Name: ....

Sydney Technical High School

Maths Class: ....

Year 12

Mathematics Extension 2

HSC Course

Assessment 3

TERM 2 2017

Time allowed: 90 minutes

General Instruction

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- BOSTES reference sheet has been supplied for your use.

Section 1 Multiple Choice
Questions 1-5
5 Marks

Allow approximately 10 minutes for this section

Section II Questions 6 - 9
49 Marks

Allow approximately 80 minutes for this section

Section 1

5 marks
Attempt Questions 1 – 5
Allow about 10 minutes for this section
Use the Multiple Choice answer sheet for questions 1 – 5

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1. If the line $y = mx + b$ is a tangent to the hyperbola $xy = c^2$, which of the following is true?

(A) $b^2 = -4mc^2$

(B) $b^2 = 4mc^2$

(C) $b = 4mc$

(D) $c^2 = 4mb$

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2. For the hyperbola, with equation, $x^2 - 4y^2 = 4$, the distance between its directrices is:

(A) $\sqrt{5}$

(B) $\frac{4}{\sqrt{5}}$

(C) $2\sqrt{5}$

(D) $\frac{8\sqrt{5}}{5}$

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3. Which expression is equal to $\int x^2 \sec^2 x dx$

(A) $2x \tan x - 2\int \tan x dx$

(B) $\frac{1}{3}(x^3 \sec^2 x - \int x^3 \tan x dx)$

(C) $x^2 \tan x - 2\int x \tan x dx$

(D) $x^2 \tan x - 2\int x \sec^2 x dx$

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4. Which of the following is the range of $f(x) = \sin^{-1} x + \tan^{-1} x$

(A) $-\pi < y < \pi$

(B) $-\pi \le y \le \pi$

(C) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

(D) $-\frac{3\pi}{4} \le y \le \frac{3\pi}{4}$

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5. Given that $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)$ +C, evaluate, $\int_1^e \frac{1}{x\sqrt{1 + (\ln x)^2}} dx$;

(A) $-\frac{\pi}{4} + \tan^{-1} e$

(B) $\ln\left(\frac{e + \sqrt{e^2 + 1}}{1 + \sqrt{2}}\right)$

(C) $\frac{\pi}{4}$

(D) $\ln\left(1 + \sqrt{2}\right)$

End of Section 1

Section II

Attempt Questions 6 – 11

Allow about 1 hour and 20 minutes for this section.

Answer each question in your answer booklet STARTING EACH QUESTION ON A NEW PAGE.

In Questions 6 – 11 your responses should include all relevant mathematical reasoning and / or calculations.

Question 6 - 12 marks

a. i. Find the derivative of $\sin^{-1}\sqrt{x}$ and state the domain for which $\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$ exists. **2**

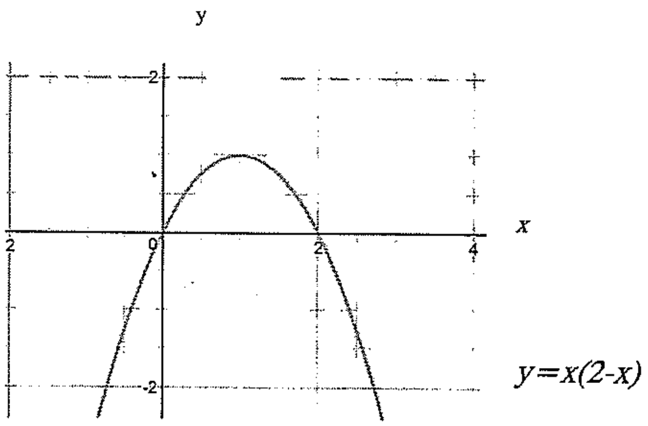
ii. Hence, evaluate $\int_{0.25}^{0.5}\frac{dx}{\sqrt{x-x^2}}$ (answer to 3 sig fig) **2**

b. Find;

i. $\int \cos^3 x \sin x\,dx$ **1**

ii. $\int \cos^3 x \sin^2 x\,dx$ **3**

c. Consider the sketch of $y=f(x)$ given below



On separate diagrams, one third of a page each, sketch;

i. $y=\ln f(x)$ **2**

ii. $y=f\left(e^x\right)$ **2**

Question 7 - 12 marks

Begin this question on a NEW PAGE in your answer booklet.

a. Evaluate $\int_0^1 \tan^{-1} x\,dx$

3

b. Find $\int \frac{2}{\sqrt{16-9x^2}}dx$ 2

c. Use an appropriate substitution, or otherwise to evaluate $\int_0^1 x\sqrt{1-x}\,dx$ 4

d. Find $\int \sqrt{\frac{6-x}{6+x}}dx$ 3

Assessment continues on the next page

Question 8 - 12 marks

Begin this question on a NEW PAGE in your answer booklet.

a. i. Find the values of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ such that,

$$\frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)}=\frac{Ax+B}{1+2x^2}+\frac{C}{1+6x}$$ 2

ii. Hence, show that,

$$\int_0^2 \frac{3-x}{\left(1+2x^2\right)\left(1+6x\right)}dx=\frac{1}{2}\ln\frac{13}{3}$$ 2

b. The point $T\left(ct,\frac{c}{t}\right)$ lies on the rectangular hyperbola $xy=c^2$. The normal at T meets the line $y=x$ at $\boldsymbol{R}$.

i. Sketch the parabola $xy=c^2$ showing the foci and asymptotes, in terms of c. 1

ii. Find the equation of the normal at T 2

iii. Hence find the coordinates of R and show that the x - coordinate of R is the sum of the co-ordinates of T.

2

c. Given $\int_0^{2a} f(x)\,dx=\int_{-a}^{a} f(a-x)\,dx$ for $a>0$

Show that, $\int_0^1 \sqrt{x(1-x)}\,dx=\frac{\pi}{8}$ 3

Assessment continues on the next page

Question 9 - 13 marks

Begin this question on a NEW PAGE in your answer booklet.

a. i. Show that $\cos^6\theta - \sin^6\theta = \cos 2\theta\left(1 - \frac{1}{4}\sin^2 2\theta\right)$

2

ii. Hence, evaluate, in exact form, $\int_0^{\frac{\pi}{12}} \cos^6\theta - \sin^6\theta \, d\theta$ **2**

b. For each integer $n \geq 0$ let $I_n = \int_0^1 x^{2n+1}e^{x^2}\,dx$

i. Show that for $n \geq 1$, $I_n = \frac{e}{2} - nI_{n-1}$ **2**

ii. Hence, or otherwise, show that $I_2 = \frac{e}{2} - 1$ **2**

c. Consider the hyperbola H with equation; $\frac{x^2}{4} - \frac{y^2}{2} = 1$

i. Find the value of the eccentricity of H **1**

ii. Write down the equations of the asymptotes of H **1**

iii. Given the tangent at the point $P(2\sec\theta, \sqrt{2}\tan\theta)$ has the equation:

$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{2}} = 1$, prove that the area bounded by this tangent and the asymptotes of H is independent of the position of P. **3**

END OF TASK

Extension 2 Task 3 2017

M/C 1. A 2. D 3. C 4. D 5. D

(Solutions at the end)

Question 6

a) $\frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-1/2}$

$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{x - x^2}}$, $0 < x < 1$

ii) $\int_{0.25}^{0.5} \frac{1}{\sqrt{x-x^2}}\,dx = 2\sin^{-1}\sqrt{x}$

$= 2\left[\sin^{-1}\sqrt{0.5} - \sin^{-1}\sqrt{0.25}\right]$

$\boxed{\text{NOTE ALL CALCULUS IS IN RADIANS}}$

$= \frac{\pi}{6} \approx 0.524$

b) i. $\int \cos^3 x \sin x \, dx$

$= -\frac{1}{4}\cos^4 x + C$

ii. $\int \cos^3 x \sin^2 x \, dx$

$\int \cos x(1 - \sin^2 x)\sin^2 x \, dx$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

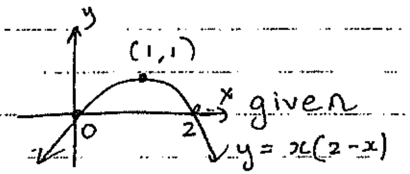
$= \int (1-u^2)u^2 \, du$

$= \int u^2 - u^4 \, du$

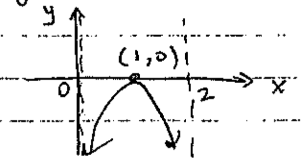
$= \frac{1}{3}u^3 - \frac{1}{5}u^5$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

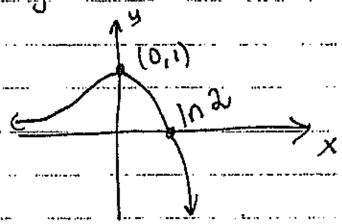
c)



i) $y = \ln f(x)$



ii) $y = e^x(2 - e^x)$



Question 7

a) Evaluate

$\int_0^1 \tan^{-1} x \, dx$

$= \int_0^1 1 \times \tan^{-1} x \, dx$

– parts –

$= x\tan^{-1} x - \int x \cdot \frac{1}{1+x^2}\,dx$

$= x\tan^{-1} x\Big]_0^1 - \frac{\ln(1+x^2)}{2}\Big]_0^1$

$= (1\tan^{-1} 1 - 0) - \frac{1}{2}\left[\ln 2 - \ln 1\right]$

$= \frac{\pi}{4} - \ln\sqrt{2}$

b) $\int \frac{2}{\sqrt{16 - 9x^2}}\,dx$

$2\int \frac{1}{\sqrt{9\left(\frac{16}{9} - x^2\right)}}\,dx$

$= 2\int \frac{1}{3\sqrt{\left(\frac{4}{3}\right)^2 - x^2}}\,dx$

$= \frac{2}{3}\sin^{-1}\frac{3x}{4} + C$

c) $\int_0^1 x\sqrt{1-x}\,dx$ let $u = 1 - x$

$x = 1$ $u = 0$

$x = 0$ $u = 1$

$\frac{du}{dx} = -1$

$-du = dx$

$= \int_1^0 (1-u)\sqrt{u} \cdot -du$

$= \int_0^1 \sqrt{u} - u\sqrt{u}\,du$

$= \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big]_0^1$

$= \frac{2}{3} - \frac{2}{5} - 0 = 4/15$

d) $\int \sqrt{\frac{6-x}{6+x}}\,dx$

$= \int \frac{\sqrt{6-x}}{\sqrt{6+x}} \times \frac{\sqrt{6-x}}{\sqrt{6-x}}\,dx$

$= \int \frac{6-x}{\sqrt{36-x^2}}\,dx$

$= \int \frac{6}{\sqrt{36-x^2}}\,dx - \int \frac{x}{\sqrt{36-x^2}}\,dx$

$= 6\sin^{-1}\frac{x}{6} + \sqrt{36 - x^2} + C$

Question 8

i)

$3 - x = (Ax+B)(1+6x) + C(1+2x^2)$

let $x = 0$

$3 = B(1) + C(1)$

let $x = 1$

$2 = (A+B)(7) + 3C$

let $x = -1/6$

$3\frac{1}{6} = \boxed{A(-\frac{1}{6}) + B}(0) + C\left(\frac{19}{18}\right)$

$C = 3 \quad \therefore B = 0$

and $A = -1$

e $\dfrac{-x}{1+2x^2} + \dfrac{3}{1+6x}$

ii) $\int_0^2 =$

$-\frac{1}{4}\ln(1+2x^2) + \frac{1}{2}\ln(1+6x)\Big]_0^2$

$= -\frac{1}{4}\ln 9 + \frac{1}{2}\ln 13 - (0+0)$

$= \frac{1}{2}\ln 13 - \frac{1}{4}\ln 9$

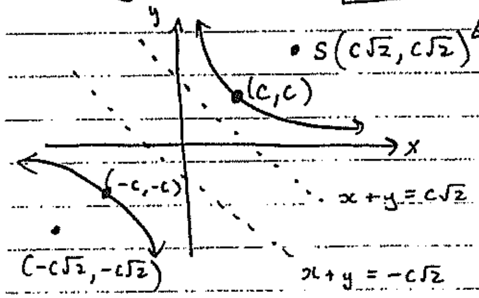
$= \frac{1}{2}\ln 13 - \frac{1}{2} \times \frac{1}{2}\ln 9$

$= \frac{1}{2}\ln 13 - \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left(\frac{13}{3}\right)$

Q8

b) $xy = c^2$ $\quad \boxed{e = \sqrt{2}}$ focus needed.



ii) $xy = c^2$

$y = \dfrac{c^2}{x}$

$y' = \dfrac{-c^2}{x^2}$ at $x = ct$

$M_T = -\dfrac{1}{t^2} \quad \therefore M_N = t^2$

equation

$y - \dfrac{c}{t} = t^2(x - ct)$

$ty - c = t^3(x - ct)$

$ty - c = t^3 x - ct^4 \quad (1)$

iii) Now if meets Line $y = x$

sub $y = x$ into (1)

$tx - c = t^3 x - ct^4$

$x(t - t^3) = c(1 - t^4)$

$x(t - t^3) = c(1-t^2)(1+t^2)$

$xt(1-t^2) = c(1-t^2)(1+t^2)$

$x = \dfrac{c(1+t^2)}{t}$

$\therefore R\left[\dfrac{c(1+t^2)}{t}, \dfrac{c(1+t^2)}{t}\right]$

$\therefore$ x-coordinate $= \dfrac{c}{t} + ct$ or $x_T + y_T$

c) Given

$\int_0^{2a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$

✱ Apply rule :

$\therefore \int_{-0.5}^{0.5} \sqrt{\left(\frac{1}{2} - x\right)\left(1 - \left(\frac{1}{2} - x\right)\right)}\,dx$

$= \int_{-0.5}^{0.5} \sqrt{\left(\frac{1}{2} - x\right)\left(\frac{1}{2} + x\right)}\,dx$

$= \int_{-0.5}^{0.5} \sqrt{\frac{1}{4} - x^2}\,dx$

let $x = \frac{1}{2}\sin\theta$

$x = 0.5 \quad \theta = \pi/2$

$x = -0.5 \quad \theta = -\pi/2$

$dx = \frac{1}{2}\cos\theta\, d\theta$

alternative :

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\sqrt{1 - \sin^2\theta} \cdot \frac{1}{2}\cos\theta\, d$

$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\, d\theta$

$= \frac{1}{8}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{1}{8}\left[\frac{\pi}{2} + 0 - \left(\frac{-\pi}{2} + 0\right)\right]$

$= \frac{\pi}{8}$

OR $\frac{1}{2}$ area of circle

$A = \frac{1}{2} \times \pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$

~~Q9~~

c H: $\frac{x^2}{4} - \frac{y^2}{2} = 1$

i) $b^2 = a^2(e^2-1)$

$\frac{2}{4} = e^2 - 1$

$\frac{6}{4} = e^2$

$e = \frac{\sqrt{6}}{2}$

ii) asymptotes

$y = \pm \frac{\sqrt{2}x}{2}$

iii) tangent:

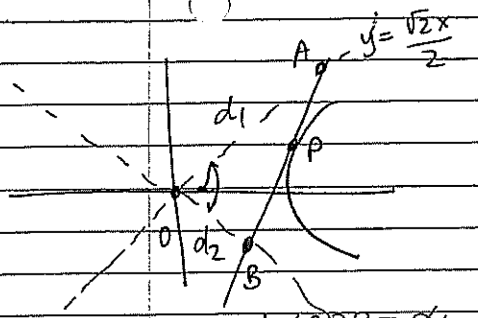
$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{2}} = 1$

Finding A: $y = \frac{\sqrt{2}x}{2}$

$A = \left[\frac{2}{\sec\theta - \tan\theta}, \frac{\sqrt{2}}{\sec\theta - \tan\theta}\right]$

Finding B: $y = \frac{-\sqrt{2}x}{2}$

$B: \left[\frac{2}{\sec\theta + \tan\theta}, \frac{-\sqrt{2}}{\sec\theta + \tan\theta}\right]$



let $\angle AOB = \alpha$

calculate dist OA

$d_1 = \frac{\sqrt{6}}{\sec\theta - \tan\theta}$

calculate dist OB

$d_2 = \frac{\sqrt{6}}{\sec\theta + \tan\theta}$

Now Area $= \frac{1}{2}ab\sin C$

$A = \frac{1}{2}\left[\frac{\sqrt{6}}{\sec\theta - \tan\theta}\right]\left[\frac{\sqrt{6}}{\sec\theta + \tan\theta}\right] \times \sin(\angle AOB)$

$= \frac{3}{\sec^2\theta - \tan^2\theta} \times \sin\alpha$

$= \frac{3}{1} \times \sin\alpha$

$= 3\sin\alpha$

which is independent of P as No θ in the result.

1. $x(mx+b) - c^2 = 0$
$mx^2 + bx - c^2 = 0$
tangent $\Delta = 0$
$b^2 - 4(m)(-c^2) = 0$
$b^2 = -4mc^2$

A

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2. $\frac{x^2}{4} - y^2 = 1$

$e \Rightarrow 1 = 4(e^2-1)$

$\frac{5}{4} = e^2$

$e = \frac{\sqrt{5}}{2}$

$\therefore$ distance $= 2 \times \frac{a}{e}$

$= 2 \times \frac{2}{\sqrt{5}/2}$

$= 2 \times 2 \div \frac{\sqrt{5}}{2}$

$= 4 \times \frac{2}{\sqrt{5}}$

$= \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}$

D

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3. $\int x^2 \sec^2 x \, dx$ Parts

$= x^2 . \tan x - \int 2x . \tan x \, dx$

:

C

$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$

But for $-1 \leq x \leq 1$

$\therefore$ you can only take the $\tan^{-1}x$ values from

$-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$

making addition. function range $-\frac{3\pi}{4} \leq y \leq \frac{3\pi}{4}$.

D

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5. let $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$

$x = e$ $u = \ln e = 1$ $du = \frac{dx}{x}$

$x = 1$ $u = \ln 1 = 0$

$\therefore \int_0^1 \frac{1}{\sqrt{1+u^2}} \cdot du$

$= \ln\left(u + \sqrt{u^2+1}\right)\Big]_0^1$

$= \ln(1+\sqrt{2}) - \ln(0+\sqrt{1})$

$= \ln(1+\sqrt{2}) - \ln 1$

$= \ln(1+\sqrt{2}) - 0$

$= \ln(1+\sqrt{2})$

D