Polynomials

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- Division of polynomials
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- Sums and Products of Roots
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Definitions and properties of polynomials

Polynomial Expression	$P(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ where $p_0 \neq 0$
Coefficients	$p_0 , p_1 , p_2 , p_3 , \ldots$
Leading term	p_nx^n
Constant	p_0
If $p_n = 1$	It is a **monic**
If $p_0 = p_2 = p_3 = 0$	Then P(x) is a **zero polynomial**

Example 1

$P(x) = 3x^4 - x^3 + 7x^2 - 2x + 3$

Coefficient of	x^4	Is	3
	x^3	Is	-1

Leading term is $3x^4$

Constant is 3

Division of polynomials

P(x)	= A(x)	× Q(x)	+ R(x)
Dividend	= Divisor	× Quotient	+ Remainder
$3x^4 - x^3 + 7x^2 - 2x + 3$	$= x - 2$	$\times\ 3x^3 + 5x^2 + 17x + 32$	$+ 67$

LONG DIVISION!!!

$$
\begin{array}{r|rrrrr}
 & & 3x^3 & +5x^2 & +17x & +32 \\
\hline
x-2 & 3x^4 & -x^3 & +7x^2 & -2x & +3 \\
 & 3x^4 & -6x^3 & & & \\
\cline{2-3}
 & & 5x^3 & +7x^2 & & \\
 & & 5x^3 & -10x^2 & & \\
\cline{3-4}
 & & & 17x^2 & -2x & \\
 & & & 17x^2 & -34x & \\
\cline{4-5}
 & & & & 32x & +3 \\
 & & & & 32x & -64 \\
\cline{5-6}
 & & & & & 67
\end{array}
$$

Example 2

Divide and find "a" such that R(x) = 0

$$
\begin{array}{r|rrrr}
 & & x^2 & +(a-2)x & +(4-a) \\
\hline
x+2 & x^3 & +ax^2 & +ax & +6 \\
 & x^3 & +2ax^2 & & \\
\cline{2-3}
 & & (a-2)x^2 & +ax & \\
 & & (a-2)x^2 & +(2z-4)x & \\
\cline{3-4}
 & & & (4-a)x & +6 \\
 & & & (4-a)x & +2(4-a) \\
\cline{4-5}
 & & & & 2a-2
\end{array}
$$

For R(x) = 0,

$$
\begin{aligned}
2a - 2 &= 0 \\
2a &= 2 \\
a &= 1
\end{aligned}
$$

Theorems

Remainder Theorem

- If a polynomial P(x) is divided by (x – a), then the remainder is P(a)

Example 1

$x - 2$; a = 2

$$
\begin{aligned}
P(2) &= 3(2)^4 - (2)^3 + 7(2)^2 - 2(2) + 3 \\
&= 48 - 8 + 28 - 4 + 3 \\
&= 67
\end{aligned}
$$

$$
\begin{array}{r|rrrrr}
 & & 3x^3 & +5x^2 & +17x & +32 \\ \hline
x-2 & 3x^4 & -x^3 & +7x^2 & -2x & +3 \\
 & 3x^4 & -6x^3 & & & \\ \cline{2-3}
 & & 5x^3 & +7x^2 & & \\
 & & 5x^3 & -10x^2 & & \\ \cline{3-4}
 & & & 17x^2 & -2x & \\
 & & & 17x^2 & -34x & \\ \cline{4-5}
 & & & & 32x & +3 \\
 & & & & 32x & -64 \\ \cline{5-6}
 & & & & & 67
\end{array}
$$

Factor Theorem

- For any polynomial P(x), if P(a) = 0, then (x – a) is a factor of P(x)

  OR
- For any polynomial P(x), if (x – a) is a factor of P(x), then P(a) = 0

Other

- For a polynomial of degree n, there exist at least k factors, where k < n
- If we have n distinct zeroes, the degree of the polynomial must be at least n degree
- Polynomials of n degree, cannot have more than n zeroes
- If a polynomial of n degree has more than n zeroes, than P(x) = 0; null polynomial
- $P_1(x)$, $P_2(x)$ ar both of degree n, the coefficients are equal

$$Ax^2 + Bx + C = 2x^2 - 3x + 5$$

A = 2

B = -3

C = 5

Sums and Products of Roots

Quadratic : $ax^2 + bx + c$

$\alpha + \beta$	$= -\frac{b}{a}$	Sum of roots 1 at a time
$\alpha\beta$	$= \frac{c}{a}$	Sum of roots 2 at a time (product of roots)

Cubic : $ax^3 + bx^2 + cx + d$

$\alpha + \beta + \gamma$	$= -\frac{b}{a}$	Sum of roots 1 at a time
$\alpha\beta + \beta\gamma + \gamma\alpha$	$= \frac{c}{a}$	Sum of roots 2 at a time
$\alpha\beta\gamma$	$= -\frac{d}{a}$	Sum of roots 3 at a time (product of roots)

Quartic : $ax^4 + bx^3 + cx^2 + dx + e$

$\alpha + \beta + \gamma + \delta$	$= -\frac{b}{a}$	Sum of roots 1 at a time
$\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \beta\delta$	$= \frac{c}{a}$	Sum of roots 2 at a time
$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta$	$= -\frac{d}{a}$	Sum of roots 3 at a time
$\alpha\beta\gamma\delta$	$= \frac{e}{a}$	Sum of roots 4 at a time (product of roots)

Approximation Methods

Half-interval

- $f(x)$ is continuous and differentiable
- $a \le x \le b$
- $f(a)$ and $f(b)$ have opposite signs
- There should be at least 1 root

Midpoint x_3 $= \dfrac{x_1 + x_2}{2}$

Midpoint x_4 $= \dfrac{x_3 + x_2}{2}$ OR $= \dfrac{x_3 + x_1}{2}$

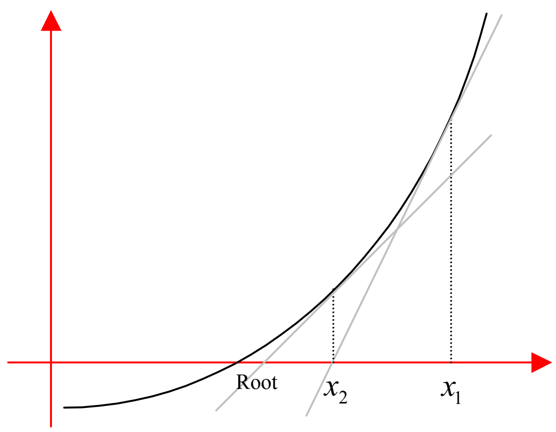
Using x_2 or x_1 depends if $f(x_3)$ is < 0 or > 0

Newton's Method of Approximation

If x_1 is close to the desired root, then x_2 is a good approximation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- If the approximation becomes further, stop!
- Can't use stationary points.



If x = a is clsoe to the root of the equation f(x) = 0, then the x-intercept x_2 of the tangent at x_1 is closer to the root.

Example 1

Using approximation methods, find the root of $P(x) = x^3 - 5x + 12$

Half-interval – Using 4 times; x = -1, 4; 2 decimal places.

$$\begin{aligned} P(-1) &= (-1)^3 - 5(-1) + 12 \\ &= -1 + 5 + 12 \\ &= 16 \\ &16 > 0 \end{aligned}$$

$$\begin{aligned} P(-4) &= (-4)^3 - 5(-4) + 12 \\ &= -64 + 20 + 12 \\ &= -32 \\ &-32 < 0 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{-1-4}{2} \\ &= -2.5 \end{aligned}$$

$$\begin{aligned} P(-2.5) &= (-2.5)^3 - 5(-2.5) + 12 \\ &= -15.63 + 12.5 + 12 \\ &= 8.87 \\ &8.87 > 0 \end{aligned}$$

$$\begin{aligned} x_4 &= \frac{-2.5-4}{2} \\ &= -3.25 \end{aligned}$$

$$\begin{aligned} P(-3.25) &= (-3.25)^3 - 5(-3.25) + 12 \\ &= -34.33 + 16.25 + 12 \\ &= -6.08 \\ &-6.08 < 0 \end{aligned}$$

$$\begin{aligned} x_5 &= \frac{-3.25-2.5}{2} \\ &= -2.88 \end{aligned}$$

$$\begin{aligned} P(-2.88) &= (2.88)^3 - 5(-2.88) + 12 \\ &= -23.89 + 14.4 + 12 \\ &= 4.51 \\ &4.51 > 0 \end{aligned}$$

$$\begin{aligned} x_6 &= \frac{-2.88-3.25}{2} \\ &= -3.07 \end{aligned}$$

$$\begin{aligned} P(-3.07) &= (-3.07)^3 - 5(-3.07) + 12 \\ &= -28.93 + 15.35 + 12 \\ &= -1.58 \end{aligned}$$

This is close to the root (–3)

Our answer after 4 times, is –3.07

Newton's Method of Approximation

Let $x_1 = -2$

$f(x) = x^3 - 5x + 12$

$f'(x) = 3x^2 - 5$

$$\begin{aligned} f(-2) &= (-2)^3 - 5(-2) + 12 \\ &= -8 + 10 + 12 \\ &= 14 \\ f'(-2) &= 3(-2)^2 - 5 \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

$$\begin{aligned} x_2 &= -2 - \tfrac{14}{7} \\ &= -4 \end{aligned}$$

$$\begin{aligned} f(-4) &= (-4)^3 - 5(-4) + 12 \\ &= -64 + 20 + 12 \\ &= -32 \\ f'(-4) &= 3(-4)^2 - 5 \\ &= 48 - 5 \\ &= 43 \end{aligned}$$

$$\begin{aligned} x_3 &= -4 - \tfrac{-32}{43} \\ &= -3.26 \end{aligned}$$

The root is close to –3.26