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Answer 1

The cube graph, denoted as Q_n, is a graph whose vertices count are the 2^n , and two vertices are adjacent if and only if their corresponding binary strings differ in exactly one position.

Let E_n be the number of edges in Q_n. We can construct a recurrence relation for E_n.
We have $E_0 = 0$ and $E_1 = 1$.
In each graph say Q_k we copy the previous graph say Q_{k-1} next to itself and connect the two copy graphs vertices to corresponding vertices, which will create 2^{k-1} New Edges. Old edges are also two copies of Q_{k-1} Which correspond to $2 * E_{k-1}$ So we can conclude our recurrence relation to be:

$$E_n = 2 * E_{n-1} + 2^{n-1}$$

Answer 2

The sequence is an arithmetic sequence with a common difference of 3. The nth term of this sequence can be expressed as:

$$a_n = 1 + 3(n - 1)$$

To find the generating function for this sequence, we'll use:

$$A(x) = 1 + 4x + 7x^2 + \dots$$

$$xA(x) = x + 4x^2 + 7x^3 + \dots$$

$$A(x) - xA(x) = 1 + 3x + 3x^2 + \dots$$

$$(1 - x)A(x) = 1 + 3x(1 + x + x^2 + \dots)(* * *)$$

We also have

$$(1 + x + x^2 + \ldots) = \frac{1}{1-x}$$

By (***) we can say

$$(1-x)A(x) = 1 + \frac{3x}{1-x}$$

$$A(x) = \frac{2x+1}{(1-x)^2}$$

So, the generating function for the sequence $< 1, 4, 7, 10, 13, \ldots >$ is $A(x) = \frac{2x+1}{(1-x)^2}$ in closed Form.

Answer 3

Let's use generating functions to solve the given recurrence relation:

$$a_n = a_{n-1} + 2^n, \quad n \geq 1$$

with the initial condition $a_0 = 1$.

Define the generating function $A(x)$ for the sequence (a_n) as follows:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Multiply both sides of the recurrence relation by x^n and sum over all valid values of n:

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + 1 + \sum_{n=1}^{\infty} 2^n x^n$$

Now, substitute the initial condition $a_0 = 1$ and simplify:

$$\sum_{n=1}^{\infty} (a_n - a_{n-1}) x^n = \sum_{n=1}^{\infty} 2^n x^n$$

Combine like terms and solve for $A(x)$:

$$A(x) - xA(x) = \frac{1}{(1-2x)}$$

$$A(x)(1-x) = \frac{1}{1-2x}$$

$$A(x) = \frac{1}{(1-2x)(1-x)}$$

Now, express $A(x)$ as partial fractions:

$$A(x) = \frac{A}{1-2x} + \frac{B}{1-x}$$

We solve A = 2 and B = -1

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

Now, recognize that the expression on the right is the generating function for the geometric series:

$$A(x) = 2 * \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} x^n$$

$$A(x) = \sum_{n=0}^{\infty} (2^{n+1} - 1) x^n$$

Now, identify the individual terms and obtain the closed-form expression for a_n:

$$a_n = 2^{n+1} - 1$$

Therefore, the solution to the given recurrence relation with the initial condition $a_0 = 1$ is $a_n = 2^{n+1} - 1$.

Answer 4

Part A)

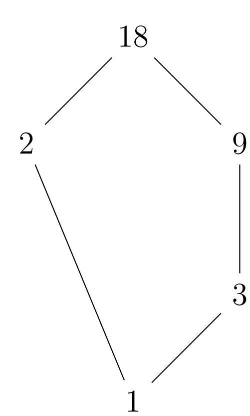

Figure 1: Hasse Diagram

Part B)

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 9 & 18 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ 9 & 0 & 0 & 0 & 1 & 1 \\ 18 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First Row and First Column of Matrix are Lables of elements in our Set A.

Part C)

Consider the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $A = \{1, 2, 3, 9, 18\}$.

1. Least Upper Bound (LUB): - For any pair (a, b) in R, the LUB would be the least common multiple (LCM) of a and b. - For example, for $a = 2$ and $b = 3$, LUB is 6 (LCM of 2 and 3). - For $a = 9$ and $b = 18$, LUB is 18 (since 18 is a multiple of 9).

2. Greatest Lower Bound (GLB): - For any pair (a, b) in R, the GLB would be the greatest common divisor (GCD) of a and b. - For example, for $a = 2$ and $b = 3$, GLB is 1 (GCD of 2 and 3). - For $a = 9$ and $b = 18$, GLB is 9 (since 9 is a divisor of both 9 and 18).

Now, let's check for each pair:

$$\begin{aligned}
&\text{For } a = 2 \text{ and } b = 3: && \text{LUB} = 6, && \text{GLB} = 1 \\
&\text{For } a = 2 \text{ and } b = 9: && \text{LUB} = 18, && \text{GLB} = 1 \\
&\text{For } a = 2 \text{ and } b = 18: && \text{LUB} = 18, && \text{GLB} = 2 \\
&\text{For } a = 3 \text{ and } b = 9: && \text{LUB} = 9, && \text{GLB} = 3 \\
&\text{For } a = 3 \text{ and } b = 18: && \text{LUB} = 18, && \text{GLB} = 3 \\
&\text{For } a = 9 \text{ and } b = 18: && \text{LUB} = 18, && \text{GLB} = 9
\end{aligned}$$

Based on this analysis, it seems that the relation R on set A defines a lattice. The LUB and GLB are well-defined for each pair of elements.

Part D)

$$R* = \begin{bmatrix} 0 & 1 & 2 & 3 & 9 & 18 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 1 & 1 & 1 \\ 9 & 1 & 0 & 1 & 1 & 1 \\ 18 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

First Row and First Column of Matrix are Lables of elements in our Set A.

Part E)

In a relation R defined on a set A, two elements a and b are said to be comparable if either (a, b)

or (b, a) (or both) is in R. Let's analyze the given relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $A = \{1, 2, 3, 9, 18\}$:

1. Integers 2 and 9: - In this case, 2 does not divide 9 since $9 = 2 \times 4 + 1$. So, $(2, 9)$ is not in R. - Therefore, 2 and 9 are not comparable.

2. Integers 3 and 18: - In this case, 3 divides 18 since $18 = 3 \times 6$. So, $(3, 18)$ is in R. - Therefore, 3 and 18 are comparable.

Answer 5

Part A)

A relation consists of a set of ordered pairs a, b. Here a can be chosen in n ways, and similarly, b can be chosen in n ways. So, totally $n \times n = n^2$ possible ordered pairs are possible for a relation. Now each of these ordered pairs can either be present in the relation or not, with 2 possibilities for each of the n^2 pairs. So, the total number of possible relations is 2^{n^2}.

Now, for a relation R to be reflexive, ordered pairs (a, a) must be present in R for each a. i.e., the relation set R must have n ordered pairs fixed. So, the number of ordered pairs possible is $n^2 - n$, and hence the total number of reflexive relations is equal to 2^{n^2-n}.

A relation becomes symmetric if, for ordered pair (a, b) in R, ordered pair (b, a) is also present in R. So, here, the total number of ordered pairs possible is reduced from n^2 to $\frac{n^2+n}{2}$.

So, the total number of possible symmetric relations is given by: $2^{\frac{n^2+n}{2}}$

in symmetric relation, the pair (a, b) is considered as equivalent to the pair (b, a), whether it is included in the relation or not. So the total number of reflexive and symmetric relations is $2^{\frac{n^2-n}{2}}$.

Part B)

A relation consists of a set of ordered pairs a, b. Here a can be chosen in n ways, and similarly, b can be chosen in n ways. So, totally $n \times n = n^2$ possible ordered pairs are possible for a relation. Now each of these ordered pairs can either be present in the relation or not, with 2 possibilities for each of the n^2 pairs. So, the total number of possible relations is 2^{n^2}.

Now, for a relation R to be reflexive, ordered pairs (a, a) must be present in R for each a. i.e., the relation set R must have n ordered pairs fixed. So, the number of ordered pairs possible is $n^2 - n$, and hence the total number of reflexive relations is equal to 2^{n^2-n}.

A relation becomes antisymmetric if, for the ordered pairs (a, b) and (b, a) in R, $a = b$. i.e., the pairs (a, b) and (b, a) cannot be simultaneously in the relation unless $a = b$.

For the n pairs (a, a) in R, they can be either present in the relation or absent. So, 2 possibilities for each, giving 2^n possible relations.

Number of pairs (a, b) in R such that $a \neq b$ equals the number of ways of selecting 2 numbers from n without repetition, which equals $\frac{n(n-1)}{2}$.

Now, for each of these pairs (a, b), there are 3 possibilities:

1. (a, b) and (b, a) not in R,
2. (a, b) in R but (b, a) not in R,
3. (a, b) not in R but (b, a) in R.

So, the total number of possibilities for all such pairs is $3^{\frac{n(n-1)}{2}}$.

And the total number of antisymmetric relations on a set of n elements becomes: $2^n \times 3^{\frac{n(n-1)}{2}}$

Finally, number of relations that are both reflexive and antisymmetric is $3^{\frac{n(n-1)}{2}}$

Answer 6

No, the transitive closure of an antisymmetric relation is not always antisymmetric.

Consider the relation $R = (a, b), (b, c), (c, a)$ on the set $A = a, b, c$. R is antisymmetric.

However, the transitive closure of R, $R* = (a, a), (b, b), (c, c), (a, b), (b, c), (c, a), (a, c), (b, a), (c, b)$, is not antisymmetric because $(a, b), (b, a) \in R*$ with $a \neq b$.