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Answer 1

PART A)
Base Case (n=1):

$$2^{3(1)} - 3^1 = 8 - 3 = 5,$$

which is divisible by 5.

Inductive Step: Assume the statement is true for some arbitrary positive integer k, i.e., $2^{3k} - 3^k$ is divisible by 5. say $2^{3k} - 3^k = 5m$ meaning $2^{3k} = 5m + 3^k$

Now, consider $n = k + 1$:

$$2^{3(k+1)} - 3^{k+1} = 2^{3k+3} - 3^{k+1} = 8 \cdot 2^{3k} - 3 \cdot 3^k$$

$$8 \cdot (5m + 3^k) - 3 \cdot 3^k = 8 * 5m + 8 * 3^k - 3 * 3^k$$

$$8 * 5m + 8 * 3^k - 3 * 3^k = 8 * 5m + 3^k(8 - 3) = 8 * 5m + 3^k * 5 = 5 * (8m + 3^k)$$

which is clearly divisible by 5.

Hence, by mathematical induction, $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$.

PART B)
Base Case (n=2):

$$4^2 - 7 \cdot 2 - 1 = 16 - 14 - 1 = 1 > 0.$$

Inductive Step: Assume the statement is true for some arbitrary large positive integer k, i.e., $4^k - 7k - 1 > 0$. say $m = 4^k - 7k - 1$ and $4^k = m + 7k + 1$.

Now, consider $n = k + 1$:

$$4^{k+1} - 7(k + 1) - 1 = 4 \cdot 4^k - 7k - 7 - 1.$$

Therefore,

$$4 \cdot 4^k - 7k - 7 - 1 = 4(m + 7k + 1) - 7k - 7 = 4m + 21k - 3.$$

Since $4m$ is positive (by our inductive assumption), and $21k - 3 \geq 0 \forall k >= 2$ then $4^{k+1} - 7(k + 1) - 1$ is also positive.

Hence, by mathematical induction, $4^n - 7n - 1 > 0$ for all integers $n \geq 2$.

Answer 2

PART A)

To find the number of bit strings of length 10 that have at least seven 1s, we can consider the different cases:

1. Exactly 7 ones: Choose 7 positions out of 10 to place the ones.

$$\binom{10}{7} = \frac{10!}{7! \cdot 3!} = 120.$$

2. Exactly 8 ones: Choose 8 positions out of 10 to place the ones.

$$\binom{10}{8} = \frac{10!}{8! \cdot 2!} = 45.$$

3. Exactly 9 ones: Choose 9 positions out of 10 to place the ones.

$$\binom{10}{9} = \frac{10!}{9! \cdot 1!} = 10.$$

4. Exactly 10 ones: Choose all 10 positions to be ones.

$$\binom{10}{10} = 1.$$

Now, summing up these cases:

$$120 + 45 + 10 + 1 = 176.$$

So, there are 176 bit strings of length 10 that have at least seven 1s in them.

PART B)

All Discrete Mathematics and all Statistical Methods are identical so we have the indistinguishable from indistinguishable situation. And at least one of each.

To find the number of ways to make a collection of 4 books with the given conditions, we can use the enumerate counting approach.

3 Discrete Mathematics and 1 Statistical Methods
2 Discrete Mathematics and 2 Statistical Methods
1 Discrete Mathematics and 3 Statistical Methods

this will add up to 3 ways.

PART C)
The number of onto functions $f : A \to B$ where $n(A) = m$ and $n(B) = n$ with $m \geq n$ is given by:

$$\text{Number of onto functions} = \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^m$$

Substituting $n = 3$ and $m = 5$, we have:

$$\sum_{r=1}^{3} (-1)^{3-r} \binom{3}{r} r^5$$

$$\Rightarrow \binom{3}{1} \cdot 1^5 - \binom{3}{2} \cdot 2^5 + \binom{3}{3} \cdot 3^5$$

$$\Rightarrow 3 - 96 + 243$$

$$\Rightarrow 243 - 93$$

$$= 150$$

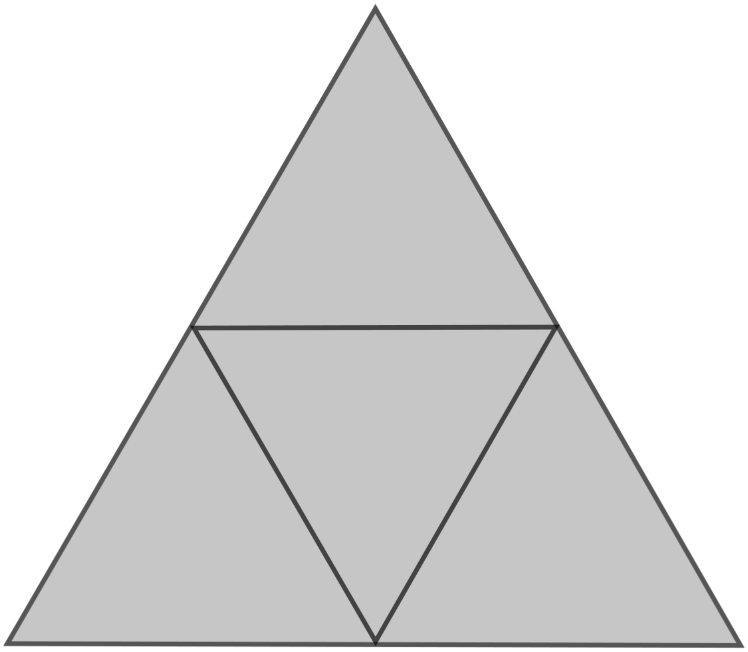
Answer 3

To prove this we use the Pigeonhole Principle, let's consider the vertices of the equilateral triangle as pigeonholes and the kids as pigeons. Each kid corresponds to a point within the triangle.

Since there are 5 kids and 3 vertices, there must be at least two kids in the same vertex (pigeonhole). Let's denote the vertices as A, B, and C.

Now, consider the distances between each pair of vertices. The maximum distance between any two vertices in an equilateral triangle with side length 500 meters is 500 meters.

4 holes 5 piegons (4 triangles 5 kids).



If both kids are at the same vertex, they are within 250 meters of each other.

If the two kids are at different vertices, the distance between them cannot exceed 500 meters. According to the Pigeonhole Principle, as soon as they move away from each other, at least one of them must be within 250 meters of the midpoint of the line segment connecting the two vertices.

Therefore, no matter how the 5 kids are distributed within the equilateral triangle, there will always be two of them within 250 meters of each other.

Answer 4

PART A)

The homogeneous part of the recurrence relation corresponds to $3a_{n-1}$, which leads to the characteristic equation:

$$r - 3 = 0$$

Solving for r, we find $r = 3$. Therefore, the homogeneous solution is $a_n^{(h)} = c \cdot 3^n$, where c is a constant.

PART B)

The particular solution corresponds to the term 5^{n-1}. Assuming a particular solution of the form $a_n^{(p)} = d \cdot 5^n$, we substitute it into the original recurrence relation:

$$d \cdot 5^n = 3 \cdot (d \cdot 5^{n-1}) + 5^{n-1}$$

Solving for d, we find $d = \frac{1}{2}$. Therefore, the particular solution is $a_n^{(p)} = \frac{1}{2} \cdot 5^n$.

By using Part A and Part B $a_n = \frac{1}{2} \cdot 5^n + c * 3^n$ and using $a_1 = 4$ we get $c = \frac{1}{2}$.

PART C)

Base Case (n=1):

$$a_1 = 4$$

(given)

Inductive Step: Assume that the solution $a_k = \frac{1}{2} \cdot 3^k + \frac{1}{2} \cdot 5^k$ holds for some arbitrary positive integer k. Now, let's prove that it holds for $k+1$:

$$a_{k+1} = 3a_k + 5^k$$

Substitute the assumed solution:

$$= 3(\frac{1}{2} \cdot 3^k + \frac{1}{2} \cdot 5^k) + 5^k$$

Simplify:

$$= \frac{1}{2} \cdot 3^{k+1} + \frac{3}{2} \cdot 5^k + 5^k$$

Combine the terms:

$$= \frac{1}{2} \cdot 3^{k+1} + \frac{5}{2} \cdot 5^k$$

Factor out 5^k:

$$= \frac{1}{2} \cdot 3^{k+1} + \frac{5}{2} \cdot 5^k$$

The above expression is of the form $\frac{1}{2} \cdot 3^{k+1} + \frac{1}{2} \cdot 5^{k+1}$, which matches our assumed solution.

Therefore, by mathematical induction, the expression $a_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot 5^n$ is a solution to the given recurrence relation $a_n = 3a_{n-1} + 5^{n-1}$ with the initial condition $a_1 = 4$.