Quantum computing
Seasons of Code - Shridhar Patil

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Linear algebra: The representation of quantum states is done with state vectors in a complex inner product space. The vectors are represented with kets: $|\psi\rangle$. These vectors have a dual bra vector $\langle\psi|$. They are adjoints of each other.

$$\langle\psi| = (\overline{|\psi\rangle})^T = |\psi\rangle^\dagger$$

Inner product: The complex inner product space is called a Hilbert space. For $|\psi\rangle$ and $|\phi\rangle$ ~~beig~~ being two state vectors of quantum systems, Inner product is defined as $\langle\phi|\psi\rangle \in \mathbb{C}$.

i) $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$

ii) $\langle\alpha\phi|\psi\rangle = \overline{\alpha}\langle\phi|\psi\rangle$

iii) $\langle\phi|\alpha\psi\rangle = \alpha\langle\phi|\psi\rangle$

iv) Norm: $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} \geq 0$: equal iff $|\psi\rangle = 0$

Outer product: Outer products of form $|\phi\rangle\langle\psi|$ will be represented as linear matrix transforms. They are useful for defining projectors : $P_\psi = |\psi\rangle\langle\psi|$. $P_\psi|\phi\rangle$ calculates component of $|\phi\rangle$ on $|\psi\rangle$. $P_\psi^2 = P_\psi$.

Operators: From abstract definition: $\hat{A}: V \to W$ takes vector $|\psi\rangle \in V$ to a vector $|\phi\rangle \in W$. ($\hat{A}|\psi\rangle = |\phi\rangle$). They are linear ~~operators~~ transforms, represented by matrices on choosing proper basis vectors.

i) $\langle\phi|\hat{A}\psi\rangle = \hat{A}\langle\phi|\psi\rangle$

ii) $\langle\hat{A}\phi|\psi\rangle = \hat{A}^\dagger\langle\phi|\psi\rangle$

$\hat{A}^\dagger$ denotes adjoint $(\overline{A})^T$.

Quantum state vectors: A state vector $|\psi\rangle$ can be represented as linear combination of basis vectors of its hilbert space. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. If measurement is made along these basis vectors, $P(|0\rangle) = |\alpha|^2$, $P(|1\rangle) = |\beta|^2$ hence:

$$|\alpha|^2 + |\beta|^2 = 1$$

$|0\rangle$ and $|1\rangle$ are orthonormal

$|0\rangle$ and $|1\rangle$ are orthonormal:

$\langle 0|0\rangle = 1 \qquad \langle 1|1\rangle = 1$

$\langle 0|1\rangle = \langle 1|0\rangle = 0$

→ 0 is zero vector present in every hilbert space. $|0\rangle$ is state vector.

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* Tensor product: ~~Represent~~ Used to combine multiple individual independent quantum state vectors into a single composite state vector. If states are $|\psi\rangle$ and $|\phi\rangle$ the composite system is $|\psi\rangle \otimes |\phi\rangle$. The method to compute tensor product on matrices representation:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{matrix} \to |0\rangle \\ \to |1\rangle \end{matrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \begin{matrix} \to |00\rangle \\ \to |01\rangle \\ \to |10\rangle \\ \to |11\rangle \end{matrix}$$

Suppose operator $\hat{A} = [a_{ij}]$ acts on $|\psi\rangle$ and $\hat{B} = [b_{ij}]$ acts on $|\phi\rangle$ then operator on $|\psi\rangle \otimes |\phi\rangle$ is given by $\hat{A} \otimes \hat{B}$:

$$(\hat{A} \otimes \hat{B})(|\psi\rangle \otimes |\phi\rangle) = (\hat{A}|\psi\rangle \otimes \hat{B}|\phi\rangle)$$

$$\hat{A} \otimes \hat{B} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & \ddots & & & \vdots \\ & & & & a_{nn} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & \ddots & & \vdots \\ & & & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ & & & \vdots \\ & & & a_{nn}B \end{bmatrix}$$

submatrix ↓ expand completely

$\therefore$ if $\hat{A}, \hat{B} \in \mathbb{C}^{n \times n}$, $\hat{A} \otimes \hat{B} \in \mathbb{C}^{n^2 \times n^2}$

\# Gates: All quantum gate matrices are unitary → ensures no information loss (remains reversible). → $(U^\dagger U = I)$

i) Hadamard: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$

* Global phase: State vectors $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ differ only by global phase, They are physically indistinguishable. (all expectation values and observations are unchanged by global phase)

Relative phase: when $|\psi\rangle$ and $|\phi\rangle$ have same linear combination amplitudes but differ only in phase $e^{i\gamma_j}$ for amplitude a_j

not intefering with external stuff

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$U^\dagger U = I$

Time evolution: A closed quantum system will undergo unitary evolution.

$$|\psi\rangle \xrightarrow{t} |\psi'\rangle = U^\dagger |\psi\rangle$$

Schrödinger's equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \longrightarrow i\hbar \frac{\partial}{\partial t}(U|\psi(0)\rangle) = H\,U|\psi(0)\rangle$$

$$\therefore\; i\hbar \frac{\partial U}{\partial t} = HU \rightarrow U(t) = e^{-iHt/\hbar}$$

use operator functions since $\hat{H}$ is hamiltonian operator.

$$H = \sum_i \lambda_i |i\rangle\langle i|$$

$$f(H) = e^{-iHt/\hbar} = \sum_i e^{\frac{-i\lambda_i t}{\hbar}} |i\rangle\langle i|$$

Matrix representation of operators: for operator $A = [a_{ij}]$

$a_{ij} = \langle w_j | A | v_i \rangle$ where $|v_i\rangle$ is i^{th} column basis and $\langle w_j|$ is j^{th} row basis vector.

$$\therefore\; A = \sum_{ij} \langle w_j | A | v_i \rangle |v_i\rangle\langle w_j|$$

Pauli matrices: (I, X, Y, Z) are 4 orthonormal matrices
$\sigma_0, \sigma_x, \sigma_y, \sigma_z$
which are very useful for analysis of quantum systems. Since $\text{span}(I, X, Y, Z) = \mathbb{C}^{2\times 2}$: $A = \alpha I + \beta X + \gamma Y + \delta Z$

$$\rho = \frac{I + \vec{V}\cdot\vec{\sigma}}{2}$$

(bloch vector) $\quad (\sigma_x, \sigma_y, \sigma_z)$

density matrix.

Measurements: A measurement of state vector quantum system $|\psi\rangle$ forces it to collapse onto a certain state. Suppose we represent energy quantum state vector as: $|\psi\rangle = \alpha|E_1\rangle + \beta|E_2\rangle$

$\xrightarrow{M} |\psi'\rangle = |E_1\rangle$ or $|E_2\rangle$ with prob^n $|\alpha|^2$ & $|\beta|^2$.

A measurement is represented by a set of measurement operators $\{M_m\}$ for each outcome 'm'.

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$\{M_m\}$ is measurement operator. For state $|\psi\rangle$, prob[n] of outcome m: $P(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$.

$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{P(m)}}$$

since $\sum P(m) = 1 \rightarrow \langle\psi| (\sum M_m^\dagger M_m) |\psi\rangle = 1$

$\therefore \sum M_m^\dagger M_m = I$.

Projective measurement: for observable, we can define observable operator $M = \sum m P_m$

$\hookrightarrow$ projector onto $|m\rangle$ state.

$\hookrightarrow P_m = |m\rangle\langle m|$.

& $P_m^2 = P_m$

$\therefore P(m) = \langle\psi| P_m^\dagger P_m |\psi\rangle$

$= \langle\psi| P_m |\psi\rangle$

$$|\psi'\rangle = \frac{P_m|\psi\rangle}{\sqrt{P(m)}}$$

Spectral decomposition of M

i) m is eigenvalues

ii) $|m\rangle$ are eigenvalues

POVM: we define POVM elements $\{E_m\} \Rightarrow E_m = \sqrt{M_m}$.

$\therefore M_m^\dagger M_m = E_m$. $\therefore P(m) = \langle\psi| E_m |\psi\rangle$; $|\psi'\rangle = \frac{\sqrt{E_m}|\psi\rangle}{\sqrt{P(m)}}$

Superdense coding: Suppose we have to send two bits of info from A to B. Classically we require 2 bits but can do with only 1 qubit. Have 2 qubits in composite state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

give one to A, other to B. A will apply some gates to his qubit as per the bits he want to send.

bits	A will apply	final
00	I (nothing)	$|\beta_{00}\rangle$
01	Z	$|\beta_{01}\rangle$
10	X	$|\beta_{10}\rangle$
11	$ZX = iY$	$|\beta_{11}\rangle$.

(final states: orthonormal)

A will now transfer this singular qubit to B. Now B has both qubits. Since final state has orthogonal states, B can differentiate all 4 final states. $\therefore$ bits to transfer can be identified.