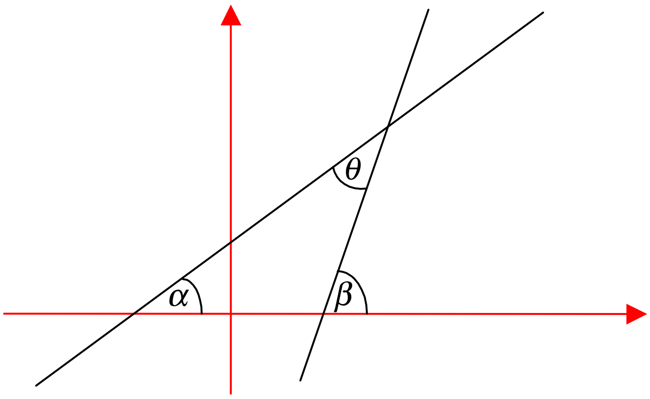
Finding the (acute) Angle Between Two Lines



$$\begin{aligned}
\theta &= \alpha - \beta \\
\tan\theta &= |\tan(\alpha - \beta)| \\
&= \left|\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right| \\
&= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|
\end{aligned}$$

$$\begin{aligned}
m_1 &= \tan\alpha \\
m_2 &= \tan\beta
\end{aligned}$$

The gradients of the lines must be taken in y = mx + b form

<u>Example 1</u>
Find the acute angle between the lines 4x + y = 6, x – 7y = 3

$$\begin{aligned}
4x + y &= 6 \\
y &= -4x + 6
\end{aligned}$$

$$\begin{aligned}
x - 7y &= 3 \\
7y &= x - 3 \\
y &= \tfrac{x}{7} - \tfrac{3}{7}
\end{aligned}$$

$m_1 = -4$ $\qquad$ $m_2 = \frac{1}{7}$

$$\begin{aligned}
\tan\theta &= \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \\
&= \left|\frac{-4 - \frac{1}{7}}{1 + (-4)(\frac{1}{7})}\right| \\
&= \left|\frac{-\frac{29}{7}}{\frac{3}{7}}\right| \\
&= \frac{29}{3} \\
\theta &= 84^\circ\, 6'
\end{aligned}$$

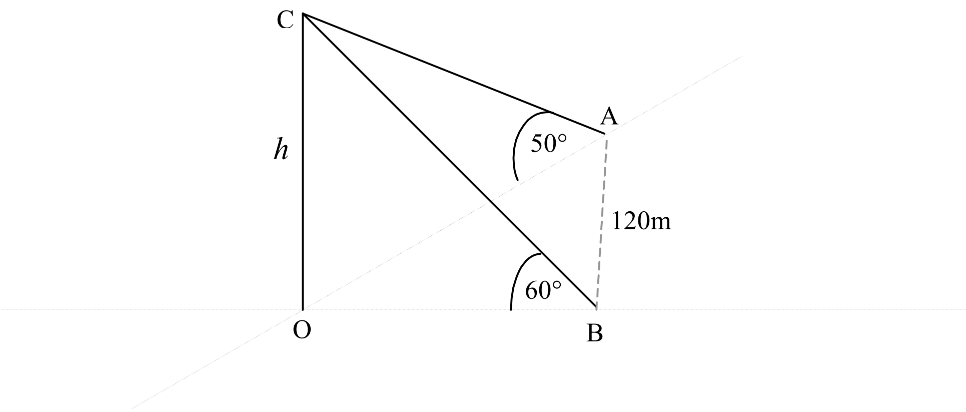
3D Trigonometry

- Draw a neat sketch
- Use $\cot x$ instead of $\dfrac{1}{\tan x}$
- Pythagoras, Cosine or Sine rule?

$a^2 = b^2 + c^2$ $\qquad$ $a^2 = b^2 + c^2 - 2bc\cos A$ $\qquad$ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

<u>Example 1</u>

The elevation of the top of a television tower is 50° from point A and 60° from point B. The points are 120m apart and A is due north and B is due east of the tower.
Find the height of the tower.



$\tan 60 = \dfrac{h}{BO}$ $\qquad$ $\tan 50 = \dfrac{h}{AO}$

$BO = h\cot 60$ $\qquad$ $AO = h\cot 50 \; h\cot 50$

$$
\begin{aligned}
120^2 &= BO^2 + AO^2 \\
14400 &= h^2\cot^2 60 + h^2\cot^2 50 \\
&= h^2(\cot^2 60 + \cot^2 50) \\
h^2 &= \frac{14400}{\cot^2 60 + \cot^2 50} \\
h &= \sqrt{\frac{14400}{\cot^2 60 + \cot^2 50}} \\
&= 117.82\text{m}
\end{aligned}
$$

Division of an Interval Given a Ratio

Finding the co-ordinates

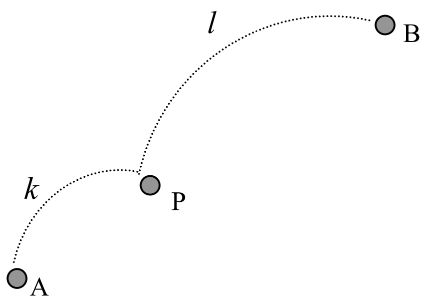
$$x = \frac{kx_2 + lx_1}{k + l} \qquad\qquad y = \frac{ky_2 + ly_1}{k + l}$$

An interval may be divided internally or externally.

<u>Internally:</u>

Co-ordinates should be chosen appropriately according to the ratio

- A (x_1, y_1)
- B (x_2, y_2)
- P (x, y)

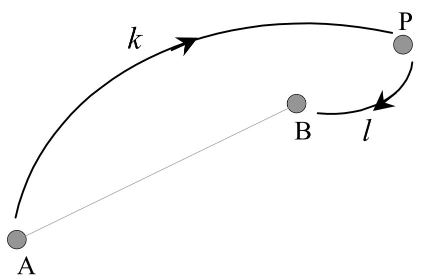


<u>Externally:</u>

If:

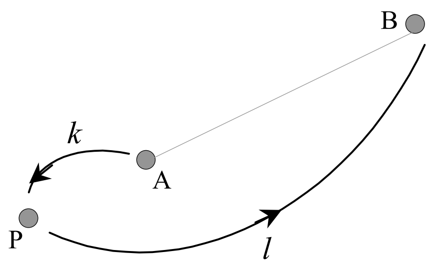
$|k : l| > 1$

$k^+ \quad l^-$

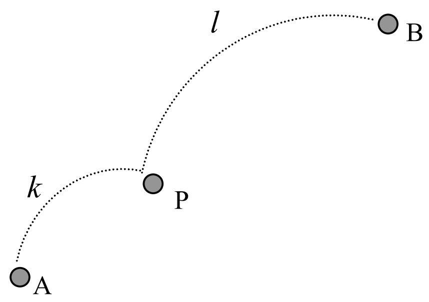


$|k : l| < 1$

$k^- \quad l^+$



Dividing the Interval **INTERNALLY**



- A (x_1, y_1)
- B (x_2, y_2)
- P (x, y)

<u>Example 1</u>
The interval AB has end points at A(-2, 3) and B(10,11).
Find the co-ordinates of the point P which divides the interval AB 3:1 internally.

$$\begin{aligned} x_p &= \frac{kx_2 + lx_1}{k+l} \\ &= \frac{3(10)+1(-2)}{3+1} \\ &= \frac{30-2}{4} \\ &= \frac{28}{4} \\ &= 7 \end{aligned}$$

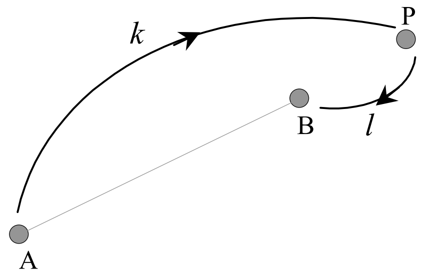
$$\begin{aligned} y_p &= \frac{ky_2 + ly_1}{k+l} \\ &= \frac{3(11)+1(3)}{3+1} \\ &= \frac{33+3}{4} \\ &= \frac{36}{4} \\ &= 9 \end{aligned}$$

The co-ordinates of P are (7, 9)

Dividing the Interval **EXTERNALLY**

$|k:l|>1$

$k^+ \quad l^-$



$|k:l|<1$

$k^- \quad l^+$

Example 2

The interval AB, where A(–1, 2) and B(2, 5) is divided externally by P in the ration 3:1. Find the co-ordinates of P.

$$\begin{aligned} x_p &= \frac{kx_2+lx_1}{k+l} \\ &= \frac{3(3)+(-1)(-1)}{3+(-1)} \\ &= \frac{9+1}{2} \\ &= \frac{10}{2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} y_p &= \frac{ky_2+ly_1}{k+l} \\ &= \frac{3(5)+(-1)(2)}{3+(-1)} \\ &= \frac{15-2}{2} \\ &= \frac{13}{2} \\ &= 6.5 \end{aligned}$$

The co-ordinates of P are $(5, 6\frac{1}{2})$

Limits of Sin, Cos, Tan

$$\lim_{x\to 0}\frac{\sin x}{x} \qquad = \lim_{x\to 0}\frac{\tan x}{x} \qquad = \lim_{x\to 0}\cos x \qquad = 1$$

<u>Example 1</u>

$$\lim_{x\to 0}\frac{\tan 4x}{\sin 2x} \qquad = \frac{\tan 4x}{4x}\times\frac{4x}{2x}\times\frac{2x}{\sin 2x}$$
$$= \frac{4}{2}$$
$$= 2$$

Mathematical Induction

- Normal addition
- Inequality
- Divisible

Proof by induction consists of three steps:

<u>Step 1</u> Show that the result is true for n = 1 or (n = 1, 2, 3)

<u>Step 2</u> Assume the result is true for n = k.
Then use the assumed result to prove that n = k + 1

<u>Step 3</u> Conclusion:
If the statement is true for n = 1
And true for n = k, n = k + 1
Then the statement holds true for all positive integer n

Normal Addition

Prove $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$

<u>Step 1:</u> Let n = 1

LHS = $1^3 = 1$

RHS = $\frac{1}{4}(1)^2((1)+1)^2 = 1$

LHS = RHS

True for n = 1

<u>Step 2:</u> Assume n = k

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{1}{4}k^2(k+1)^2$$

Prove n = k + 1

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 &= \tfrac{1}{4}(k+1)^2\left((k+1)+1\right)^2 \\
\tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 &= \tfrac{1}{4}(k+1)^2(k+2)^2 \\
\tfrac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] & \\
\tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) & \\
\tfrac{1}{4}(k+1)^2(k+2)^2 & \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

<u>Step 3:</u>

True for n = 1

True for n = k, n = k+1

∴ True for all positive integer n

Divisible

Prove that $3^{4n} - 1$ is divisible by 80

Step 1: Let n = 1

$$3^{4(1)} - 1 = 81 - 1 = 80 \text{ is divisible by } 80$$

True for n = 1

Step 2: Assume n = k

Assume $3^{4k} - 1$ is divisible by 80

So: $3^{4k} - 1 = 80M$ where M is an integer

Prove n = k + 1

$$\begin{aligned} 3^{4(k+1)} - 1 &= 3^{4k+4} - 1 \\ &= 3^4.3^{4k} - 1 \\ &= 81.3^{4k} - 1 \\ &= 81(1 + 80M) - 1 \\ &= 81 - 1 + 81.80M \\ &= 80 + 81.80M \\ &= 80(1 + 81M) \quad \text{is divisible by } 80 \end{aligned}$$

Step 3:

True for n = 1

True for n = k, n = k + 1

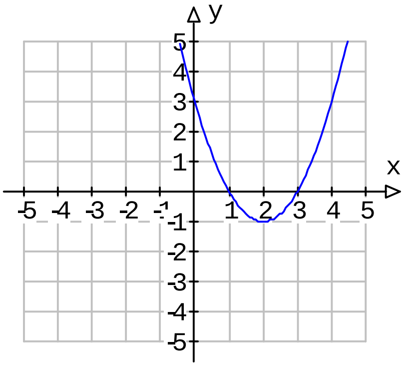
∴ True for all positive integer n

Inequalities

1. Find asymptotes
2. Multiply both sides by denominator^2
3. Equate to 0
4. Factorize
5. Solve
6. State and graph solution

<u>Example 1</u>

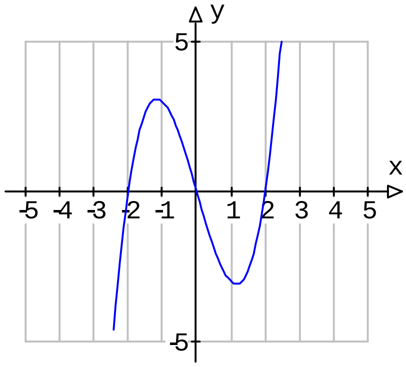
$$\begin{aligned}
\frac{2}{x-1} &\leq 1 \\
\frac{(x-1)^2 2}{x-1} &\leq 1\,(x-1)^2 \\
2(x-1) &\leq (x-1)^2 \\
2(x-1)-(x-1)^2 &\leq 0 \\
(x-1)[2-(x-1)] &\leq 0 \\
(x-1)(3-x) &\leq 0
\end{aligned}$$



$x \neq 1$ | $3-x \leq 0$
$x-1 < 0$ | $3 \leq x$
$x < 1$ | $x \geq 0$

<u>Example 2</u>

$$\begin{aligned}
\frac{x^2-4}{x} &> 0 \\
\frac{x^2(x^2-4)}{x} &> 0 \\
x(x-2)(x+2) &> 0
\end{aligned}$$



$x = 0$ | $x-2>0$ | $x+2>0$
 | $x>2$ | $x>-2$

$-2 < x < 0$ & $x > 2$