Sydney Technical High School

Name: ....

Maths Class: ....

Year 11

Mathematics Extension 1

Preliminary Course

Assessment 3

September, 2017

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1
Multiple Choice
Questions 1-10
10 Marks

Section II
Questions 11-16
60 Marks

Section 1 Multiple Choice (10 marks)
Use the multiple choice answer sheet for Question 1-10

1. If the exact value of $\cos x$ is $\frac{1}{\sqrt{5}}$, what is the exact value of $\cos 2x$?

(A) $-\frac{3}{5}$

(B) $-\frac{2}{\sqrt{5}}$

(C) $\frac{3}{5}$

(D) $\frac{2}{\sqrt{5}}$

2. The coordinates of the focus of the parabola $x^2 = 4ay$ are

(A) (0, -a)

(B) (0, a)

(C) (0, 1)

(D) (0, 4a)

3. Given $f(x) = 3x^2 - 5x + 2$, find $f(a+1)$

(A) $3a^2 - 5a + 3$

(B) $3a^2 + 11a$

(C) $3a^2 + a + 1$

(D) $3a^2 + a$

4. If $x = \frac{1}{2}at$ and $y = 2at^2$ which of the following is an expression for $\frac{dy}{dx}$?

(A) $8t$

(B) $4at$

(C) $2t$

(D) t

5. Which statement is true of the quadratic expression $2x^2 + 6x + 9$?

(A) It is positive definite

(B) It has two unreal roots

(C) It is a perfect square

(D) The zeros add to 3

6. If $\sin 25^0 = \cos(x - 45^o)$ find x if $45^o < x < 135^o$

(A) 45^o

(B) 70^o

(C) 110^o

(D) 135^o

7. The correct solution of $\frac{x}{x-3} > 0$ *is*:

(A) $x < 0$ or $x > 3$

(B) $0 < x < 3$

(C) $x > 0$

(D) $x > 0$ or $x > 3$

8. Which is the correct condition for $y = mx + b$ to be a tangent to $x^2 = 4ay$?

(A) $am + b = 0$

(B) $am^2 + b = 0$

(C) $am - b = 0$

(D) $am^2 - b = 0$

9. Which of the following functions does NOT have a horizontal asymptote $y = 1$?

(A) $y = 1 + 2^x$

(B) $y = \frac{x^2+1}{x^2-1}$

(C) $y = 3 - \frac{2x+1}{x+1}$

(D) $y = \frac{3x^2+1}{3x+1}$

10. What is the total number of solutions of the equation $3cosx + 4sinx = 5$ for $0° \leq x \leq 360°$?

(A) 0

(B) 1

(C) 2

(D) 3

Section II Total Marks 60
Attempt Questions 11 – 16. Answer each question in your writing booklet.
In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (10 Marks) Use a Separate Sheet of paper

(a) A parabola has equation $x^2 = -12y + 24$.

(i) Give the coordinates of its focus. **1**

(ii) Give the equation of its directrix. **1**

(iii) Sketch the parabola, showing its main features. **2**

(b) Solve the equation $\sin 2x + \cos x = 0$ over the Domain $0^o \leq x \leq 360^o$ **3**

(c) Let $P(x) = 2x^3 - 3x^2 - 3x + 2$

(i) Show that $(x + 1)$ is a factor of $P(x)$ **1**

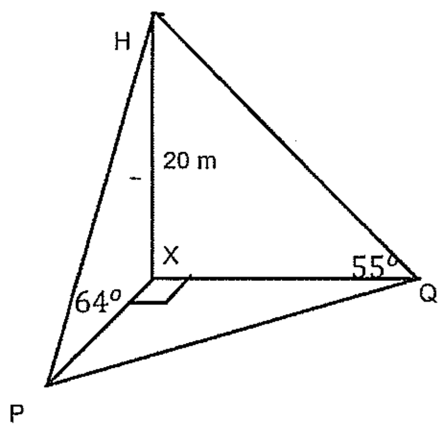
(ii) Hence express $P(x)$ as a product of three linear factors. **2**

Question 12 (10 Marks) Use a Separate Sheet of paper

(a) Solve the equation: $\frac{3x+1}{x-3} \leq 4$ **3**

(b) Find the co-ordinates of the point P(x,y) which divides the interval joining A$(-3,-7)$ To the point B$(-1,-4)$ externally in the ratio 4:3 **2**

(c) Two sailors P and Q, floating in the ocean, spot a helicopter above. From P the angle of elevation to the helicopter is 64^o, while from Q the angle of elevation is 55^o. Using a point X immediately below the helicopter, the triangle PQX is right angled at X.



(i) Show that XQ $= \frac{20}{tan55^o}$ **1**

(ii) How far apart are P and Q, to 3 significant figures? **2**

(d) The equation $x^3 + 3x^2 + 2x + 1 = 0$ has roots α, β and γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

Question 13 **(10 Marks)** Use a Separate Sheet of paper

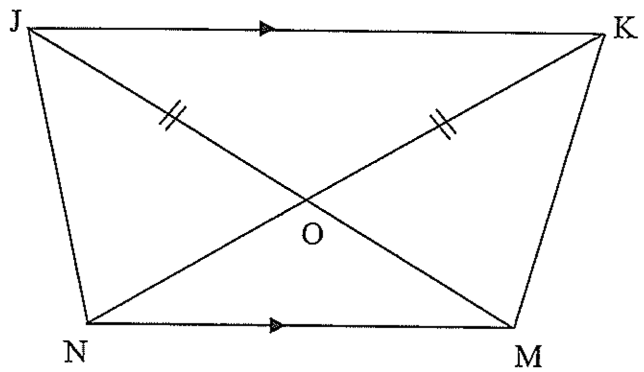
(a) Find the derivative of these expressions in simplified form :

(i) $x(2x^2-4)^5$ **2**

(ii) $\dfrac{x+1}{(x-1)^2}$ **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to solve the equation $2 + cosx - 2\sin x = 0$ for $0° \le x \le 360°$, giving answers correct to the nearest degree. **3**

(c) In the diagram below, JK || NM, JO = KO.



Redraw the diagram into your answer booklet

(i) Prove that Δ JOK ||| Δ NOM **2**

(ii) Hence prove that $\Delta JON \equiv \Delta KOM$ **1**

Question 14 (10 Marks) Use a Separate Sheet of paper

(a) Find the values of A, B and C if

$$2x^2 - 3x + 5 \equiv A(x-1)(x-2) + B(x-1) + C$$ **3**

(b) (i) Find the Cartesian equation of the curve which has the parametric equations **2**

$$x = 3 + t$$
$$y = 2t^2 - 2$$

(ii) Describe Geometrically the curve found in part (i) **1**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$ with focus S.

(i) Show that the normal to the parabola at P has equation $x + py - 2ap - ap^3 = 0$ **2**

(ii) Hence find the coordinates of the three points on the parabola such that the normals to the parabola at these three points pass through the point $(0, 6a)$. **2**

Question 15 **(10 Marks)** Use a Separate Sheet of paper

(a) For the parabola with parametric equations $x = 10t$ and $y = 5t^2$,

(i) Find the Cartesian equation of the parabola **1**

(ii) Find the coordinates of the focus. **1**

(iii) Sketch the parabola showing the focus, vertex and directrix. **1**

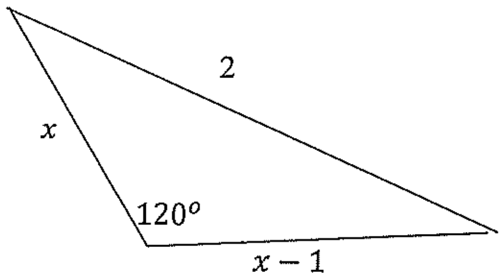
(iv) Show that the focal chord that passes through the point on the parabola where $t = 2$ has equation $3x - 4y + 20 = 0$. **2**

(b) For the polynomial $G(x) = x^2(1-x)(x+3)$ draw a sketch of $y = G(x)$. **3**

(c) Three tangents to the curve $y = 3x^4 + 4x^3 - 12x^2 + x + 3$ are parallel to the line $y = x$. Find the x value of the point of contact for each of these three tangents. **2**

Question 16 (10 Marks) Use a Separate Sheet of paper

(a)



In the triangle above, find the <u>exact</u> value of x. **3**

(b) (i) Find the acute angle between the lines $y = \frac{1}{\sqrt{3}}x$ and $y - \sqrt{3}x + 4\sqrt{3} = 0$. **2**

(ii) Prove that the lines and x-axis form an isosceles triangle. **2**

(c) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$ with focus S

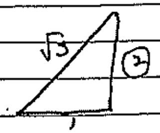
The normal to the parabola at P has equation $x + py - 2ap - ap^3 = 0$ and cuts the y axis at N. Show that $PS = NS$. **3**

End of Examination

YEAR 11 2017 FINAL EXAMS

EXTENSION 1 SOLUTIONS

MULTIPLE CHOICE

Q1. 

$\cos 2x = 2\cos^2 x - 1$

$= 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1$

$= -3/5$ so (A)

Q2. (B)

Q3. $f(a+1) = 3(a+1)^2 - 5(a+1) + 2$

$= 3a^2 + 6a + 3 - 5a - 5 + 2$

$= 3a^2 + a$ (D)

Q4. Either: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ OR $y = 2a\left(\frac{2x}{a}\right)^2$

$= 4at \times 2/a$ $\quad = 8x^2/a$

$= 8t$ (A) $\quad \therefore \frac{dy}{dx} = 16x/a$

$= 8at/a$

$= 8t$

Q5. $\Delta = 36 - 4(2)(9)$

$= -36$

$\therefore$ POSITIVE DEF. (A)

Q6. $\sin 25 = \cos 65$

$\therefore 65 = x - 45$

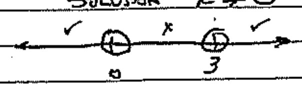
$\therefore x = 110^\circ$ (C)

Q.9 (D) as $x \to \infty$

$y \to x$

Q7. C.V. $x \neq 3$

Solution $x \neq 0$



$\therefore x < 0$ or $x > 3$ (A)

Q.10 (B)

Q8. $x^2 = 4a(mx + b)$

$x^2 - 4amx - 4ab = 0$

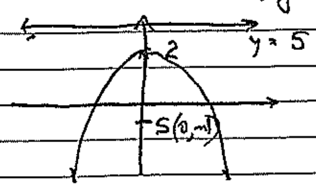
$\Delta = 16a^2m^2 + 16ab$

For tangency $\Delta = 0$

$am^2 + b = 0$ (B)

QUESTION 11:

(a) $x^2 = -12(y-2) \Rightarrow a = 3$ (or $a = -3$)



Vertex is $(0, 2)$

(i) S is $(0, -1)$

(ii) d is $y = 5$

(iii) above.

(b) $2\sin x\cos x + \cos x = 0$

$\therefore \cos x(2\sin x + 1) = 0$

$\therefore \cos x = 0$ or $\sin x = -\frac{1}{2}$

$\therefore x = 90^\circ, 270^\circ$ or $x = 210^\circ, 330^\circ$

(c) $P(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2$

$= -2 - 3 + 3 + 2$

$= 0$

$\therefore (x+1)$ is a factor

$$\begin{array}{r|l} & 2x^2 - 5x + 2 \\ x+1 & 2x^3 - 3x^2 - 3x + 2 \\ & 2x^3 + 2x^2 \\ \hline & -5x^2 - 3x + 2 \\ & -5x^2 - 5x \\ \hline & 2x + 2 \end{array}$$

$\Rightarrow P(x) = (x+1)(2x^2 - 5x + 2)$

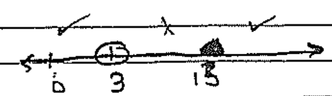
$= (x+1)(2x-1)(x-2)$

QUESTION 12:

(a) $\frac{3x+1}{x-3} = 4x - 12$

C.V. $x \neq 3$

$\therefore x = 13$



$\therefore x < 3$ or $x \geq 13$

(b) $A(-3, -7)$ $B(-1, -4)$

$-4 : 3$

M is $\left(\frac{-9+4}{-1}, \frac{-21+16}{-1}\right) = (5, 5)$

Q12

(c) (i) In $\triangle HXQ$, $\frac{HX}{XQ} = \tan 55°$

$\therefore XQ = \frac{20}{\tan 55°}$.

(ii) Similarly $XP = \frac{20}{\tan 64°}$

By Pythagoras, in $\triangle PQX$,

$$\left(\frac{20}{\tan 55}\right)^2 + \left(\frac{20}{\tan 64}\right)^2 = PQ^2$$

$\therefore PQ^2 = 196.116 + 95.153$

$\therefore PQ = 17.07$ m.

(d)

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= (-3)^2 - 2(2)$$

$$= 5$$

QUESTION 13:

(a)(i) $(2x^2-4)^5 \cdot 1 + x \cdot 20x\,(2x^2-4)^4$

$= (2x^2-4)^4\left[2x^2-4+20x^2\right]$

$= (2x^2-4)^4(22x^2-4)$

(ii) $\frac{(x-1)^2 \cdot 1 - (x+1)\cdot 2(x-1)}{(x-1)^4} = \frac{(x-1)-2(x+1)}{(x-1)^3}$

$= \frac{-x-3}{(x-1)^3}$

(b)

$2 + \frac{1-t^2}{1+t^2} - \frac{4t}{1+t^2} = 0$

$\therefore 2+2t^2+1-t^2-4t = 0$

$t^2-4t+3 = 0$

$\therefore (t-3)(t-1) = 0$

$0 \le x \le 360$

$0 \le x/2 \le 180$

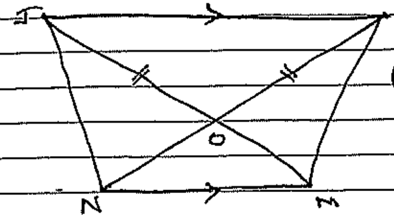
$\therefore t = 3$ OR $t = 1$

$\therefore \tan x/2 = 3$, $\tan x/2 = 1$

$\therefore x/2 = 71°34'$ OR $x/2 = 45°$

$\therefore x = 143°8'$ OR $x = 90°$

Q13 (c)



(i) In $\triangle JOK$ and $\triangle NOM$,

$\angle JOK = \angle NOM$ (vertically opposite angles)

$\angle KJO = \angle NMO$ (alternate angles $JK \parallel NM$)

$\therefore \triangle JOK \,|||\, \triangle NOM$ (equiangular)

(ii) Since the triangles are similar the sides are in ratio

$\therefore \frac{JO}{MO} = \frac{KO}{NO}$ OR $\frac{JO}{KO} = \frac{MO}{ON}$

$\therefore \frac{MO}{ON} = 1$ (since $JO = KO$)

$\therefore MO = NO$.

In $\triangle JON$ and $\triangle KOM$

$MO = NO$ (above)

$JO = KO$ (given)

$\angle JON = \angle KOM$ (vertically opposite)

$\therefore \triangle JON \equiv \triangle KOM$ (SAS)

QUESTION 14:

(a) By equating x^2: $A = 2$

let $x=1$ $4 = C$

let $x=2$ $7 = B$

(b) (i) $t = x-3$ $y = 2t^2-2$

$= 2(x-3)^2-2$

$\therefore (x-3)^2 = \frac{1}{2}(y+2)$

(ii) A parabola, vertex $(3,-2)$ focus $(3,-1\frac{7}{8})$

(c) (i) $dy/dx = x/2a$ At P, $m_T = p$

$\therefore m_N = -1/p$

Equation of normal:

$y - ap^2 = -1/p(x-2ap)$

$py - ap^3 = -x+2ap$

$x + py = ap^3+2ap$

Q 14 (c) (ii) Passes through $(0, 6a)$

$\therefore\ 6ap - 2ap - ap^3 = 0$

$\therefore\ 4ap - ap^3 = 0$

$\therefore\ ap(4 - p^2) = 0$

$\therefore\ p = 0$ or $p = 2$ or $p = -2$

$\therefore$ Points are

$(0,0)$, $(4a, 4a)$ and $(-4a, 4a)$

QUESTION 15:

(a) (i) $y = 5\left(\frac{x}{10}\right)^2$

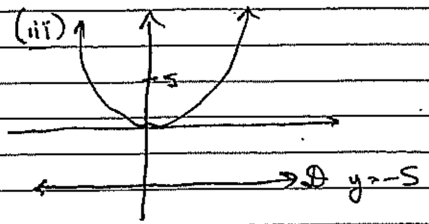
$\therefore\ y = x^2/20$

(ii) $x^2 = 20y$

$= 4(5)y$

$\therefore$ focus is $(0, 5)$

(iii)



(iv) At $t = 2$, P is $(20, 20)$

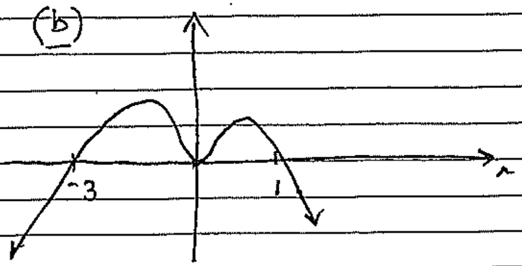
$m_{PS} = \frac{15}{20} = \frac{3}{4}$

$\therefore\ y - 20 = \frac{3}{4}(x - 20)$

$4y - 80 = 3x - 60$

$3x - 4y + 20 = 0$

(b)



(c) $12x^3 + 12x^2 - 24x + 1 = 1$ ← slope of $y = x$

$\therefore\ 12x(x^2 + x - 2) = 0$

$\therefore\ 12x(x+2)(x-1) = 0$

$\therefore\ x = 0$ or $x = 1$ or $x = -2$

QUESTION 16:

(a). By cosine rule, $2^2 = x^2 + (x-1)^2 - 2x(x-1)\cos 120°$

$\therefore\ 4 = 2x^2 - 2x + 1 - 2x(x-1)(-\frac{1}{2})$

$\therefore\ 4 = 2x^2 - 2x + 1 + x^2 - x$

$\therefore\ 3x^2 - 3x - 3 = 0$

$x^2 - x - 1 = 0$

$\therefore\ x = \frac{1 \pm \sqrt{1+4}}{2}$

Since $x > 0$, $x = \frac{1 + \sqrt{5}}{2}$

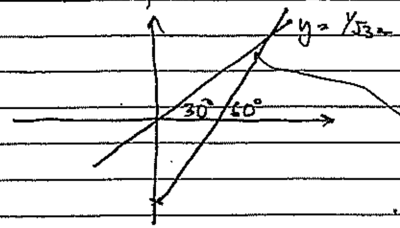
(b) (i) $m_1 = \frac{1}{\sqrt{3}}$, $m_2 = \sqrt{3}$.

$\therefore\ \tan\theta = \left|\frac{1/\sqrt{3} - \sqrt{3}}{1 + 1}\right|$

$= \left|\frac{1-3}{2\sqrt{3}}\right|$

$= 1/\sqrt{3} \implies \theta = 30°$

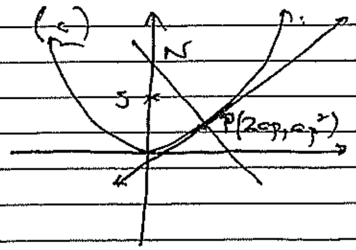
(ii) The slopes give tan of the angle made with the x-axis, so we have



By exterior angles, or by part (i) the missing angle is 30°

$\therefore$ Isosceles triangle.

(c)



Normal is $x + py - 2ap - ap^3 = 0$

$\therefore$ N is $(2a + ap^2)$

S is $(0, a)$

$\therefore\ NS = a + ap^2$

$PS^2 = (2ap)^2 + (ap^2 - a)^2$

$= 4a^2p^2 + a^2p^4 + a^2 - 2a^2p^2$

$= (ap^2 + a)^2$

$\therefore\ PS = ap^2 + a = NS$