Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Extension Mathematics

HSC Course

Assessment 1

November, 2015

Time allowed: 70 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Reference formulae is provided at the rear of this booklet, and may be removed at any time.

Section 1	Multiple Choice Questions 1-5 5 Marks
Section II	Questions 6-11 50 Marks

Section 1

5 marks
Attempt Questions 1-5
Allow 7 minutes for this section.

Use the multiple-choice answer sheet for Question 1-10

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1. Find the values of x for which the geometric series $2 + 4x + 8x^2 + \ldots$ has a limiting sum.

   (a) $x < \frac{1}{2}$

   (b) $x \geq \frac{1}{2}$

   (c) $|x| \leq \frac{1}{2}$

   (d) $|x| < \frac{1}{2}$

2. What is the remainder when the polynomial $p(x) = x^3 + 2x^2 - 5x - 6$ is divided by $(x - 2)$?

   (a) -12

   (b) -6

   (c) 0

   (d) 4

3. The statement $7^n - 3^n$ *is always divisible by 10* is true for

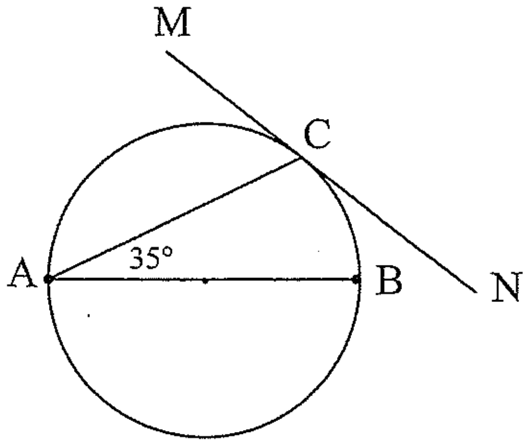
   (a) all integers $n \geq 1$

   (b) all integers $n \geq 2$

   (c) all odd integers $n \geq 1$

   (d) all even integers $n \geq 2$

4. In the diagram, AB is a diameter of the circle and MCN is the tangent to the circle at C. $\angle CAB = 35°$. What is the size of $\angle MCA$?



(a) 35°

(b) 45°

(c) 55°

(d) 65°

5. Find the gradient of the normal to the parabola $x = 6t,\ y = 3t^2$ at the point where $t = -2$.

(a) -2

(b) $-\dfrac{1}{2}$

(c) $\dfrac{1}{2}$

(d) 2

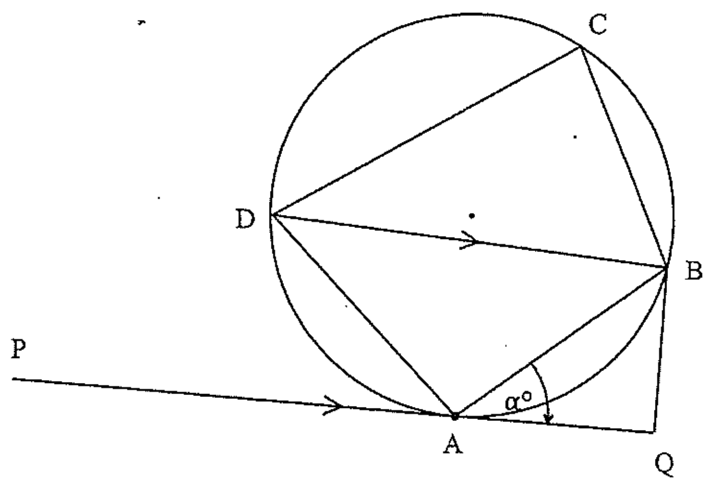
Section II

60 marks
Attempt Questions 6-11
Allow about 1 hour and 3 minutes for this section.

Answer each question in the answer booklet provided
In Questions 6-11, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(7 Marks)**

a) The tangents from Q touch the circle at A and B. PC and PQ are straight lines
$\angle BAQ = \alpha$



Copy or trace the diagram into your writing booklet.

(i) Given PD = 5cm and DC = 7cm, calculate the exact length of AP **1**

(ii) Show that $\angle BCD = 2\alpha$ **3**

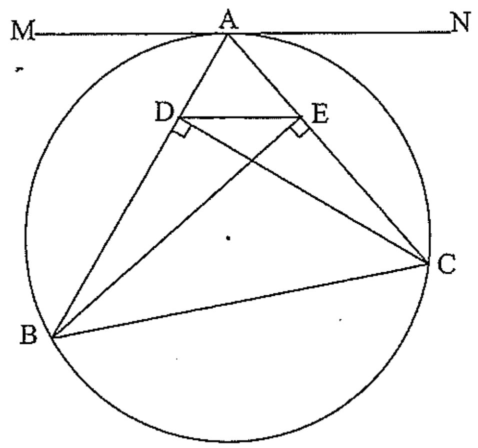
(iii) Show that PQBC is a cyclic quadrilateral **3**

Question 7 Start a new Page **(8 Marks)**

(a) On 1st July 2015, Mikaela invested \$18 000 in a bank account that paid interest at a rate of 5% p.a. compounded annually.

(i) How much would be in the account after the payment of interest on 1st July 2025 if no additional deposits were made? **1**

(ii) Consider if Mikaela made additional deposits of \$1500 to her account on the 1st July each year, beginning on 1st July 2016. After the payment of interest and her deposit on 1st July 2025, how much was in her account? **3**



(b) ABC is a triangle inscribed in a circle. MAN is the tangent at A to the circle ABC. CD and BE are altitudes of the triangle.

Copy the diagram into your answer booklet

(i) Give a reason why BCED is a cyclic quadrilateral **1**

(ii) Hence show that DE is parallel to MAN **3**

Question 8 (Start a new Page) **(9 Marks)**

a) The point $P(6p,3p^2)$ is a point on the parabola $x^2 = 12y$

(i) Find the equation of the tangent at P. **2**

(ii) The tangent at P cuts the y-axis at B.
The point A divides PB internally in the ratio 1:2.
Find the locus of the point A as P varies. **3**

(b) Use Mathematical induction to show that for all positive integers $n \geq 1$ **3**

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n.$$

(c) Evaluate $\sum_{n=1}^{5} \frac{1}{2^n}$ **1**

Question 9 (Start a new Page) **(7 Marks)**

(a) Helen borrows \$30000 over 4 years to purchase a 4wd from a car dealership. The dealer offers an 'interest free' period for the first 6 months of the loan.

After 6 months, the remainder of the loan is charged at 18% p.a. with interest calculated each month, just before each repayment.

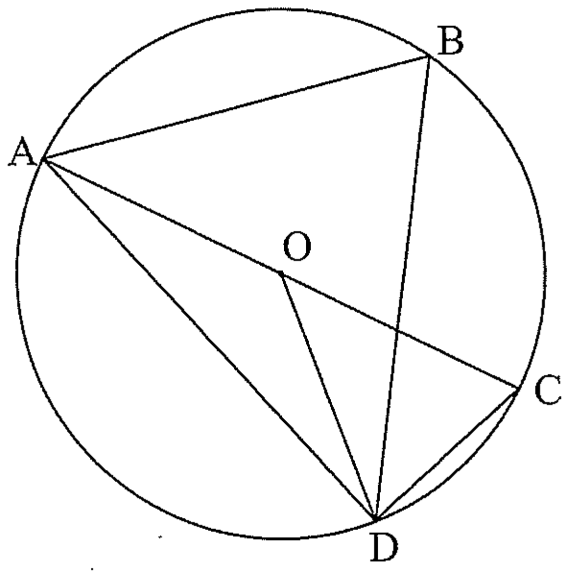
The loan is to be repaid in 48 equal monthly repayments of \$M. Let A_n be the amount owing after the nth repayment.

i) Find an expression for A_6 **1**

ii) Show that $A_8 = (30\,000 - 6M)(1.015)^2 - M(1 + 1.015)$ **2**

iii) Find the value of Helen's monthly repayment \$M **2**

(b) Consider the circle below where O is the centre and AC is a diameter. The points A, B, C and D all lie on the circumference of the circle.



Prove $\angle DCA = 90° - \angle DBC$ **2**

Question 10 (Start a new Page) **(10 Marks)**

(a) The polynomial $p(x) = x^3 + ax + b$ has $(x - 5)$ as one of its factors and has a remainder of -60 when divided by $(x + 5)$. Find the values of a and b. **3**

(b) Find the sum of the multiples of 6 between 1 and 400 **3**

(c) The polynomial equation $2x^3 - 4x^2 + 5x - 1 = 0$ has 3 roots α, β *and* γ.

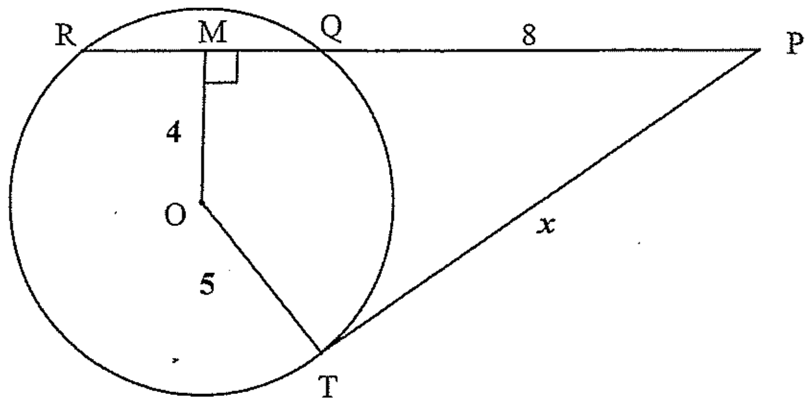
(i) Find $2\alpha\beta + 2\beta\gamma + 2\alpha\gamma$. **2**

(ii) Find $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$ **2**

Question 11 (Start a new Page) **(9 Marks)**

(a) PT is a tangent to the circle, centre O. OM is perpendicular to the secant RQ.

Find the value of x. **2**



(b) A parabola has parametric equations

$$x = t^2 + 1$$
$$y = 2(2t + 1)$$

(i) Sketch the parabola showing its orientation, the vertex and the focus. (Hint: use a ruler) **2**

(ii) Point **P** is the point on the parabola where $t = p$
Point P' is the point on the parabola where $t = -p$
Find the equation of the locus of the midpoint of PP' and state its geometrical significance **2**

(iii) A line with gradient m passes through (0,5) and cuts the parabola at distinct points Q and R. Find the range of possible values for m. **3**

END OF EXAMINATION

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Sydney Technical High School

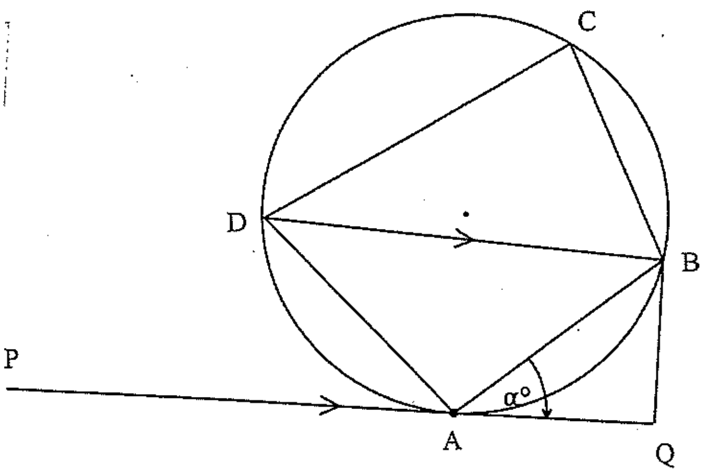
Extension One Mathematics

2015 - HSC Assessment Task 1

Multiple Choice

1. D
2. C
3. D
4. C
5. A

Question 6



i) $AP^2 = PC \times PD$

$AP^2 = 12 \times 5$

$AP^2 = 60$

$AP = \sqrt{60}$

$AP = 2\sqrt{15}$

ii) $\angle ABD = \angle BAQ = \alpha$

(alternate angles, $DB \parallel AQ$)

$\angle ADB = \angle BAQ = \alpha$ (alternate segment the

$\angle BAD = 180 - 2\alpha$ (angle sum of ΔABD)

$\angle BAD + \angle BCD = 180°$

(opposite angles of cyclic quadrilateral are supplementary)

$\therefore 180 - 2\alpha + \angle BCD = 180°$

$\therefore \angle BCD = 2\alpha$

iii) $QA = QB$ (tangent to circle from external point are equal)

$\angle QBA = \angle QAB = \alpha$ (equal angles opposite equal sides in isosceles trian

$\angle AQB = 180° - 2\alpha$ (angle sum of triangl

$\angle AQB + \angle BCD = 180° - 2\alpha + 2\alpha$

$= 180°$

$\therefore$ PQBC is a cyclic quadrilateral as opposite angles are supplementary

Question 7

2i) $r = 5\%$ $\quad A = P(1+r)^n$

$n = 10$ $\quad = 18\,000\left(1+\frac{5}{100}\right)^{10}$

$P = 18000$ $\quad = \$29\,320.10$

i) $A_1 = 1500\,(1.05)^9$

$A_2 = 1500(1.05)^8$

$A_3 = 1500(1.05)^7$

$\vdots$

$A_{10} = 1500$

$A = A_1 + A_2 + A_3 + \ldots + A_{10}$

$= 1500(1 + 1.05 + 1.05^2 + \ldots + 1.05^9)$

$= 1500\left[\frac{1(1.05^{10}-1)}{1.05-1}\right]$

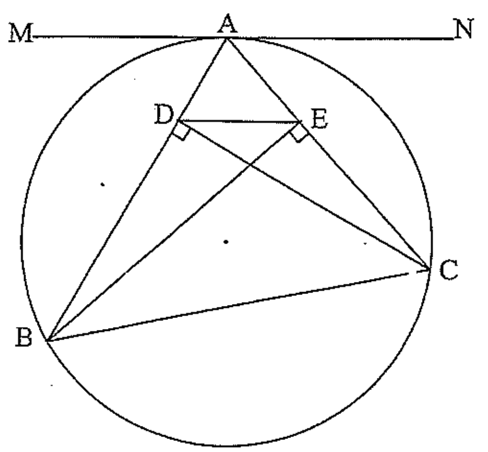
$= \$18\,866.84$

The total amount

$= \$29\,320.10 + 18\,866.84$

$= \$48\,186.94$

b



i) BC subtends equals angles at D & E

$\angle ABC = \angle AED$

(exterior angle of a cyclic quadrilatera is equal to the opposite interior angle)

ii) $\angle ABC = \angle NAC$

(angle between a chord and tangent is equal to the angle subtended by the chord at the circumference in the alternate segment)

$\therefore \angle AED = \angle NAC$ (both equal to $\angle ABC$)

$\therefore MAN \parallel DE$ (alternate angles are equal).

Question 8

ai) $P(6p, 3p^2)$ $\quad y = \frac{x^2}{12}$

$x^2 = 12y$ $\quad y' = \frac{x}{6}$

at $x = 6p$ $\quad m = \frac{6p}{6} = p$

$$y - y_1 = m(x - x_1)$$
$$y - 3p^2 = p(x - 6p)$$
$$y - 3p^2 = px - 6p^2$$
$$y = px - 3p^2$$

ii) Cuts y-axis at B $x = 0$ $y = -3p^2$

$B(0, -3p^2)$

$P(6p, 3p^2)$

$m : n = 1 : 2$

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{1 \times 0 + 2 \times 6}{3}, \frac{1 \times -3p^2 + 2 \times 3p^2}{3}\right)$$

$$= \left(\frac{1 + 12p}{3}, \frac{-3p^2 + 6p^2}{3}\right)$$

$A(4p, p^2)$

$x = 4p$ $\quad p = \frac{x}{4}$

$y = p^2$ $\quad y = \frac{x^2}{16}$

$x^2 = 16y$ is the locus

$$S_n = 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n$$

Step 1 = let $n = 1$

$L \cdot H \cdot S = 1 \times 2^0 = 1$

$R \cdot H \cdot S = 1 + (1-1) \times 2^1 = 1$

$\therefore$ true for $n = 1$

Step 2 Assume true for $n = k$

$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} = 1 + (k-1)2^k$

Step 3 Consider $S(k+1)$

$L \cdot H \cdot S = 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} \stackrel{?}{=} 1 + [(k+1)-1]2^{k+1}$

$= 1 + (k-1)2^k + (k+1)2^k$

$= 1 + (k - 1 + k + 1)2^k$

$= 1 + 2k \cdot 2^k$

$= 1 + [(k+1) - 1]2^{k+1}$

$= R \cdot HS$

Step 4 Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true then $S(3)$ is true and so on. $\therefore S(n)$ is true for all positive integers n

c) $\sum_{n=1}^{5} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

$= \frac{31}{32}$

Question 9

ai) $A_1 = 30\,000 - M$

$A_2 = 30\,000 - 2M$

$A_3 = 30\,000 - 3M$

$A_6 = 30\,000 - 6M$

aii) $A_7 = \left[30\,000 - 6M\right]1{\cdot}015 - M$

$A_8 = A_7(1{\cdot}015) - M$

$= \left[(30000 - 6M)1{\cdot}015 - M\right]1{\cdot}015 - M$

$= (30\,000 - 6M)\,1{\cdot}015^2 - M(1{\cdot}015) - M$

$= (30\,000 - 6M)1{\cdot}015^2 - M(1 + 1{\cdot}015)$

iii) $A_{48} = (30\,000 - 6M)1{\cdot}015^{42} - M(1 + 1{\cdot}015 + \ldots + 1{\cdot}015^{41})$

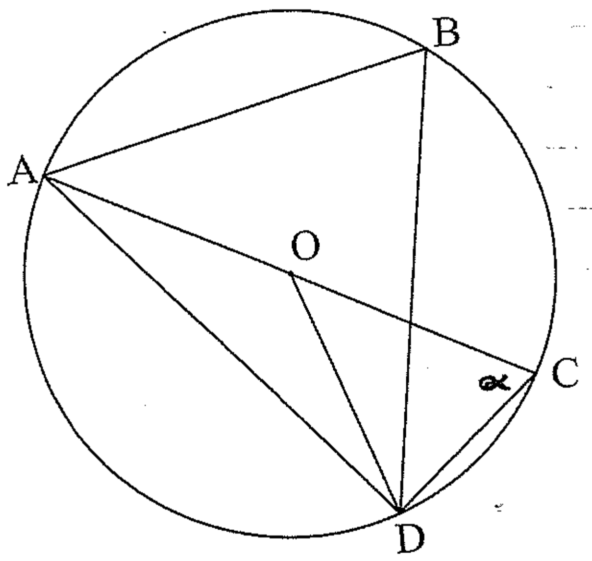
$0 = (30\,000 - 6M)(1{\cdot}015)^{42} - M\left(\dfrac{1{\cdot}015^{42} - 1}{1{\cdot}015 - 1}\right)$

$M\left(\dfrac{1{\cdot}015^{42} - 1}{1{\cdot}015 - 1}\right) = 30\,000(1{\cdot}015)^{42} - 6M(1{\cdot}015)^{42}$

$M\left[\dfrac{1{\cdot}015^{42} - 1}{0{\cdot}015} + 6(1{\cdot}015)^{42}\right] = 30\,000\,(1{\cdot}015)^{42}$

$M = \$811$

b)



Join BC

Let $\angle DCA = \alpha$

$\angle ABC = 90°$ (angle at the circumference is a semicircle)

$\angle ABD = \angle DCA = \alpha$. (angles at the circumference subtended by the same arc are equal)

$\angle DBC = 90° - \alpha$

$90 - \angle DBC = 90 - (90 - \alpha)$

$= \angle DCA$

$\therefore$ Proven as required that

$\angle DCA = 90° - \angle DBC$

Question 10

a) $p(x) = x^3 + ax + b$

$p(5) = 0$

$0 = 125 + 5a + b \quad \cdots\cdots (1)$

$p(-5) = -60$

$-125 - 5a + b = -60$

$-5a + b - 65 = 0 \quad \cdots\cdots (2)$

Solving simultaneously to find a and b

$-5a + b + 125 = 0$

$-5a + b - 65 = 0$

$2b + 60 = 0$

$b = -30$

$5a - 30 + 125 = 0$

$5a = -95$

$a = -19$

b) $S_{66} = \frac{66}{2}(6 + 396)$

$= 13\ 266$

$2x^3 - 4x^2 + 5x - 1$

$a = 2$

$b = -4$

$c = 5$

$d = 1$

i) $2\alpha\beta + 2\beta\gamma + 2\alpha\gamma$

$= 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 2 \times \frac{5}{2}$

$= 5$

ii) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$

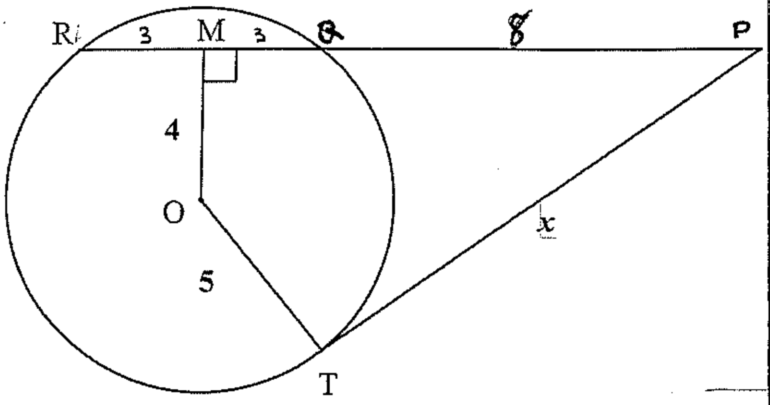
$= \frac{2\beta\gamma + 2\alpha\gamma + 2\alpha\beta}{\alpha\beta\gamma}$

$= \frac{2(\beta\gamma + \alpha\gamma + \alpha\beta)}{\alpha\beta\gamma}$

$= \frac{2 \times 5}{\frac{1}{2}}$

$= 20$

Question 11



a) $RM = \sqrt{5^2 - 4^2}$
$= 3$

Line is perpendicular through the centre of a circle perpendicular to a chord, bisecting the chord.

$RQ = 6$

$PR = 14$

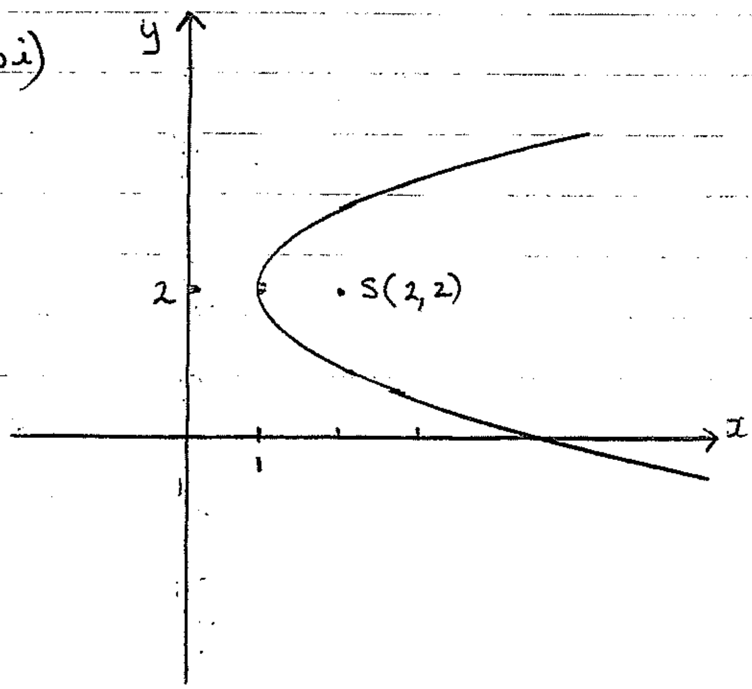
$(PT)^2 = PQ \cdot PR$

$x^2 = 8 \times 14$

$x^2 = 112$

$x = \sqrt{112}$ or 10.6 units

bi)



ii) $M = \left[\dfrac{(p^2+1)+(-p)^2+1}{2}\right], \left[\dfrac{2(2p+1)+2(-p)+1}{2}\right]$

$= \dfrac{2p^2+2}{2}, \dfrac{4p+2-4p+2}{2}$

$= (p^2+1, 2)$

$y = 2 \quad x \geq 1$

this is the axis of the parabola

iii) Equation of the line $y = xm + 5$

$(t^2+1), 2(2t+1)$

$2(2t+1) = m(t^2+1) + 5$

$4t + 2 = mt^2 + m + 5$

$0 = mt^2 - 4t + m + 3$

$\Delta > 0$

$(-4)^2 - 4(m)(m+3) > 0$

$16 - 4m^2 - 12m > 0$

$m^2 + 3m - 4 < 0$

$(m+4)(m-1) < 0$

$-4 < m < 1$

but $m \neq 0$