Name: .................................................... Maths Class: ......................

# SYDNEY TECHNICAL HIGH SCHOOL

## Year 12

# Mathematics Extension 2

## HSC Course

## Assessment 2

## March, 2015

*Time allowed: 70 minutes*

***General Instructions:***

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

| | |
|---|---|
| Section 1 | Multiple Choice<br>Questions 1-5<br>5 Marks |
| Section II | Questions 6-9<br>40 Marks |

**Section I**
**5 marks**
**Attempt Questions 1-5**

Use the multiple choice answer sheet for Questions 1 – 5.

1. A square root of $8 + 6i$ is :

(A) $3 - i$ (B) $5 - 3i$

(C) $-3 - i$ (D) $-3 + i$

2. The equation of a curve is given by $x^2 + xy + y^2 = 9$. Which of the following expressions will provide the value of $\frac{dy}{dx}$ at any point on the curve?

(A) $\frac{-2x - y}{2y}$ (B) $\frac{-2x - y}{x + 2y}$

(C) $\frac{-2x + y}{2y}$ (D) $\frac{-2x + y}{x + 2y}$

3. The equation of an hyperbola is given by $9x^2 - 4y^2 = 36$. The foci and the directrices of this hyperbola are:

(A) $(\pm\sqrt{13}, 0)$ and $x = \pm\frac{4\sqrt{13}}{13}$.

(B) $(0, \pm\sqrt{13})$ and $x = \pm\frac{4\sqrt{13}}{13}$.

(C) $(\pm\sqrt{13}, 0)$ and $y = \pm\frac{4\sqrt{13}}{13}$.

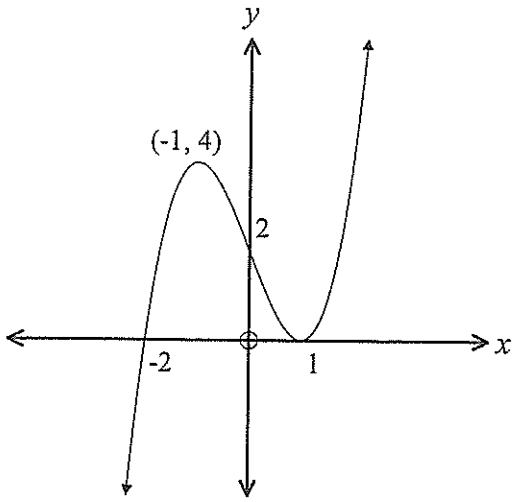
(D) $(0, \pm\sqrt{13})$ and $y = \pm\frac{4\sqrt{13}}{13}$.

4. The area bounded by the curves $y = x^2$ and $x = y^2$ is rotated about the $x$ – axis. The volume of the solid of revolution formed in cubic units is:

(A) $\frac{9\pi}{70}$ (B) $\frac{3\pi}{10}$

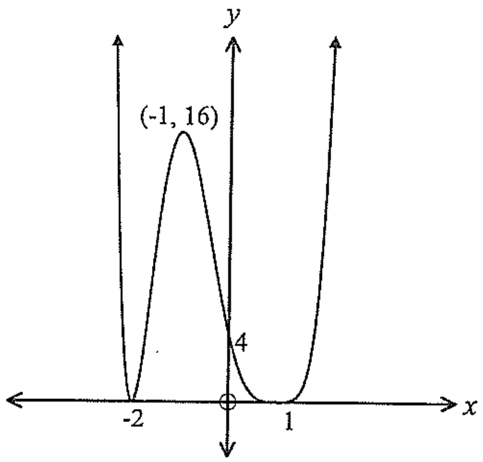
(C) $\frac{7\pi}{10}$ (D) $\frac{3\pi}{2}$

5. The graph of the function $y = f(x)$ is drawn below:

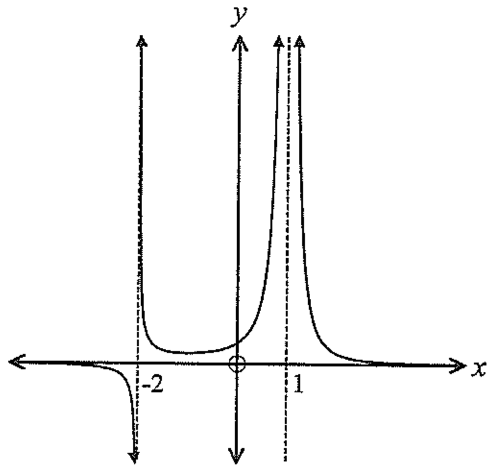



Which of the following graphs best represents the graph $y = \sqrt{f(x)}$ ?

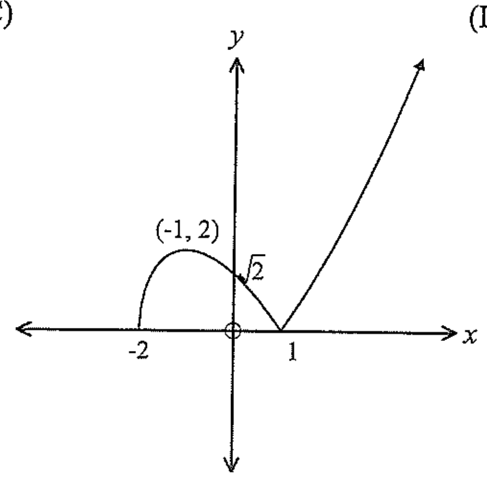
(A)



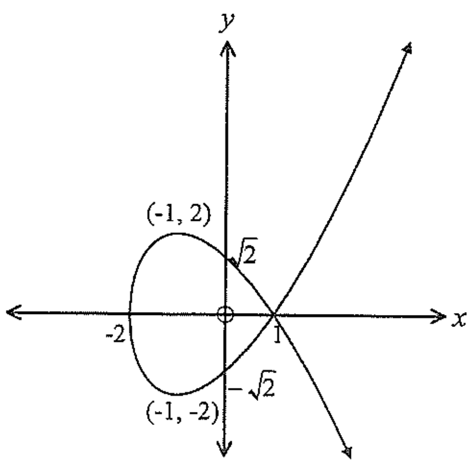

(B)




(C)




(D)




**End of Section I**

**Section II**

**Total marks (40)**
**Attempt Questions 6 - 9**

**Question 6 (10 marks)** **Marks**

a) An ellipse $E$ has equation $\frac{x^2}{4} + \frac{y^2}{2} = 1$

(i) Show that the equation of $E$ can be written in the parametric form **2**

$$x = 2cos\theta, y = \sqrt{2}sin\theta$$

(ii) Assuming the perimeter of $E$ is given by the formula **2**

$$p = 2\int_0^{\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta\ ,$$

show that $p = 2\sqrt{2}\ \int_0^{\pi} \sqrt{2 - cos^2\theta}\ d\theta$

b) (i) If $w = \frac{1+i\sqrt{3}}{2}$ show that $w^3 = -1$ **1**

(ii) Hence calculate $w^{12}$ **1**

(iii) Find all the cube roots of -1, both Real and Complex. **2**

c) Given that one root of the equation $x^4 - 5x^3 + 5x^2 + 25x - 26 = 0$ is $3 + 2i$, solve the equation. **2**

**Question 7 (10 marks) Start a new page**

a) If $f(x) = -x^2 + 7x - 10$, on separate diagrams and without using calculus, sketch the following graphs, indicating the intercepts with the axes and any asymptotes for each sketch:

(i) $y = f(x)$ **1**

(ii) $y = |f(x)|$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = -f(x+2)$ **2**

b) Find all the roots of $18x^3 + 3x^2 - 28x + 12 = 0$, given that two roots are equal. **3**

**Question 8 (10 marks) Start a new page**

a) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers **2**

find $\dfrac{z_1}{z_2}$ in modulus-argument form

b) Given that the Argand Diagram for $|z-2| + |z-4| = 10$ is an ellipse,

(i) Find the co-ordinates of the centre of this ellipse and the lengths of the major and minor axes **3**

(ii) On an Argand Diagram, show the region for which z satisfies the inequalities **3**

$$z + \bar{z} \le 6 \quad \text{and} \quad |z-2| + |z-4| \le 10$$

c) Find the perimeter of the shape in the Argand Diagram described by **2**

$$|z-1| \le 1 \qquad \text{and} \qquad 0 \le \arg z \le \frac{\pi}{6}$$

**Question 9 (10 marks) Start a new page**

a) Find the equation of the tangent to $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ at the point $P(4\cos\theta, 5\sin\theta)$. **2**

b) P $(2p, \dfrac{2}{p})$ is a variable point on the hyperbola xy=4.

The normal to the hyperbola at P meets the hyperbola again at Q $(2q, \dfrac{2}{q})$.

M is the midpoint of PQ.

(i) Show that the equation of the normal at P is given by $p^3x - py = 2(p^4 - 1)$ **2**

(ii) Show that $q = -\dfrac{1}{p^3}$ **1**

(iii) Show that M has coordinates $[\frac{1}{p}\left(p^2 - \frac{1}{p^2}\right),\ p\left(\frac{1}{p^2} - p^2\right)]$ **2**

(iv) Show that, as P moves on the curve xy = 4, the locus of M is given by **3**

$$(x^2 - y^2)^2 = -x^3y^3$$

***End of Examination***

SOLUTIONS S.T.H.S. YR12 EXT 2. ASS. 2 MAR 2015

SECTION I

1. C 2 B 3. A 4. B 5. C

5×1 = 5 MARKS

SECTION II

6 a) i) $x = 2\cos\theta$ & $y = \sqrt{2}\sin\theta$

$x^2 = 4\cos^2\theta$ $y^2 = 2\sin^2\theta$

$\therefore \frac{x^2}{4} + \frac{y^2}{2} = \cos^2\theta + \sin^2\theta$

$= 1$ as req'd

$\therefore$ E can be written parametrically as

$x = 2\cos\theta$ $y = \sqrt{2}\sin\theta$

2

(ii) If $x = 2\cos\theta$ & if $y = \sqrt{2}\sin\theta$

$\frac{dx}{d\theta} = -2\sin\theta$ $\frac{dy}{d\theta} = \sqrt{2}\cos\theta$

$\therefore p = 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ becomes

$= 2\int_0^{\pi}\sqrt{4\sin^2\theta + 2\cos^2\theta}\, d\theta$

$= 2\int_0^{\pi}\sqrt{4(1-\cos^2\theta) + 2\cos^2\theta}\, d\theta$

$= 2\int_0^{\pi}\sqrt{4 - 2\cos^2\theta}\, d\theta$

2

$= 2\sqrt{2}\int_0^{\pi}\sqrt{2-\cos^2\theta}\, d\theta$ as req'd

b) i) $w = \frac{1}{2}(1+i\sqrt{3})$

$\therefore w^3 = \frac{1}{8}(1+i\sqrt{3})^3$

$= \frac{1}{8}(1+i\sqrt{3})(1+i\sqrt{3})^2$

$= \frac{1}{8}(1+i\sqrt{3})(-2+2\sqrt{3}i)$

$= -\frac{2}{8}(1+i\sqrt{3})(1-i\sqrt{3})$

$= -\frac{1}{4} \times 4$ (1)

$= -1$ as req'd

ii) $w^{12} = (w^3)^4$

$= (-1)^4$ (1)

$= 1$

(iii) Cube roots of $-1$ are solutions of $w^3 = -1$

Let $w = \cos\theta + i\sin\theta$

$\therefore w^3 = \cos 3\theta + i\sin 3\theta$

Thus $\cos 3\theta + i\sin 3\theta = -1$ w $0 \leq 3\theta \leq 6\pi$

$\therefore 3\theta = \pi, 3\pi, 5\pi$

$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Thus $w_1 = \underline{\text{cis}\,\frac{\pi}{3}}$ or $\underline{\frac{1}{2} + \frac{i\sqrt{3}}{2}}$ (as given)

$w_2 = \underline{\text{cis}\,\pi}$ or $\underline{-1}$ (real)

$w_3 = \underline{\text{cis}\,\frac{5\pi}{3}}$ or $\underline{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$

(2)

c) If $3+2i$ is a root, then $3-2i$ is also a root

$\therefore (x-3-2i)(x-3+2i)$ is a factor

$\therefore (x-3)^2+4$ " " "

$\therefore x^2-6x+13$ " " "

Using long division

$$\begin{array}{r|l} & x^2 + x - 2 \\ x^2-6x+13 & x^4-5x^3+5x^2+25x-26 \\ & \underline{x^4-6x^3+13x^2} \\ & \quad x^3-8x^2+25x \\ & \quad \underline{x^3-6x^2+13x} \\ & \qquad -2x^2+12x-26 \\ & \qquad \underline{-2x^2+12x-26} \\ & \qquad \cdot \quad \cdot \quad \cdot \end{array}$$

$\therefore P(x) = (x^2-6x+13)(x^2+x-2)$

$= (x-3-2i)(x-3+2i)\,(x+2)(x-1)$

$\therefore P(x) = 0$ has solutions

$\underline{3+2i,\ 3-2i,\ -1,\ 1}$

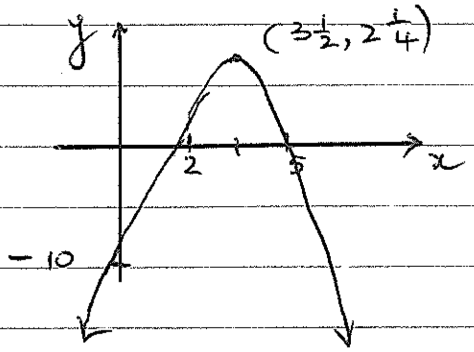
2

7. a) $f(x) = -x^2 + 7x + 10$

$= -(x^2 - 7x + 10)$

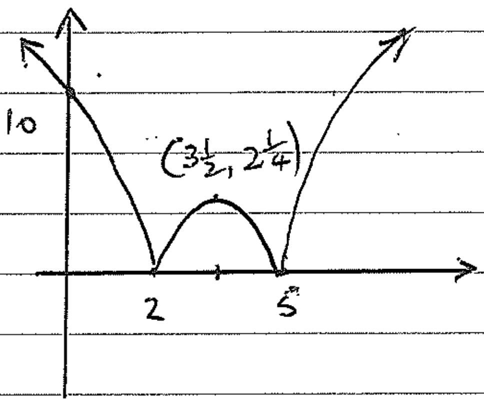
$= -(x-2)(x-5)$

(i) $y = f(x)$



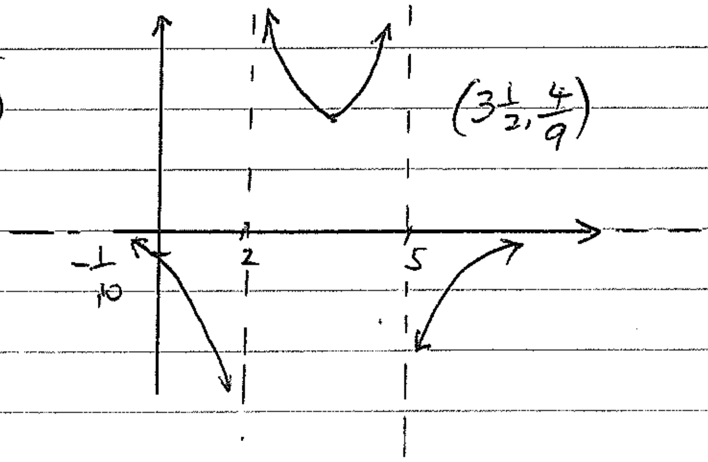

1

(ii) $y = |f(x)|$



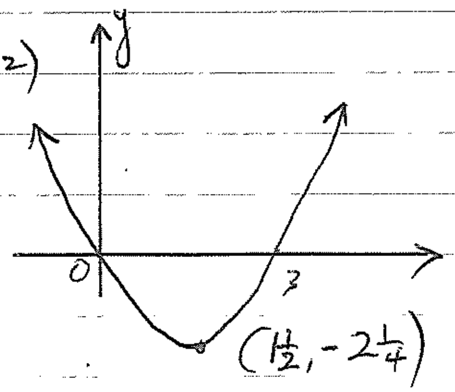

2

(iii) $y = \dfrac{1}{f(x)}$




2

(iv) $y = -f(x+2)$




2

(b) $P(x) = 18x^3 + 3x^2 - 28x + 12$

Solve $P'(x) = 54x^2 + 6x - 28 = 0$

$\therefore \quad 27x^2 + 3x - 14 = 0$

$\therefore x = \dfrac{-3 \pm \sqrt{9 + 4 \cdot 27 \times 14}}{54}$

$= \dfrac{-3 \pm 39}{54}$

$= \dfrac{36}{54}$ & $\dfrac{-42}{54}$

$= \dfrac{2}{3}$ & $\dfrac{-7}{9}$

So one of these is a repeated root of $P(x)$

$P\left(\frac{2}{3}\right) = 0$

$\therefore \quad x = \frac{2}{3}$ is a double root of $P(x)$

$\therefore (3x-2)^2$ is a factor

$\therefore 9x^2 - 12x + 4$ is a factor

$$
\begin{array}{r}
2x + 3 \\
9x^2 - 12x + 4 \overline{\smash{)}\, 18x^3 + 3x^2 - 28x + 12} \\
\underline{18x^3 - 24x^2 + 8x \quad\quad} \\
27x^2 - 36x + 12 \\
\underline{27x^2 - 36x + 12}
\end{array}
$$

$\therefore P(x) = (3x-2)^2(2x+3)$

which has roots

$\dfrac{2}{3}$ & $-\dfrac{3}{2}$ ONLY

3

Q8

a) $z_1 = 1 + i\sqrt{3}$ & $z_2 = 1 - i$

$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $= \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= 2 \operatorname{cis} \frac{\pi}{3}$ $= \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

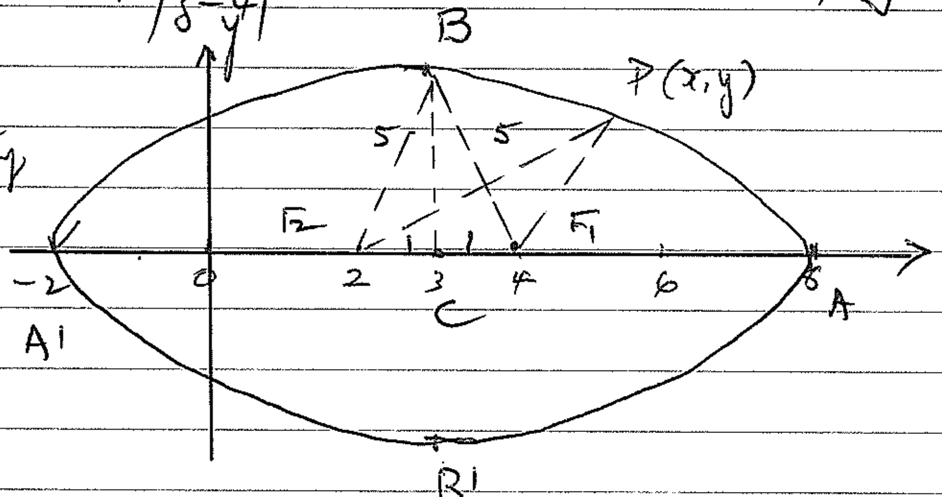
$\therefore \frac{z_1}{z_2} = \frac{2 \operatorname{cis} \frac{\pi}{3}}{\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)}$

(2)

$= \sqrt{2} \operatorname{cis}\left(\frac{7\pi}{12}\right)$ IN MOD-ARG FORM

b) i) Given $|z-2|$ is distance from $z$ to $x=2$ & $|z-4|$ " " " " to $x=4$ } on Argand diagram

Using symmetry




Since P is on curve such that $F_1P + F_2P = 10$

$\therefore$ A must be $(8,0)$ A' must be $(-2,0)$

Length of major axis is $AA' = 10$ units

Centre is at C which must be $(5,0)$

$BC^2 + 1^2 = 5^2$

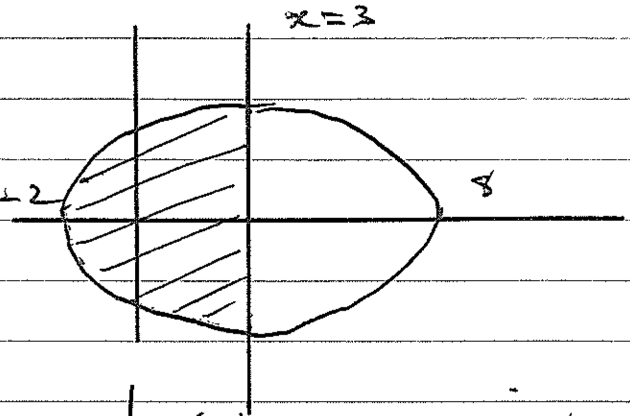
(3)

$\therefore BC = \sqrt{24}$

$= 2\sqrt{6}$

$\therefore$ Length of minor axis is $BB' = 4\sqrt{6}$

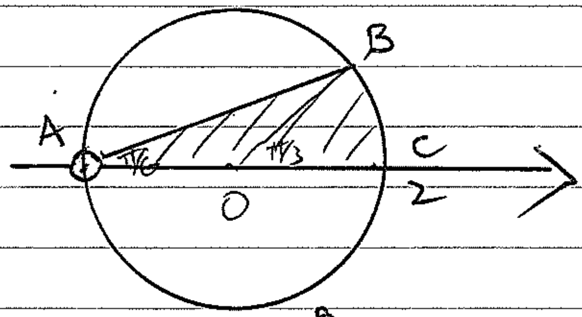
(ii) $z + \bar{z} \le 6$ is $2x \le 6$, $x \le 3$ & $|z-2| + |z-4| \le 10$ is region inside ellipse above.




(3)

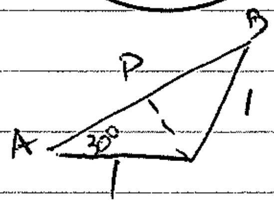
(c) $|z-1| \le 1$ is region inside circle centre $(1,0)$ radius 1

& $0 \le \arg z \le \pi/6$ as shown
[Actually $0 < \arg z \le \pi/6$]




Shape ABC

$\triangle ABD$




P is midpoint of AB
$AP = \cos 30° = \frac{\sqrt{3}}{2}$

$\therefore AB = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

(2)

Arc BC $= l = r\theta$
$= 1 \times \pi/3$
$= \frac{\pi}{3}$

$\therefore P = AB + \text{arc } BC + 2$
$= \sqrt{3} + \frac{\pi}{3} + 2$ units

9. a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ Diff. implicitly

$$\frac{2x}{16} + \frac{2y}{25}\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = -\frac{25x}{16y}$$

$$\therefore m_T = \frac{-25}{16} \cdot \frac{4\cos\theta}{5\sin\theta}$$

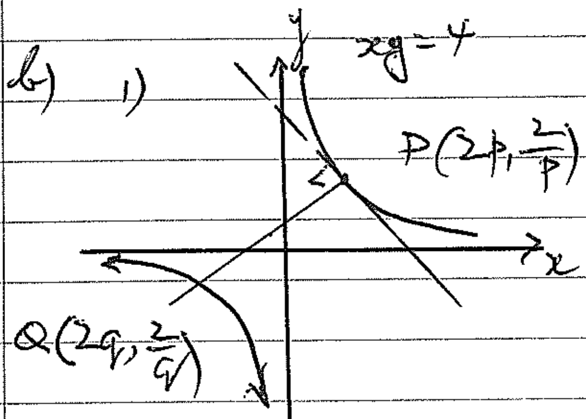
$$= -\frac{5}{4}\frac{\cos\theta}{\sin\theta}$$

Eqn of tang is $y - 5\sin\theta = -\frac{5\cos\theta}{4\sin\theta}(x - 4\cos\theta)$

$$4\sin\theta\, y - 20\sin^2\theta = -5\cos\theta\, x + 20\cos^2\theta$$

OR $\underline{\frac{\cos\theta}{4}x + \frac{\sin\theta}{5}y = 1}$ 2

b) 1)




$$xy = 4$$
$$y = 4x^{-1}$$
$$y' = -\frac{4}{x^2}$$

at P $m_T = -\frac{4}{(2p)^2} = -\frac{1}{p^2}$

$$\therefore m_N = +p^2$$

Eqn of NORMAL is

$$y - \frac{2}{p} = p^2(x - 2p) \quad \text{—} (1)$$

(1) × p $\qquad py - 2 = p^3x - 2p^4$

OR $\underline{p^3x - py = 2(p^4 - 1)}$ as req'd 2

(ii) Reverting to ① & substituting $Q\left(2q, \frac{2}{q}\right)$

OR/ ALSO by $m_{PQ} = p^2$

$$\frac{2}{q} - \frac{2}{p} = p^2(2q - 2p)$$

$$\frac{1}{q} - \frac{1}{p} = p^2(q - p)$$

$$\frac{\cancel{(p-q)}}{pq} = -p^2\cancel{(p-q)}$$

& Noting $p \neq q$

$$\frac{1}{pq} = -p^2 \qquad ①$$

OR $q = -\frac{1}{p^3}$ as req'd

(iii) Midpt of $PQ$ is $M\left(\frac{2p+2q}{2}, \frac{\frac{2}{p}+\frac{2}{q}}{2}\right)$

OR $\left(\frac{p+q}{1}, \frac{1}{p}+\frac{1}{q}\right)$

But $q = -\frac{1}{p^3}$ $\therefore M\left(p - \frac{1}{p^3}, \frac{1}{p} - p^3\right)$ ②

$M$ is $\left[\frac{1}{p}\left(p^2 - \frac{1}{p^2}\right), p\left(\frac{1}{p^2} - p^2\right)\right]$

(IV) Checking $(x^2-y^2)^2 = -x^3y^3$

$$LHS = \left[\frac{1}{p^2}\left(p^2-\frac{1}{p^2}\right)^2 - p^2\left(\frac{1}{p^2}-p^2\right)^2\right]^2$$

$$= \left[\frac{1}{p^2}\left(\frac{1}{p^2}-p^2\right)^2 - p^2\left(\frac{1}{p^2}-p^2\right)^2\right]^2$$

$$= \left[\left(\frac{1}{p^2}-p^2\right)^2\left(\frac{1}{p^2}-p^2\right)\right]^2$$

$$= \left(\frac{1}{p^2}-p^2\right)^6$$

$$RHS = -\frac{1}{\cancel{p^3}}\left(p^2-\frac{1}{p^2}\right)^3 \cdot \cancel{p^3}\left(\frac{1}{p^2}-p^2\right)^2$$

$$= +\left(\frac{1}{p^2}-p^2\right)^3\left(\frac{1}{p^2}-p^2\right)^3$$

$$= \left(\frac{1}{p^2}-p^2\right)^6$$

As LHS = RHS we have confirmed Locus of M is

$$(x^2-y^2)^2 = -x^3y^3$$

3