$

(c)(i) $(1^2-3^2)+(5^2-7^2)+\ldots+\left[(4n-3)^2-(4n-1)^2\right]$

$= -8-24-40-\ldots+(-16n+8)$

$= -8\left[1+3+5+\ldots+(2n-1)\right]$.

$= -8n^2$

(ii) $T_{2n+1} = (4n+1)^2 = 16n^2+8n+1$

$\therefore S_{2n+1} = -8n^2+16n^2+8n+1$

$= 8n^2+8n+1$