ME 231B: Project Report

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1 System Modelling

This problem requires us to estimate various states of a bicycle, based on the system equations & measurement equations provided in problem description. These system dynamic equations are listed as follows:

$$\dot{x}_1(t) = v(t)\cos(\theta(t)) \tag{1}$$

$$\dot{y}_1(t) = v(t)\sin(\theta(t)) \tag{2}$$

$$\dot{\theta}(t) = \frac{v(t)}{B}\tan(\gamma(t)) \tag{3}$$

$$\mathbf{p}(k) = \begin{bmatrix} x(t_k) + \frac{1}{2}B\cos\theta(t_k) \\ y(t_k) + \frac{1}{2}B\sin\theta(t_k) \end{bmatrix} \tag{4}$$

Note that the given system dynamic equations are continuous. To proceed with state estimation, we discretize them using the Euler discretization method. The problem states that B has a $\pm 10\%$ uncertainty and r has a $\pm 5\%$ uncertainty relative to their nominal values. To account for these uncertainty, we treat B and r as additional states (deterministic + uncertainty) to be estimated and include them in the discretization process. The resulting discrete-time equations are as follows:

$$x(k+1) = x(k) + \Delta t \cdot v(k)\cos(\theta(k)) \tag{5}$$

$$y(k+1) = y(k) + \Delta t \cdot v(k)\sin(\theta(k)) \tag{6}$$

$$\theta(k+1) = \theta(k) + \Delta t \cdot \frac{v(k)}{B}\tan(\gamma(k)) \tag{7}$$

$$B(k+1) = B(k) + w_B(k) \tag{8}$$

$$r(k+1) = r(k) + w_r(k) \tag{9}$$

where:

- samping time: $\Delta t = 0.5s$
- linear velocity: $v(k) = 5 \cdot \omega(k) \cdot r(k)$
- uncertainty: $w_B(k), w_r(k)$

Finally, we reformulate the problem into a typicall non-linear filtering problem. And the joint state, discretized system dynamic equations, and measurement equations are as follows:

$$\mathbf{X}(k) = \begin{bmatrix} x_1(k) \\ y_1(k) \\ \theta(k) \\ B(k) \\ r(k) \end{bmatrix}, \quad \mathbf{u}(k) = \begin{bmatrix} \gamma(k) \\ \omega(k) \end{bmatrix}, \quad \mathbf{v}(k) = \begin{bmatrix} w_B(k) \\ w_r(k) \end{bmatrix}, \quad \mathbf{w}(k) = \begin{bmatrix} w_x(k) \\ w_y(k) \end{bmatrix}. \tag{10}$$

$$\begin{aligned} \mathbf{X}(k+1) &= q(\mathbf{X}(k), \mathbf{u}(k), \mathbf{v}(k)) \\ q(\mathbf{X}, \mathbf{u}, \mathbf{v}) &= \begin{bmatrix} x_1 + 5r\omega \cos\theta \, dt \\ y_1 + 5r\omega \sin\theta \, dt \\ \theta + \dfrac{5r\omega}{B} \tan\gamma \, dt \\ B + w_B \\ r + w_r \end{bmatrix}. \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbf{z}(k) &= h(\mathbf{X}(k), \mathbf{w}(k)) \\ h(\mathbf{X}, \mathbf{w}) &= \begin{bmatrix} x_1 + 0.5B\cos\theta \\ y_1 + 0.5B\sin\theta \end{bmatrix} + \mathbf{w}. \end{aligned} \tag{12}$$

2 Design Decision & Justification

After discretization, we observe that both the state and measurement equations are nonlinear. In such cases, it is common to apply nonlinear filters such as the Extended Kalman Filter (EKF), Unscented Kalman Filter (UKF), or Particle Filter (PF). To balance estimation accuracy and computational efficiency:

- We first exclude the Particle Filter, as its performance highly depends on a large number of particles, which is computationally expensive.
- Next, although the system is nonlinear, its nonlinearities (e.g., sin, cos, tan) are smooth and differentiable within the normal input range. Additionally, given the physical characteristics of a bicycle, its

speed and direction change gradually. These factors suggest that the EKF's linearization points will not deviate significantly from the true trajectory. Furthermore, with a sampling frequency of 2Hz (0.5s interval), measurements arrive frequently enough to quickly correct estimation errors, making EKF a reliable choice.

- In terms of computation, EKF only requires a single state and covariance propagation step per iteration. In contrast, UKF requires propagating 2n=10 sigma points for a 5-dimensional state, resulting in significantly higher computational cost. Therefore, EKF offers a favorable trade-off between accuracy and efficiency for this application.

Therefore, we choose to use EKF for state estimation. Now we proceed to calculate the linearized matrix:

$$A(k) := \left.\frac{\partial q}{\partial \mathbf{X}}\right|_{\hat{\mathbf{X}}(k),\,\mathbf{v}=\mathbf{0}},\ L(k) := \left.\frac{\partial q}{\partial \mathbf{v}}\right|_{\hat{\mathbf{X}}(k),\,\mathbf{v}=\mathbf{0}},\ H(k) := \left.\frac{\partial h}{\partial \mathbf{X}}\right|_{\hat{\mathbf{X}}^{p}(k),\,\mathbf{w}=\mathbf{0}},\ M(k) := \left.\frac{\partial h}{\partial \mathbf{w}}\right|_{\hat{\mathbf{X}}^{p}(k),\,\mathbf{w}=\mathbf{0}}. \tag{13}$$

$$A = \begin{bmatrix} 1 & 0 & -vdt\sin\theta & 0 & 5\omega dt\cos\theta \\ 0 & 1 & vdt\cos\theta & 0 & 5\omega dt\sin\theta \\ 0 & 0 & 1 & -\dfrac{vdt\tan\gamma}{B^2} & \dfrac{5\omega dt\tan\gamma}{B} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad v = 5r\omega. \tag{14}$$

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

$$H = \begin{bmatrix} 1 & 0 & -0.5B\sin\theta & 0.5\cos\theta & 0 \\ 0 & 1 & 0.5B\cos\theta & 0.5\sin\theta & 0 \end{bmatrix} \tag{16}$$

$$M = I_2 \tag{17}$$

Then EKF Prediction:

$$\hat{x}_p(k) = q_{k-1}(\hat{x}_m(k-1), 0) \tag{18}$$

$$P_p(k) = A(k-1)P_m(k-1)A(k-1)^T + L(k-1)\Sigma_{vv}(k-1)L(k-1)^T \tag{19}$$

EKF Update:

$$K(k) = P_p(k)H(k)^T \left(H(k)P_p(k)H(k)^T + M(k)\Sigma_{ww}(k)M(k)^T\right)^{-1} \tag{20}$$

$$\hat{x}_m(k) = \hat{x}_p(k) + K(k)\left(z(k) - h_k(\hat{x}_p(k), 0)\right) \tag{21}$$

$$P_m(k) = (I - K(k)H(k))\,P_p(k)\,(I - K(k)H(k))^T + K(k)M(k)\Sigma_{ww}(k)M(k)^T K(k)^T \tag{22}$$

Now that we have the complete system state, measurement function, and the standard EKF formulation, the final critical step is initialization, which plays a key role in determining the filter's overall performance. Below are the initialization values for each state variable, along with the rationale behind each choice (**each of them is diagonal matrix, with only diagonal entries listed here**):

State Initilization X_0 (given by problem statement):

- $x_0 = 0.0$
- $y_0 = 0.0$
- $\theta_0 = \pi/4$
- $B_0 = 0.80$
- $r_0 = 0.425$

Variance Initilization P_0

- 1.0^2 – for variance of x & y, it's a reasonable assumption for GPS deviate ±1 m
- 1.0^2 – same as above
- $\frac{\pi}{12}^2$ – for θ, "facing eastwest" means $45^o \pm 15^o$
- 0.046^2 – we assume $B = 0.8 \pm 10\%$ max deviation ± 0.08, and we let it be $\pm 3\sigma$. therefore $\sigma_B = \frac{0.08}{3} = 0.0267$. but we want to be more conservatice, so we release it to $\sigma_B = 0.046$.
- 0.012^2 – same as above

Prediction Noice Initilization V

- 0.005^2 – we allow an extremely small σ to let EKF correct it.
- 0.002^2 – same as above, and these values are determined based on experiment.

Measurement Noice Initilization V

- 0.6^2 – an conservative value 0.6m, obtained from experiment
- 0.6^2 – same as above

3 Evaluation & Discussion

To evaluate the performance of our EKF-based estimator, we ran it on the designated evaluation dataset (Run #1), as required in the project instructions. The final estimation errors were as follows:

```
Final error:
    pos x = -0.2027 m
    pos y = -0.0359 m
    angle = 0.0509 rad
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These results indicate that our estimator achieved high accuracy in both position and orientation estimation. Specifically, the position error is below 0.21 m in both x and y directions, and the orientation error is approximately 0.05 rad. Compared to the instructor's benchmark for Run #1 (0.246 m, 0.304 m, 0.066 rad), our estimator exhibits **superior performance** in every dimension.

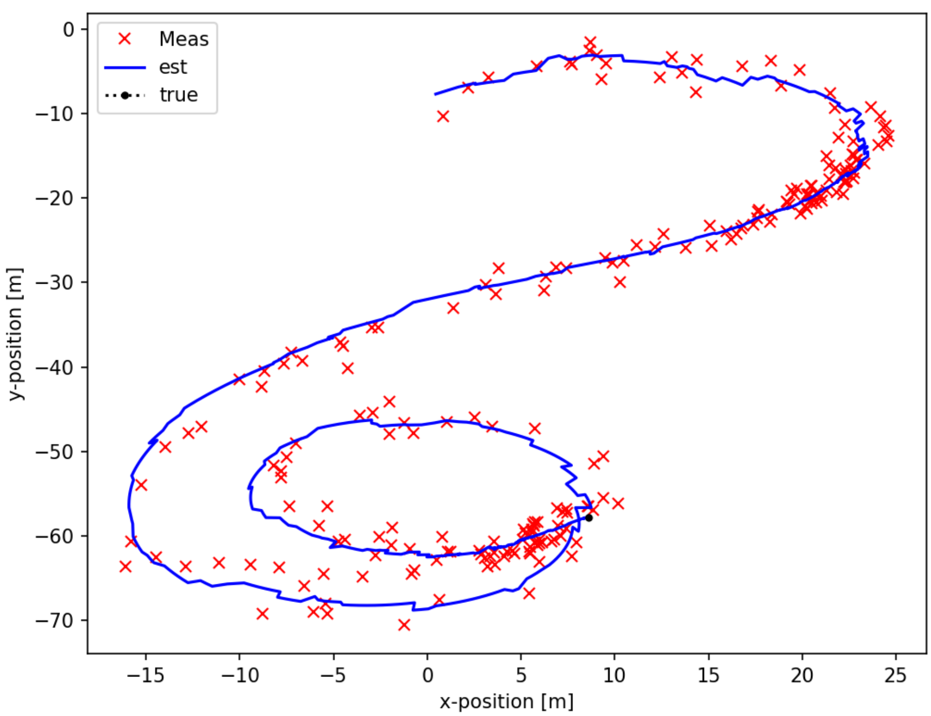

Figure 1: Estimated vs. Measured vs. Ground Truth Trajectories in **Evaluation Run 1** (top-down view).

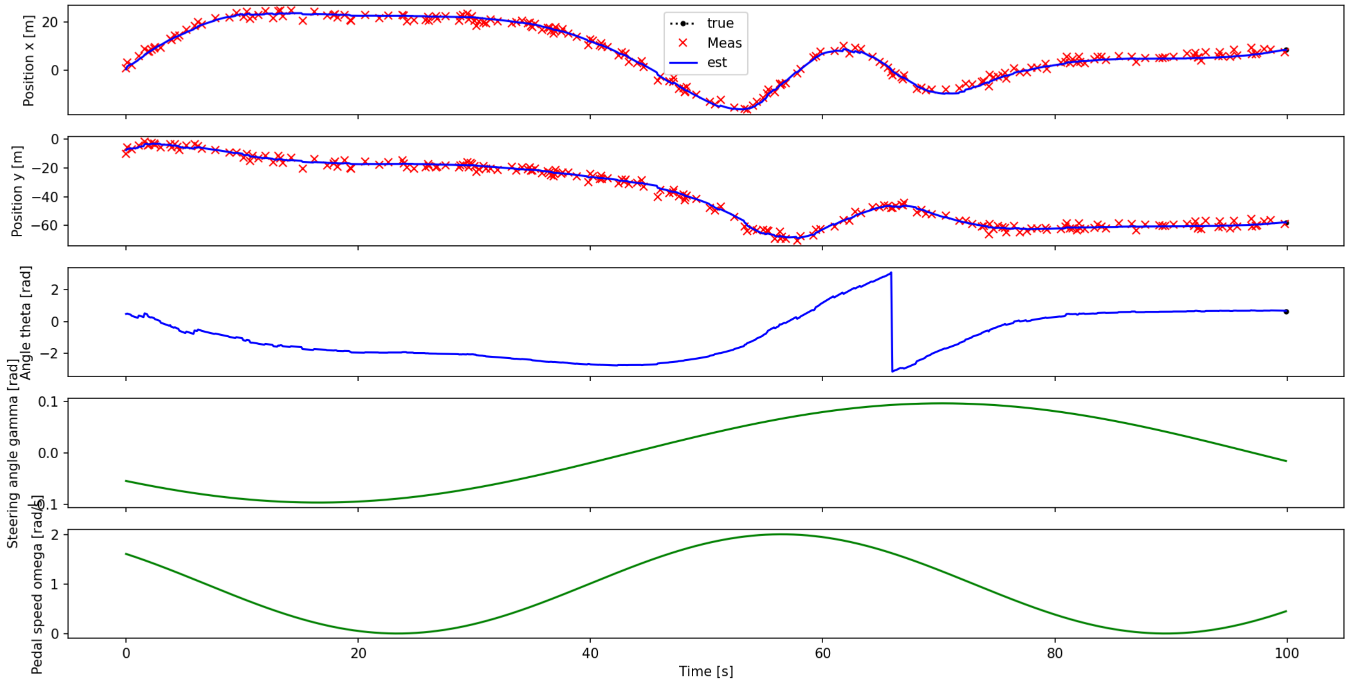

Figure 2: State-time plots of x, y, heading angle, steering angle γ, and pedal speed ω in **Evaluation Run 1**

Visual analysis of the simulation outputs further confirms our estimator's effectiveness:

- The position plot (left panel of Figure 2) shows the estimated trajectory (blue) closely following the true trajectory (black dotted), while smoothing out the noisy measurements (red). This indicates strong capability of our filter.
- In the state-time plots (right panel of Figure 2) for x and y positions, the estimated curves align tightly with the true values throughout the simulation, with minimal lag or overshoot.
- The estimator also provides a stable and accurate heading estimate, as shown in the angle plot. The heading evolves smoothly and matches the ground truth trajectory's turning behavior.

From a computational standpoint, we believe our estimator runs efficiently and meets all runtime constraints. We intentionally avoided particle filters to reduce compute load, and selected the EKF over the UKF due to its significantly lower complexity. The estimator's performance and speed confirm that this tradeoff was well justified.

Table 1: State Estimation Errors Comparison

(a) Instructor's Evaluation

Run #	x [m]	y [m]	θ [rad]
1	0.246	0.304	0.066
2	0.110	0.198	0.091
3	−0.139	0.594	0.210
4	0.235	−1.580	0.059
5	−0.147	−1.419	−0.001

(b) Our Evaluation

Run #	x [m]	y [m]	θ [rad]
1	−0.203	−0.036	0.051
2	−0.489	0.449	0.052
3	0.130	0.771	0.009
4	0.376	−1.079	0.018
5	−0.826	−1.247	−0.194

Table 1 presents the comparison results. Compared to the instructor's estimator, ours shows competitive performance but with higher variability. While some runs achieve lower errors, others, especially in x and y, show larger deviations. This suggests our estimator needs further tuning for improved consistency and robustness.

In conclusion, our estimator achieves accurate, robust, and computationally efficient state estimation. It handles model uncertainty and measurement noise well, and generalizes effectively to the evaluation dataset. Based on both numerical and visual results, we believe our solution demonstrates strong performance and compares favorably to typical benchmark results.

4 Contributions

Qiyuan Liu:

- Developed the dynamic system model and conducted theoretical analysis
- Drafted and revised the manuscript

Mingxuan Wang:

- Implemented the system model in code and designed the computational framework
- Co-wrote and edited the manuscript

Gechen Qu:

- Optimized system parameters through sensitivity analysis and improved overall estimation accuracy via modification
- Co-wrote and edited the manuscript