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Answer 1

PART A)
Let $m = 4$,

Let x_1, x_2, x_3, x_4 belong to set C, and all are distinct,
Let $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0$ in R,
Such that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$.
We aim to demonstrate that $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \in C$.
We will now employ the definition of a convex set:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = \lambda_1(x_1) + \lambda_2(x_2) + \lambda_3(x_3) + \lambda_4(x_4)$$

The above expression is a linear combination of x_1, x_2, x_3, x_4. Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$, it can be interpreted as a convex linear combination. Thus, by the definition of a convex set, $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 \in C$.

PART B)
Consider the function $f(x) = -x$. To establish its convexity, the following condition must hold for $\lambda \in [0, 1]$ and $x_1, x_2 \in x$:

$$\lambda(-x_1) + (1 - \lambda)(-x_2) \geq -(\lambda x_1 + (1 - \lambda)x_2)$$
$$-\lambda x_1 + (\lambda - 1)x_2 \geq -\lambda x_1 + (\lambda - 1)x_2$$
$$0 \geq 0$$

The above inequality always holds, demonstrating that $f(x) = -x$ is a convex function.

Now, let's examine the function $g(x) = x^2$. To establish its convexity, the following condition must be satisfied for $\lambda \in [0, 1]$ and $x_1, x_2 \in x$:

$$\lambda x_1^2 + (1 - \lambda)x_2^2 \geq (\lambda x_1 + (1 - \lambda)x_2)^2$$
$$\lambda(\lambda - 1)(x_1 - x_2)^2 \leq 0$$

Since $\lambda(\lambda - 1)$ is always negative for $\lambda \in (0, 1)$ and $(x_1 - x_2)^2$ is always positive, the inequality $\lambda(\lambda - 1)(x_1 - x_2)^2 \leq 0$ always holds. Thus, $g(x) = x^2$ is a convex function.

Now, consider the composition $(f \circ g)(x) = -x^2$. To establish its convexity, the following condition must hold for $\lambda \in [0, 1]$ and $x_1, x_2 \in x$:

$$\lambda x_1^2 + (1 - \lambda)x_2^2 \leq (\lambda x_1 + (1 - \lambda)x_2)^2$$

However, since we previously proved that $\lambda x_1^2 + (1-\lambda)x_2^2 \geq (\lambda x_1 + (1-\lambda)x_2)^2$ always holds for convex functions, it contradicts the condition for convexity. Therefore, $(f \circ g)(x) = -x^2$ is not a convex function.

Answer 2

PART A)

The set of all σ-algebras $\mathcal{X}$ such that $X - U$ is either finite or EMPTY. X is in this set because $X - X = EMPTY).IfA$ is in this set, then $X - A$ is either finite or EMPTY. However, the complement of A, $X - A$, is not necessarily in this set because if $X - A$ is finite, then A might be infinite. This set is not closed under countable unions. If we take countable unions of sets where $X - U$ is finite, the result could be a set where $X - U$ is infinite. So, this set is not a σ-algebra on X.

PART B)
The set of all σ-algebras $\mathcal{X}$ such that $X - U$ is either countable or is all of X - X is in this set because $X - X = EMPTY$. If A is in this set, then $X - A$ is either countable or X. So, A's complement is also in this set because the complement of a countable set or X is also countable or X. - This set is closed under countable unions because the countable union of countable sets is countable. So, this set is a σ-algebra on X.

PART C)
The set of all σ-algebras $\mathcal{X}$ such that $X \cup U$ is infinite or X - X is in this set because $X - X = EMPTY$. - If A is in this set, then $X - A$ is either infinite or EMPTY or X. However, the complement of A, $X - A$, is not necessarily in this set because if $X - A$ is infinite, then A might be finite. - This set is not closed under countable unions. If we take countable unions of sets where $X \cup U$ is infinite, the result could be a set where $X \cup U$ is finite. So, this set is not a σ-algebra on X.

Answer 3

PART A)
A congruence like $ax \equiv b \pmod{p}$ can be solved for x if the greatest common divisor (GCD) of a and p is a divisor of b. If x can be found: Suppose we can solve the congruence for x. By the rules governing congruences, there exists some integer k making the equation $ax = b + kp$ true. Based on a mathematical principle (Bezout's identity), the GCD of a and p can be expressed in terms of a and p. This leads us to conclude that this GCD divides b. If the GCD divides b: Conversely, if the GCD of a and p is a factor of b, then using Bezout's identity, we can rearrange and find an x that satisfies our initial congruence. This is a valid statement. The existence of a solution for the congruence is tightly linked to whether the GCD of a and p is a factor of b.

PART B)
Two congruences, $a_1x \equiv b_1 \pmod{p_1}$ and $a_2x \equiv b_2 \pmod{p_2}$, can be resolved together if p_1 and p_2 have no common factors and the GCDs of a_1, p_1 and a_2, p_2 are factors of b_1 and b_2, respectively. The fact that p_1 and p_2 are coprime (their GCD is 1) implies they share no common divisors. Each of these congruences is solvable independently because the respective GCDs appropriately divide b_1 and b_2. By a special mathematical rule (Chinese Remainder Theorem) for numbers that are coprime, like p_1 and p_2, a solution exists that works for both congruences simultaneously. This is a correct interpretation. These conditions guarantee a shared solution for the pair of congruences.

PART C)
In a more complex system of congruences $a_ix \equiv b_i \pmod{p_i}$ for $i = 1, \ldots, k$, there's a solution $x \equiv c \pmod{\Pi}$, where $\Pi = p_1p_2 \ldots p_k$. This holds when each p_i is coprime with the others, and every a_i and p_i pair meets a specific divisibility condition with b_i. With each p_i being coprime to the others, no pair of these numbers shares any factors. The divisibility condition for each a_i and p_i with respect to b_i means individual solutions are viable for each congruence. According to a crucial theorem in number theory (Chinese Remainder Theorem), when dealing with such coprime numbers, there's a unique solution that aligns with all the given congruences. This assertion is accurate. Under the stated conditions, the system has a unique solution, represented as specified, due to the pairwise coprimality and the specific divisibility criteria.

Answer 4

PART A)
a) The Cartesian product of a countable set with itself is still countable. In this case, X is a set of 29 letters, and you're taking the Cartesian product of X with itself, i.e., $X \times X$. Each element of $X \times X$ is an ordered pair (x, y), where x and y are elements from X. Since X is countable, $X \times X$ is also countable.

However, if you consider the infinite Cartesian product $\prod_{i \in Z} X$, where you are taking the Cartesian product of X with itself an infinite number of times, the result is not countable. This is because the set of all sequences of elements from X, each corresponding to a different index in Z, can be put in a one-to-one correspondence with the power set of X^Z (the set of all functions from Z to X), which is uncountable. This demonstrates that $\prod_{i \in Z} X$ is not countable.

PART B)
You have a family of sets $\{Y_i\}$ where each Y_i is countably infinite. You want to determine whether the union $\bigcup_{i \in Z^+} Y_i$ is countable or not.

If each Y_i is countably infinite, then for each i, there exists a surjective function $f_i : Z^+ \to Y_i$, indicating that Y_i is countable. Now, you want to consider the union of all these countable sets.

The union of countably many countable sets is still countable. To show this, you can arrange the elements of each Y_i in a sequence, and then enumerate all the elements in the union as follows:

- Take the first element from Y_1.

- Take the first element from Y_2.
- Take the second element from Y_1.
- Take the first element from Y_3.
- Take the second element from Y_2.
- Take the third element from Y_1.
- Take the first element from Y_4.
- Take the second element from Y_3.
- Take the third element from Y_2.
- Take the fourth element from Y_1.
- And so on.

This process will enumerate all the elements of the union, and since you can establish a bijection between this enumeration and the set of positive integers Z^+, the union $\bigcup_{i \in Z^+} Y_i$ is indeed countable.