SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1
November 2016

Mathematics Extension 2

Name ....

Teacher ....

General Instructions

- Working Time - 75 min.
- Write using a blue or black pen.
- Board approved calculators may be used.
- The reference sheet is provided at the back of this paper.
- In Questions 6-9, show relevant mathematical reasoning and /or calculations.
- Begin each question on a fresh sheet of paper.

Total marks (53)

- Attempt Questions 1-9.
- All questions are of equal value.

Section 1
Multiple Choice (5 marks)
Use the multiple choice answer sheet for Question 1-5

1. What is $-\sqrt{3} + i$ expressed in modulus-argument form?

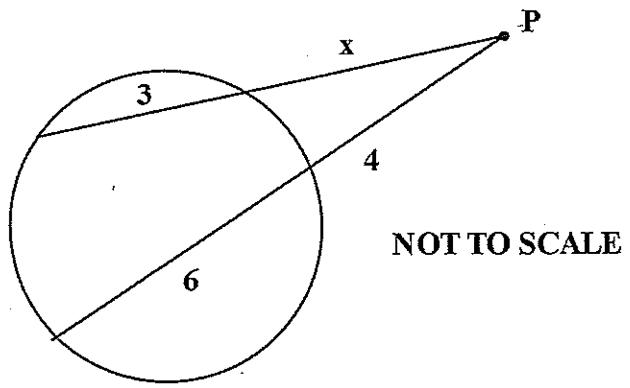
(A) $\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(B) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(C) $\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(D) $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

2. Two secants from the point P intersect a circle as shown in the diagram.



What is the value of x?

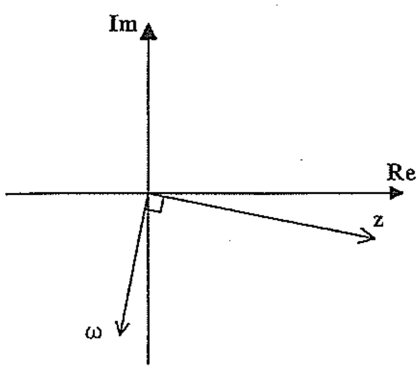
(A) 2

(B) 5

(C) 7

(D) 8

3. Consider the following diagram, drawn to scale, showing the complex numbers z and ω.



Which statement is false?

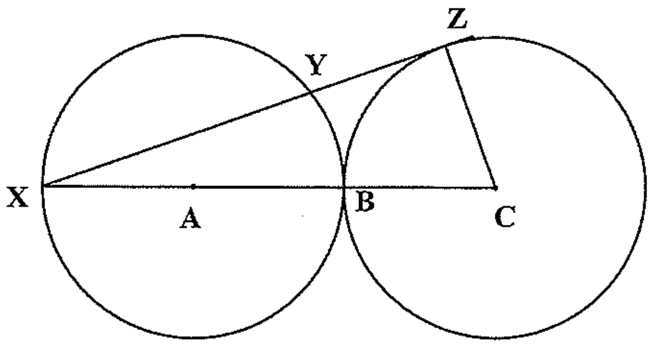
(A) $|z + \omega| = |z - \omega|$

(B) $Re\left(\dfrac{\omega}{z}\right) = 0$

(C) $-\pi < \arg(z - \omega) < 0$

(D) $z^2 = k\omega^2$ for some real number k

4. Two equal circles touch externally at B. XB is a diameter of one circle. XZ is the tangent from X to the other circle and cuts the first circle at Y.



Which is the correct expression that relates to XZ to XY?

(A) 2XZ = 3XY

(B) XZ = 2XY

(C) 3XZ = 4XY

(D) 2XZ = 5XY

5. What is the value of $S = \sum_{r=1}^{\infty} rp(1-p)^{r-1}$ where $0 < p < 1$?

(A) $S = 1$

(B) $S = \frac{1}{p}$

(C) $S = \frac{1}{1-p}$

(D) $S = \frac{1}{p(1-p)}$

Section II
Total Marks (48)
Attempt Questions 6 – 9.

Answer each question in your writing booklet.
In Questions 6-9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(12 Marks)** Use a Separate Sheet of paper

a) Let $A = 3 - 4i$ and $B = 2 + i$
Find in the form $x + iy$

i) $A - B$ **1**

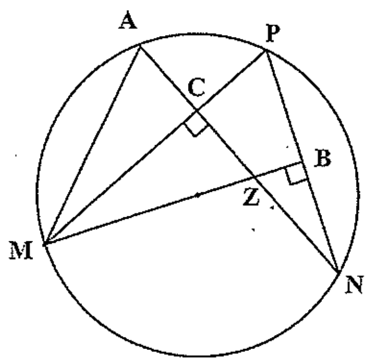
ii) $\overline{A}B$ **1**

iii) $\dfrac{5}{A}$ **2**

iv) $\dfrac{iB}{\overline{B}}$ **2**

b) Use Mathematical Induction to prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for an integer $n > 1$ **3**

c) M, N and P are three points on a circle.
The altitudes MB and NC in the acute-angles triangle MNP meet at the point Z.
NC produced meets the circle at A as shown below. **3**



Copy the diagram into your answer booklet.

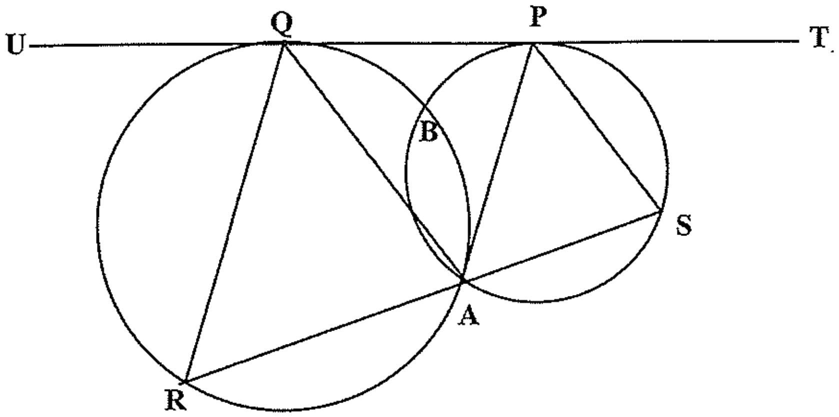
Prove that $AC=CZ$

End of Question 6

Question 7 **(12 Marks)** Use a Separate Sheet of paper

a) If $P = \sqrt{3} + i$, find P^4 in modulus-argument form. **3**

b) In the diagram below, two circles intersect at A and B. The common tangent TU touches the circles at P and Q respectively. A line through A cuts the left hand circle at R and the right hand at S
It is found that PQRS is a cyclic quadrilateral.



Copy the diagram into your answer booklet.

i) Give a reason why $\angle UQR = \angle PSA$ **1**

ii) Prove that $PS \parallel AQ$ **2**

iii) Hence show that $\Delta PAS \ ||| \ \Delta QRA$ **2**

c) Given that $z = w + \dfrac{1}{w}$ where $w = 2(\cos\theta + i\sin\theta)$,

i) Express the real and imaginary parts of z in terms of θ. **2**

ii) Show that the point representing z in the Argand diagram lies on the curve with Cartesian equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = \dfrac{1}{4}$. **2**

End of Question 7

Question 8 **(12 Marks)** Use a Separate Sheet of paper

a) Given that A and B represent the complex numbers $\sqrt{3}+i$ and $1-i\sqrt{2}$ respectively, find:

i) $\dfrac{A}{B}$ in the form $x+iy$ **2**

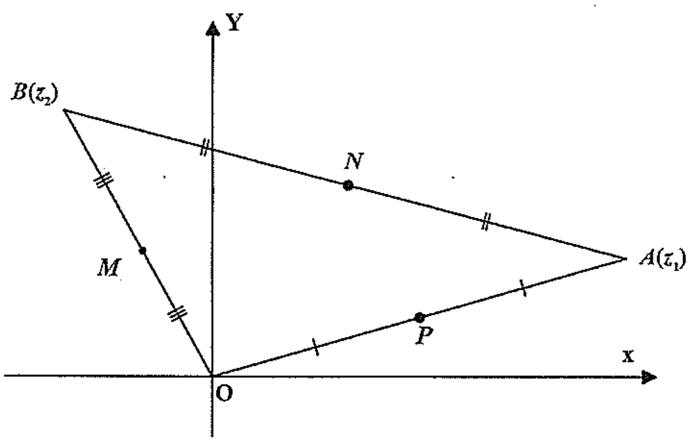
ii) the modulus and the argument of A. **2**

b) Considering the sum of an arithmetic series, show that:

i) $(1+2+3+\ldots+n)^2=\frac{1}{4}n^2(n+1)^2$ **1**

ii) By mathematical induction prove that:
$1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2, n\geq 1$ **3**

c) ΔABO lies on the Argand diagram.
Points A and B represent the complex numbers z_1 and z_2 respectively.
M, N and P are the midpoints of OB, AB and OA respectively.



i) Which complex number is represented by N? **1**

ii) Express $\overrightarrow{AM}$ and $\overrightarrow{BP}$ in terms of z_1 and z_2. **2**

iii) Hence simplify $\overrightarrow{ON}+\overrightarrow{AM}+\overrightarrow{BP}$. **1**

End of Question 8

Question 9 **(12 Marks)** Use a Separate Sheet of paper

a) Sketch the following loci on an Argand diagram:

i) $|z| \leq |z-2+i|$ **2**

ii) $|(z-1)+i| = 2$ **2**

iii) $\arg\left(\dfrac{z-2}{z-1}\right) = \dfrac{\pi}{3}$ **2**

b) Find the complex square roots of $7 + 6i\sqrt{2}$ giving your answer in the form $+ib$, where a and b are real. **3**

c) Let n be a positive integer and z a complex root of the equation

$$(z-1)^n + (z+1)^n = 0 \text{ ....(1)}$$

Show that $|z-1| = |z+1|$ and by letting $z = \lambda + i\alpha$ show that all the roots of (1) can be written in the form $z = i\alpha_k$, where α_k is real and $k = 1,2,\ldots,n$. **3**

End of Examination

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Sydney Technical High School
Mathematics - Extension 2
H.S.C Assessment 1 - 2016

Multiple Choice

1. D
2. B
3. C
4. A
5. B

Question 6.

i) $A - B = (3-4i) - (2+i)$

$= 1 - 5i$

ii) $\bar{A}B = (3+4i)(2+i)$

$= 6 + 3i + 8i - 4$

$= 2 + 11i$

iii) $\frac{5}{A} = \frac{5}{3-4i} \times \frac{3+4i}{3+4i}$

$= \frac{5(3+4i)}{25}$

$= \frac{3}{5} + \frac{4}{5}i$

iv) $\frac{iB}{\bar{B}} = \frac{i(2+i)}{2-i} \times \frac{2+i}{2+i}$

$= \frac{i(4+4i-1)}{5}$

$= \frac{i(3+4i)}{5}$

$= \frac{3i}{5} - \frac{4}{5}$

b) Step 1 When $n=1$

$3^{2\times1+4} - 2^{\times1} = 3^6 - 2^2$

$= 729 - 4$

$= 725$

which is divisible by 5

Hence result is true for $n=1$

Step 2

Assume the result is true for $n=k$. $3^{2k+4} - 2^{2k} = 5P$ where P is an integer

Step 3

Prove the result is true for $n=k+1$ $3^{2k+6} - 2^{k+2} = 5Q$ where Q is an integer.

$3^{2k+6} - 2^{k+2} = 3^2(3^{2k+4}) - 2^{2k+2}$

$= 9(5P + 2^{2k}) - 2^2(2^{2k})$

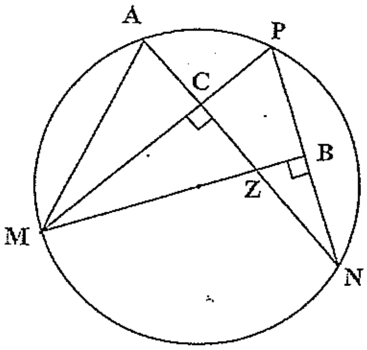
$= 45P + 9 \times 2^{2K} - 4 \times (2^{2}$

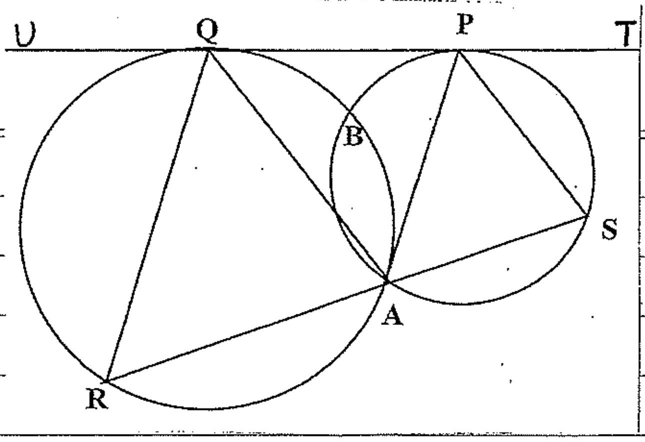
$= 45P + 5 \times 2^{k}$

$= 5(9P + 2^{k})$

$= 5Q$

Where $Q = 9P + 2^{2k}$ which is an integer. Hence the result is true for $n=k+1$, if it is true for $n=k$. Since the result is true for $n=1$, then it is true for all positive integers by induction.





$\angle MAN = \angle MPN$ (angles in the same segment)
CPBZ is a cyclic quadrilateral (angle C and angle B are supplementary)
$\angle MZC = \angle CPB$ (exterior angle of cyclic quadrilateral is equal to the interior opposite angle)

$\angle MAZ = \angle MZA$
ie ΔMAZ is isosceles
MC is perpendicular to AZ and therefore bisects AZ
$AC = CZ$

b.i) exterior angle of a cyclic quadrilateral equal the opposite interior angle.

ii) $\angle UQR = \angle QAR$ (alternate segment theorem)
$\angle PSA = \angle QAR$ (exterior angle of a cyclic quad. equals the opposite interior angle)

iii) In ΔPAS and ΔQRA
1) $\angle PSA$ and $\angle QAR$
2) $PA \parallel QP$, $\angle PAS = \angle QRA$ (corresponding angles, $PA \parallel QR$
$\therefore \Delta PAS \parallel\!\mid \Delta QRA$ (equiangula

Question 7

a) $P = \sqrt{3} + i$
$P = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$
$P = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

$P^4 = 2^4\left(\cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6}\right)$

$= 16\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

c i) $Z = w + \frac{1}{w}$
$= w + \frac{\overline{w}}{|w|^2}$
$= 2(\cos\theta + i\sin\theta) + \frac{2(\cos\theta - i\sin}{4}$

$= \frac{5}{2}\cos\theta + \frac{3}{2} i \sin\theta$

Real $= \frac{5}{2}\cos\theta$

Imaginary $= \frac{3}{2}\sin\theta$

ii) $x = \frac{5}{2}\cos\theta$ $\quad y = \frac{3}{2}\sin\theta$

$\therefore \frac{4x^2}{25} = \cos^2\theta$ $\quad \frac{4y^2}{9} = \sin^2\theta$

$\frac{4x^2}{25} + \frac{4y^2}{9} = \cos^2\theta + \sin^2\theta$

$\frac{4x^2}{25} + \frac{4y^2}{9} = 1$

$4\left(\frac{x^2}{25} + \frac{y^2}{9}\right) = 1$

$\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{4}$

Question 8

ai) $\frac{A}{B} = \frac{\sqrt{3}+i}{1-\sqrt{2}i} \times \frac{1-\sqrt{2}i}{1+\sqrt{2}i}$

$= \frac{\sqrt{3}+\sqrt{6}i + i - \sqrt{2}}{3}$

$= \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{6}+1}{3} i$

i) $|A| = 2$

$A = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\text{Arg } A = \theta$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\text{Arg } A = \frac{\pi}{6}$

bi) $(1+2+3+\ldots+n)^2 = \frac{1}{4}n^2(n+1)^2$

$1+2+3+\ldots+n^2$

$a=1$ $\quad S_n = \frac{n}{2}[2a+(n-1)d]$

$d=1$ $\quad = \frac{n}{2}[2+n-1]$

$= \frac{n}{2}[n+1]$

$[S_n]^2 = \left[\frac{n}{2}(n+1)\right]^2$

$[S_n]^2 = \frac{n^2}{4}(n+1)^2$

bii) $S_n = 1^3 + 2^3 + \ldots + n^3$

Step 1 $\quad S_n = \frac{n^2}{4}(n+1)^2$

$L.H.S = 1^3 = 1$

$R.H.S = 1$

$\therefore LHS = RHS$

Step 2 Assume it is true for

$n=k$ $\quad S_k = \frac{k^2}{4}(k+1)^2$

Step 3 $\quad S_{k+1} = S_k + T_{k+1}$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= (k+1)^2\left[\frac{1}{4}k^2 + k+1\right]$

$= (k+1)^2[k^2+4k+4]$

$= \frac{1}{4}(k+1)^2(k+2)^2$

The statement is true for $n=k+$ and by principle of mathematica induction it is true for all $n \geq 1$

i) $N = \dfrac{z_1 + z_2}{2}$

ii) $\overrightarrow{AM} = \dfrac{z_2}{2} - z_1$

$\overrightarrow{BP} = \dfrac{z_1}{2} - z_2$

iii) $\overrightarrow{ON} + \overrightarrow{AM} + \overrightarrow{BP}$

$= \dfrac{z_1 + z_2}{2} + \dfrac{z_2 - 2z_1}{2} + \dfrac{z_1 - 2z_2}{2}$

$= \dfrac{z_1 + z_2 + z_2 - 2z_1 + z_1 - 2z_2}{2}$

$= 0$

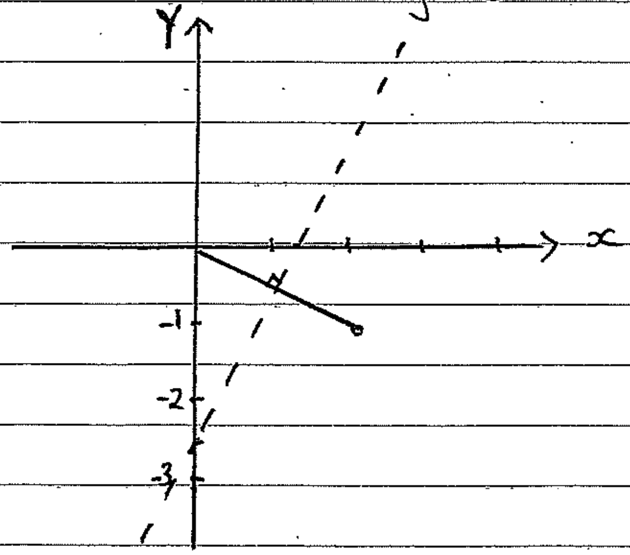
Question 9

i) $|z| < |z - 2 + i|$

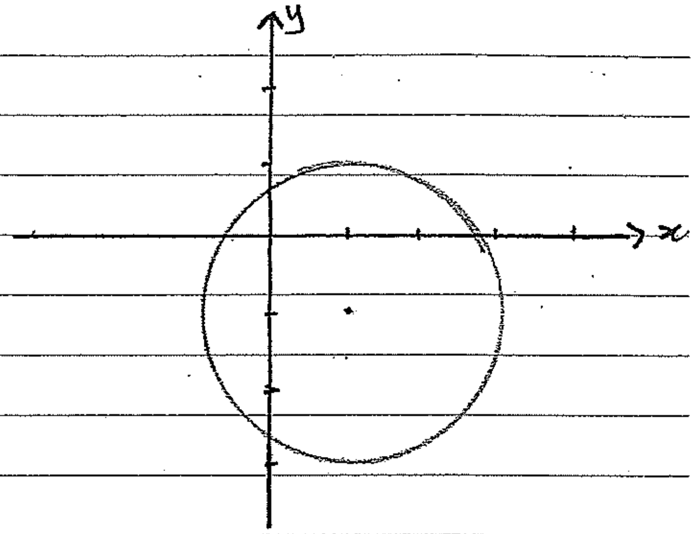
$x^2 + y^2 < (x-2)^2 + (y+1)^2$

$x^2 + y^2 < x^2 - 4x + 4 + y^2 + 2y + 1$

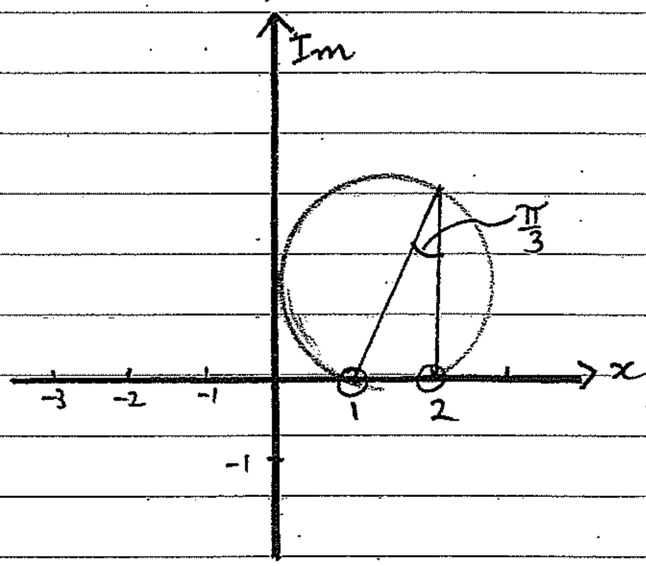
$0 < -4x + 2y + 5$



b) Circle centre $(1,-1)$ radius 2.



c) $\arg\left(\dfrac{z-2}{z+1}\right) = -\dfrac{\pi}{3}$



2.) $7+6i\sqrt{2} = (a+ib)^2$

$= a^2-b^2+2abi$

Real $= a^2-b^2=7$

Im $= 2ab = 6\sqrt{2}$

$ab = 3\sqrt{2}$

$\therefore a^2+b^2 = 11$

$2a^2 = 18$

$a^2 = 9$

$a = \pm 3$

$b = \pm 3$

$z = \pm(3+\sqrt{2}i)$

$(z-1)^n + (z+1)^n = 0$

$(z-1) = -(z-1)^n$

$\frac{(z+1)^n}{(z-1)^n} = -1 \qquad (z \neq 1)$

$\frac{|z+1|^n}{|z-1|^n} = |-1| = 1$

$\left|\frac{z+1}{z-1}\right|^n = 1$ since $\left|\frac{z+1}{z-1}\right| > 0$

$\therefore |z+1| = |z-1|$

let $z = \lambda + ia$

$|\lambda + i\alpha + 1| = |\lambda + i\alpha - 1|$

$\sqrt{(\lambda+1)^2+\alpha^2} = \sqrt{(\lambda-1)^2+\alpha^2}$

$\lambda^2+2\lambda+1+\alpha^2 = \lambda^2-2\lambda+1+\alpha^2$

$4\lambda = 0$

$\lambda = 0$

$z = i\alpha_K \qquad K=1,2,3\ldots n$