Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics
TRIAL HSC

August, 2015

Time allowed: 3 hours plus 5 minutes reading time

General Instructions:

- **Reading time – 5 minutes**
- **Working time – 3 hours**
- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- *Begin each question on a new page*
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Total marks - 100

Section 1

10 Marks

* Attempt Questions 1-10 on the sheet provided
* Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

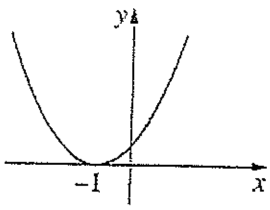
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

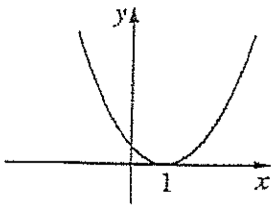
Use the multiple-choice answer sheet for Questions 1-10

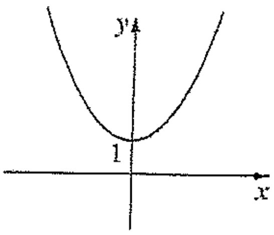
---

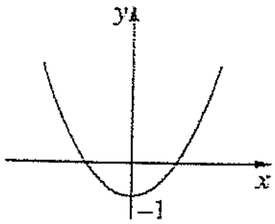
1. What is the value of $\frac{\sqrt[3]{3}}{2\pi}$, correct to 3 significant figures ?
   A. 0.23
   B. 0.230
   C. 0.229
   D. 0.22

2. Which graph best represents $y = x^2 + 2x + 1$?

(A)


(B)


(C)


(D)


3. What is the solution to the equation $\log_3(x + 1) = 4$?
   A. 11
   B. 81
   C. 80
   D. 12

4. Which equation represents the line parallel to $2x - 3y = 8$, passing through the point (-1, 2) ?
   A. $3x + 2y - 1 = 0$
   B. $3x + 2y - 8 = 0$
   C. $2x - 3y - 8 = 0$
   D. $2x - 3y + 8 = 0$

5. Which expression is a factorisation of $8x^3 - 27$?

A. $(2x-3)(4x^2+12x-9)$

B. $(2x+3)(4x^2-12x+9)$

C. $(2x-3)(4x^2+6x+9)$

D. $(2x+3)(4x^2-6x+9)$

6. The correct solutions to the equation $2\sin^2 x - 1 = 0$ for $-\pi \le x \le \pi$ are ?

A. $\frac{\pi}{4}, \frac{3\pi}{4}$

B. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

C. $\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

D. $\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

7. The value of $\sum_{o}^{\infty} 2 \times \left(\frac{3}{5}\right)^n$ is ?

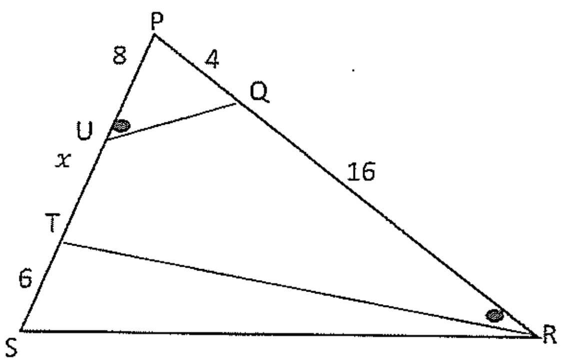
A. $\frac{6}{5}$

B. 2

C. 5

D. 3

8. The length of *PS* in the following diagram is :



A. 2

B. 24

C. 32

D. 16

9. A parabola with a directrix $x = 2$ has a focus at (-4, 3). The focal length of this parabola is?

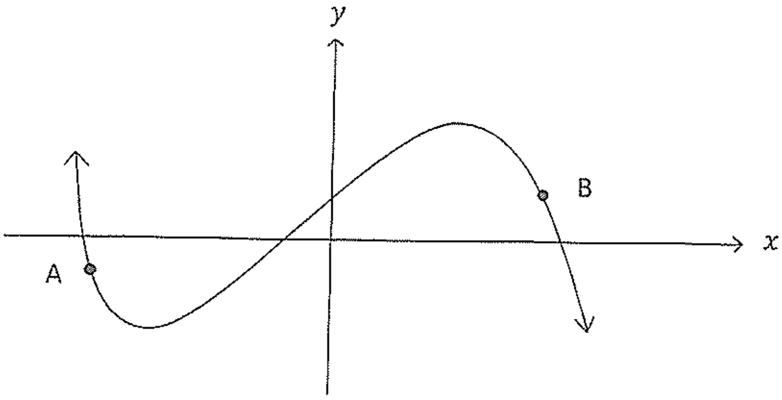
A. -6

B. -3

C. 6

D. 3

10. For the curve $y = f\ (x)$, which of the following statements is correct?



A. $f^{/}(x) > 0$ at A and $f^{//}(x) < 0$ at B
B. $f^{/}(x) < 0$ at A and $f^{//}(x) < 0$ at B
C. $f^{/}(x) > 0$ at A and $f^{//}(x) > 0$ at B
D. $f^{/}(x) < 0$ at A and $f^{//}(x) > 0$ at B

End of Section 1

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question on the appropriate writing page. Extra pages are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

a) Rationalise the denominator of $\frac{4}{\sqrt{6}-2}$ 2

b) Factorise $9x^2 - 37x + 4$ 2

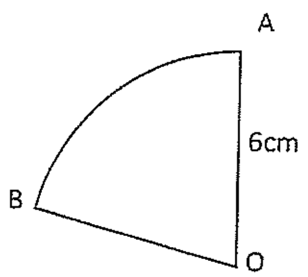
c) Differentiate $\frac{x+1}{x^3}$ 2

d) Find $\int \frac{dx}{(2x+1)^3}$ 2

e) Find $\int \cos\left(\frac{x}{2}\right) dx$ 2

f) Find the equation of the normal to the curve $y = 2\sqrt{x}$ at the point $x = 9$. 3

g) The perimeter of the sector AOB is 15cm. Calculate the size of angle AOB , correct to the nearest degree. 2

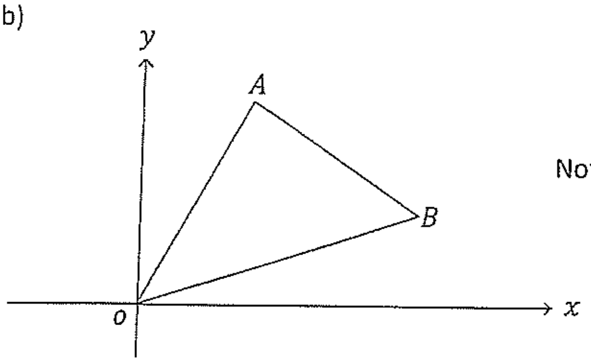


Question 12 (15 marks) ***Start a new page.***

a) Consider the series 106 + 97 + 88 + ...

Find the sum of all the positive terms belonging to this series. 3

b)



Not to scale

Consider the points $A(3,5)$, $O(0,0)$ and $B(6,2)$ in the diagram above.

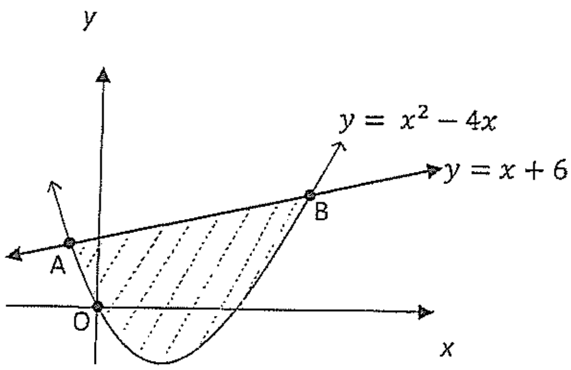
I. Find the equation of BO. 2

II. Show that the distance of the interval BO is $2\sqrt{10}$. 1

III. Show that the area of triangle AOB is 12 square units. 2

IV. Hence, or otherwise, find perpendicular distance from the point O to the line AB. 2

c) The parabola $y = x^2 - 4x$ and the line $y = x + 6$ intersect at the points A and B.



I. Find the x co-ordinate of the points A and B. 2

II. Calculate the area enclosed by the parabola $y = x^2 - 4x$ and the line

$y = x + 6$. 3

Question 13 (15 marks) *Start a new page.*

a) i) Write down the exact value of $\tan 2x$ when $x = \frac{\pi}{6}$ 1

ii) Give the exact value of $\int_0^{\frac{\pi}{6}} sec^2 2x \, dx$ 2

b) Simplify the expression $\frac{\cos(\frac{\pi}{2} - \theta)}{\sin(\pi + \theta)}$ 2

c) Consider the curve $y = 2x^3 + 3x^2 - 12x - 9$

i. Find the co-ordinates of any stationary points and determine their nature. 3

ii. Show that a point of inflexion exists and state its co-ordinates. 2

iii. Sketch the curve $y = f\ (x)$ in the domain $-3 \leq x \ \leq 3$, showing the

y –intercept. 2

iv. For what values of x, in the domain given in part (iii), is the curve both increasing

and concave down? 2

v. Write down the minimum value for $y = f\ (x)$ in the interval $-3 \leq x \ \leq 3$. 1

Question 14 (15 marks) ***Start a new page.***

a) i) Differentiate $2xe^{-x}$ 2

ii) Hence find $\int_0^1 xe^{-x}\ dx$ 2

b) The roots of the quadratic equation $3x^2 - kx + 18 = 0$ are $\propto$ and β

i) Find the value of $\propto\beta$. 1

ii) Given that $\propto^2 + \beta^2 = 4$, find the value/s of k. 2

c) The region bounded by the curve $y = 1 + x^2$ and the x-axis between $x = 0$ and $x = 3$ is rotated about the x-axis to form a solid. Find the volume of the solid. 3

d) The quantity of bacteria in a culture is growing according to the equation

$$\frac{dM}{dt} = kM$$

where M is the mass of the bacteria present in the culture in mg, t is the time in hours and k is a constant.

i) Show that $M = Ae^{kt}$ is a solution to the equation, where A is a constant. 1

ii) The time for the bacteria to double in mass is calculated to be 12 minutes.

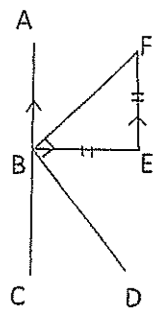
Write down the value of k, correct to 4 significant figures. 2

iii) If the initial amount of bacteria is 8mg, after how many minutes does the bacteria reach 1 gram? (Answer to one decimal place). 2

Question 15 (15 marks) ***Start a new page.***

a) Solve $2\log_5 x - \log_5(x+2) = \frac{2}{3}\log_5 125$ 3

b)



In the diagram BEF is a triangle with BE = EF. BF is perpendicular to BD and the line AC through B is parallel to EF.

i) Copy the diagram into your answer booklet.

ii) Prove that BF bisects angle ABE. 2

iii) Prove that BD bisects angle EBC. 2

c) Water is being released from a rainwater tank. The rate of flow, R litres per minute is given by $R = t\,(t-12)^2$, where t is the number of minutes since the water began to flow.

i) For how long does the water flow? 1

ii) Find the maximum rate of flow. 2

iv) What is the total volume of water released from the tank? 3

d) Given $y = e^{kx}$, find the value of k such that $y = 2\frac{dy}{dx} - \frac{d^2y}{dx^2}$ 2

Question 16 (15 marks) ***Start a new page.***

a) Allied Lending is offering a special on loans of \$50,000.00 or more. The terms offered are a reducible interest rate of only ½ % per month with the first six months interest free.

Paddington takes out a loan of \$80,000.00 to start his marmalade shop and agrees to the terms set out by Allied Lending.

Paddington agrees to repay the loan in equal monthly instalments of \$M, over 10 years with the first repayment due at the end of the first month. Let A_n be the amount owing at the end of the nth repayment.

i) Write down an expression for the amount Paddington owes at the end of the first six months. 1

ii) If the interest is calculated immediately before each repayment is made, show that the amount owing at the end of 8 months is,

$A_8 = (80000 - 6M)(1.005)^2 - M(1.005 + 1)$ 2

iii) Hence show that $A_{120} = (80000 - 6M)(1.005)^{114} - 200M(1.005^{114} - 1)$ 2

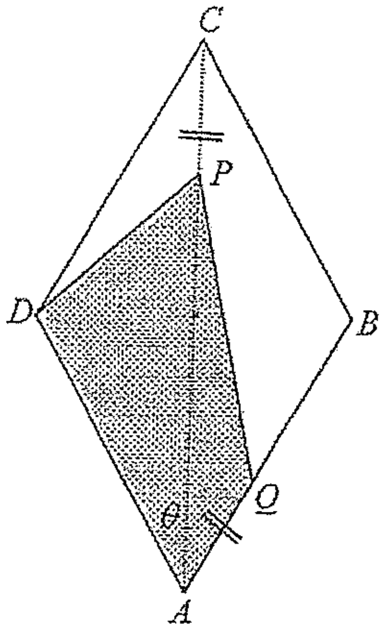
iv) Find the value of each monthly repayment correct to the nearest dollar. 2

b) ABCD is a rhombus of side 2cm.

P and Q are points on AC and AB respectively such that

$CP = AQ = x cm.$ $\angle DAP = \theta$ $(where\ 0 < \theta < \frac{\pi}{2})$ and θ is a constant.

$Let\ the\ area\ of\ the\ shaded\ area\ PDAQ\ be\ S\ cm^2.$



(i) Show that $S = \frac{\sin\theta}{2}(4\cos\theta - x)(2 + x)$ 3

(ii) If $\frac{dS}{dx} = 0$, find x in terms of θ 2

(iii) Find $\frac{d^2S}{dx^2}$ in terms of θ 1

(iv) Suppose that $\theta = \frac{\pi}{6}$, show that S attains its maximum when

$\frac{PC}{AC} = \frac{\sqrt{3}-1}{2\sqrt{3}}$ 2

End of Assessment task

STHS 2unit Trial

Solutions

1. B
2. A
3. C
4. D
5. C
6. C
7. C
8. D
9. D
10. B

Question 11

a) $\frac{4}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2}$

$= \frac{4(\sqrt{6}+2)}{6-4}$

$= 2(\sqrt{6}+2)$
$= 2\sqrt{6}+4$ either

b) $9x^2 - 37x + 4$

$= (9x-1)(x-4)$

c) $\frac{x+1}{x^3} = x^{-2} + x^{-3}$

$\frac{d}{dx} = -2x^{-3} - 3x^{-4}$

$= \frac{-2}{x^3} - \frac{3}{x^4}$

d) $\int \frac{dx}{(2x+1)^3} = \int (2x+1)^{-3}\,dx$

$= \frac{(2x+1)^{-2}}{-2\times 2} + C$

$= \frac{-1}{4(2x+1)^2} + C$

e) $\int \cos(\tfrac{1}{2}x)\,dx$

$= 2\sin(x/2) + C$

f) $x = 9 \quad y = 2\sqrt{9}$
$= 6$ $\therefore (9,6)$

$y' = x^{-1/2}$

$m_T = 1/\sqrt{9}$

$= 1/3 \quad \therefore M_N = -3$

Eq: $y-6 = -3(x-9)$

$y - 6 = -3x + 27$

$3x + y - 33 = 0$

g) $P = r\theta + 12$

$r\theta = 3$

$6\theta = 3$

$\theta = 1/2$

$\therefore \theta = \frac{1}{2} \times \frac{180}{\pi}$

$\doteqdot 29°$

Question 12

a) $106 + 97 + 88 + \dots$

AP $a = 106$

$d = -9$

$T_n < 0$

$106 + (n-1)(-9) < 0$

$-9n < -115$

$n > 12.\dot{7}$

$\therefore n = 13$

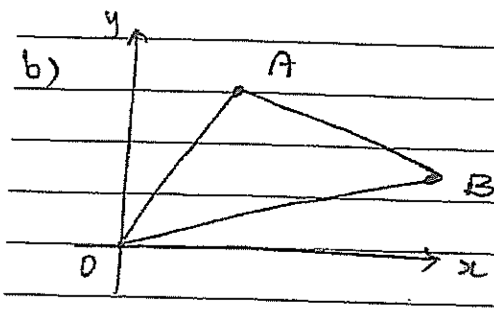
$T_{12} = 7$

$S_n = \frac{n}{2}(a + l)$ or $\frac{n}{2}(2a + (n-1)d)$

$= \frac{12}{2}(106 + 7)$

$= 678$

b)



(i) Equation $M_{BO} = \frac{0-2}{0-6}$

$= \frac{1}{3}$

$\therefore y = \frac{1}{3}x \longrightarrow 3y = x$

(ii) $B(6,2)$ $O(0,0)$

$d_{BO} = \sqrt{36 + 4}$

$= \sqrt{40}$

$= 2\sqrt{10}$

(iii) $A(3,5)$ $d_\perp = 12/\sqrt{10}$

$A = \frac{1}{2}hb$

$= \frac{1}{2} \times 12/\sqrt{10} \times 2\sqrt{10}$

$= 12 \, u^2$

AB $(3,5)$ $(6,2)$

$d_{AB} = \sqrt{(6-3)^2 + (2-5)^2}$

$= \sqrt{9 + 9}$

$= \sqrt{18}$

$= 3\sqrt{2}$

$\therefore \frac{1}{2}hb = \frac{1}{2} \times h \times 3\sqrt{2}$

$12 = \frac{1}{2} \times h \times 3\sqrt{2}$

$8 = h\sqrt{2}$

$h = \frac{8}{\sqrt{2}}$ $(4\sqrt{2})$

c) Solving

$x^2 - 4x = x + 6$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6, \quad x = -1$

(ii) $A = \int_{-1}^{6} (x+6) - (x^2 - 4x) \, dx$

$= \int_{-1}^{6} (5x + 6 - x^2) \, dx$

$= \left[\frac{5x^2}{2} + 6x - \frac{x^3}{3} \right]_{-1}^{6}$

$= \frac{5}{2}(36) + 36 - \frac{6^3}{3} - \left(\frac{5}{2} - 6 + \frac{1}{3} \right)$

$= 57\frac{1}{6} \, u^2$

Question 13

i) $\tan 2x = \tan 2(\pi/6)$

$= \tan \pi/3$

$= \sqrt{3}$

ii) $\int_0^{\pi/6} \sec^2 2x \, dx = \frac{1}{2} \tan 2x \Big]_0^{\pi/6}$

$= \frac{1}{2}\left[\tan \pi/3 - \tan 0\right]$

$= \frac{1}{2}\left[\sqrt{3} - 0\right]$

$= \sqrt{3}/2$

b) $\dfrac{\cos(\pi/2 - \theta)}{\sin(\pi + \theta)} = \dfrac{\sin\theta}{-\sin\theta}$

$= -1$

c) $y = 2x^3 + 3x^2 - 12x - 9$

$dy/dx = 6x^2 + 6x - 12$

$d^2y/dx^2 = 12x + 6$

i) Stat pts

$6(x^2 + x - 2) = 0$

$6(x+2)(x-1) = 0$

$x = -2$, $y = 11$ | $x = 1$, $y = -16$

$\therefore (-2, 11)$ & $(1, -16)$

Nature

x	-3	-2	-1
y'	12	0	-12
	/	—	\

x	-1	1	2
y'	-12	0	24
	\	—	/

$\therefore$ MAX TP & MIN TP

$(-2, 11)$ $(1, -16$

ii. inflexion $y'' = 0$

$12x + 6 = 0$

$x = -1/2$

$y = -5/2$

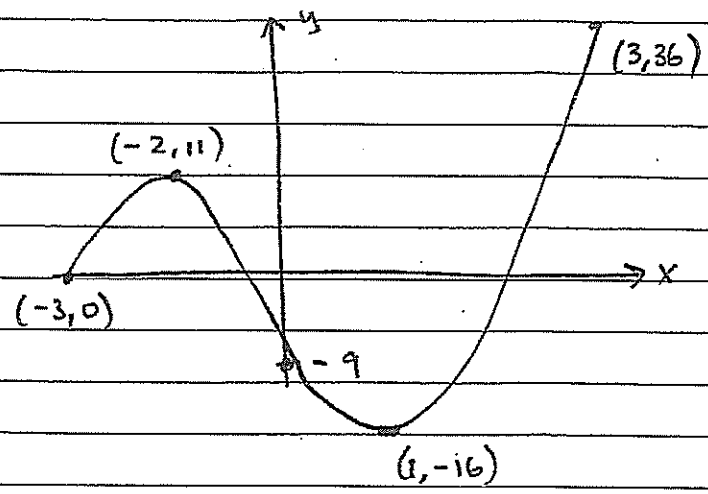
concavity check

$x = -1$ $x = -1/2$ $x = 0$

$y'' = -6$ 0 $y'' = 6$

$\therefore$ change in concavity @ $\left(-\frac{1}{2}, -\frac{5}{2}\right)$

iii. end pts $(-3, 0)$ & $(3, 36)$



iv) ↗ and ∩ $-3 \le x < -2$

accept $-3 < x < -2$

$\boxed{\text{donot accept } -3 \le x \le -2}$

v) min value is $\boxed{-16}$

donot accept $(1, -16)$

Question 14

a) $\frac{d}{dx}(2xe^{-x})$

$= 2e^{-x} - 2xe^{-x}$ 2

ii) $\frac{1}{2}\frac{d}{dx}(2xe^{-x}) = e^{-x} - xe^{-x}$

$\therefore \int xe^{-x}\,dx = \left[\int e^{-x} - \int \frac{1}{2}\frac{d}{dx}(2xe^{-x})\right] dx$

$= \int e^{-x}\,dx - xe^{-x}$

$= -e^{-x} - xe^{-x}$

$\therefore \int_0^1 xe^{-x}\,dx = \left[-e^{-x} - xe^{-x}\right]_0^1$

$= -e^{-1} - 1e^{-1} - (-e^0 - 0)$

$= -1/e - 1/e + 1$

$= 1 - 2/e$

$= \boxed{1 - \frac{2}{e}}$ 2 $\frac{e-2}{e}$

b) i. $\alpha\beta = c/a$

$= 6$ 1

ii. $\alpha^2 + \beta^2 = 4$ $\alpha + \beta = \frac{K}{3}$

$(\alpha+\beta)^2 - 2\alpha\beta = 4$

$(\alpha+\beta)^2 - 12 = 4$

$(K/3)^2 = 16$

$\frac{K}{3} = 4$ or -4

$\therefore K = \pm 12$ 2

c) $V_x = \pi \int y^2\,dx$

$= \pi \int_0^3 (1 + x^2)^2\,dx$

$= \pi \int_0^3 1 + 2x^2 + x^4 \; dx$

$= \pi \left[x + \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^3$

$= \pi \left[3 + \frac{2.27}{3} + \frac{3^5}{5} - (0)\right]$

$= \frac{348\pi}{5} u^3$ 3

d) $M = Ae^{kt}$

$\frac{dM}{dt} = k.Ae^{kt}$

$= k.M \quad \therefore$ a soln 1

ii. $2A = Ae^{k(12)}$ $2 = e^{\frac{1}{5}k}$

$2 = e^{12k}$ $\ln 2 = \frac{1}{5}k$

$\ln 2 = 12k$ $k = 3.466$

$k = 1/12 \ln 2$ 2

$\doteqdot 0.05776$ (4 sig fig)

iii. 1 gram = 1000 mg

$1000 = 8e^{kt}$ using $k = 0.0577$

$125 = e^{kt}$

$\ln 125 = kt$

$t = \ln 125 \div k$

$= 83.592\ldots$

$\therefore 83.6$ minutes. 2

Question 15

a) $2\log_5 x - \log_5(x+2) = \frac{2}{3}\log_5 125$

$$\log_5\left(\frac{x^2}{x+2}\right) = \log_5 (5^3)^{2/3}$$

$$\log_5\left(\frac{x^2}{x+2}\right) = \log_5 5^2$$

$$\frac{x^2}{x+2} = 25$$

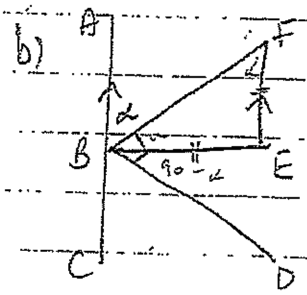
$$x^2 - 25x - 50 = 0$$

$$x = \frac{25 \pm \sqrt{825}}{2}$$

as $x > 0$

$$x = \frac{25+\sqrt{825}}{2} \text{ only}$$

b)



i. let $\angle ABF = \alpha$

$\angle BFE = \angle ABF$ (alternate angles $AC \parallel EF$)

$= \alpha$

$\angle FBE = \angle BFE$ (equal angles opposite equal sides $FE = BE$ given)

$= \alpha$

$\therefore \angle ABF = \angle FBE \ (= \alpha)$

$\therefore$ BF bisects angle ABE

ii. $\angle EBD = 90 - \alpha$ (adjacent complementary with $\angle FBE$)

$\angle ABE + \angle EBD + \angle CBD = 180$ (straight line)

$\therefore \angle CBD = 90 - \alpha$

$= \angle EBD$

and BD bisects $\angle EBC$

c) $R = dV/dt = t(t-12)^2$

i) $R \geq 0 \quad \therefore \quad 0 \leq t \leq 12$

$\therefore$ flows for 12 minutes

ii) MAX flow rate $\frac{dR}{dt} = 0$

$R = t(t^2 - 24t + 144)$

$= t^3 - 24t^2 + 144t$

$dR/dt = 3t^2 - 48t + 144 = 0$

$t^2 - 16t + 48 = 0$

$(t-12)(t-4) = 0$

as $0 \leq t \leq 12$

test $t = 4$

t	3	4	5
$\frac{dR}{dt}$	27	0	−21

max flow rate is $R = 4(4-12)^2$

$= 256$ L/min

iii) $V = \int_0^{12} t^3 - 24t^2 + 144t \, dt$

$= \left[\frac{t^4}{4} - 8t^3 + 72t^2\right]_0^{12}$

$= \left[\frac{12^4}{4} - 8(12)^3 + 72(12)^2 - 0\right]$

$= 1728$ L.

d) $y = e^{kx}$ $\quad y' = ke^{kx}$ $\quad y'' = k^2e^{kx}$

$y = 2y' - y''$

$e^{kx} = 2(ke^{kx}) - (k^2e^{kx})$

$1 = 2k - k^2$

$k^2 - 2k + 1 = 0$

$(k-1)^2 = 0$

$\therefore k = 1$

Question 16

a)

(i) $A_6 = 80\,000 - 6m$

(ii) $A_7 = (80\,000 - 6M)(1.005) - m$

$A_8 = \left[(80\,000 - 6M)(1.005) - m\right](1.005) - m$

$= (80\,000 - 6M)(1.005)^2 - m(1.005) - m$

$= (80000 - 6M)(1.005)^2 - M\left[1.005 + 1\right]$

$= (80000 - 6m)(1.005)^2 - M(2.005)$

(iii) $A_{120} = (80\,000 - 6M)(1.005)^{114} - M\left[1 + 1.005 + \dots\right]$

G.P $a = 1$

$r = 1.005$

$n = 114$

$= (80\,000 - 6M)(1.005)^{114} - M\left[\frac{a(r^n - 1)}{r - 1}\right]$

$= (80\,000 - 6m)(1.005)^{114} - M\left[\frac{1.005^{114} - 1}{0.005}\right]$

$= (80\,000 - 6M)(1.005)^{114} - 200M(1.005^{114} - 1)$

iv) $A_{120} = 0$ Find M

$0 = 80000(1.005)^{114} - M\left[6(1.005)^{114} + 200(1.005)^{114} - 200\right]$

$M = 80\,000(1.005)^{114} \div 163.746\ldots$

$= \$862.681\ldots$

∴ repayment is \$863.

(i) Let X be the intersection of the diagonals.

$\angle XAD = \angle XAC = \theta$ [property of rhombi]

$AX = 2\cos\theta \Rightarrow AC = 4\cos\theta$

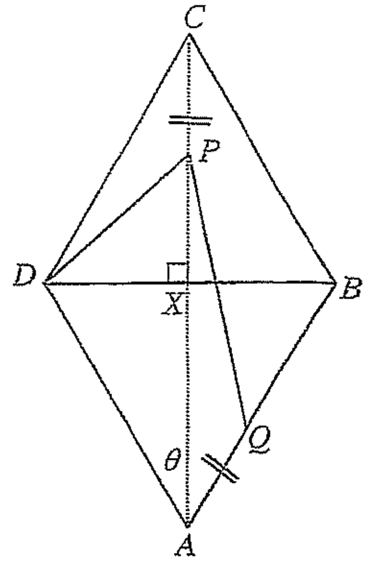
$\therefore AP = 4\cos\theta - x$

The shaded area is the sum of triangles ADP and

$$S = \frac{1}{2}\times 2\times(4\cos\theta - x)\sin\theta + \frac{1}{2}\times(4\cos\theta - x)\times$$

$$= \frac{\sin\theta}{2}(4\cos\theta - x)(x+2)$$

[NB S is a concave down parabola in x]



(ii) $$S = \frac{\sin\theta}{2}\left[8\cos\theta + (4\cos\theta - 2)x - x^2\right]$$

$$\frac{dS}{dx} = \frac{\sin\theta}{2}\left[(4\cos\theta - 2) - 2x\right] = \sin\theta(2\cos\theta - 1 -$$

$$\therefore \frac{dS}{dx} = 0 \Rightarrow x = 2\cos\theta - 1 \qquad [\because \sin\theta \neq 0]$$

(iii) $$\frac{dS}{dx} = \sin\theta(2\cos\theta - 1 - x)$$

$$\therefore \frac{d^2S}{dx^2} = -\sin\theta \qquad \left[< 0 \text{ for } 0 < \theta < \frac{\pi}{2}\right]$$

(iv) $$\theta = \frac{\pi}{6}$$

$$\frac{dS}{dx} = 0 \Rightarrow x = 2\cos\left(\frac{\pi}{6}\right) - 1 = \sqrt{3} - 1$$

$$\frac{d^2S}{dx^2} < 0 \Rightarrow S \text{ is a maximum}$$

$$AC = 4\cos\left(\frac{\pi}{6}\right) = 2\sqrt{3}$$

$$\therefore \frac{PC}{AC} = \frac{\sqrt{3} - 1}{2\sqrt{3}}$$