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SCALING OF MULTIPLICITY DISTRIBUTIONS IN HIGH ENERGY HADRON COLLISIONS

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Abstract: It is shown that asymptotically $\langle n\rangle\,\sigma_n(s)$ is only a function of $n/\langle n\rangle$, where $\sigma_n(s)$ is the multiplicity distribution. The essential assumption in deriving this result is that multiparticle inclusive cross sections obey the scaling law. It is also pointed out that the available data below 30 GeV may possibly indicate rather long-range correlations in multibody reactions.

1. INTRODUCTION

There exists already a considerable amount of experimental support for scaling [1] in inclusive reactions in which a single particle in the final state is measured. It is therefore reasonable to expect scaling to be a good principle in hadron physics and it becomes interesting to derive further consequences of this principle.

In the present paper we show that multiparticle scaling implies a very simple regularity for the multiplicity distribution at asymptotic energies. Our result can be expressed by saying that the normalized multiplicity distribution keeps its form independently of the beam energy and just scales up as $\ln s$.

Previously, Mueller [2] has considered the multiplicity distribution from the point of view of scaling in inclusive reactions. Assuming conventional Reggeism and short-range correlations, a scheme was set up which allows one to relate the moments of the multiplicity distribution to the single-particle distribution and to the correlation functions in many-particle inclusive reactions**. Our scheme is rather similar and in some sense more limited, since we consider asymptotic energies and take the leading term only. On the other hand, we do not assume any conventional dynamical mechanism with respect to multiparticle production, nor do we assume that the correlations are necessarily of short-range character. We feel that one should have an open mind with respect to such questions, and if possible try to derive as

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** Similar conclusions have been obtained by the authors in ref. [2a].

much as one can from the data. Actually one finds an empirical regularity in the data at energies up to 30 GeV, which will support long-range correlations, if it remains valid at higher energies.

2. ASYMPTOTIC RESULTS

We are going to derive the asymptotic result that the multiplicity distribution

$$P_n(s) = \frac{\sigma_n(s)}{\sigma_{\text{tot}}(s)}, \tag{2.1}$$

where $\sigma_n(s)$ is the cross section for the multiplicity being n at a c.m.s. energy $\sqrt{s}$, has the form

$$P_n(s) = \frac{1}{\langle n \rangle} \psi\left(\frac{n}{\langle n \rangle}\right), \tag{2.2}$$

where $\langle n \rangle$ is the average multiplicity at the c.m.s. energy $\sqrt{s}$ and where $\psi(z)$ is independent of s except through the variable

$$z = \frac{n}{\langle n \rangle}. \tag{2.3}$$

The main assumption is only the scaling [1] with the scaling functions

$$f^{(q)}(x_1, p_{\perp 1}; x_2, p_{\perp 2}; ...; x_q, p_{\perp q}) = \frac{1}{\sigma_{\text{tot}}} \frac{\mathrm{d}\sigma}{\frac{\mathrm{d}^3 p_1}{\omega_1} \frac{\mathrm{d}^3 p_2}{\omega_2} \cdots \frac{\mathrm{d}^3 p_q}{\omega_q}} \tag{2.4}$$

being non-singular at $x_1 = x_2 = ... = x_q = 0$. We assume that scaling is reached so quickly as s approaches infinity that the multiplicity distribution $P_n(s)$ can be calculated to the highest order in $\ln s$ as if the scaling is exactly true. It is thus important for our argument that for large s the Feynman functions $f^{(q)}(x_1, p_{\perp 1}; ...; x_q, p_{\perp q})$ are approaching a limit that is constant and does not even have variations of the form $\ln s \times$ constant.

In reality we do not need the scaling in the transverse momentum dependence but can do by assuming it only for the distribution function integrated over the transverse momentum,

$$\tilde{f}^{(q)}(x_1, ..., x_q) = \int \mathrm{d}^2 p_{\perp 1} \cdots \mathrm{d}^2 p_{\perp q} f^{(q)}(x_1, p_{\perp 1}; ...; x_q, p_{\perp q}). \tag{2.5}$$

There we need, however, the extra assumption that the transverse momentum is limited as s goes to infinity.

We recall that the Feynman parameters are defined by

$$x_j = \frac{2p_{\parallel j}}{\sqrt{s}} .$$

The argument consists in finding all the moments of the multiplicity distribution $P_n(s)$ to the highest order in $\ln s$ and thus determining the distribution. It is easily seen that

$$\langle n(n-1) \dots (n-q+1) \rangle \equiv \sum_n^{\infty} P_n(s) n(n-1) \dots (n-q+1)$$

$$= \int f^{(q)}(x_1, p_{\perp 1}; \dots ; x_q, p_{\perp q}) \frac{\mathrm{d}^3 p_1}{\omega_1} \dots \frac{\mathrm{d}^3 p_q}{\omega_q}$$

$$= \int f^{(q)}(x_1, p_{\perp 1}; \dots ; x_q, p_{\perp q}) \frac{\mathrm{d}x_1\, \mathrm{d}^2 p_{\perp 1}}{\sqrt{x_1^2 + \frac{p_{\perp 1}^2 + m^2}{\frac{1}{4}s}}} \dots \frac{\mathrm{d}x_q\, \mathrm{d}^2 p_{\perp q}}{\sqrt{x_q^2 + \frac{p_{\perp q}^2 + m^2}{\frac{1}{4}s}}} , \qquad (2.6)$$

where m is the mass of the kind of particles produced. (It is only for simplicity that we restrict ourselves to the case where only one kind of particles is produced.)

We now apply partial integration to the x integration in (2.6), which is seen to diverge logarithmically if we just put $s = \infty$ in it.

Thus we transform expression (2.6) into

$$2 \int \left\{ \left[f^{(q)}(x_1, p_{\perp 1}, \dots, x_q P_{\perp q}) \ln \left(x_1 + \sqrt{x_1^2 + \frac{p_{\perp 1}^2 + m^2}{\frac{1}{4}s}} \right) \right]_{\mathrm{o}}^{1} \right.$$

$$\left. - \int \mathrm{d}x_1 \frac{\partial}{\partial x_1} f^{(q)}(x_1, p_{\perp 1}, \dots, x_q, p_{\perp q}) \ln \left(x_1 + \sqrt{x_1^2 + \frac{p_{\perp 1}^2 + m^2}{\frac{1}{4}s}} \right) \right\}$$

$$\times\, \mathrm{d}^2 p_{\perp 1} \frac{\mathrm{d}x_2 \mathrm{d}^2 p_{\perp 1}}{\sqrt{x_2^2 + \frac{p_{\perp 2}^2 + m^2}{\frac{1}{4}s}}} \dots \frac{\mathrm{d}x_q \mathrm{d}^2 p_{\perp q}}{\sqrt{x_q^2 + \frac{p_{\perp q}^2 + m^2}{\frac{1}{4}s}}}$$

$$= \int \ln \frac{s}{(p_{\perp 1}^2 + m^2)} \left[f^{(q)}(0, p_{\perp 1}, x_2, p_{\perp 2}, \dots, x_q, p_{\perp q}) \right.$$

$$\left. - \frac{g}{\ln \frac{s}{(p_{\perp 1}^2 + m^2)}} \right] \mathrm{d}^2 p_{\perp 1} \frac{\mathrm{d}x_2\, \mathrm{d}^2 p_{\perp 2}}{\sqrt{x_2^2 + \frac{p_{\perp 2}^2 + m^2}{\frac{1}{4}s}}} \dots \frac{\mathrm{d}x_q\, \mathrm{d}^2 p_{\perp q}}{\sqrt{x_q^2 + \frac{p_{\perp q}^2 + m^2}{\frac{1}{4}s}}} , \qquad (2.7)$$

where

$$g = \int \frac{\partial}{\partial x_1} f^{(q)}(x_1, p_{\perp 1}, ..., x_q, p_{\perp q}) \ln \left(x_1 + \sqrt{x_1^2 + \frac{p_{\perp 1}^2 + m^2}{\frac{1}{4}s}} \right) \tag{2.8}$$

has the property of going to a constant in the limit $s \to \infty$, because the integral (2.8) converges. By applying the same type of partial integration to the other x integrations, we obtain

$$\begin{aligned}\langle n(n-1) ... (n-q+1) \rangle &= \int \ln \frac{s}{(p_{\perp 1}^2 + m^2)} ... \ln \frac{s}{(p_{\perp q}^2 + m^2)} \\ &\quad \times \left[f^{(q)}(0, p_{\perp 1}; 0, p_{\perp 2}; ...; 0, p_{\perp q}) + \mathrm{O}\left(\frac{1}{\ln s}\right) \right] \mathrm{d}^2 p_{\perp 1} ... \mathrm{d}^2 p_{\perp q} \\ &= \tilde{f}^{(q)}(0, ..., 0) (\ln s)^q + \mathrm{O}((\ln s)^{q-1}) ,\end{aligned} \tag{2.9}$$

where $\mathrm{O}((\ln s)^{q-1})$ means terms that at most go like $(\ln s)^{q-1}$. Using formula (2.9) for lower values of q, we easily see that the average of any $(q-1)$-order polynomial is

$$\langle a_{q-1} n^{q-1} + a_{q-2} n^{q-2} + ... + a_0 \rangle = \mathrm{O}((\ln s)^{q-1}) , \tag{2.10}$$

so that we have

$$\langle n^q \rangle = \tilde{f}^{(q)}(0, ..., 0) (\ln s)^q + \mathrm{O}((\ln s)^{q-1}) . \tag{2.11}$$

That is to say

$$\sum_n P_n(s) n^q \approx \int_0^\infty n^q P_n(s)\, \mathrm{d}n = \tilde{f}^{(q)}(0, ..., 0) (\ln s)^q + \mathrm{O}((\ln s)^{q-1}) . \tag{2.12}$$

Dividing both sides of this eq. (2.12) by $(\ln s)^q\, [\tilde{f}^{(1)}(0)]^q$, we obtain

$$\int_0^\infty z^q \tilde{f}^{(1)}(0) \ln s\, P_n(s)\, \mathrm{d}z = \frac{\tilde{f}^{(q)}(0, ..., 0)}{[\tilde{f}^{(1)}(0)]^q} + \mathrm{O}\left(\frac{1}{\ln s}\right) = c^q + \mathrm{O}\left(\frac{1}{\ln s}\right) , \tag{2.13}$$

where

$$z = \frac{n}{\tilde{f}^{(1)}(0) \ln s} .$$

Now, we invoke the assumption that the moments (2.13) determine the function $\tilde{f}^{(1)}(0)P_n(s)\ln s = P_{z\langle n\rangle}(s)\ln s\tilde{f}^{(1)}(0)$ uniquely. This puts a rather weak restriction on the moments

$$\frac{\tilde{f}^{(q)}(0, ..., 0)}{(\tilde{f}^{(1)}(0))^q} + O(1/\ln s)$$

and will be discussed in the next section. For the time being, we assume this and thus obtain

$$\tilde{f}^{(1)}(0)P_{z\langle n\rangle}(s)\ln s = \psi(z) + O\left(\frac{1}{\ln s}\right) . \tag{2.14}$$

So we have to the highest order in $\ln s$ the scaling result

$$P_n(s) = \frac{1}{\tilde{f}^{(1)}(0)\ln s}\,\psi\left(\frac{n}{\tilde{f}^{(1)}(0)\ln s}\right) + O\left(\frac{1}{(\ln s)^2}\right) . \tag{2.15}$$

Since

$$\langle n\rangle = \sum_n P_n(s)\,n \approx \int_0^\infty nP_n(s)\,dn = \int_0^\infty z\,\psi(z)\,dz\,\tilde{f}^{(1)}(0)\ln s \approx \tilde{f}^{(1)}(0)\ln s \tag{2.16}$$

by eqs. (2.14) and (2.13) for $q = 1$, we can rewrite (2.15) as

$$P_n(s) = \frac{1}{\langle n\rangle}\,\psi\left(\frac{n}{\langle n\rangle}\right) + O\left(\frac{1}{\langle n\rangle^2}\right) , \tag{2.17}$$

which is our asymptotic scaling result.

The functional form $\psi(z)$ is a priori unknown and can be different for different reactions and it may depend upon what kind of particle is measured. So there may a priori exist one function $\psi_{\pi^0}(z)$ for the π^0 multiplicity in pp and another one for the charged prong multiplicity in π^-p, etc.

We remark that our normalizations imply the conditions

$$\int_0^\infty \psi(z)\,dz = \int_0^\infty P_n(s)\,dn = 1 \tag{2.18}$$

and

$$\int_0^\infty z\,\psi(z)\,\mathrm{d}z = \frac{\int_0^\infty n P_n(s)\,\mathrm{d}n}{\langle n\rangle} = 1\ . \tag{2.19}$$

It is easy to generalize our result to the case where we are interested in the cross section for obtaining n_1 particles of type 1, n_2 of type 2, etc., in the final state. That is, we want to obtain a scaling rule for the probability

$$P_{n_1 n_2 \dots n_k}(s) = \frac{\sigma_{n_1 n_2 \dots n_k}(s)}{\sigma_{\mathrm{tot}}(s)}. \tag{2.20}$$

for yielding n_i particles of type $i = 1, 2, \dots, k$ in the final state.

Quite analogously to the above arguments we find

$$\langle n_1^{q_1} n_2^{q_2} \dots n_k^{q_k}\rangle = \tilde{f}^{(q_1 \dots q_k)}(0, 0, \dots, 0)(\ln s)^q + \mathrm{O}((\ln s)^{q-1})\ , \tag{2.21}$$

where

$$q = q_1 + q_2 + \dots + q_k \tag{2.22}$$

and

$$\tilde{f}^{(q_1 q_2 \dots q_k)}(x_1 x_2 \dots x_q) = \int f^{(q_1 \dots q_k)}(x_1 p_{\perp 1}, \dots, x_q, p_{\perp q})\,\mathrm{d}^2 p_{\perp 1} \dots \mathrm{d}^2 p_{\perp q} \tag{2.23}$$

are the scaling functions corresponding to the inclusive reactions in which

$$\mathrm{a} + \mathrm{b} \to \underbrace{\mathrm{c} + \dots + \mathrm{d}}_{q_1 \text{ particles of type } 1} + \underbrace{\mathrm{e} + \dots + \mathrm{f}}_{q_2 \text{ particles of type } 2}$$

$$+ \dots + \underbrace{\mathrm{g} + \dots + \mathrm{h}}_{q_k \text{ particles of type } k} + \text{anything}\ . \tag{2.24}$$

Further, by expressing $\langle n_1^{q_1} \dots n_k^{q_k}\rangle$ in terms of $P_{n_1 \dots n_k}(s)$ and dividing by $(\ln s)^q$

$$\int_0^\infty \dots \int_0^\infty \mathrm{d}z_1 \dots \mathrm{d}z_k (\ln s)^k P_{n_1 n_2 \dots n_k}(s)\, z_1^{q_1} \dots z_k^{q_k}$$

$$= \tilde{f}^{(q_1 q_2 \dots q_k)}(0 \dots 0) + \mathrm{O}\left(\frac{1}{\ln s}\right)\ . \tag{2.25}$$

From this equation and using the multidimensional moment problem, we can obtain by appropriate normalization

$$P_{n_1 \dots n_k}(s) = \frac{1}{\langle n_1 \rangle \langle n_2 \rangle \dots \langle n_k \rangle} \, \psi \left(\frac{n_1}{\langle n_1 \rangle}, \frac{n_2}{\langle n_2 \rangle}, \dots, \frac{n_k}{\langle n_k \rangle} \right) , \tag{2.26}$$

where the function ψ only depends upon its k arguments but not on s explicitly.

It is straightforward to see from the above result that a joint distribution of various particles (e.g. the multiplicity distribution of the charged prongs) obeys the scaling law (2.17).

One may speculate whether the functions $\psi(z_1 \dots z_k)$ depend on the types of particles or whether they are universal, so that there is no dependence on the initial particles either. We can in fact provide some arguments for these two kinds of universality of ψ. The argument is based on Mueller's generalized optical theorem [3] and a Regge- or dual model with pomeron exchange. According to Mueller, the inclusive cross section for producing q particles is given by a discontinuity of a $(2q + 4)$-point function. Now, if this $(2q + 4)$-function has the same functional form irrespective of the type of the q-particles [as is the case in the Veneziano model (when we only consider unexcited particles)], we can invoke factorization to determine the normalization for a diagram like fig. 1.

From this we find

$$f^{(q_1 q_2 \dots q_k)}(x_1, p_{\perp 1}; \dots; x_q, p_{\perp q}) \propto g_1^{q_1} g_2^{q_2} \dots g_k^{q_k} \tag{2.27}$$

(and this is independent of the initial particles because the dependence is divided out by σ_{tot}).

When we put (2.27) into (2.25) [or (2.13) rather], we find that the moments c_q

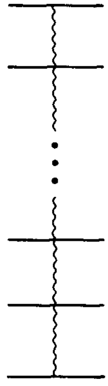

Fig. 1.

of ψ become independent of the type of particle considered and also independent of the initial state particles. We stress that this result of the complete universality of ψ is on a less strong basis than the s-independence of ψ, which essentially only stems from the scaling.

An interesting feature of the derived asymptotic scaling law for multiplicity is that this law is "invariant under resonance decay". By this we mean the following:

Suppose a kind of particle (π^+ say) is produced solely as the decay product of some types of resonances ($N^*(1238)$, ρ, etc.). Then, if the multiplicity distribution for the resonances obeys the multi-dimensional scaling law (2.26), that of the decay product will also obey the scaling law, (2.17) say.

This result is true only if the average multiplicities $\langle n_{N^*}\rangle$, $\langle n_\rho\rangle$, etc., for the resonances rise with energy in the same way, e.g. if they are all proportional to $\ln s$. Further it is true only to the lowest order in $1/\langle n\rangle$ if some of the resonances can decay into a variable number of π^+ mesons. However, it is only to this order anyhow that the scaling law is expected to work, and thus it is only this accuracy that is wanted.

The "invariance of the multiplicity scaling law under resonance decay" can be shown by noticing that, when there are many resonances, the distribution of π^+ multiplicities is approximately that of the linear combination $b_{N^*}n_{N^*} + b_\rho n_\rho + ...$, where $b_{N^*}, b_\rho, ...$ are the average numbers of π^+ produced by the decay of an N^*, ρ, ... Then the result follows directly from (2.26).

We recall that the scaling is also "invariant under resonance decay".

An example of a model in which our scaling principle works with ψ being a proper function (i.e., not a delta function as in the case of short range correlation models) is provided by the geometric distribution law [4]

$$P_n(s) = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n . \tag{2.28}$$

This is approximately (for $\langle n\rangle \gg 1$)

$$P_n(s) = \frac{1}{\langle n\rangle}\exp\left(-\frac{n}{\langle n\rangle}\right) . \tag{2.29}$$

That is to say,

$$\psi(z) = e^{-z} . \tag{2.30}$$

3. THE MOMENT PROBLEM

In the previous section, it has been shown that the normalized multiplicity distribution $P_n(s)$ can be expressed in the form

$$P_n(s) = \frac{1}{\langle n \rangle} \psi \left(\frac{n}{\langle n \rangle} \right) , \tag{3.1}$$

where the function ψ satisfies the equation

$$\int_0^\infty z^q \psi(z)\, \mathrm{d}z = c_q \, . \tag{3.2}$$

Eq. (3.1) is an asymptotic result, valid when $s \to \infty$. The coefficient c_q is determined in terms of the Feynman distribution function $\tilde{f}^{(q)}$ in the region where $x_1, ..., x_q$ are small.

In principle, we can imagine that all the inclusive distribution functions $\tilde{f}^{(q)}$ have been measured, in which case the constants c_q in eq. (3.2) are known for all values of q. Then the interesting question arises as to whether the moments c_q determine the function $\psi(z)$ uniquely. From the point of view of physics we know, of course, that a unique $P_n(s)$ exists for all n. However, the moments c_q might still not be able to determine $P_n(s)$ uniquely.

To show that the moment problem is not entirely academic we shall discuss an example which is due to Stieltjes [5] and which shows that in general the moment problem (3.2) does not have a unique solution. Consider the integral

$$I = \int_0^\infty z^q \, \mathrm{e}^{-z^{\frac{1}{4}}} \sin(z^{\frac{1}{4}})\, \mathrm{d}z \, . \tag{3.3}$$

With the substitution $u = z^{\frac{1}{4}}$ we obtain

$$\begin{aligned} I &= \frac{2}{i} \int_0^\infty u^{4q+3} [\mathrm{e}^{-u(1-i)} - \mathrm{e}^{-u(1+i)}]\, \mathrm{d}u \\ &= \frac{2}{i} (4q+3)! \left[\frac{1}{(1-i)^{4q+4}} - \frac{1}{(1+i)^{4q+4}} \right] = 0 \, . \end{aligned} \tag{3.4}$$

Thus, if we take

$$\psi(z) = \tfrac{1}{24} \mathrm{e}^{-z^{\frac{1}{4}}} [1 + \epsilon \sin(z^{\frac{1}{4}})] \, , \; |\epsilon| < 1 \, , \tag{3.5}$$

the moment problem

$$\int_0^\infty z^q\,\psi(z)\,\mathrm{d}z = \tfrac{1}{6}(4q+3)! \tag{3.6}$$

has an infinite number of solutions corresponding to the infinite number of values that ϵ can assume.

The moment problem (3.6) is characterized by a very rapid increase of c_q, $c_q \sim q^{4q}$. There is a general theorem by Boas (see chapter III of ref. [6]), which states that, if c_q increases very rapidly with increasing q, there exist at least two essentially distinct distribution functions $\psi(z)$.

Since the example (3.5) of a non-unique class of solutions of the moment problem (3.2) does not look too pathological, it is important for us to make plausible that the physical problem does indeed have a unique solution. To see this, let us introduce the Fourier transform* $\phi(t)$ of $\psi(t)$,

$$\phi(t) = \int_0^\infty \mathrm{e}^{-izt}\,\psi(z)\,\mathrm{d}z\ . \tag{3.7}$$

If we expand the exponent and assume convergence of the series obtained by integrating term by term, we obtain by use of eq. (3.2)

$$\phi(t) = \sum_{q=0}^\infty \frac{c_q}{q!}(-it)^q\ . \tag{3.8}$$

If the series (3.8) converges, $\phi(t)$ is an analytic function of t, and hence the Fourier transform $\psi(z)$ is uniquely determined. The example (3.5) does clearly not satisfy this condition, since $c_q \sim q^{4q}$ for $q \to \infty$.

In the actual physical problem, energy conservation puts a restriction on $\psi(z)$, namely that $\psi(z) = 0$ for $z > \sqrt{s}/\ln s$. The moments (3.2) are therefore restricted by

$$c_q \leqslant (\sqrt{s}/\ln s)^q\ . \tag{3.9}$$

It follows that the series (3.8) converges, and hence that the moments c_q determine the distribution function ψ uniquely through eq. (3.8) and

$$\psi(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}t\ \mathrm{e}^{izt}\,\phi(t)\ . \tag{3.10}$$

* In the theory of statistics, $\phi(t)$ is called "the characteristic function".

One could argue that, since eq. (3.2) is an asymptotic equation, the upper limit (3.9) is infinite* in principle. However, we can also look at eq. (3.2) as a first order perturbation expression, and s is then considered as being finite. We shall now show that the next term in the perturbation expansion can also be obtained by solving a moment problem.

From the exact equation

$$\sum_{n=q}^{\infty} n(n-1)\dots(n-q+1)P_n(s) = \int \frac{\mathrm{d}^3p_1}{\omega_1}\dots\frac{\mathrm{d}^3p_q}{\omega_q} f^{(q)}(p_1\dots p_q)\,, \tag{3.11}$$

we obtain by expanding the right-hand side in powers of logs and replacing the sum over n by an integral

$$\int_0^\infty \mathrm{d}z\, z^q \langle n\rangle P_n(s) \prod_{j=1}^{q-1}\left(1-\frac{j}{z\langle n\rangle}\right) = c_q + \frac{\mathrm{d}q}{\langle n\rangle} + \mathrm{O}\left(\frac{1}{\langle n\rangle^2}\right)\,, \tag{3.12}$$

where $\langle n\rangle \sim \ln s$ for s large. Here, $\mathrm{d}q$ can in principle be obtained from $f^{(q)}$. However, eq. (3.12) is not a moment problem of the simple type (3.2). We shall show that, to order $1/\langle n\rangle$, we can reduce eq. (3.12) to a simple moment problem.

Expanding the product in the integrand, we obtain to order $1/\langle n\rangle$

$$\int_0^\infty \mathrm{d}z\, z^q \langle n\rangle P_n(s)\left(1-\frac{q(q-1)}{2z\langle n\rangle}\right) = c_q + \frac{\mathrm{d}q}{\langle n\rangle} + \mathrm{O}\left(\frac{1}{\langle n\rangle^2}\right)\,. \tag{3.13}$$

Making the perturbative ansatz

$$\langle n\rangle P_n(s) = \psi(z) + \frac{1}{\langle n\rangle}\psi^{(1)}(z) \tag{3.14}$$

* We mention that there exist criteria for uniqueness even if the interval of integration is infinite. If the series

$$\sum_0^\infty 1/(c_q)^{\frac{1}{2q}}$$

diverges, then the distribution function $\psi(z)$ is determined uniquely in terms of the moments. Also, if

$$\psi(z) < Cz^{1-\epsilon}\exp(-az^\lambda)\,, \qquad \lambda > \tfrac{1}{2}\,,$$

where $\epsilon > 0$ and C = const., and where z is large, then the moments determine a unique function $\psi(z)$.

where $\psi(z)$ is the solution of eq. (3.2), we obtain to order $1/\langle n\rangle$,

$$\int_0^\infty \mathrm{d}z\, z^q \left[\psi(z) + \frac{1}{\langle n\rangle}\psi^{(1)}(z) - \frac{q(q-1)}{2z\langle n\rangle}\psi(z)\right] = c_q + \frac{\mathrm{d}q}{\langle n\rangle}, \tag{3.15}$$

or, by use of eq. (3.2),

$$\int_0^\infty \mathrm{d}z\, z^q \left[\psi^{(1)}(z) - \frac{q(q-1)}{2z}\psi(z)\right] = \mathrm{d}q\ . \tag{3.16}$$

This is not yet a moment problem of the type (3.2) because of the factor $q(q-1)$. However, we notice that, since all the moments of ψ exist, we have that ψ goes to zero very rapidly as $z \to \infty$. Hence, by two partial integrations,

$$q(q-1) \int_0^\infty z^{q-1}\psi(z)\,\mathrm{d}z = \int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}z^2}(z\,\psi(z))\,z^q\,\mathrm{d}z\ , \tag{3.17}$$

since the surface terms vanish. Eq. (3.16) can then be written

$$\int_0^\infty \mathrm{d}z\, z^q \left(\psi^{(1)}(z) - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z)\right) = \mathrm{d}q\ . \tag{3.18}$$

The set of moments $\mathrm{d}q$ therefore determines the function $\psi^{(1)}(z)$. In principle, the perturbative expansion can be continued to higher orders in $1/\langle n\rangle$, but in practice the resulting equations are rather complicated. We shall therefore not discuss the perturbation expansion further.

As a concrete example of the use of the Fourier transformation technique (3.8) and (3.10) for determining $\psi(z)$ from c_q, let us mention

$$c_q = 1\ (\text{all}\ q)\ . \tag{3.19}$$

This example corresponds to no correlations at all in the inclusive distribution function $\tilde{f}^{(q)}$ in the central region of phase space. We find from eq. (3.8)

$$\phi(t) = \mathrm{e}^{-it} \tag{3.20}$$

and hence from eq. (3.10)

$$\psi\left(\frac{n}{\langle n\rangle}\right) = \delta\left(\frac{n}{\langle n\rangle} - 1\right)\ . \tag{3.21}$$

This is the continuous analogue of the Poisson distribution.

Instead of using the infinite set of moments c_q ($q = 0, 1, 2, ...$) to specify the function $\psi(z)$, there exists an alternative characterization of $\psi(z)$ in terms of the so-called cumulants (or the "semi-invariants") introduced by Thiele [7] (see ref. [8] for an extensive discussion). The cumulants μ_q are formally defined by

$$\ln \left[\int_0^\infty e^{zt} \psi(z) \, dz \right] = \sum_{q=1}^{\infty} \frac{\mu_q}{q!} t^q \,. \tag{3.22}$$

Since $\psi(z)$ is normalized, there is no constant term on the right-hand side.

It is worth noticing that these numbers μ_q multiplied by $(\ln s)^q$ just represent the asymptotic behaviour of the q-particle correlation functions integrated over the phase space, i.e.

$$\mu_q \langle n \rangle^q \approx \int \frac{d^3 p_1}{\omega_1} \cdots \frac{d^3 p_q}{\omega_q} \rho^{(q)}(p_1 \ldots p_q) \,. \tag{3.23}$$

In the case of short-range correlation, μ_q vanishes to leading order. The proof of this relation is already contained in the papers of Mueller [2] and Predazzi and Veneziano [9], and we shall therefore not give the proof*.

From eq. (3.23) it is straightforward to evaluate the asymptotic behaviour of integrated correlation functions in cases where $\psi(z)$ is given in a simple form.

For example, for the high-energy limit of the Poisson distribution (3.21) we have

$$\mu_q^{\text{Poisson}} = 0 \; (q \geqslant 2) \,. \tag{3.24}$$

All the correlation integrals thus vanish.

Another case is the geometric distribution (2.28). Here

$$\mu_q^{\text{Geometric}} = (q-1)! \tag{3.25}$$

so that

$$(\mu_q / c_q)^{\text{Geometric}} = 1/q \,. \tag{3.26}$$

We conclude this section by noticing that it is an important problem to investigate experimentally the moments (or the cumulants) in order to gain insight into the actual nature of the distribution function $\psi(z)$, since we have shown that $\psi(z)$

* If one identifies Mueller's variable h with $t/\langle n \rangle$ and takes the limit $\langle n \rangle \to \infty$, it is easy to obtain eq. (3.23) from Mueller's work [2].

can indeed be determined uniquely in terms of its moments because of the cutoff in n provided by energy conservation.

4. EXPERIMENTAL SITUATION

Experimental information on the multiplicity distributions of the charged particles and their energy dependence in the high energy πp and pp collisions is summarized in the rapporteur talks of Wróblewski [10] and Deutschmann [11]. The global feature of the multiplicity distribution is similar to that of the Poisson distribution, but a definite deviation from the latter has been noticed also [10, 12]. The increase of the average multiplicity $\langle n\rangle$ with energy is slow and the logarithmic dependence seems to be valid, although there is an indication that at lower energies a certain power law may fit better [11].

Of particular interest to us is the work of Czyżewski and Rybicki [12], quoted in Wróblewski's review [10]. These authors have made an extensive study of relatively accurate data on multiplicity distributions of charged prongs in π^-p and pp collisions (mainly with hydrogen bubble-chamber data below 30 GeV) and found, among other things, two conspicuous features of the multiplicity distributions in this energy range.

(i) The ratio $\langle n\rangle/D$ is approximately constant, where D^2 denotes the mean square deviation:

$$D^2 = \langle(n - \langle n\rangle)^2\rangle = \langle n^2\rangle - \langle n\rangle^2 \,. \tag{4.1}$$

See tables 1 and 2.

(ii) When the normalized cross sections P_n, defined by (2.1), are expressed in terms of $\langle n\rangle$ and D utilizing the variables

$$x = \frac{n - \langle n\rangle}{D}\,, \qquad y = DP_n\,, \tag{4.2}$$

then all the data (even including zero-prong events in π^-p collisions) lie nearly on a single universal curve

$$y = f(x)\,.$$

In order to express this curve, these authors choose a generalized Poisson distribution

$$f(x) = 2d\, e^{-d^2} \frac{d^{2(xd+d^2)}}{\Gamma(xd + d^2 + 1)}$$

Table 1
π^-p collision.

P_{lab} (GeV/c)	$\langle n \rangle$	D	$\langle n \rangle/D$	$F^{(2)}$	$R^{(2)}$	$R^{(2)}/F^{(2)}$
4.0	2.63 ± 0.02	1.21 ± 0.02	2.18 ± 0.03	5.75 ± 0.10	−1.17 ± 0.05	−0.203 ± 0.010
4.0	3.08 ± 0.01	1.22 ± 0.01	2.76 ± 0.03	7.90 ± 0.06	−1.59 ± 0.03	−0.201 ± 0.004
6.8*	3.15 ± 0.09	1.38 ± 0.06	2.29 ± 0.12	8.7 ± 0.5	−1.2 ± 0.2	−0.14 ± 0.02
8.0	3.74 ± 0.01	1.44 ± 0.01	2.60 ± 0.02	12.32 ± 0.07	−1.67 ± 0.03	−0.136 ± 0.003
10.0	3.61 ± 0.03	1.64 ± 0.02	2.19 ± 0.03	12.1 ± 0.02	−0.92 ± 0.07	−0.076 ± 0.006
11.0	3.81 ± 0.04	1.63 ± 0.02	2.34 ± 0.04	13.4 ± 0.2	−1.15 ± 0.08	−0.086 ± 0.005
16.0	4.19 ± 0.05	1.91 ± 0.04	2.18 ± 0.05	17.0 ± 0.4	−0.54 ± 0.16	−0.032 ± 0.004
16.0	4.33 ± 0.12	1.71 ± 0.07	2.54 ± 0.12	17.3 ± 0.8	−1.4 ± 0.3	−0.081 ± 0.015
20.0	4.60 ± 0.04	1.92 ± 0.02	2.00 ± 0.03	20.2 ± 0.3	−0.91 ± 0.09	−0.045 ± 0.004
25.0	4.86 ± 0.04	2.11 ± 0.02	2.30 ± 0.03	23.2 ± 0.4	−0.41 ± 0.09	−0.018 ± 0.004
60.0**	6.64 ± 0.16	2.13 ± 0.12	2.12 ± 0.10	42.0 ± 2.0	−2.10 ± 0.5	−0.050 ± 0.013

* Propane bubble-chamber data.
** Emulsion data.

The values of $\langle n \rangle$, D and $\langle n \rangle/D$ are taken from the table 1 of Czyżewski and Rybicki's work [12].

Table 2
p–p collision.

P_{lab} (GeV/c)	$\langle n \rangle$	D	$\langle n \rangle / D$	$F^{(2)}$	$R^{(2)}$	$R^{(2)}/F^{(2)}$
4.0	2.54 ± 0.03	0.90 ± 0.01	2.81 ± 0.05	4.72 ± 0.12	−1.73 ± 0.03	−0.345 ± 0.006
5.5	2.71 ± 0.01	1.03 ± 0.01	2.64 ± 0.02	5.69 ± 0.05	−1.65 ± 0.02	−0.290 ± 0.005
10.0	3.22 ± 0.06	1.33 ± 0.02	2.41 ± 0.05	8.9 ± 0.3	−1.45 ± 0.08	−0.163 ± 0.007
12.9	3.58 ± 0.03	1.51 ± 0.01	2.63 ± 0.02	11.52 ± 0.19	−1.30 ± 0.04	−0.113 ± 0.003
18.0	3.93 ± 0.02	1.72 ± 0.01	2.28 ± 0.02	14.47 ± 0.14	−0.97 ± 0.04	−0.067 ± 0.003
19.0	4.02 ± 0.02	1.75 ± 0.02	2.29 ± 0.03	15.20 ± 0.16	−0.96 ± 0.07	−0.063 ± 0.005
21.1	4.21 ± 0.02	1.85 ± 0.01	2.27 ± 0.02	16.93 ± 0.15	−0.79 ± 0.04	−0.047 ± 0.002
24.0	4.31 ± 0.06	1.88 ± 0.03	2.29 ± 0.05	17.8 ± 0.5	−0.78 ± 0.13	−0.044 ± 0.007
24.1	4.41 ± 0.02	1.90 ± 0.01	2.23 ± 0.02	18.65 ± 0.16	−0.80 ± 0.04	−0.043 ± 0.002
28.4	4.58 ± 0.02	2.08 ± 0.02	2.20 ± 0.02	20.73 ± 0.18	−0.25 ± 0.09	−0.012 ± 0.004
28.5	4.42 ± 0.03	2.00 ± 0.02	2.21 ± 0.03	19.12 ± 0.3	−0.42 ± 0.09	−0.022 ± 0.004

The values of $\langle n \rangle$, D and $\langle n \rangle / D$ are taken from the table 1 of Czyżewski and Rybicki's work [12].

with a constant

$$d = 1.8 \ .$$

When we combine these two empirical facts, we directly obtain the scaling law proposed in the present work*. It is remarkable that this scaling behaviour, which is expected to set in at sufficiently high energy, seems to work already at these moderately high-energy regions, although one cannot of course say much from this, since the average multiplicity changes in the studied region only by a factor less than 2.

From the values of $\langle n \rangle$ and D tabulated by Czyzewski and Rybicki [12] we can readily evaluate the integral of the two-particle distribution function and two-particle correlation function.

$$F^{(2)} = \int\int f^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) \frac{\mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2}{\omega_1 \omega_2} = D^2 + \langle n \rangle^2 - \langle n \rangle, \tag{4.3}$$

$$R^{(2)} = \int\int \rho^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2) \frac{\mathrm{d}^3 p_1}{\omega_1} \frac{\mathrm{d}^3 p_2}{\omega_2} = D^2 - \langle n \rangle \, . \tag{4.4}$$

The results are given in tables 1 and 2**.

As we see from these tables, the integrated two-particle correlation function is negative in these energy ranges and its absolute magnitude decreases with energy. (This trend is of course consistent with the empirical fact (1), i.e. $D/\langle n \rangle \approx$ const.). Presumably it will become vanishingly small at a certain critical energy (say at somewhere in 50–100 GeV). A crucial test of various models will be the behaviour of the value (4.4) above the latter energy. If the Poisson distribution is a good approximation at high energy, as in the case of the uncorrelated jet model [13] or the multiperipheral model [14], the value (4.4) will remain vanishingly small. This means that the empirical law $\langle n \rangle / D \approx$ const. breaks down above the critical energy. If, on the other hand, the latter law continues to be valid, which will be the case with any model that scales with $\mu_2 \neq 0$, e.g. the non-equilibrium model [4, 15], then the value (4.4) will become positive and increase with energy logarithmically. Such a behaviour requires long-range correlations as one can see from eq. (4.4). The behaviour $\langle n \rangle / D \approx$ const. therefore suggests that we may have long-range correlations. This would imply that a multi-Regge mechanism does not work.

Information provided by Drs. O.Czyżewski and K.Rybicki on their analysis of experimental data has been very helpful and stimulating to our work. We thank Dr. Rybicki for giving us permission of using their results. We are also grateful to S. Barshay and Y.A.Chao for useful comments and discussions.

* Note that our scaling law (2.17) leads directly to $D/\langle n \rangle$ = const. [provided ψ is not a delta function, as in eq. (3.21)].

** Some further implications of Czyżwski and Rybicki's analysis will be discussed in a separate note [16].

NOTE ADDED IN PROOF

In the work of Czyzewski and Rybicki [12] utilized in sect. 4, the cross sections for producing n charged particles are normalized to the total *inelastic cross* section. Consequently, the correlation integral R_2 is also evaluated with this normalization. Because the inelastic diffractive cross section is presumably not large, this normalization is approximately equal to taking non-diffractive production processes only. We are very grateful to Dr. A. Białas for calling our attention to these points.

In the meantime, the correlation integral of the Echo Lake cosmic ray data (90 – 500 GeV) has been evaluated by Białas and Zaleski. (Cracow preprint, TPJU-22/71. We are grateful to these authors for sending us their results). We have also been informed of earlier cosmic ray results (100 GeV – 10 TeV) of Yamada and Koshiba, Phys. Rev. 157 (1967) 1279. (We are grateful to Dr. A. Giovanni for this information.) Both these cosmic ray results seem to indicate that the dispersion D continues to be comparable with the average number $\langle n \rangle$.

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