Sydney Technical High School

Name: ....

Maths Class: ....

Year 12

Mathematics

HSC Course

Assessment Task 3

June 2017

Time allowed: 90 minutes

General Instructions:

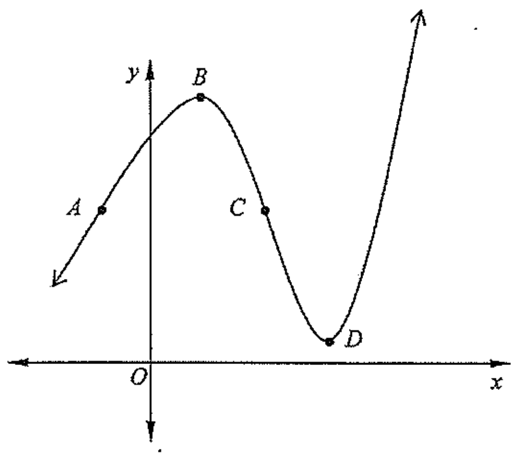
- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Section I	Multiple Choice Questions 1-5 5 Marks
Section II	Questions 6-9 52 Marks

Section I

5 marks
Allow approximately 10 minutes for this section.
Use Multiple Choice answer sheet for questions 1 – 5.

Question 1



The diagram shows the points A, B, C and D on a curve. At which point is $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$?

A) A
B) B
C) C
D) D

Question 2

What is the period of the function $y = 4 - 5\sin 2x$?
A) 2π
B) π
C) 4
D) 5

Question 3

What is the greatest value taken by the function $f(x) = 3 - \sin x$?
A) 2
B) 3
C) 4
D) 6

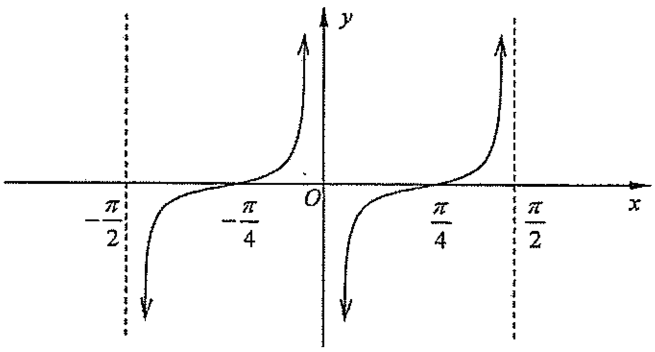
Question 4

Which of the following correctly evaluates the definite integral $\int_1^3 x^{-2}dx$?

A) $\frac{26}{81}$
B) $-\frac{4}{3}$
C) $-\frac{2}{3}$
D) $\frac{2}{3}$

Question 5

Part of the graph y = f(x) is shown below.



The equation of $f(x)$ could be:
A) $f(x) = \tan(2x - \frac{\pi}{2})$
B) $f(x) = \tan(2x - \frac{\pi}{4})$
C) $f(x) = \tan x$
D) $f(x) = \tan(x + \frac{\pi}{4})$

End of section I

Section II

52 marks
Allow approximately 1 hour and 20 minutes for this section.
Answer each question in your answer booklet.

Question 6 (13 marks) **Start a new page.**

a) Differentiate:

(i) $\frac{1}{\sqrt{x}} + \frac{5x^2}{2}$ 2

(ii) $(5 - 4x^2)^6$ 2

b) Evaluate the sum of the series $2 + 0 - 2 + \ldots -30$ 2

c) Express 210° in radians, in terms of π. 1

d) If $\sin\theta = -\frac{2}{3}$ and $\cos\theta > 0$, find the value of $\tan\theta$ (in surd form). 2

e) Find a primitive of $(3x + 2)^5$ 1

f) Use Simpson's Rule with five function values to approximate $\int_0^2 3^x dx$, correct to 3 decimal places. 3

Question 7 (13 marks) **Start a new page.**

a) Find the exact value of $tan\frac{11\pi}{3}$ 1

b) Can there be an infinite geometric series with a limiting sum of $\frac{5}{8}$ and a first term of 2? (All working and reasoning must be shown) 2

c) The first term of an arithmetic sequence is 4 and the fifth term is four times the third term. Find the common difference. 2

d) Find the equation of the normal to the curve $y = x^2 - 3x + 5$ at the point $(3, 5)$. 2

e) Find $\int \frac{dx}{(6x+1)^2}$ 2

f) (i) Sketch $y = 1 + \sin x$ for $0 \leq x \leq 2\pi$, showing all essential features. 2

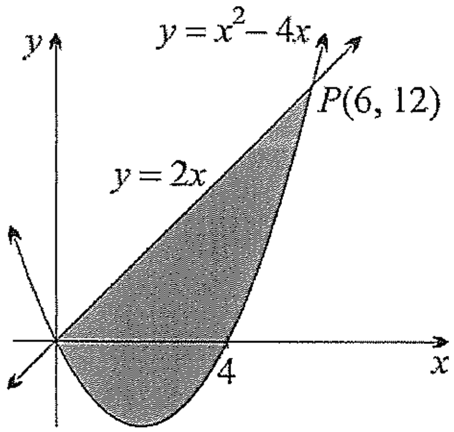
(ii) Find the values of x where the graph $y = 1 + \sin x$ intersects with $y = 1\frac{1}{2}$ for $0 \leq x \leq 2\pi$ 2

Question 8 (13 marks) Start a new page.

a) Solve $2sin^2x = 1$ for $0 \leq x \leq 2\pi$ 3

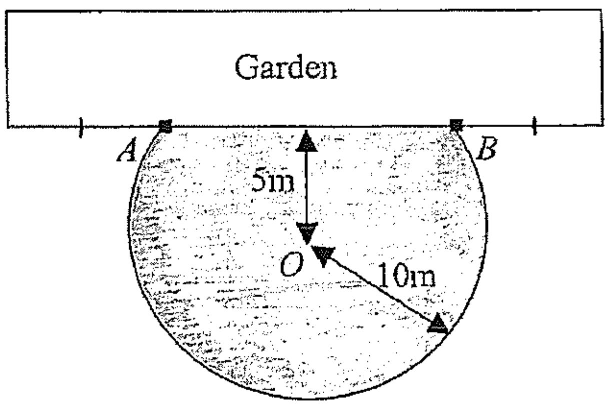
b) Find the equation of the curve $y = f(x)$, given that $\frac{d^2y}{dx^2} = 2x + 1$ and that there is a stationary point at (1, -2). 3

c)



The graphs $y = 2x$ and $y = x^2 - 4x$ are drawn above. They intersect at the origin and the point P(6, 12). Find the shaded area. 2

d)



A water sprinkler covers a circular lawn area of radius 10 metres, as shaded. The sprinkler (O) is placed 5 metres from a rectangular garden bed.

(i) Garden stakes are placed at A and B. Show that $\angle AOB = \frac{2\pi}{3}$ radians. 1

(ii) What area of **lawn** will the sprinkler cover? (Answer to 1 d.p.) 2

(iii) What is the total perimeter of the **lawn**? (Answer to 1 d.p.) 2

Question 9 (13 marks) **Start a new page.**

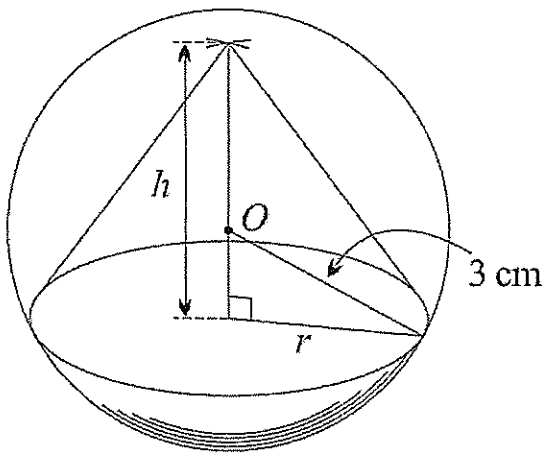
a) (i) State the domain and range of the function $y = \sqrt{9-x}$ 2

(ii) Sketch a graph of this function, labelling important features. 1

(iii) Calculate the volume of the solid generated when the area bounded by the curve and the coordinate axes in the first quadrant is rotated about the y axis. 3

b) Show that $\sqrt{\dfrac{cosec^2x - cot^2x - cos^2x}{cos^2x}} = \tan x$ 2

c)



A right circular cone of height h and base radius r is inscribed in a sphere of radius 3cm, as shown above.

[Note: Volume sphere = $\frac{4}{3}\pi r^3$ Volume cone = $\frac{1}{3}\pi r^2 h$]

(i) Show that the volume of the cone is given by $V = \frac{\pi}{3}(6h^2 - h^3)$. 1

(ii) Find the dimensions of the cone so that its volume is maximised. 3

(iii) What fraction of the sphere is occupied by this cone? 1

End of section II

End of examination ☺

STHS Year 12 Mathematics Task 3 2017

Multiple Choice

1. B
2. B
3. C
4. D
5. A

Question 6

a)(i) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}} + \frac{5x^2}{2}\right)$

$= \frac{d}{dx}\left(x^{-\frac{1}{2}} + \frac{5}{2}x^2\right)$

$= -\frac{1}{2}x^{-\frac{3}{2}} + 5x$

$= \frac{-1}{2\sqrt{x^3}} + 5x$

(ii) $\frac{d}{dx}(5-4x^2)^6$

$= 6(5-4x^2)^5 \cdot -8x$

$= -48x(5-4x^2)^5$

b) $2+0-2+\ldots-30$

AP $a=2$ $d=-2$

$T_n = a+(n-1)d$

$-30 = 2-2(n-1)$

$\therefore n=17$

$S_n = \frac{n}{2}[2a+(n-1)d]$

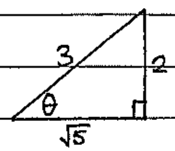
$= \frac{17}{2}[2(2)+16(-2)]$

$= -238$

c) $210 \times \frac{\pi}{180} = \frac{7\pi}{6}$ radians

d) $\sin\theta = -\frac{2}{3}$ and $\cos\theta > 0$

3rd, 4th 1st, 4th



In 4th quadrant, $\tan\theta < 0$

$\therefore \tan\theta = -\frac{2}{\sqrt{5}}$

e) $\int (3x+2)^5\,dx$

$= \frac{(3x+2)^6}{18} + C$

f)

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$f(x)$	1	1.732	3	5.196	9

$\int_0^2 3^x\,dx \doteq \frac{\left(\frac{1}{2}\right)}{3}\left[1+9+4(1.732+5.196)+2(3)\right]$

$= 7.285$ (3d.p.)

Question 7

a) $\tan\frac{11\pi}{3} = -\tan\frac{\pi}{6}$

$= -\sqrt{3}$

b) $S_\infty = \frac{a}{1-r}$ only exists if $-1<r<1$

$\frac{5}{8} = \frac{2}{1-r}$

$5-5r = 16$

$5r = -11$

$r = -\frac{11}{5}$

Since $r<-1$, there is no limiting sum.

c) $a=4$

$T_5 = 4T_3$

$T_5 = a+4d$

$T_3 = a+2d$

$\therefore a+4d = 4(a+2d)$

$a+4d = 4a+8d$

$-3a = 4d$ (sub $a=4$)

$4d = -12$

$\therefore d = -3$

d) $y = x^2-3x+5$

$y' = 2x-3$

at $x=3$, $y' = 2(3)-3$

$= 3$

$\therefore$ gradient of the tangent is 3.

$m_1 \times m_2 = -1$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Equation of normal:

$y-5 = -\frac{1}{3}(x-3)$

$y = -\frac{1}{3}x+6$ OR $x+3y-18=0$

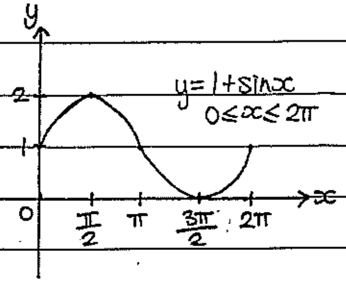
e) $\int \frac{dx}{(6x+1)^2}$

$= \int (6x+1)^{-2}\,dx$

$= \frac{(6x+1)^{-1}}{-6} + C$

$= \frac{-1}{6(6x+1)} + C$

7. f)(i)



(ii) $1 + \sin x = \frac{3}{2}$

$\sin x = \frac{1}{2}$

acute $x = \frac{\pi}{6}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Question 8

a) $2\sin^2 x = 1 \qquad 0 \le x \le 2\pi$

$\sin^2 x = \frac{1}{2}$

$\sin x = \pm\frac{1}{\sqrt{2}}$

S ✓	A ✓
T ✓	C ✓

acute $x = \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b) $\frac{d^2y}{dx^2} = 2x+1$

$\frac{dy}{dx} = x^2 + x + C$

When $x = 1$, $\frac{dy}{dx} = 0$

$1^2 + 1 + C = 0$

$\therefore C = -2$

$\therefore \frac{dy}{dx} = x^2 + x - 2$

$y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + k$

when $x = 1, y = -2$

$-2 = \frac{1}{3} + \frac{1}{2} - 2 + k$

$\therefore k = -\frac{5}{6}$

$\therefore y = \frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{5}{6}$

c) Area $= \int_0^6 2x - (x^2 - 4x)\, dx$

$= \int_0^6 6x - x^2\, dx$

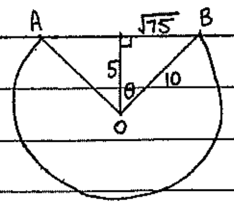
$= \left[3x^2 - \frac{x^3}{3}\right]_0^6$

$= [108 - 72 - (0)]$

$= 36$

$\therefore$ Area is 36 units^2

d)



8. d)(i) $\cos\theta = \frac{5}{10}$

$\theta = \frac{\pi}{3}$

$\angle AOB = 2\theta$

$\therefore \angle AOB = 2 \times \frac{\pi}{3}$

$= \frac{2\pi}{3}$ radians

(ii) Area of lawn = area of circle − area of minor segment

$= \pi \times 10^2 - \left[\frac{1}{2} \times 10^2 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)\right]$

$= 252.7$ (1 d.p.)

$\therefore$ Area of lawn is 252.7m^2 (1 d.p.)

(iii) Perimeter $= \left(2\pi - \frac{2\pi}{3}\right) \times 10 + 2 \times \sqrt{75}$

$= \frac{4\pi}{3} \times 10 + 2\sqrt{75}$

$= 59.2$ (1 d.p.)

$\therefore$ Perimeter of lawn is 59.2m (1 d.p.)

Question 9

a)(i) $y = \sqrt{9-x}$

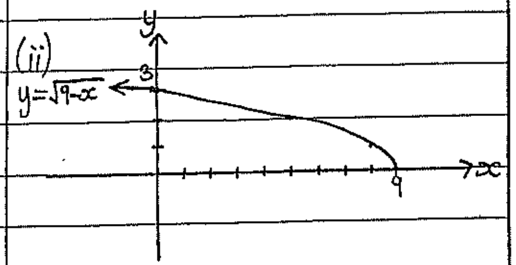
$9 - x \ge 0$

$x \le 9$

Domain: $x \le 9$

Range: $y \ge 0$

(ii)



(iii) $V = \pi \int_0^3 x^2\, dy \qquad y = \sqrt{9-x}$

$y^2 = 9 - x$

$x = 9 - y^2$

$x^2 = (9-y^2)^2$

$V = \pi \int_0^3 81 - 18y^2 + y^4\, dy$

$= \pi \left[81y - 6y^3 + \frac{y^5}{5}\right]_0^3$

$= \pi\left[\left(243 - 162 + \frac{243}{5}\right) - 0\right]$

$= \frac{648\pi}{5}$

$\therefore$ Volume is $\frac{648\pi}{5}$ units^3

b) LHS $= \sqrt{\frac{\operatorname{cosec}^2 x - \cot^2 x - \cos^2 x}{\cos^2 x}}$

$= \sqrt{\frac{1 - \cos^2 x}{\cos^2 x}}$

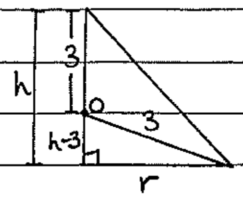
$= \sqrt{\frac{\sin^2 x}{\cos^2 x}}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

$=$ RHS

9.c)(i) $V_{cone} = \frac{1}{3}\pi r^2 h$



Now $r^2 = 3^2 - (h-3)^2$

$= 9 - (h^2 - 6h + 9)$

$= 6h - h^2$

$\therefore V = \frac{1}{3}\pi(6h - h^2)h$

$= \frac{\pi}{3}(6h - h^3)$ as required.

(ii) $V = \frac{\pi}{3}(6h - h^3)$

$= 2\pi h^2 - \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = 4\pi h - \pi h^2$

$= \pi h(4-h)$

Max. volume occurs when $\frac{dV}{dh} = 0$

$0 = \pi h(4-h)$

$\therefore h = 4$, $h = 0$ ($h > 0$ so ignore $h = 0$)

Check that there is a max. at $h = 4$

$\frac{d^2V}{dh^2} = 4\pi - 2\pi h$

when $h = 4$, $\frac{d^2V}{dh^2} = -12.566\ldots$

< 0

$\therefore$ Maximum at $h = 4$

$\therefore$ Dimensions of cone are $\underline{h = 4}$

$r = \sqrt{6h - h^2}$

$= \sqrt{6(4) - 4^2}$

$= \sqrt{8}$

$\underline{r = 2\sqrt{2}}$

(iii) $V_{cone} = \frac{1}{3}\pi(8)(4)$

$= \frac{32\pi}{3}$

$\therefore$ Volume of cone is $\frac{32\pi}{3}$ units^3

$V_{sphere} = \frac{4}{3}\pi(3)^3$

$= 36\pi$

$\therefore$ Volume of sphere is 36π units^3

$\therefore$ Cone occupies $\dfrac{\left(\frac{32\pi}{3}\right)}{36\pi}$ of sphere

$= \frac{8}{27}$ of the sphere.