Sydney Technical High School

Mathematics Department

TRIAL H.S.C. - MATHEMATICS 2 UNIT

AUGUST 2013

General Instructions

- Reading time – 5 minutes
- Working Time – 180 minutes.
- Approved calculators may be used.
- Write using blue or black pen.
- A table of Standard Integrals is provided at the back of this paper.
- In Question 11-16, show relevant mathematical reasoning and/or calculations.
- Begin each question on a new side of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.

NAME ____________________

TEACHER ____________________

Total Marks – 100

SECTION 1 **Pages 2 – 5**
10 marks

- **Attempt Questions 1 – 10**
- **Allow about 15 minutes.**

SECTION 2 **Pages 6 – 12**
90 marks

- **Attempt Questions 11 – 16**
- **Allow about 2 hours 45 mins.**

Question 1

For what values of k does the equation $x^2 - 6x - 3k = 0$ have real roots?

A. $k \geq -3$

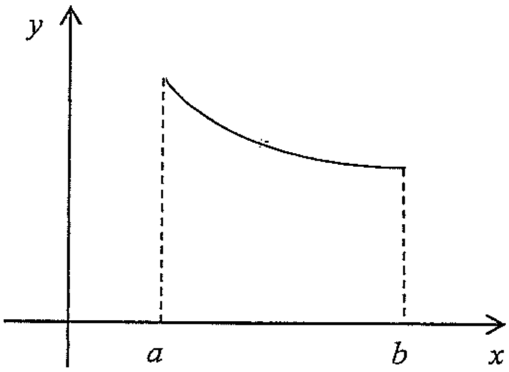
B. $k \leq -3$

C. $k \geq 3$

D. $k \leq 3$

Question 2

For the function $y = f(x),\ a < x < b$ graphed below:



which of the following is true?

A. $f'(x) > 0\ and\ f''(x) > 0$

B. $f'(x) > 0\ and\ f''(x) < 0$

C. $f'(x) < 0\ and\ f''(x) > 0$

D. $f'(x) < 0\ and\ f''(x) < 0$

Question 3

An infinite geometric series has a first term of 8 and a limiting sum of 12.

What is the common ratio?

A. 1/6

B. 5/3

C. 1/2

D. 1/3

Question 4

What are the domain and range of the function $f(x) = \sqrt{4 - x^2}$?

A. Domain: $-2 \leq x \leq 2$, Range: $0 \leq y \leq 2$

B. Domain: $-2 \leq x \leq 2$, Range: $-2 \leq y \leq 2$

C. Domain: $0 \leq x \leq 2$, Range: $-4 \leq y \leq 4$

D. Domain: $0 \leq x \leq 2$, Range: $0 \leq y \leq 4$

Question 5

What is the maximum value of $6 + 2x - x^2$?

A. 6

B. 1

C. 7

D. cannot be determined.

Question 6

The sine curve with amplitude 3 units and period 4π units has equation:

A. $y = 4 \sin 3x$

B. $y = 3 \sin 4x$

C. $y = 3 \sin 2x$

D. $y = 3 \sin \frac{x}{2}$

Question 7

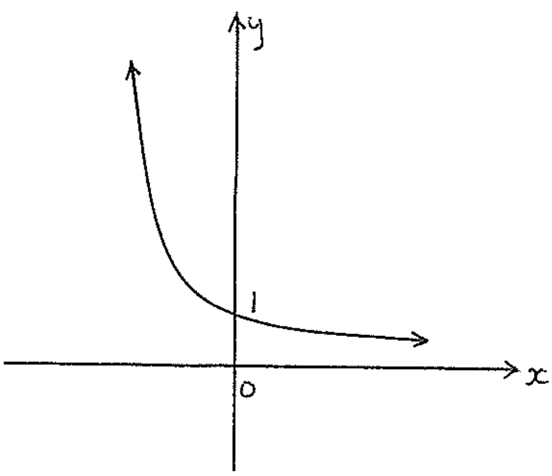
The illustrated graph could be:

A. $y = 2^x$

B. $y = -2^{-x}$

C. $y = (\frac{1}{2})^x$

D. $y = (\frac{1}{2})^{-x}$



Question 8

Janet works out the sum of n terms of an arithmetic series. Her answer, which is correct, could be:

A. $S_n = 2(2^n - 1)$

B. $S_n = 9 - 2n$

C. $S_n = 8n - n^2$

D. $S_n = 7 \times 2^{n-1}$

Question 9

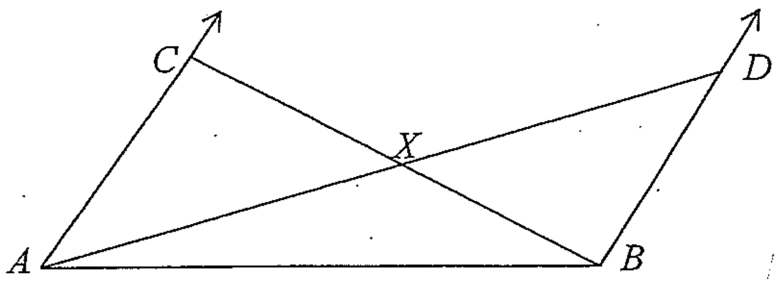


Figure not to scale

In the diagram above: $AC \parallel BD$, $\angle CAX = 2\angle BAX$, $\angle DBX = 2\angle ABX$.

$\angle AXB = ?$

A. 150°

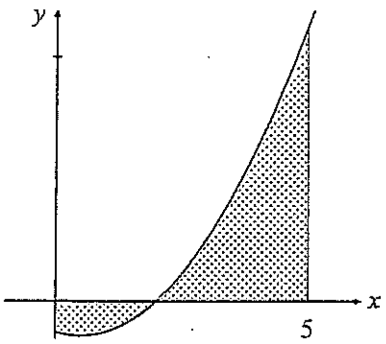
B. 120°

C. 160°

D. 135°

Question 10

Which expression below will give the area of the shaded region bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 0$ and $x = 5$?



A. $A = \left|\int_0^1 (x^2 - x - 2)dx\right| + \int_1^5 (x^2 - x - 2)dx$

B. $A = \int_0^1 (x^2 - x - 2)dx + \left|\int_1^5 (x^2 - x - 2)dx\right|$

C. $A = \left|\int_0^2 (x^2 - x - 2)dx\right| + \int_2^5 (x^2 - x - 2)dx$

D. $A = \int_0^2 (x^2 - x - 2)dx + \left|\int_2^5 (x^2 - x - 2)dx\right|$

END OF SECTION 1

SECTION 2

90 marks

Attempt Question 11 – 16

Allow about 2 hours 45 minutes for this section.

Answer each question in the writing book provided. Start each question on a new page. All necessary working should be shown. Full marks cannot be given for illegible writing.

Question 11 (15 marks)

			Marks
a)	Differentiate:		
	(i)	$x \sin 2x$	2
	(ii)	$e^{4x} + \frac{1}{x}$	2
	(iii)	$\frac{x+1}{3+2x}$	2
b)	Find $\int (4x+2)^6\, dx$		2
c)	Solve for x: $3^{1-x} = \frac{1}{\sqrt{27}}$		2
d)	Solve $(\sin x + 1)(2\sin x + 1) = 0$ for $0 \leq x \leq 2\pi$		3
e)	Evaluate $\sum_{n=1}^{50}(2n+3)$		2

Question 12 (15 marks) Marks

a) Solve $|x+2| = 3x$ 2

b) Use a change of base to evaluate $\log_2 50$ correct to 2 decimal places. 1

c) Find the gradient of the curve $y = e^{\sin x}$ at the point where $x = 0$. 2

d) If α and β are the roots of $x^2 + 4x + 1 = 0$, find without solving:

i) $\alpha + \beta$ and $\alpha\beta$. 1

ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ 2

e) Differentiate:

i) $\ln(x^2 + 3)$ 1

ii) $tan^2 4x$ 2

f) Given the parabola $4y = x^2 - 12$, find the:

i) focal length. 1

ii) coordinates of the focus. 1

g) Use Simpson's Rule and the five function values in the table below to estimate $\int_2^4 f(x)dx$. 2

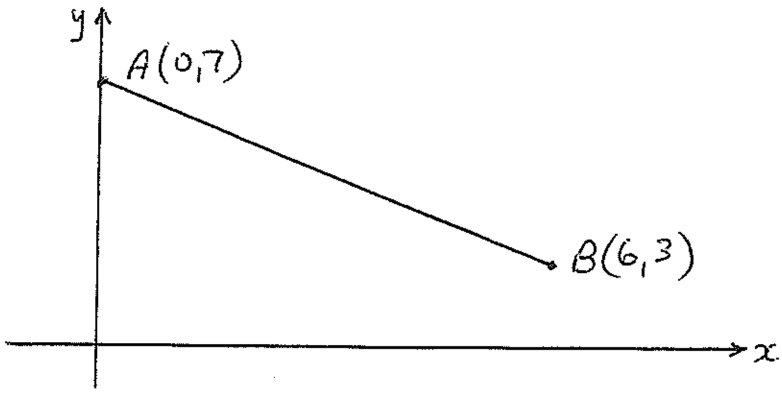
x	2	2.5	3	3.5	4
$f(x)$	4	1	-2	3	8

Question 13 (15 marks) Marks

a) i) Factorise $24 + 2m - m^2$ 1

ii) Hence solve $24 + 2m - m^2 < 0$ 1

b)



A(0,7) and B(6,3) are points on the number plane and the equation of AB is $2x + 3y - 21 = 0$.

i) Find the length of AB. 1

ii) Find the gradient of AB. 1

iii) Show that the equation of the perpendicular from D(-2,0) to AB is $3x - 2y + 6 = 0$. 2

iv) Find the perpendicular distance from D to AB. 2

v) Find the coordinates of a point C such that ABCD is a parallelogram. 1

c) An amount of money doubles in value over a period of n months. Interest is compounded at the rate of 1% per month. Use the compound interest formula to find the number of months required, correct to the nearest month. 2

d) i) Find $\frac{d}{dx}(\operatorname{cosec} x)$ 2

ii) Hence evaluate $\int_{\pi/3}^{\pi/2} \cot x \operatorname{cosec} x \, dx$. Give your answer in exact form. 2

8

Question 14 (15 marks) Marks

a) Find the angle that the line $3x + 5y + 2 = 0$ makes with the positive direction of the x-axis. 2

b) Find: i) $\int \sin\frac{2x}{3}\,dx$ 2

ii) $\int \frac{x^2e^{x^2}+1}{x}\,dx$ 2

c) Prove that $\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} = 2\sec\theta$ 2

d) Solve for m: $\log_m 8 + 3\log_m 4 = 6$. Leave your answer in exact form. 3

e)

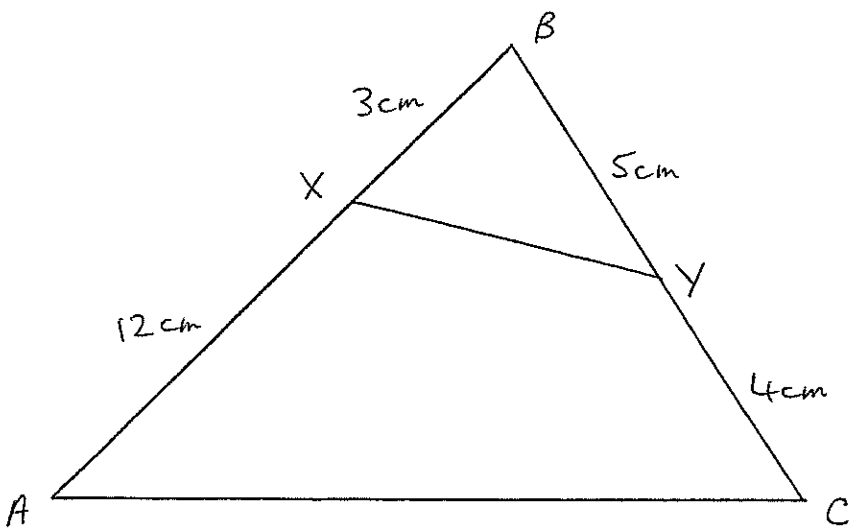


Figure not to scale

i) Prove that ΔBXY is similar to ΔABC. 2

ii) If angle A is 35°, use the Sine Rule to find the size of angle C, correct to the nearest degree. 2

Question 15 (15 marks) Marks

a)

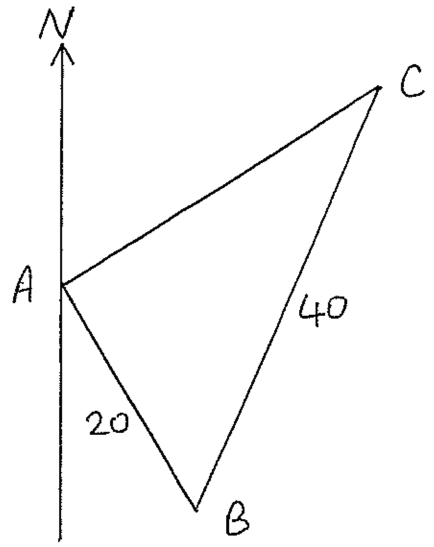


Figure not to scale

Two geologists on a large level area of land drive 20 km from point A on a bearing of 150°T to a point B. They then drive 40 km on a bearing of 020°T to point C.

i) Copy the above diagram into your answer booklet, and find the size of $\angle ABC$. 1

ii) Use the Cosine Rule to find the distance AC to the nearest kilometre. 2

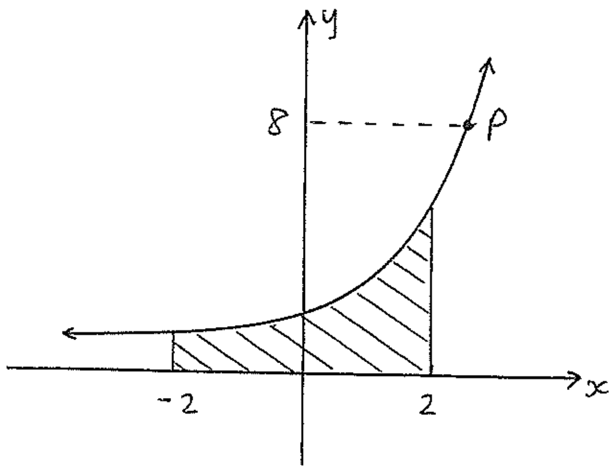
b) Consider the curve defined by $y = 4 - \cos 2x$.

i) State the amplitude <u>and</u> period of this curve. 2

ii) Sketch the curve for $0 \leq x \leq \pi$. Show clear, relevant information on the axes. 2

iii) Find the area between the curve and the line $y = 2$ for $0 \leq x \leq \pi$. 3

c) The diagram shows the curve $y = e^x$, a shaded area from $x = -2$ to $x = 2$, and a point P on the curve.



Not to scale

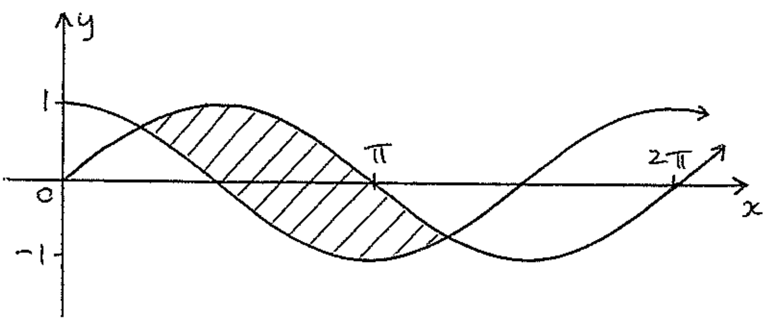
i) The point P has a y coordinate of 8. Find its x coordinate. 1

ii) The shaded area is rotated about the x-axis. Find the volume of the generated solid, giving your answer correct to 3 significant figures. 3

d) Factorise $x^2 + 2xy + y^2 - 1$ 1

Question 16 (15 marks) Marks

a)



The diagram shows the curves $y = \sin x$ and $y = \cos x$.

Write an appropriate integral expression to represent the shaded area above. 2

DO NOT EVALUATE THIS INTEGRAL.

b) Given the curve $y = x \log x - x$, for $x > 0$.

i) Find where the curve crosses the x-axis. 2

ii) Find any stationary points and determine their nature. 2

iii) Write a statement for the concavity of this curve. 1

iv) Find y when $x = e^2$, and sketch the curve for $0 < x \leq e^2$ 2

c) A man has 1 million (10^6) dollars in a bank account. The account earns a steady $\frac{1}{2}$% interest per month, compounded monthly.

At the same time, however, a bank employee is stealing a constant amount \$M per month from this account, immediately after the month's interest is added to the man's account.

Let A_n be the amount remaining in the man's account at the end of n months.

i) Write an expression for A_1, and show that 2

$A_2 = 10^6(1.005)^2 - M(1.005 + 1)$

ii) Write a simplified expression for A_n 2

iii) Determine the value of \$M that is stolen each month, such that the man will have only \$20 remaining in his account after 10 years. 2

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Solutions.

① $b^2 - 4ac \geq 0$

$36 + 12k \geq 0$

$k \geq -3$

$\underline{\underline{A}}$

② $\underline{\underline{C}}$

③ $\frac{a}{1-r} = 12$

$8 = 12 - 12r$

$12r = 4$

$r = \frac{1}{3}$

$\underline{\underline{D}}$

④ $\underline{\underline{A}}$

⑤ $x = \frac{-2}{-2} = 1$

max. value $= 7$

$\underline{\underline{C}}$

⑥ $y = 3\sin\frac{x}{2}$ $\underline{\underline{D}}$

⑦ $y = 2^{-x}$

$= (2^{-1})^x$

$= (\frac{1}{2})^x$

$\underline{\underline{C}}$

⑧ $\underline{\underline{C}}$

⑨ $3x + 3y = 180$

$x + y = 60$

$\underline{\underline{B}}$

⑩ $(x-2)(x+1)$

$x = 2, \cancel{\times}$

$\underline{\underline{C}}$

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⑪ a) i) $y' = 1 \times \sin 2x + 2\cos 2x \times x$

$= \sin 2x + 2x\cos 2x$

ii) $y' = 4e^{4x} - \frac{1}{x^2}$

iii) $y' = \frac{1(3+2x) - 2(x+1)}{(3+2x)^2}$

$= \frac{3 + \cancel{2x} - \cancel{2x} - 2}{(3+2x)^2}$

$= \frac{1}{(3+2x)^2}$

b) $\frac{(4x+2)^7}{28} + c$

c) $3^{1-x} = 3^{-\frac{3}{2}}$

$1 - x = -\frac{3}{2}$

$x = 2\frac{1}{2}$

d) $\sin x = -1$ or $\sin x = -\frac{1}{2}$

$\therefore x = \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

e) $S_{50} = \frac{50}{2}(5 + 103)$

$= 2700$

(12) a) $x+2=3x$ or $-(x+2)=3x$

$x=1$ ✓ $\quad -x-2=3x$

$4x=-2$

$x=-\frac{1}{2}$ ✗

only solution is $x=1$

b) $\frac{\log 50}{\log 2} \doteqdot 5.64$

c) $y'=e^{\sin x}\times\cos x$

when $x=0$, $m_T=e^0\times\cos 0$

$=1$

d) i) $\alpha+\beta=-4$, $\alpha\beta=1$

ii) $\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$=\frac{16-2}{1}$

$=14$

e) i) $y'=\frac{2x}{x^2+3}$

ii) $y'=2\tan 4x\times\sec^2 4x\times 4$

$=8\tan 4x\sec^2 4x$

f) $x^2=4y+12$

$=4(y+3)$

i) focal length $a=1$

ii) vertex at $(0,-3)$ $\therefore$ focus at $(0,-2)$

(12) g) $\int_2^4 f(x)\,dx \doteqdot \frac{0.5}{3}\left(4 + 4\times 1 + 2\times(-2) + 4\times 3 + 8\right)$

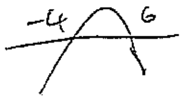
$= \frac{1}{6}\left(4 + \cancel{4} - \cancel{4} + 12 + 8\right)$

$= \frac{1}{6}\times 24$

$= 4$

*

(13) a) i) $(6-m)(4+m)$

ii) 

$\therefore m < -4$ or $m > 6$

b) i) $d_{AB} = \sqrt{36+16}$

$= \sqrt{52}$ or $2\sqrt{13}$

ii) $m_{AB} = \frac{-4}{6}$

$= -\frac{2}{3}$

iii) Use $m_\perp = \frac{3}{2}$

$\therefore y - 0 = \frac{3}{2}(x+2)$

$2y = 3x + 6$

$3x - 2y + 6 = 0$ as reqd.

iv) p.d. $= \frac{|-4+0-21|}{\sqrt{2^2+3^2}}$

$= \frac{25}{\sqrt{13}}$

v) C is $(4,-4)$

c) $2P = P(1+r)^n$

$2 = 1.01^n$

$\log 2 = n \log 1.01$

$n = \frac{\log 2}{\log 1.01}$

$\doteqdot 70$ months

d) $\frac{d}{dx}\left[(\sin x)^{-1}\right]$

$= -(\sin x)^{-2} \times \cos x$

$= \frac{-\cos x}{\sin^2 x}$

$= -\cot x \operatorname{cosec} x$

ii) $\left[-\operatorname{cosec} x\right]_{\pi/3}^{\pi/2}$

$= \frac{-1}{\sin \pi/2} - \left(-\frac{1}{\sin \pi/3}\right)$

$= \frac{-1}{1} + \frac{1}{\sqrt{3}/2}$

$= -1 + \frac{2}{\sqrt{3}}$

(14) a) $5y = -3x - 2$

$y = -\frac{3}{5}x - \frac{2}{5}$

$\therefore$ grad. $= -\frac{3}{5}$

$\tan\theta = -\frac{3}{5}$

$\therefore \theta = 149^\circ$

b) i) $-\cos\frac{2x}{3} \times \frac{3}{2} + c$

$= -\frac{3}{2}\cos\frac{2x}{3} + c$

ii) $\int\left(xe^{x^2} + \frac{1}{x}\right)dx$

$= \frac{e^{x^2}}{2} + \log x + c$

c) $LHS = \frac{\cos\theta(1-\sin\theta) + \cos\theta(1+\sin\theta)}{1-\sin^2\theta}$

$= \frac{2\cos\theta}{\cos^2\theta}$

$= 2\sec\theta$

$= RHS$

d) $\log_m 8 + \log_m 64 = 6$

$\log_m 512 = 6$

$m^6 = 512$

$\therefore m = \sqrt[6]{512}$

(or $2\sqrt{2}$)

e) $\frac{BX}{BC} = \frac{3}{9} = \frac{1}{3}$

$\frac{BY}{BA} = \frac{5}{15} = \frac{1}{3}$

and $\angle B$ is common

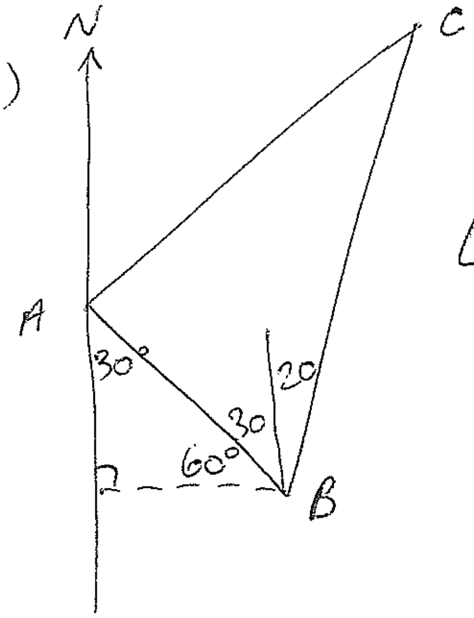
$\therefore \triangle BXY \,|||\, \triangle ABC$ (equal ratio of sides about equal incl. ang.)

ii) $\frac{9}{\sin 35^\circ} = \frac{15}{\sin C}$

$\sin C = \frac{15\sin 35}{9}$

$\therefore C \doteq 73^\circ$

(15) a) i)



$\angle ABC = 50^\circ$

ii) $AC^2 = 20^2 + 40^2 - 2 \times 20 \times 40 \times \cos 50^\circ$

$= 400 + 1600 - 1600 \cos 50^\circ$

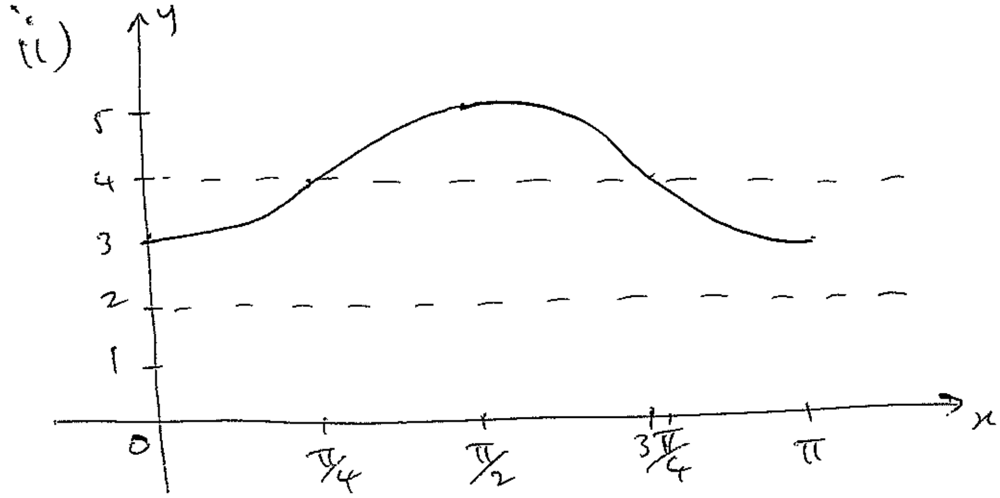
$= 971.54$

$\therefore AC \doteqdot 31$ km

b) i) amp. $= 1$, period $= \frac{2\pi}{2}$

$= \pi$ units

ii)



iii) Area $= \int_0^{\pi} (4 - \cos 2x)\, dx - 2\pi$

$= \left[4x - \frac{\sin 2x}{2} \right]_0^{\pi} - 2\pi$

$= (4\pi - 0) - (0 - 0) - 2\pi$

$= 2\pi\ u^2$

(15) c) i) $8 = e^x \Rightarrow x = \log_e 8$ or $\ln 8$ or 2.079

ii) $V = \pi \int_{-2}^{2} (e^x)^2 \, dx$

$= \pi \int_{-2}^{2} e^{2x} \, dx$

$= \pi \left[\frac{e^{2x}}{2} \right]_{-2}^{2}$

$= \frac{\pi}{2} (e^4 - e^{-4})$

$\doteq 85.7 \; u^3$

d) $(x+y)^2 - 1 = (x+y+1)(x+y-1)$

(16) a) Area $= \left(\int_{\pi/4}^{\pi} \sin x \, dx - \int_{\pi/4}^{\pi/2} \cos x \, dx \right) \times 2$

b) i) $y = 0 \Rightarrow x \log x - x = 0$

$x(\log x - 1) = 0$

$x = 0$ (no sol.) or $\log x - 1 = 0$

$\log x = 1$

$\therefore \underline{x = e}$ only

ii) S.P.'s when $y' = \log x + \frac{1}{x} \times x - 1 = 0$

$\therefore \log x + \cancel{1} - \cancel{1} = 0$

$\log x = 0$

$\therefore x = 1$

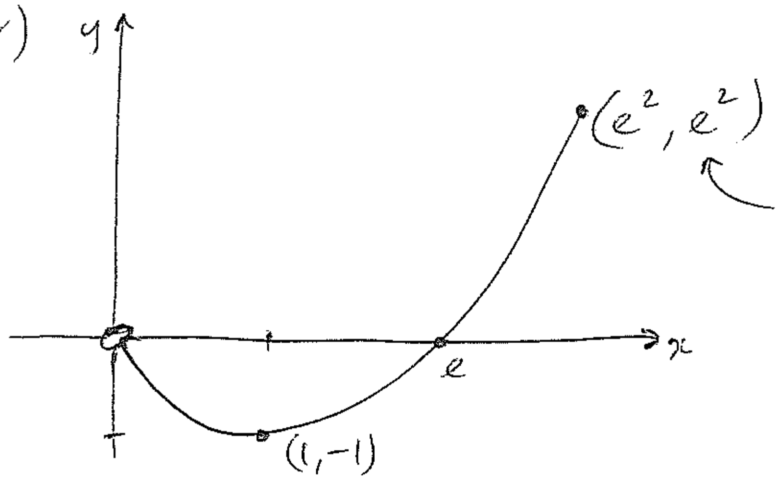
$y'' = \frac{1}{x}$

when $x = 1$, $y'' > 0 \Rightarrow$ minimum turning pt at $(1, -1$

iii) For all $x>0$, $y''>0$

$\therefore$ curve is always concave up.

iv)



$x = e^2$

$\therefore y = e^2 \log(e^2) - e^2$

$= e^2 \times 2 - e^2$

$= e^2$

c) i) $A_1 = 10^6(1.005) - M$

$A_2 = [10^6(1.005) - M] \times 1.005 - M$

$= 10^6(1.005)^2 - 1.005M - M$

$= 10^6(1.005)^2 - M(1.005 + 1)$ as reqd.

ii) $A_n = 10^6(1.005)^n - M(1.005^{n-1} + 1.005^{n-2} + \dots + 1)$

$= 10^6(1.005)^n - M \times 1\dfrac{(1.005^n - 1)}{1.005 - 1}$

$= 10^6(1.005)^n - M\dfrac{(1.005^n - 1)}{0.005}$

iii) $A_{120} = 20$

$\therefore 20 = 10^6(1.005)^{120} - 200M(1.005^{120} - 1)$

$\therefore M = \dfrac{10^6(1.005)^{120} - 20}{200(1.005)^{120} - 1}$

$= \$11,101.93$ (accept 101 or 102)