SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 3

JUNE 2015

Mathematics Extension 2

General Instuctions

- Working time - 90 minutes
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in questions 6 to 9
- Full marks may not be awarded for careless work or illegible writing
- Start each question on a new page
- All answers are to be in the writing booklet provided
- Marks for each question are indicated on the question
- A table of standard integrals is provided at the back of this paper

Total marks - 55

Section 1 - 5 marks

Attempt Questions 1 – 5.
Allow about 10 minutes for this section.

Section 2 - 50 marks

Attempt Questions 6 – 9.
Allow about 80 minutes for this section.

Name : ______________________

Teacher : ______________________

Section 1 (5 marks)

Attempt Questions 1 – 5

Use the multiple-choice answer sheet in your answer booklet for Questions 1 – 5.
Do not remove the multiple-choice answer sheet from your answer booklet.

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1. The polynomial $P(x)$ of degree 4 has real coefficients.
$P(x)$ has roots α, β, γ and δ and it is known that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -10$.

Which of the following must be true ?

(A) $P(x)$ has all its roots real.

(B) $P(x)$ has one real and three imaginary roots.

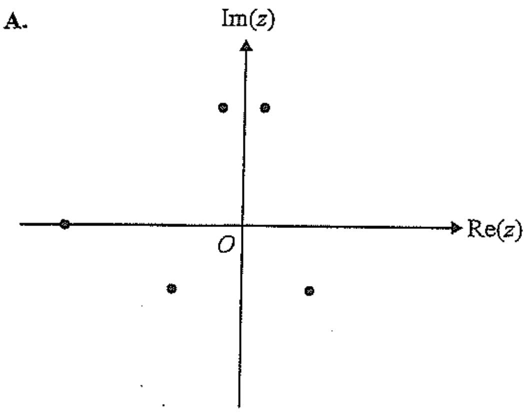
(C) $P(x)$ has two real and two imaginary roots.

(D) $P(x)$ has at least two imaginary roots.

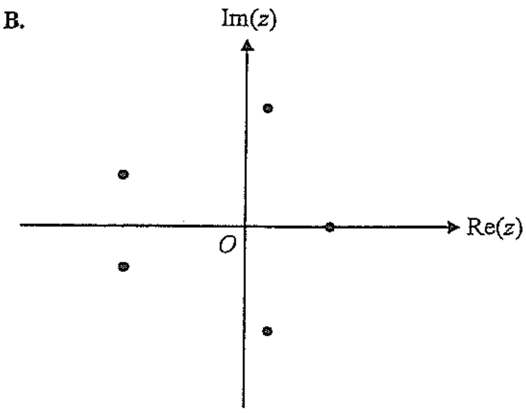
2. Which one of the following diagrams could represent the location of the roots of

$z^5 + z^2 - z + c = 0$ in the complex plane, given that c is real ?

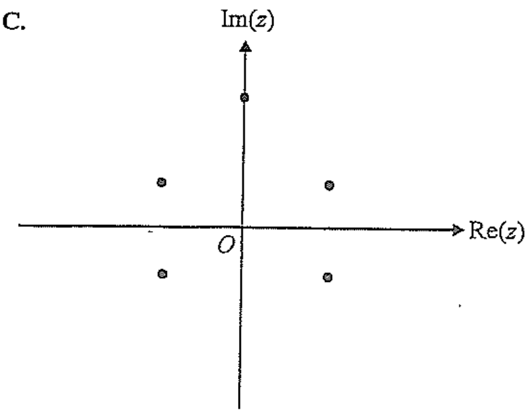
A.



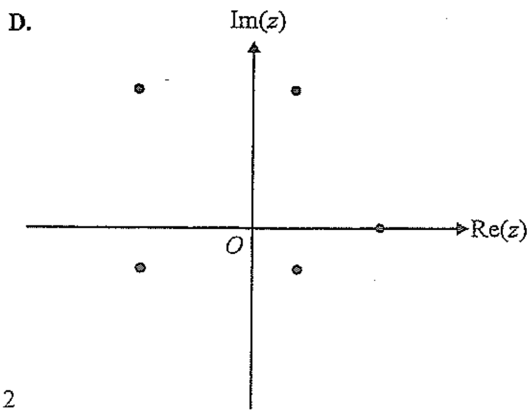
B.



C.



D.



3. With a suitable substitution, $\int_0^{\frac{\pi}{3}} cos^2x\ sin^3x\ dx$ can be expressed as

(A) $\int_{0.5}^{1} u^2 - u^4\ du$

(B) $\int_{1}^{0.5} u^2 - u^4\ du$

(C) $\int_0^{\frac{\pi}{3}} u^2 - u^4\ du$

(D) $-\int_0^{\frac{\sqrt{3}}{2}} u^2 - u^4\ du$

4. Which one of the following is a primitive function of $\frac{6}{\sqrt{1-4x^2}}$?

(A) $3\sin^{-1}(2x)$

(B) $6\sin^{-1}(2x)$

(C) $12\sin^{-1}(\frac{x}{2})$

(D) $3\sin^{-1}(\frac{x}{2})$

5. $\int_0^{a}(sin^2(\frac{3x}{2}) - cos^2\left(\frac{3x}{2}\right))dx$ is equal to

(A) $-\frac{4}{3}\sin(\frac{3a}{4})$

(B) $-\frac{1}{3}\sin(3a)$

(C) $\frac{1}{3}\sin(3a)$

(D) $\frac{1}{3}(1-\sin(3a))$

Section 2 (50 marks)

Attempt Questions 6 – 9
Start each question on a new page

Question 6 (12 marks)

(a) Given the polynomial $P(x) = 3x^4 - 14x^3 + 12x^2 + 24x - 32$ has a triple root, solve the equation $P(x) = 0$. **3**

(b) Find $\int \frac{dx}{x^2-6x+11}$. **3**

(c) Use the substitution $u = -x$ to evaluate $\int_{-1}^{1} \frac{dx}{e^x+1}$. **3**

(d) Using the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, or otherwise, solve the polynomial equation $8x^3 - 6x + 1 = 0$, giving your answers correct to 3 decimal places. **3**

Question 7 (13 marks) (Start a new page in your answer booklet)

(a) Find $\int \frac{dx}{x^2+6x-7}$. 3

(b) Use the substitution $x = 3\sin\theta$ to evaluate $\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\,dx$. . 4

(c) The polynomial $P(x) = x^3 - 5x^2 + 8x + b$, where b is a constant,

has a factor in the form $(x-k)^2$.

(i) Show that the possible values of k are $\frac{4}{3}$ and 2. 3

(ii) For $k = 2$, find the value of b and hence fully factorise $P(x)$. 3

Question 8 (13 marks) (Start a new page in your answer booklet)

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cos x-4\sin x}$ using the substitution $t = tan\frac{x}{2}$. 4

(b) If α, β and γ are the roots of the equation $x^3 + 4x^2 + 3x - 3 = 0$

find the polynomial equation whose roots are

(i) $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$. 2

(ii) $\alpha\beta - 1$, $\alpha\gamma - 1$ and $\beta\gamma - 1$ 3

(c) (i) If $I_n = \int_1^e (1 - \ln x)^n \, dx$, $n \geq 0$ 2

show that $\mathrm{I_n} = -1 + \mathrm{n\,I_{n-1}}$, $n \geq 1$.

(ii) Hence, or otherwise, evaluate $\int_1^e (1 - \ln x)^3 \, dx$ 2

Question 9 (12 marks) (Start a new page in your answer booklet)

(a) (i) Use the substitution $x = u^2$, $u > 0$, to show that 4

$$\int_4^{16} \frac{\sqrt{x}}{x-1}\, dx = 4 + 2\ln 3 - \ln 5.$$

(ii) Hence use integration by parts to evaluate 2

$$\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\, dx$$

(b) (i) Solve the equation $z^5 + 1 = 0$ over the complex field, 2

giving the complex roots in the form $r(cos\theta + isin\theta)$.

(ii) If α is the complex root of $z^5 + 1 = 0$ with smallest positive argument, 2

show that the other complex roots can be expressed as $-\alpha^2$, α^3 and $-\alpha^4$.

(iii) If α is the complex root of $z^5 + 1 = 0$ with smallest positive argument, 2

form the quadratic equation with roots $\alpha - \alpha^4$ and $\alpha^3 - \alpha^2$,

giving your answer in the form $ax^2 + bx + c = 0$.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXTENSION 2 SOLUTIONS - JUNE 2015

1. D 2. B 3. A 4. A 5. B

6. a)

$$P(x) = 3x^4 - 14x^3 + 12x^2 + 24x - 32$$
$$P'(x) = 12x^3 - 42x^2 + 24x + 24$$
$$P''(x) = 36x^2 - 84x + 24$$

Solving $36x^2 - 84x + 24 = 0$

$$3x^2 - 8x + 2 = 0$$
$$(3x-1)(x-2) = 0$$
$$x = \frac{1}{3}, 2$$

$$P'(\tfrac{1}{3}) \neq 0 \quad , \quad P'(2) = 0$$

$\therefore$ triple root is $x = 2$

$$\therefore \quad 2 + 2 + 2 + \alpha = \frac{14}{3}$$
$$\alpha = -1\tfrac{1}{3}$$

$\therefore$ solutions $2, 2, 2, -1\frac{1}{3}$

b) $$\int \frac{dx}{x^2 - 6x + 11}$$

$$= \int \frac{dx}{(x-3)^2 + 2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + c$$

c) $$\int_{-1}^{1} \frac{dx}{e^{x}+1}$$

$u = -x$

$du = -dx$

$$= \int_{1}^{-1} \frac{-du}{e^{-u}+1}$$

$$= \int_{-1}^{1} \frac{du}{\frac{1}{e^{u}}+1}$$

$$= \int_{-1}^{1} \frac{e^{u}\,du}{1+e^{u}}$$

$$= \ln(1+e^{u}) \Big]_{-1}^{1}$$

$$= \ln(1+e) - \ln(1+e^{-1})$$

$$= \ln\left(\frac{1+e}{1+e^{-1}}\right)$$

$$= \ln\left(\frac{1+e}{\frac{1+e}{e}}\right)$$

$$= \ln e$$

$$= 1$$

d) $8x^3 - 6x = -1$

$2(4x^3 - 3x) = -1$ let $x = \cos\theta$

$2(4\cos^3\theta - 3\cos\theta) = -1$

$\cos 3\theta = -\frac{1}{2}$

$3\theta = 120°, 240°, 480°$

$\theta = 40°, 80°, 160°$

$\therefore\ x = \cos 40°, \cos 80°, \cos 160°$

$= 0.766, 0.174, -0.940$

Q7

a) $$\int \frac{dx}{x^2+6x-7}$$

$$= \int \frac{dx}{(x+7)(x-1)}$$

$$\frac{1}{(x+7)(x-1)} = \frac{A}{x+7} + \frac{B}{x-1}$$

=

$$\therefore \quad 1 = A(x-1) + B(x+7)$$

$x=1$: $1 = 8B$

$B = \frac{1}{8}$

$x=-7$: $1 = -8A$

$A = -\frac{1}{8}$

$$= \int \frac{-\frac{1}{8}}{x+7} + \frac{\frac{1}{8}}{x-1} \, dx$$

$$= -\frac{1}{8}\ln(x+7) + \frac{1}{8}\ln(x-1)$$

$$= \frac{1}{8}\ln\left(\frac{x-1}{x+7}\right) + c$$

b) $$\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\, dx$$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int_0^{\frac{\pi}{4}} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{4}} \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{4}} 1+\cos 2\theta\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{9}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

c) i) $(x-k)^2$ is a factor $\Rightarrow$ k is a double root

$$\therefore \quad P'(x) = 3x^2 - 10x + 8$$
$$= (3x-4)(x-2)$$

$\therefore$ double root is $\frac{4}{3}$ or 2

$\therefore \quad k = \frac{4}{3}$ or 2

ii) when $k=2$

$$2^3 - 5(2)^2 + 8(2) + b = 0$$
$$b = -4$$

$\therefore$ Sum of roots $\quad 2 + 2 + \alpha = 5$

$$\alpha = 1$$

$$\therefore \quad P(x) = (x-2)^2(x-1)$$

Q 8

a) $$\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 3\cos x - 4\sin x}$$

$$= \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{5 + 3\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2\,dt}{5(1+t^2) + 3(1-t^2) - 4(2t)}$$

$$= \int_0^1 \frac{2\,dt}{8 - 8t + 2t^2}$$

$$= \int_0^1 (t-2)^{-2}\,dt$$

$$= \left[-(t-2)^{-1}\right]_0^1$$

$$= \frac{1}{2}$$

b) i) $y = \frac{x}{2} \Rightarrow x = 2y$

$\therefore \ P(2y) = 0$ is required polynomial equation

$$(2y)^3 + 4(2y)^2 + 3(2y) - 3 = 0$$

$$8y^3 + 16y^2 + 6y - 3 = 0.$$

ii) $\alpha\beta - 1 = \frac{\alpha\beta\gamma}{\gamma} - 1$ $\qquad \alpha\gamma - 1 = \frac{3}{\beta} - 1$

$= \frac{3}{\gamma} - 1$ $\qquad \beta\gamma - 1 = \frac{3}{\alpha} - 1$

$$\therefore \ y = \frac{3}{x} - 1$$

$$\frac{3}{x} = y + 1$$

$$x = \frac{3}{y+1}$$

$\therefore \ P\left(\frac{3}{y+1}\right) = 0$ is required polynomial equation

$$\left(\frac{3}{y+1}\right)^3 + 4\left(\frac{3}{y+1}\right)^2 + 3\left(\frac{3}{y+1}\right) - 3 = 0$$

$$27 + 36(y+1) + 9(y+1)^2 - 3(y+1)^3 = 0$$

$$27 + 36y + 36 + 9y^2 + 18y + 9 - 3y^3 - 9y^2 - 9y - 3 = 0$$

$$69 + 45y - 3y^3 = 0$$

$$y^3 - 15y - 23 = 0$$

c) i) $I_n = \int_1^e (1-\ln x)^n \, dx$

$u = (1-\ln x)^n \qquad v = x$

$u' = n(1-\ln x)^{n-1}\left(\frac{-1}{x}\right) \qquad v' = 1$

$\therefore \; I_n = x(1-\ln x)^n \Big]_1^e + \int_1^e n(1-\ln x)^{n-1}\left(\frac{1}{x}\right) dx$

$= e(1-\ln e) - 1(1-\ln 1) + n\int_1^e (1-\ln x)^{n-1} \, dx$

$= -1 + n I_{n-1}$

ii) $I_3 = -1 + 3I_2$

$= -1 + 3[-1 + 2I_1]$

$= -4 + 6I_1$

$= -4 + 6[-1 + I_0] \qquad I_0 = \int_1^e (1-\ln x)^0 \, dx$

$= -10 + 6(e-1) \qquad = [x]_1^e$

$= 6e - 16 \qquad = e - 1$

Q9

a) i) $\int_4^{16} \frac{\sqrt{x}}{x-1}\, dx$ $\qquad x = u^2$

$dx = 2u\, du$

$= \int_2^4 \frac{u}{u^2-1} \cdot 2u\, du$

$= \int_2^4 \frac{2u^2\, du}{u^2-1}$

$= 2\int_2^4 \frac{u^2-1+1}{u^2-1}\, du$

$= 2\int_2^4 1 + \frac{1}{u^2-1}\, du$ $\qquad \frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$\therefore 1 = A(u+1) + B(u-1)$

$A = \frac{1}{2},\ B = -\frac{1}{2}$

$= 2\int_2^4 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\, du$

$= \left[2u + \ln(u-1) - \ln(u+1)\right]_2^4$

$= (8 + \ln 3 - \ln 5) - (4 + \ln 1 - \ln 3)$

$= 4 + 2\ln 3 - \ln 5$

ii) $\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\, dx$ $\qquad u = \ln(x-1) \qquad v = 2x^{\frac{1}{2}}$

$u' = \frac{1}{x-1} \qquad v' = x^{-\frac{1}{2}}$

$= 2\sqrt{x}\ln(x-1)\Big]_4^{16} - 2\int_4^{16} \frac{\sqrt{x}}{x-1}\, dx$

$= (8\ln 15 - 4\ln 3) - 2(4 + 2\ln 3 - \ln 5)$

$= 8\ln 15 - 8\ln 3 + 2\ln 5 - 8$

$= 10\ln 5 - 8$

b) i) $z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$z_3 = -1$

$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$ or $\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$ or $\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

ii) $\alpha = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$$\begin{aligned} z_2 &= \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} \\ &= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^3 \\ &= \alpha^3 \end{aligned}$$

$$\begin{aligned} z_4 &= \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} \\ &= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^7 \\ &= \alpha^7 \\ &= \alpha^5 . \alpha^2 \\ &= -1 \times \alpha^2 \\ &= -\alpha^2 \end{aligned}$$

$$\begin{aligned} z_5 &= \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} \\ &= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^9 \\ &= \alpha^9 \\ &= \alpha^5 \times \alpha^4 \\ &= -1 \times \alpha^4 \\ &= -\alpha^4 \end{aligned}$$

iii) quadratic with roots $\alpha - \alpha^4$ and $\alpha^3 - \alpha^2$

$$\text{Sum} = \alpha - \alpha^4 + \alpha^3 - \alpha^2$$
$$= 1 \qquad \text{as } -1 + \alpha + \alpha^3 - \alpha^4 - \alpha^2 = 0$$
sum of roots of $z^5 + 1 = 0$

$$\text{product} = (\alpha - \alpha^4)(\alpha^3 - \alpha^2)$$
$$= \alpha^4 - \alpha^3 - \alpha^7 + \alpha^6$$
$$= \alpha^4 - \alpha^3 - \alpha^5 \cdot \alpha^2 + \alpha^5 \cdot \alpha \qquad (\alpha^5 = -1)$$
$$= \alpha^4 - \alpha^3 + \alpha^2 - \alpha$$
$$= -(\alpha - \alpha^2 + \alpha^3 - \alpha^4)$$
$$= -1$$

$\therefore$ required quadratic is $x^2 - (\text{sum of roots})\,x + \text{product} = 0$

$$x^2 - x - 1 = 0$$