UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: June 2013

Module Number: IS3304 Module Name: Numerical Methods

[Three hours]

[Answer **all questions**, each question carries twelve marks]

Q1.

a) Clearly mentioning the assumptions, prove the Newton-Raphson formula,

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$ using Taylor's expansion. [2 Marks]

b) A loan of 'A' rupees is repaid by making n equal monthly payments of M rupees, starting a month after the loan is made. It can be shown that if the monthly interest rate is r, then

$$Ar = M\left(1 - \frac{1}{(1+r)^n}\right).$$

A car loan of Rs.1,000,000.00 was repaid in 60 monthly payments of Rs.25,000.00. Use Newton-Raphson method to find the monthly interest rate (%) with an accuracy of 0.001.

[7 Marks]

c) Consider the function, $f(x) = x^2 - x - e^{-x}$ shown in Figure Q1. Two equations can be derived by using simple algebraic manipulations to transform the function $f(x) = 0$ to the form of $x = g(x)$ as bellow.

$$g_1(x) = x^2 - e^{-x} \quad ; \quad g_2(x) = \sqrt{e^{-x} + x}$$

i.) Find the convergence/divergence of fixed-point iteration schemes derived with the functions $g_1(x)$ and $g_2(x)$.

(Hint: You may estimate $|g_1'(x)|$ and $|g_2'(x)|$)

ii.) Use either $g_1(x)$ or $g_2(x)$ (whichever converges) to estimate the root of the function $f(x) = x^2 - x - e^{-x}$ by using fixed-point iteration method with an accuracy of 0.001.

[3 Marks]

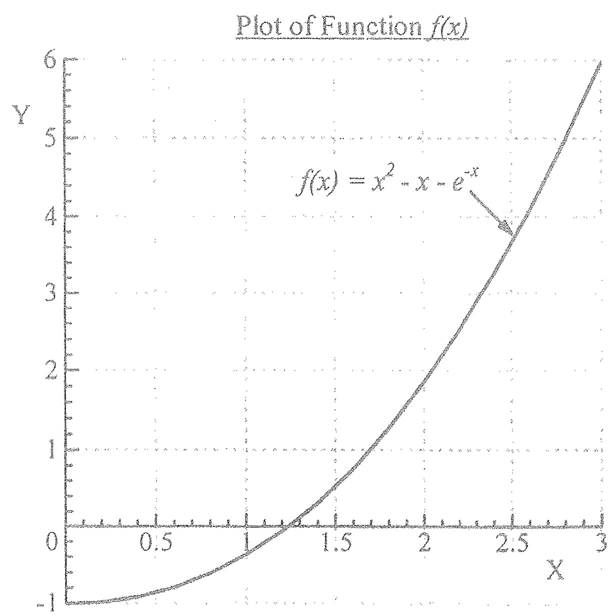

Figure Q1

Q2.

a) The Lagrangian interpolating polynomial of degree n that passes through $n+1$ data points (x_0,y_0), (x_1,y_1),…………, (x_{n-1},y_{n-1}), (x_n,y_n) is defined as $P_n(x)=\sum_{i=0}^{n} y_i L_i(x)$.

where, $L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}$.

Show that $L_i(x_j)=1$ when $i=j$ and
$L_i(x_j)=0$ when $i\neq j$.

[2 Marks]

b) If (x_0,y_0), (x_1,y_1), (x_2,y_2) and (x_3,y_3) are distinct node points, obtain the

i.) linear interpolation polynomial, $P_1(x)$

ii.) quadratic interpolation polynomial, $P_2(x)$

iii.) cubic interpolation polynomial, $P_3(x)$

by using above node points.

[2 Marks]

c) i.) Write down the general form of the Newton's divided difference interpolation polynomial.

[1 Mark]

ii.) The specific heat of water is given as a function of temperature below.

Temperature, T (°C)	42	52	82	100
Specific heat, C_p (J/kg.°C)	4179	4186	4199	4217

Determine the value of the specific heat at T= 61°C using Newton's divided difference method of interpolation with a 3rd order polynomial.

[6 Marks]

iii.) If the exact value of specific heat at T= 61°C is 4191.2 J/kg.°C, find the absolute relative true error (%) for the 3rd order polynomial approximation.

[1 Mark]

Q3.

a) The upward velocity of a rocket is given at three different times below.

Time, t (s)	5	8	12
Velocity, V (m/s)	106.8	177.2	279.2

The velocity data is approximated by a quadratic polynomial as,

$V(t) = a_1t^2 + a_2t + a_3 \quad ; \quad 5 \le t \le 12$

The coefficients a_1, a_2 and a_3 for the above expression are given by,

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Find the values of a_1, a_2 and a_3 using Gauss elimination method. Then, find the velocities of the rocket at $t = 6$ and 10 seconds, by using quadratic polynomial.

[6 Marks]

b) Human vision has the remarkable ability to infer 3D shapes from 2D images. In order to replicate some of these abilities on a computer, it is required to integrate the following vector field,

$$I = \int_{30}^{100} f(x)dx$$

Where,

$$f(x) = 0 \qquad ; 0 < x < 30$$
$$= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778 \qquad ; 30 \leq x \leq 172$$
$$= 0 \qquad ; 172 < x < 200$$

Use three-point Gaussian quadrature method to find the value of the integral.

Table 1: Weighting factors and function arguments used in Gaussian quadrature Formulas

Points	2	3	4
Weighting Factors	$c_1 = 1.0000$ $c_2 = 1.0000$	$c_1 = 0.5555$ $c_2 = 0.8888$ $c_3 = 0.5555$	$c_1 = 0.3478$ $c_2 = 0.6521$ $c_3 = 0.6521$ $c_4 = 0.3478$
Function Arguments	$t_1 = -0.5773$ $t_2 = 0.5773$	$t_1 = -0.7746$ $t_2 = 0.0000$ $t_3 = 0.7746$	$t_1 = -0.8611$ $t_2 = -0.3399$ $t_3 = 0.3399$ $t_4 = 0.8611$

[6 Marks]

Q4.

a) Briefly explain the following topics giving an example for each item.

i.) Ordinary differential equations (ODE)
ii.) Partial differential equations (PDE)
iii.) Initial value problem (IVP)
iv.) Boundary value problem (BVP)

[3 Marks]

b) A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 10^5 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.08C = 0, \qquad C(0) = 10^7$$

Using the Runge-Kutta 2nd order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

[6 Marks]

c) Using the following ordinary differential equation with given initial conditions and step size of h, show that the error of 4th order Runge-Kutta method is of order 5 [$O(h^5)$].

$$\frac{dy}{dx} = y, \qquad y(0) = 1$$

[3 Marks]

Q5.

a) The distance x of a downhill skier from a fixed point is measured at time intervals of 0.25 s as below.

Time, t (s)	0.0	0.25	0.5	0.75	1.0	1.25	1.5
Distance, x (m)	0.0	4.3	10.2	17.2	26.2	33.1	39.1

By using central difference method, approximate the skier's velocity and acceleration at t = 0.5 s and 1.25 s, with an accuracy of 0.1.

[3 Marks]

b) Classify the following equations as linear or non-linear, and state their order.

i.) $\dfrac{\partial^2 u}{\partial x^2} = c^2 \dfrac{\partial^2 u}{\partial t^2}$

ii.) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = 1$

iii.) $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w\dfrac{\partial w}{\partial x} = 0$

[3 Marks]

c) Solve the heat equation,

$$2\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0 \quad \text{for } 0 < x \le 1 \text{ and } 0 < x \le 0.05$$

with the initial conditions,

$u(x,0) = f(x) = x - x^2$

and the boundary conditions,

$u(0,t) = 0$
$u(1,t) = t.$

Use, $h = 0.25$ and $k = 0.025$, where h and k are step sizes along x and t axes respectively.

[6 Marks]