Sydney Technical High School

Mathematics - Extension One

HSC Assessment Task 1

December 2012

Name .... Teacher ....

General Instructions

- Working Time – 70 minutes.
- Write using a blue or black pen.
- Approved calculators may be used.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**55**)

- Attempt Questions 1-11.
- Marks indicated are a guide.
- All answers must be written in your answer book

Section 1

5 marks

Attempt Questions 1-5

Use the multiple-choice answer sheet for Questions 1-5

1. An infinite geometric series has a first term of 8 and a limiting sum of 12. What is the common ratio?

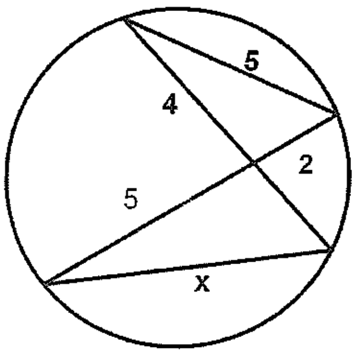
   (A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

2. The equation of the normal to the parabola $x^2 = 4\alpha y$ the variable point $P(2\alpha\rho, \alpha\rho^2)$ is given by $x+\rho y = 2\alpha\rho + \alpha\rho^3$.

   How many different values of ρ are there such that the normal passes through the focus of the parabola?

   (A) 0
   (B) 1
   (C) 2
   (D) 3

3.

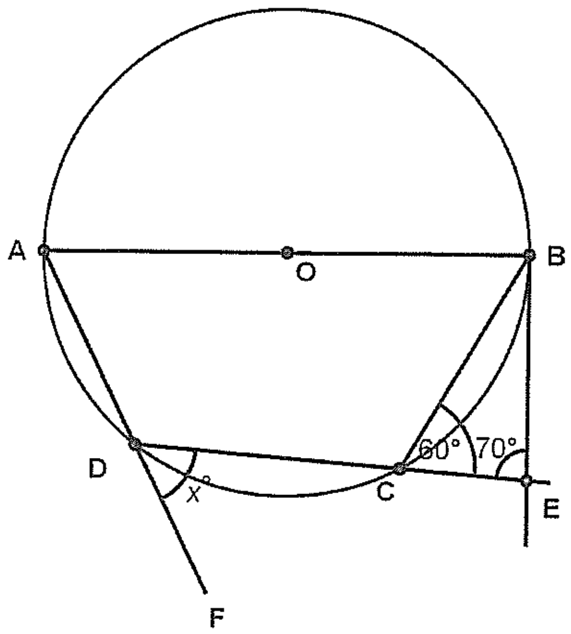


NOT TO SCALE

The value of x is

(A) 2.5 (B) 6.25 (C) 10 (D) 12.5

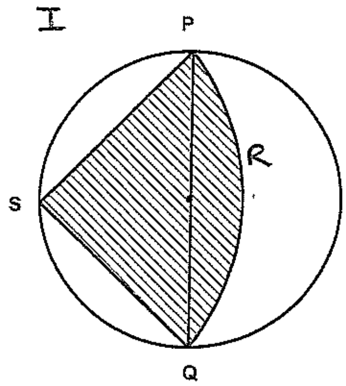
4.

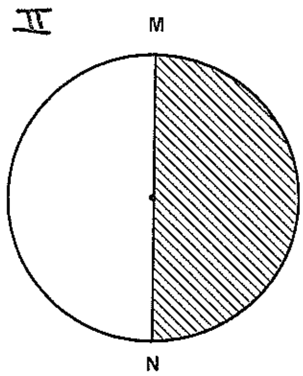


O is the centre of the circle.
AB is a diameter.
BE is a tangent to the circle.
Find the value of x.

(A) 40
(B) 50
(C) 60
(D) 70

5. In the circles below, diameter PQ = diameter MN.
In diagram I, PRQ is an arc of a circle centre S.





In which diagram is the greater area shaded?

A. Diagram I

B. Diagram II

C. The shaded areas in both diagrams are the same.

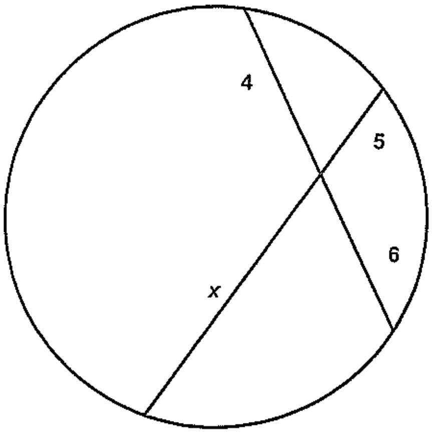
D. Cannot be determined from the information provided.

Section II. **Marks**

Question 6-11

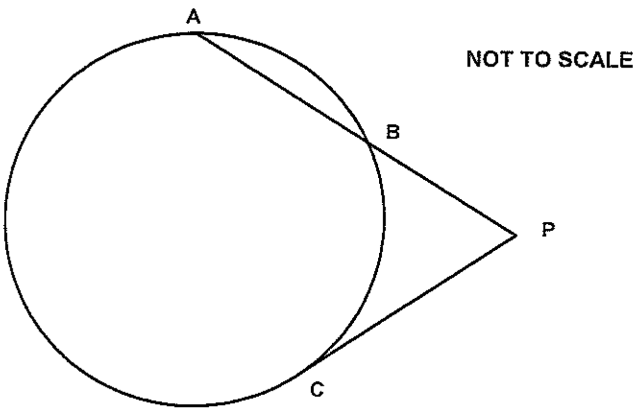
Question 6 (7 Marks)

a) Find the value of x. (reason required) **2**



b) Over 7 years \$125 grows to \$164.49. **2**

Find the compound interest rate as a percentage per annum.



c)

In the diagram the points A,B and C lie on the circle and AB produced meets the tangent from C at the point P.

(i) Given that $PC = 12\text{cm}$, $AB = 7\text{cm}$ and $PB = x$, find x. (reason not required) **2**

(ii) PC is the diameter of the circle passing through P, B and C.

Find the length of BC. (in exact form) **1**

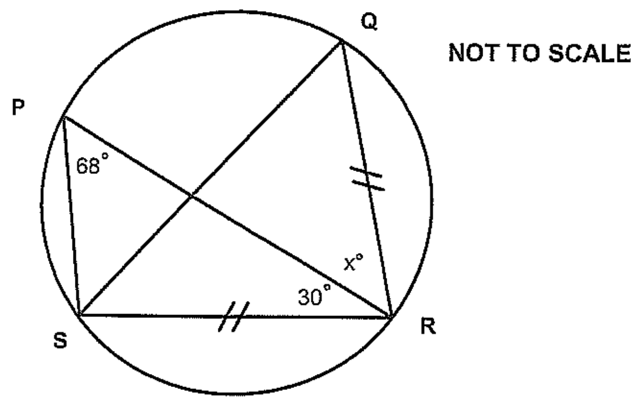
Question 7 (8 marks) **Start a new page**

a) A gardener plants a bed of roses. The bed is planted so that the first row has 24 rose plants. The second row has 29 rose plants. Each succeeding row has 5 more rose plants than the previous row.

(i) Calculate the number of roses in the eighth row. **1**

(ii) Which row would be the first to contain more than 150 rose plants? **2**

(iii) The gardener has planted 2895 roses. Assuming that the above pattern has been continued, how many rows were planted? **2**



b) The diagram shows a circle. The points P, Q, R and S lie on the circumference of the circle. Find the value of x? (reasons required) **3**

Question 8 **(8 marks)** **Start a new page** **Marks**

a) With the drought ever worsening, James and Theodore design a counting generator that can simulate the number of rain drops per minute that fall over a river during a storm. The rain drops falling per minute forms the series

$$1 + 1 + 3 + 9 + 23 + \ldots$$

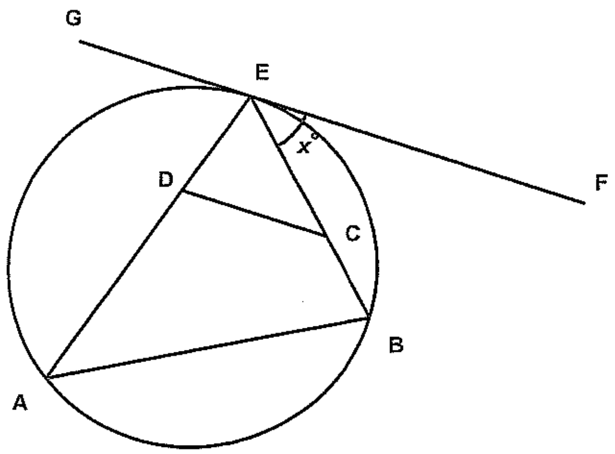
with the nth term given by the formula $R_n = 1 - 2n + 2^n$, where n represents the number of minutes.

(i) Which term of the series is 115? **1**

(ii) Find the total amount of rain drops which fall over the river in the first twenty five minutes. **3**

(iii) If the surface area of the river is 250m^2 find the average number of drops over the per cm^2 first twenty five minutes. (to the nearest drop) **1**

b)

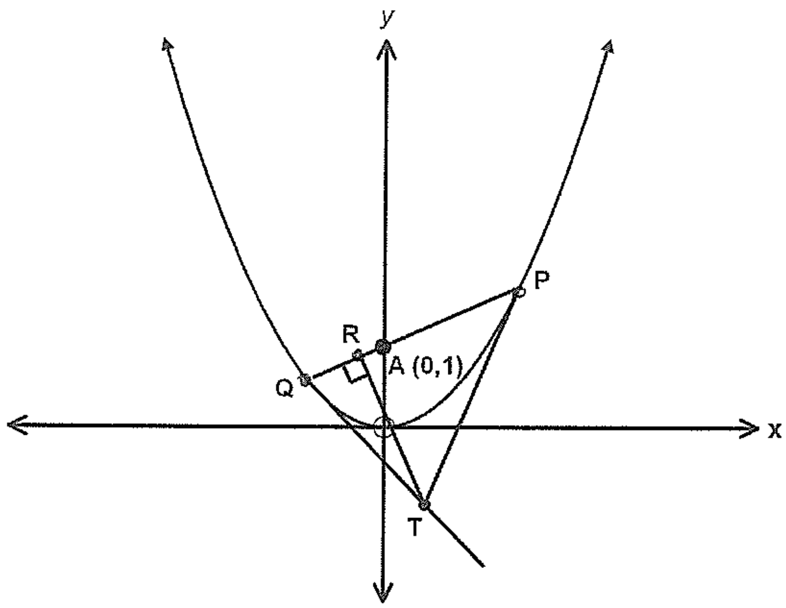


GF is a tangent to the circle at E and ABCD is a cyclic quadrilateral **3**

$\angle FEC = x^\circ$

Prove DC//GF

Question 9 (9 marks) **Start a new page**



PQ is a chord of the parabola $x^2 = 8y$ passing through the point A(0, 1) where P is $(4p, 2p^2)$ and Q is $(4q, 2q^2)$.

The tangents to the parabola at P and Q meet at the point T.

R is a point on the chord PQ with $RT \perp PQ$.

a) Show the equation of the tangent at P is given by $y - px + 2p^2 = 0$ and write the equation of the tangent at Q. **3**

b) Show the co-ordinates of the point T are $x = 2(p+q)$, $y = 2pq$ **2**

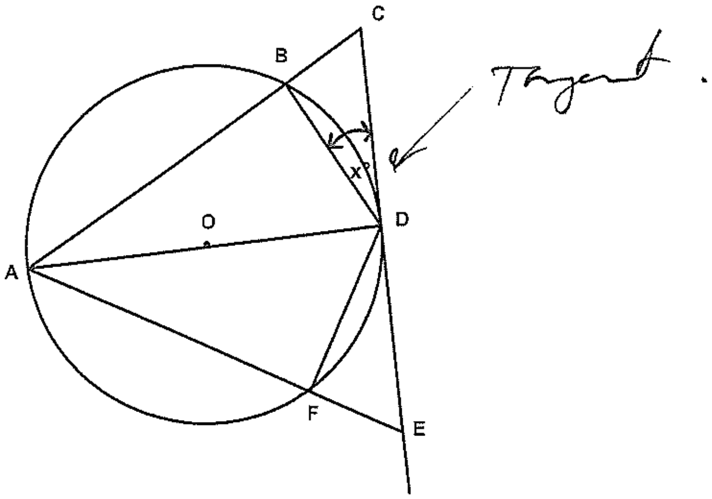
c) Show that the equation of the chord PQ is given by $2y = (p+q)x - 4pq$ **2**

d) Show that $pq = -\frac{1}{2}$ **1**

e) Find the equation of RT **1**

Question 10 **(8 marks)** **Start a new page**

a)



Copy or trace this diagram onto your Answer booklet.
Let CDB $= x$
Prove BCEF is a cyclic quadrilateral.
O is the centre of the circle. (reasons required) 3

b) Prove by mathematical induction that the following is true for all positive integers n.

$$\sum_{r=1}^{n} r(2^r) = (n-1) \cdot 2^{n+1} + 2$$ 5

Question 11 (10 marks) **Start a new page**

a) The sum of the first n terms of a series is given by $S_n = \frac{n}{3}(n+1)(n+2)$.

i) Show that the nth term is given by $T_n = n(n+1)$. **2**

ii) Find the sum of <u>the second</u> 50 terms. **2**

b) Stella sets up a prize fund with a single investment of \$1000 to provide her school with an annual prize of \$72.00. The fund accrues interest at a rate of 6% per annum, compounded annually. The first prize is awarded one year after the investment is set up.

i) Calculate the balance in the fund at the beginning of the second year, after the first prize has been awarded. **1**

ii) Let $\$B_n$ be the balance in the fund at the end of n years (after the *nth* Prize has been awarded and while funds are still available).
Show that $B_n = 1200 - 200\,(1.06)^n$ **2**

iii) At the end of the tenth year (after that prize has been awarded), it is decided that the prize will henceforward be increased to \$90. **3**

Show that the fund can only award the full prize for 14 more years.

S.T.H.S HSC TASK 1 - EXT 1 - DECEMBER 2012

Q.1 C

Q.2 B

Q.3 B

Q.4 A

Q.5 C

Question 6

a) $\frac{4}{6} = \frac{5}{x}$

$4x = 30$

$x = 7\frac{1}{2}$

b) $164.49 = 125(1+R)^7$

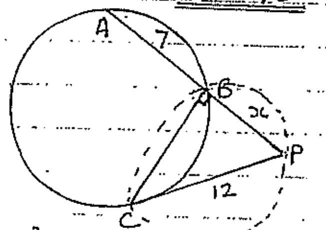
$\frac{164.49}{125} = (1+R)^7$

$\therefore 1+R = \left(\frac{164.49}{125}\right)^{1/7}$

$R = .03999867\ldots$

$\therefore R\% = 4.00\%$

c)



i) $12^2 = (7+x)x$

$144 = 7x + x^2$

$x^2 + 7x - 144 = 0$

$(x+16)(x-9) = 0$

$\therefore x = 9 \qquad x > 0$

ii) $12^2 = BC^2 + x^2$ (angle in semi circle $90°$)

$144 = BC^2 + 81$

$BC = \sqrt{63}$ cm

Question 7

) 24, 29, 34 .....

i) $T_8 = 24 + 7\times5$

$T_8 = 59$

ii) $T_n > 150$

$24 + (n-1)\times5 > 150$

$24 + 5n - 5 > 150$

$5n > 131$

$n > 26.2$

$\therefore$ Row 27 has more than 150 plant

iii) $S_n = 2895$

$2895 = \frac{n}{2}(48 + (n-1)\times5)$

$5790 = n(48 + 5n - 5)$

$5790 = n(5n + 43)$

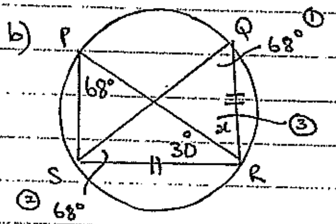
$0 = 5n^2 + 43n - 5790$

$n = \frac{-43 \pm \sqrt{43^2 - 4.5.-5790}}{10}$

$n = 30 \quad , n \geq 0$

$\therefore$ 30 rows planted

b)



$S\hat{Q}R = 68°$ (angles in same segment)

$Q\hat{S}R = 68°$ (angles opposite equal sides in isosceles triangle)

$x = 14°$ (angle sum of ΔQSR)

Question 8

a) i) Guess / Check $n = 7$

$R_7 = 1 - 2\times7 + 2^7$

$R_7 = 115 \quad \therefore 7^{th}$ term

ii) $R_n = \underbrace{(1-2n)}_{AP} + \underbrace{2^n}_{G.P.}$

AP: $T_n = 1 - 2n$

$-1, -3, -5, \ldots \quad a = -1 \quad d = -2$

$S_{25} = \frac{25}{2}(-2 + 24\times-2)$

$= -625$

GP $T_n = 2^n$

$2, 4, 8 \ldots \quad a = 2 \quad r = 2$

$S_{25} = \frac{2(2^{25}-1)}{2-1}$

$= 67108862$

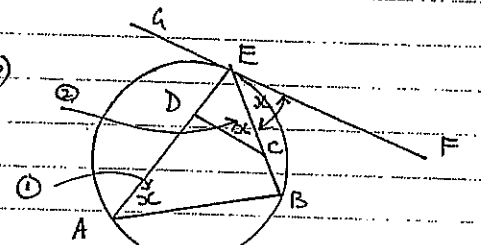
$\therefore$ Total rain in first 25 min

67,108,237 drops

iii) $250 m^2 = 2,500,000 cm^2$

$\therefore \frac{67108237}{2500,000} \div 27$ drops / cm^2

b)

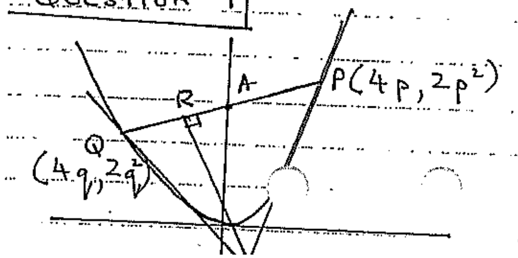


$E\hat{A}B = x$ (alternate segment theorem)

$E\hat{C}D = x$ (exterior angle of cyclic quad ABCD)

Since $F\hat{E}C = D\hat{C}E$ and they are alternate then $GF \parallel DC$

Question 9



a) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8}$

$dy/dx = \frac{x}{4}$

$\therefore$ grad of tangent at $P(4p, 2p^2)$

$m_p = p$

eqn of tangent at P

$y - 2p^2 = p(x - 4p)$

$y - 2p^2 = px - 4p^2$

$y - px + 2p^2 = 0$ — (

eqn tangent at Q

$y - qx + 2q^2 = 0$ — ②

b) sim eq — ① and ②

$px - 2p^2 = qx - 2q^2$

$px - qx = 2p^2 - 2q^2$

$x(p-q) = 2(p-q)(p+q)$

$x = 2(p+q)$

$y = p\,2(p+q) - 2p^2$

$y = \cancel{2p^2} + 2pq - \cancel{2p^2}$

$y = 2pq$

$\therefore T\ (2(p+q), 2pq)$

c) chord PQ

$m_{PQ} = \frac{2p^2 - 2q^2}{4p - 4q} = \frac{2(p-q)(p+q)}{4(p-q)}$

$m_{PQ} = \frac{p+q}{2}$

$\therefore$ chord PQ: $y - 2p^2 = \frac{(p+q)}{2}(x - 4p)$

$2y - \cancel{4p^2} = px - \cancel{4p^2} + xq - 4pq$

$2y = x(p+q) - 4pq$

d). sub $A(0,1)$ into chord PQ

$2 = -4pq$

$pq = -\frac{1}{2}$

e) $m_{RT} = \dfrac{-2}{p+q}$ $T(2(p+q), 2pq)$

eqn of RT:

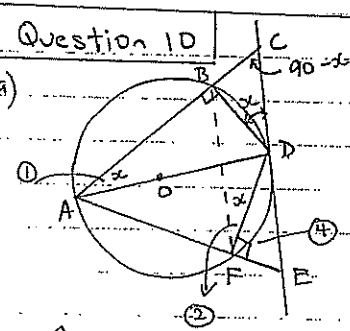
$y - 2pq = \dfrac{-2}{p+q}(x - 2(p+q))$ since $pq = -\frac{1}{2}$

$y - 1 = \dfrac{-2}{(p+q)}x + 2$

$y = \dfrac{-2}{(p+q)}x + 3$

Question 10

a)



$B\hat{A}D = x$ (alternate segment theorem)

$B\hat{F}D = x$ (angles in same segment)

$D\hat{F}E = 90°$ (angle in semi circle is $90°$ and angles on a straight line)

$\therefore B\hat{F}E = (x + 90°)$

$B\hat{C}D = (90 - x)$ (angle sum of $\triangle BCD$ $90°$ - angle in semi circle is $90°$)

since $B\hat{F}E + B\hat{C}D = x + 90 + 90 - x$

$= 180°$

$\therefore$ opposite angles add to $180°$

$\therefore$ BCEF is a cyclic quad.

b). Step ① Show true for $n=1$

$LHS = 1.2^1 = 2$

$RHS = (1-1)2^2 + 2 = 2$

$\therefore$ true for $n=1$

Step② Assume true for $n=k$ some positive integer

$\therefore S_k = (k-1)2^{k+1} + 2$

Step ③ Show true for $n = k+1$

ie $S_{k+1} = k \cdot 2^{k+2} + 2$

since

$S_{k+1} = S_k + T_{k+1}$

$= (k-1) \cdot 2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}(k - 1 + k + 1) + 2$

$= 2^{k+1}(2k) + 2$

$= k \cdot 2^1 \cdot 2^{k+1} + 2$

$= k \cdot 2^{k+2} + 2$

Step ④ Since true for $n=1$ and if assumed true for $n=k$ (a positive integer) we have shown, by M.I., true for $n=k+1$

$\therefore$ true for all positive integers $n \geq 1$

Question 11

ai) $T_n = S_n - S_{n-1}$

$= \frac{n}{3}(n+1)(n+2) - \frac{(n-1)(n)(n+1)}{3}$

$= \frac{n}{3}(n+1)(n + 2 - (n-1))$

$= \frac{n}{\cancel{3}}(n+1)(\cancel{3})$

$T_n = \frac{n}{3}(n+1)$

ii) $S_{\text{2nd 50 terms}} = S_{100} - S_{50}$

$= \frac{100}{3}(101)(102) - \frac{50}{3}(51)(52)$

$= 299\,200$

b) i) Let B_n be amount in account after n payments

$B_1 = 1000\left(1 + \frac{6}{100}\right)^1 - 72 = \$988$

ii) $B_2 = (1000(1.06)^1 - 72)(1.06)^1 - 72$

$= 1000(1.06)^2 - (1.06)^1 72 - 72$

$\therefore B_n = 1000(1.06)^n - (1.06)^{n-1}.72 - \dots - 72$

$= 1000(1.06)^n - 72\underbrace{(1 + (1.06)^1 + \dots + (1.06)^{n-1})}_{\text{G.P. } a=1}$

$\left.\begin{matrix} \text{G.P. } a=1 \\ r = 1.06 \\ N = n \end{matrix}\right\}$

$\therefore B_n = 1000(1.06)^n - 72\left(\dfrac{1.06^n - 1}{1.06 - 1}\right)$ **

$= 1000(1.06)^n - 1200(1.06^n - 1)$

$= 1000(1.06)^n - 1200(1.06)^n + 1200$

$\therefore B_n = 1200 - 200(1.06)^n$

iii) After 10 years B_{10} is

$1200 - 200(1.06)^{10} = \underline{841.83}$

prize now increases to \$90

from **

$B_n = B_{10} \cdot (1.06)^n - 90\left(\dfrac{1.06^n - 1}{.06}\right)$

Fund used when $B_n = 0$

$0 = 841.83(1.06)^n - 1500(1.06)^n + 1500$

$1500 = 658.17(1.06)^n$

using logs

$\log_{10}\left(\dfrac{1500}{658.17}\right) = n \log_{10} 1.06$

$\therefore n = \dfrac{\log_{10}\left(\frac{1500}{658.17}\right)}{\log_{10}(1.06)}$

$n = 14$

or by direct subst.

$1500 \doteq 658.17(1.06)^{14}$

$RHS = 1488.05$

$\therefore$ not enough for next prize of \$90