A Computational Approach to Classification of Additive Smooth Fano Polytopes

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1. Problem statement, objectives

The asymptotic distribution of rational solutions to polynomial equations is a topic that has been extensively studied in number theory and geometry.

Important developments in the area by Chambert-Loir, Hassett, and Tschinkel between 1999 and 2002 led to the discovery of additive varieties, irreducible varieties over an algebraically closed field that admit a "good enough" action of the commutative unipotent group $\mathbb{G}_a^n$.

1. Problem statement

There are only finitely many classes of smooth Fano toric varieties, which may be obtained algorithmically [Øbro, 2007]. They have been classified as additive and non-additive by [Huang and Montero, 2020] up to dimension 3 using algebro-geometric means.

Recent criteria by [Arzhantsev and Romaskevich, 2017] and [Dzhunusov, 2022] may be used to design and implement computational methods to this effect.

General objectives

To design and implement algorithms that apply the criteria from [Arzhantsev and Romaskevich, 2017] and [Dzhunusov, 2022] to classify additive smooth Fano toric varieties of any dimension, or up to the highest feasible dimension, thus generalising theoretical classification results obtained for small dimensions. Then, to follow guidelines indicated by [Grayson et al., 1993] to submit a new package with these functionalities to Macaulay2, a computational algebraic geometry software system.

Specific objectives

1. To analyse and synthesise existing relevant literature on toric varieties and Fano varieties, and to identify combinatorial aspects in the calculations needed for their classification.
2. To design algorithms that may be used to classify additive smooth Fano polytopes of any dimension, or up to the highest feasible dimension. This algorithm must be able to systematically replicate known results, and be able to classify new additive smooth Fano toric varieties of dimension 4 or higher.

3. To first implement these algorithms in any general-purpose programming language, and later (at least the algorithm related to [Arzhantsev and Romaskevich, 2017]) according to the standards of Macaulay2, a computational algebraic geometry software system, in order to submit a new package that may eventually be distributed on its official repository.
4. To classify new additive smooth Fano toric varieties of dimensions 4 to 6 using these implementations, as they are difficult to classify by theoretical means. To obtain, as a result, a list of all additive smooth Fano polytopes of dimensions 4 to 6.

2. Preliminaries

2. Preliminaries - Polytopes

Let M, N be dual lattices with associated vector spaces $M_{\mathbb{R}}, N_{\mathbb{R}}$ of dimension $n \in \mathbb{Z}^+$.

A **polytope** $P \subset M_{\mathbb{R}}$ is the convex hull of a finite set $S \subset M_{\mathbb{R}}$. In this situation, we say S is a **$\mathcal{V}$-representation** of P. We are concerned only with **full-dimensional** $(dim(P) = n)$ **lattice polytopes** $(S \subset M)$.

Let $u \in N_{\mathbb{R}}, u \neq 0, b \in \mathbb{R}$. The **affine hyperplane** $H_{u,b}$ is the subset

$$\{m \in M_{\mathbb{R}} : \quad \langle u, m \rangle = b\}.$$

The **closed half-space** $H_{u,b}^+$ is the subset

$$\{m \in M_{\mathbb{R}} : \quad \langle u, m \rangle \geq b\}.$$

A **face** $Q \subset P$ of P is a subset such that

$$Q = H_{u,b} \cap P, \quad P \subset H^+_{u,b}.$$

A **facet** is a face of codimension 1, an **edge** (resp., **vertex**) is a face of dimension 1 (resp., 0).

Let $P \subset M_{\mathbb{R}}$ be a polytope. If there exists a finite collection of half-spaces $(H^+_{u_i,b_i})_{1\leq i\leq s}$ such that

$$P = \bigcap_{i=1}^{s} H^+_{u_i,b_i},$$

then we say $(u_i, -b_i)_{1\leq i\leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$ is an $\boldsymbol{\mathcal{H}}$**-representation** of P.

2. Preliminaries - Minkowski-Weyl's representation theorem

Theorem [Minkowski, Weyl]

Let $P \subset M_{\mathbb{R}}$ be a polytope, let V (resp., $\mathcal{F}$) be the set of its vertices (resp., facets). Then,

1. $V \subset M_{\mathbb{R}}$ is the only inclusion minimal $\mathcal{V}$-representation of P.
2. There exist $\mathcal{H}$-representations of P.
3. For each facet $F \in \mathcal{F}$, there exist unique H_F, H_F^+ associated to F. If we write

$$\begin{aligned} H_F &= \{m \in M_{\mathbb{R}} : \quad \langle u_F, m \rangle = -a_F\}, \\ H_F^+ &= \{m \in M_{\mathbb{R}} : \quad \langle u_F, m \rangle \geq -a_F\}, \end{aligned}$$

then $(u_F, -a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ is unique up to multiplication by $\lambda \in \mathbb{R}^+$.

2. Preliminaries - Minkowski-Weyl's representation theorem

Theorem [Minkowski, Weyl] (Continuation)

iv) The collection $(u_F, -a_F)_{F \in \mathcal{F}} \subset N_{\mathbb{R}} \times \mathbb{R}$ is the unique (up to multiplication by $(\lambda_F)_{F \in \mathcal{F}} \subset \mathbb{R}^+$) inclusion minimal $\mathcal{H}$-representation of P. We may choose u_F primitive and $a_F \in \mathbb{Z}$.

Corollary

There exist facet and vertex enumeration algorithms (of unknown computational complexity in the general case).

Let $P \subset M_{\mathbb{R}}$ be a polytope such that $0 \in \mathrm{int}(P)$, and let $\mathcal{F}$ be the set of its facets. The **dual** or **polar polytope** $P^\circ \subset N_{\mathbb{R}}$ of P is the polytope given by the $\mathcal{V}$-representation $((1/a_F)u_F)_{F \in \mathcal{F}} \subset N_{\mathbb{R}}$. *The polar polytope of a lattice polytope may not be a lattice polytope!*

2. Preliminaries - Toric varieties, polytopes

Recall that there exists a bijective correspondence between (normal) toric varieties and fans of polyhedral cones.

Let $P \subset M_{\mathbb{R}}$ be a polytope (wlog, such that $0 \in \mathrm{int}(P)$). The **normal fan** of P is the complete fan Δ in $N_{\mathbb{R}}$ "based on" the faces of $P^\circ \subset N_{\mathbb{R}}$ (see the Figure).

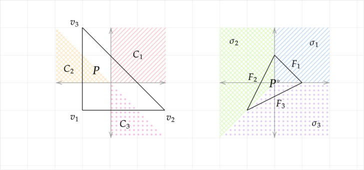

Figure: The polytope, dual polytope, and normal fan of $\mathbb{P}^2$.

Thus, there exists a bijective correspondence between n-dimensional projective toric varieties and polytopes.

2. Preliminaries - Reflexive, smooth Fano polytopes

Let X be a complete normal algebraic variety. We say X is a **Gorenstein Fano variety** if the anticanonical divisor $-K_X$ is Cartier and ample. We say X is a **smooth Fano variety** if it is Gorenstein Fano and smooth.

Let $P \subset M_{\mathbb{R}}$ be a polytope such that $0 \in \operatorname{int}(P)$. We say P is a **reflexive polytope** if $P^\circ \subset N_{\mathbb{R}}$ is also a lattice polytope.

Let $P \subset N_{\mathbb{R}}$ be a polytope such that $0 \in \operatorname{int}(P)$. We say P is a **smooth Fano polytope** if the vertices on each facet form a basis of N. *Surprisingly, this is equivalent to $P^\circ \subset M_{\mathbb{R}}$ being reflexive and smooth!*

Theorem (cf. [Cox et al., 2011])

There is a bijective correspondence between reflexive polytopes (resp., smooth Fano polytopes) and Gorenstein Fano toric varieties (resp., smooth Fano toric varieties).

There is a finite number of equivalence classes of reflexive polytopes, but they grow quickly with $n \in \mathbb{Z}^+$ (16, 4319, 473 million, ...) [Kreuzer and Skarke, 2004].

It is more interesting, then, to study smooth Fano polytopes (5, 18, 124, 866, 7622, ...), which have been classified by Mori and Mukai ($n = 3$), Batyrev, Sato ($n = 4$), and Øbro ($n \geq 5$).

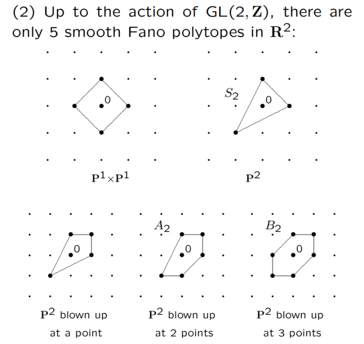

Figure: Smooth Fano polytopes of dimension 2.

Source: [Debarre, 2002]

Let K be an algebraically closed field of characteristic 0, let X be an irreducible algebraic variety of dimension n over K, and let $\mathbb{G}_a = (K, +)$ be the additive group underlying K.

An **additive action** on X is an (effective, regular) action of the commutative unipotent group $\mathbb{G}_a^n$ on X with an open orbit. We say X is an **additive variety** (resp., **uniquely additive variety**) if it admits an additive action (resp., an additive action unique up to isomorphism).

Example [Hassett and Tschinkel, 1999]

There are two distinct additive actions on $\mathbb{P}^2$ given, for all $a = (a_1, a_2) \in \mathbb{G}_a^2$ and $x = [x_1 : x_2 : x_3] \in \mathbb{P}^2$, by:

$$\tau(a)(x) = \begin{bmatrix} 1 & 0 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \rho(a)(x) = \begin{bmatrix} 1 & a_1 & a_2 + \frac{1}{2}a_1^2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

2. Preliminaries - Arzhantsev and Romaskevich's existence criterion

Let $P \subset M_{\mathbb{R}}$ be a very ample polytope, and let $\mathcal{F}$ be the set of its facets. We say P is **inscribed in a rectangle** if there exists a vertex $v_0 \in P$ of P such that:

1. The primitive vectors on the edges $E_i \subset P$ of P starting at v_0 form a basis $e_1, \ldots, e_n \in M$ of M.
2. For all $F \in \mathcal{F}$ and $i \in \{1, \ldots, n\}$, if $v_0 \notin F$, then $\langle -u_F, e_i \rangle \geq 0$.

Theorem [Arzhantsev and Romaskevich, 2017]

Let $P \subset M_{\mathbb{R}}$ be a very ample polytope. Then, the projective variety X_P associated to P is additive if and only if P is inscribed in a rectangle.

2. Preliminaries - Dzhunusov's uniqueness criterion

Let $P \subset M_{\mathbb{R}}$ be a polytope given by the primitive $\mathcal{H}$-representation $(u_i, -a_i)_{1 \leq i \leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$. For each $i \in \{1, \ldots, s\}$, define the set:

$$\mathfrak{R}_i := \{x \in M : \quad \langle u_i, x \rangle = -1, \quad (\forall j \in \{1, \ldots, s\} \setminus \{i\} : \quad \langle u_j, x \rangle \geq 0)\}.$$

A **Demazure root** is an element of the set $\cup_{i=1}^{s} \mathfrak{R}_i \subset M$.

Theorem [Dzhunusov, 2020]

Let $P \subset M_{\mathbb{R}}$ be an additive polytope given by an ordered primitive $\mathcal{H}$-representation $\mathcal{P} = (p_i, -a_i)_{1 \leq i \leq s} \subset N_{\mathbb{R}} \times \mathbb{R}$. Let $\mathcal{P}^* = (p_i^*)_{1 \leq i \leq s} \subset M_{\mathbb{R}}$ be the dual basis of $(p_i)_{1 \leq i \leq s}$ (in the linear algebraic sense), then X_P is uniquely additive if and only if for each $i \in \{1, \ldots, n\}$, one has $\mathfrak{R}_i = \{-p_i^*\}$.

2. Preliminaries - Dzhunusov's uniqueness criterion

The reflexive case is very graphical, as the Figures show.

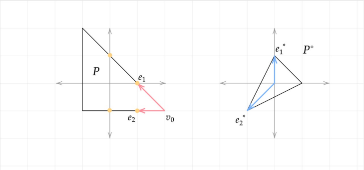

Figure: The polytope, dual polytope, and Demazure roots of $\mathbb{P}^2$.

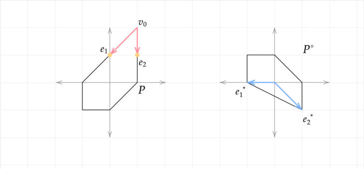

Figure: The polytope, dual polytope, and Demazure roots of $Bl_{p,q}(\mathbb{P}^2)$.

3. Solution, results

We used Selenium and WebDriver to web scrape the Smooth Toric Fano Varieties (SFTV) section of the Graded Ring Database (GRDB) by Gavin Brown and Alexander Kasprzyk, with polytope data of all 8635 such varieties of dimension $d \leq 6$. This data is provided by Mikkel Øbro, and is spread across 864 pages.

We ran the web scraping algorithm thrice (each execution took 50 min), then a simple error correction algorithm twice, and finally parsed the data using the following regular expression:

```
(\d+)
Smooth ([^\n]+)?toric Fano (?:[^\n]+) X = X\(Q\) with degree \(-KX\)(\d+) = (\d+)(?:
Zero barycentre: (\w+))?(?:
Zero dual barycentre: (\w+))?
Vol\(Q\): (\d+)
#Vertices: (\d+)
#Facets: (\d+)
Vertices: ([^\n]+)
Dual: ([^\n]+)
```

3. Solution - Find edges from vertex

Let $P \subset M_{\mathbb{R}}$ be a polytope given by both its $\mathcal{V}$ and $\mathcal{H}$-representations. Let $|I|$ be the number of vertices, and $|J|$ the number of facets.

One step of the existence algorithm will require us to find all (primitive) vectors on the edges of P starting at a given vertex of P. By duality results, this is not hard if and only if the vertex is smooth. However, we studied the more general case, following Henk and Ziegler, and produced an algorithm which compares favourably to the literature (it runs in time $O(|I|^2||J|)$ in the dense case, or $O(|I|\#VF)$ in the sparse case).

If $d \leq 4$, it can be shown that vertices that share $d-1$ facets form an edge. *If $d \geq 5$, there exist counterexamples!*

Idea: Two vertices form an edge if and only if they are the only vertices in the facets that contain them both.

3. Solution - Find edges from vertex

Define the **vertices-facets matrix** $VF := (vf_{ij})_{i \in I, j \in J} \in \mathcal{M}_{|I| \times |J|}(\mathbb{Z})$ as the vertices-facets incidences. Now define the **shared facets vector** associated to $n \in \mathbb{Z}^+$ vertices $m_{i_0}, \dots m_{i_n}$ as:

$$S^{i_0,\dots,i_n} := (s_i^{i_0,\dots,i_n})_{i \in I} := (VF)(((vf_{i_0 j})_{j \in J})^t \circ \dots \circ ((vf_{i_n j})_{j \in J})^t) \in \mathcal{M}_{|I| \times 1}(\mathbb{Z}).$$

The following is real world example:

$$S^{1,0} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Lemma

Fix $i_0, i_1 \in I, i_0 \neq i_1$. There exists an edge E of P from m_{i_0} to m_{i_1} if and only if the following two conditions hold:

1. $s_{i_1}^{i_0} \geq d - 1$.
2. For each $i \in I \setminus \{i_0, i_1\}$ such that $s_i^{i_0} \geq d - 1$ we have $s_i^{i_0,i_1} < s_{i_1}^{i_0}$.

3. Solution - Existence algorithm

Algorithm 4 Algorithm to decide if the dual of a smooth Fano polytope given by both its vertices and primitive inward-pointing facet normals is additive or not.

Precondition: $P^\circ \subset M_{\mathbb{R}}$ is the dual of a smooth Fano polytope. $V[|I|][d]$ are the vertices of P°. $H[|J|][d]$ are the primitive inward-pointing facet normals of P°.

Postcondition: The algorithm returns True if P° is additive, and False if not.

$VF \leftarrow$ A zero array of dimension $|I| \times |J|$. ▷ See Definition 3.2.1.
for $0 \leq j < |J|$ **do** ▷ Fill in VF.
    IndicesInFacet $\leftarrow Argmin_{0 \leq i < |I|}\{\langle H[j], V[i] \rangle\}$
    for $i \in$ IndicesInFacet **do**
        $VF[i, j] \leftarrow 1$
    end for
end for
for $0 \leq i < |I|$ **do** ▷ Check each vertex.
    if CheckVertexCorollary2.4.6(V, H, VF, i) **then** ▷ See Algorithm 5.
        Return True.
    end if
end for
Return False.

3. Solution - Existence algorithm

Algorithm 5 Function to check if a vertex satisfies the condition described in Corollary 2.4.6 or not.

Precondition: $V[|I|][d]$, $H[|J|][d]$, $VF[|I|][|J|]$, i are as in Algorithm 4.
Postcondition: The function returns True if $V[i]$ satisfies the condition described in Corollary 2.4.6, and False if not.

```
Edges ← FindEdgesFromVertex(V, VF, i)                ▷ See Algorithm 2.
for 0 ≤ j < |J| do
    if VF[i][j] == 0 then
        for 0 ≤ k < # Edges do
            if ⟨H[j], Edges[k]⟩ > 0 then
                Return False.
            end if
        end for
    end if
end for
Return True.
```

Theorem

For fixed d, Algorithms 4 and 5 are correct and have time complexity $O(|I|^3|J|)$, if VF is dense, or $O(|I|^2\#VF)$, where $\#VF$ is the number of non-zero entries of VF, if VF is sparse.

3. Solution - Uniqueness algorithm

Algorithm 6 Algorithm to decide if the dual of an additive smooth Fano polytope given by its primitive inward-pointing facet normals is uniquely additive or not.

Precondition: $\varepsilon > 0$ is a small tolerance. $P^\circ \subset M_\mathbb{R}$ is the dual of an additive smooth Fano polytope. $H[d][|J|]$ are the primitive inward-pointing facet normals of P°. *Note:* H *is the transpose of its homonym in Algorithm 4!*

Postcondition: The algorithm returns True if P° is uniquely additive, and False if not.

PossibleBases $\leftarrow$ Subsets(H, d) ▷ Read description for more details.
for $B \in$ PossibleBases **do**
  if $||det(B)| - 1| \leq \varepsilon$ **then** ▷ If B is a basis.
    SortedH $\leftarrow$ Sort$^*(H, B)$ ▷ Read description for more details.
    $R \leftarrow$ SortedH$[:, d:]$ ▷ Slice vectors in H that are not in B.
    $R \leftarrow B^{-1}R$ ▷ Change of basis.
    if IsNonPositive(R) **then** ▷ Read description for more details.
      $B^* \leftarrow (B^{-1})^t$ ▷ Linear algebraic dual of B.
      for $0 \leq i < d$ **do**
        $S_0 \leftarrow -B^*[:, i]$ ▷ A priori solution.
        Model $\leftarrow$ DefineLPModel(Maximise, $x = (x_j)_{0 \leq j < d}$, $f(x)$, A, b) ▷ Read description for more details.
        $S_1 \leftarrow$ LPSolve(Model)
        if Model.status != "Optimal" or $\|S_1 - S_0\| \geq \varepsilon$ **then**
          Return False.
        end if
      end for
      Return True.
    end if
  end if
end for

For each $i \in \{0, \ldots, d-1\}$, define the following ILP problem $\mathcal{P}_i$:

ILP problem $\mathcal{P}_i$

Let $x = (x_j)_{0 \le i < d} \in M$. Maximise

$$f(x) = \sum_{\substack{j=0 \\ j \neq i}}^{d-1} \langle H[:, j], x \rangle,$$

subject to

$$\begin{aligned} \langle H[:, i], x \rangle &= -1, \\ \langle H[:, j], x \rangle &\geq 0, \quad \forall j \in J, j \neq i. \end{aligned}$$

Then, we have that $S_1 = S_0 = -B^*[:, i] \in M$ if and only if $\mathfrak{R}_i = \{S_0\}$.

3. Results - Summary

The previous algorithms were implemented in Google Colab, and later in the Macaulay2 framework. The following table summarises the results of classifying all additive smooth Fano polytopes of dimensions 2 to 6. The column labels indicate, in order:

1. **d**: Dimension.
2. **#NAP**: Number of non-additive polytopes.
3. **#(NU)AP**: Number of (not uniquely) additive polytopes.
4. **#UAP**: Number of uniquely additive polytopes.

d	**#NAP**	**#(NU)AP**	**#UAP**	**Total**	**Note**
2	1	2	2	**5**	
3	4	12	2	**18**	#(NU)AP + #UAP = 14 obtained by [H. and M., 2020].
4	45	75	4	**124**	
5	396	466	4	**866**	
6	4194	3420	8	**7622**	
Total	**4640**	**3975**	**20**	**8635**	

3. Results - Data visualisation

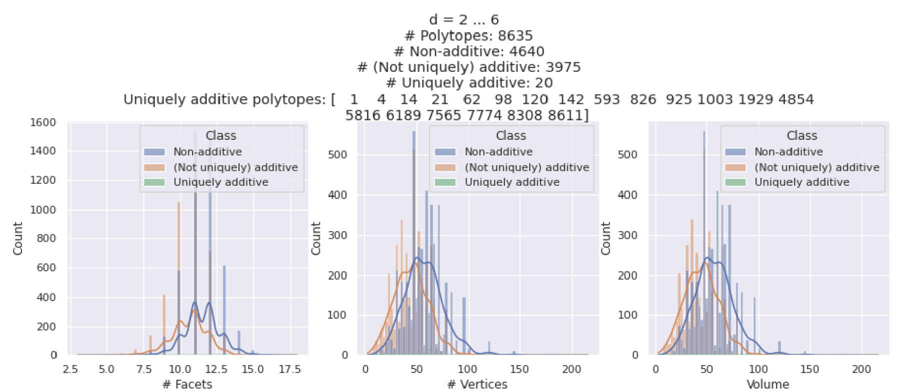

Figure: Distribution of the three classes of smooth Fano polytopes of dimensions 2 to 6 with respect to various quantities.

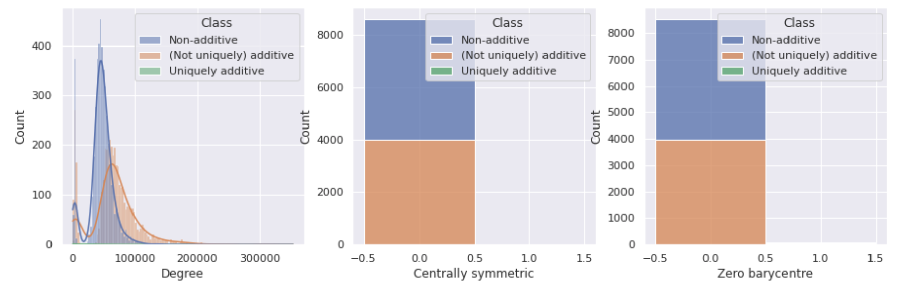

Figure: Distribution of the three classes of smooth Fano polytopes of dimensions 2 to 6 with respect to various quantities.

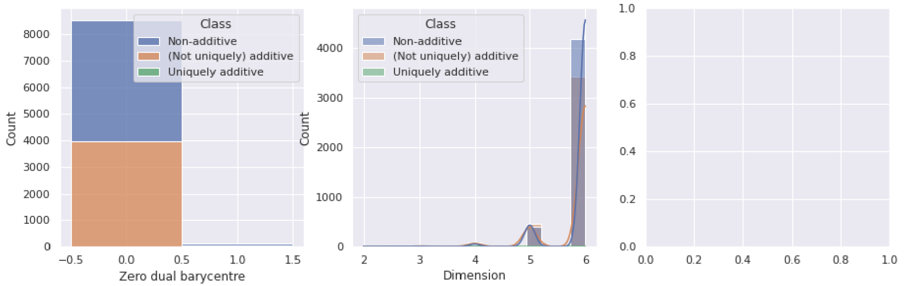

Figure: Distribution of the three classes of smooth Fano polytopes of dimensions 2 to 6 with respect to various quantities.

We created AdditiveProjectiveToricVarieties, a new Macaulay2 package with methods for working with additive actions on projective toric varieties. The package is still a wip, but may already be submitted for community review.

It includes a pre-processed database which classifies smooth Fano toric varieties up to dimension 5 as per the database on the NormalToricVarieties package. It depends on the packages NormalToricVarieties and Polyhedra, and exports the methods isAdditive, listAdditiveSmoothFanoToricVarieties, and randomAdditiveSmoothFanoToricVariety.

3. Results - Macaulay2 package

isAdditive -- check if a projective toric variety admits an additive action

Synopsis

- Usage:
```
isAdditive(P)
isAdditive(X)
isAdditive(d,n)
```
- Inputs:
  - `P`, a convex polyhedron, a full-dimensional lattice polytope associated to the projective toric variety (i. e., the normal fan spans the faces of the polar polytope)
  - `X`, a normal toric variety, a projective normal toric variety
  - `d`, an integer, the dimension (in 1 ... 6)
  - `n`, an integer, the index of a smooth Fano toric variety as per the database on the NormalToricVarieties package
- Outputs:
  - a Boolean value, true if the projective toric variety admits an additive action, false if not

Description

Checks if the projective toric variety admits an additive action by checking if the associated very ample full-dimensional lattice polytope is inscribed in a rectangle (see [1]).

If the input is a Polyhedron, the method first verifies that it is compact, full-dimensional, and its vertices lie on the lattice. If the polytope is not very ample, it also scales it by d - 1, so that it is, as the algorithm requires it.

If the input is a NormalToricVariety, the method first verifies that it is projective.

In either of the previous two cases, the projective toric variety may be of any dimension.

If the input is (d, n), the method first verifies that the values are valid and, if d <= 5, fetches the output from the pre-processed database.

Warning! For some polytopes of dimension greater than six this method may be slow; this will be fixed in the future.

Projective 2-space admits an additive action [3]:

```
i1 : PP2 = toricProjectiveSpace 2;

i2 : isAdditive(PP2)

o2 = true
```

The projective variety associated to the polar of the Del Pezzo polygon does not admit an additive action:

```
i3 : V = transpose matrix {{1, 0}, {0, 1}, {-1, 0}, {0, -1}, {1, 1}, {-1, -1}};

                2       6
o3 : Matrix ZZ  <--- ZZ
```

Figure: Automatically built HTML file corresponding to documentation for the method *isAdditive*.

4. Results validation

4. Results validation

We matched our results with existing classifications of smooth Fano polytopes by comparing polytope data such as IDs, degree, number of vertices, facets, and rational cohomology of the associated projective variety.

Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope, and let X_{P° be its associated projective variety. By Poincaré duality, its even Betti numbers are equal in pairs, and its odd Betti numbers vanish. As P is simplicial, we may calculate its even Betti numbers in terms of its face numbers (in particular, if $n \leq 4$, in terms of its number of vertices and facets). We have the following formula:

$$h_p = b_{2p} = \sum_{i=p}^{d} (-1)^{i-p} \binom{i}{p} f_{d-i-1}.$$

This yielded four long tables, and a lot of insight!

5. Conclusions

5. Conclusions

Our work yielded:

- Yet another introduction to toric geometry, polytope theory, and their intertwinings.
- A short review of some recent results by Arzhantsev, Dzhunusov, and Romaskevich.
- An algorithm to obtain the edges of a polytope of any dimension which compares favourably to existing algorithms.
- A complete classification of additive smooth Fano polytopes of dimension up to 6.
- A new Macaulay2 package with methods for working with additive actions on projective toric varieties.
- We identified a small error in the initial classification of smooth Fano toric fourfolds by Batyrev.

Possible future lines of work include:

- Trying to generalise our approach to produce an algorithm to obtain the face lattice of a polytope, instead of only the edges.
- Classifying sevenfolds and eightfolds.
- Implementing algorithms for calculating Demazure roots and uniqueness on the Macaulay2 package. Submitting the package for community review.
- Working towards justification for qualitative observations resulting from data visualisation.

Another very interesting immediate line of research is given by:

Problem

Characterise all uniquely additive smooth Fano toric varieties.

Define:

1. The $\mathbb{P}^1$ **polytope** is the polytope $[-1,1] \subset \mathbb{R}$.
2. If n is even, the **Del Pezzo polytope** is the polytope $P \subset N_{\mathbb{R}}$ given by the $\mathcal{V}$-representation:

$$\left\{ \pm e_1, \ldots, \pm e_n, \pm \sum_{i=1}^{n} e_i \right\}.$$

3. If n is even, the **pseudo Del Pezzo polytope** is the polytope $P \subset N_{\mathbb{R}}$ given by the $\mathcal{V}$-representation:

$$\left\{ \pm e_1, \ldots, \pm e_n, - \sum_{i=1}^{n} e_i \right\}.$$

5. Conclusions

The following improves on a theorem of Ewald:

Theorem [Casagrande, 2003]

Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. P has n linearly independent pairs of centrally-symmetric vertices if and only if P splits into $\mathbb{P}^1$, Del Pezzo, and pseudo Del Pezzo polytopes.

We have verified the following conjecture to be true up to dimension 5:

Conjecture

Let $P \subset N_{\mathbb{R}}$ be a smooth Fano polytope. Then, P is uniquely additive if and only if P splits into $\mathbb{P}^1$ and pseudo Del Pezzo polytopes.

A simple dimensional analysis tells us that there is an extra uniquely additive sixfold that is not accounted for by the previous conjecture.

Thank you for your time & attention! :)