Lecture 1:

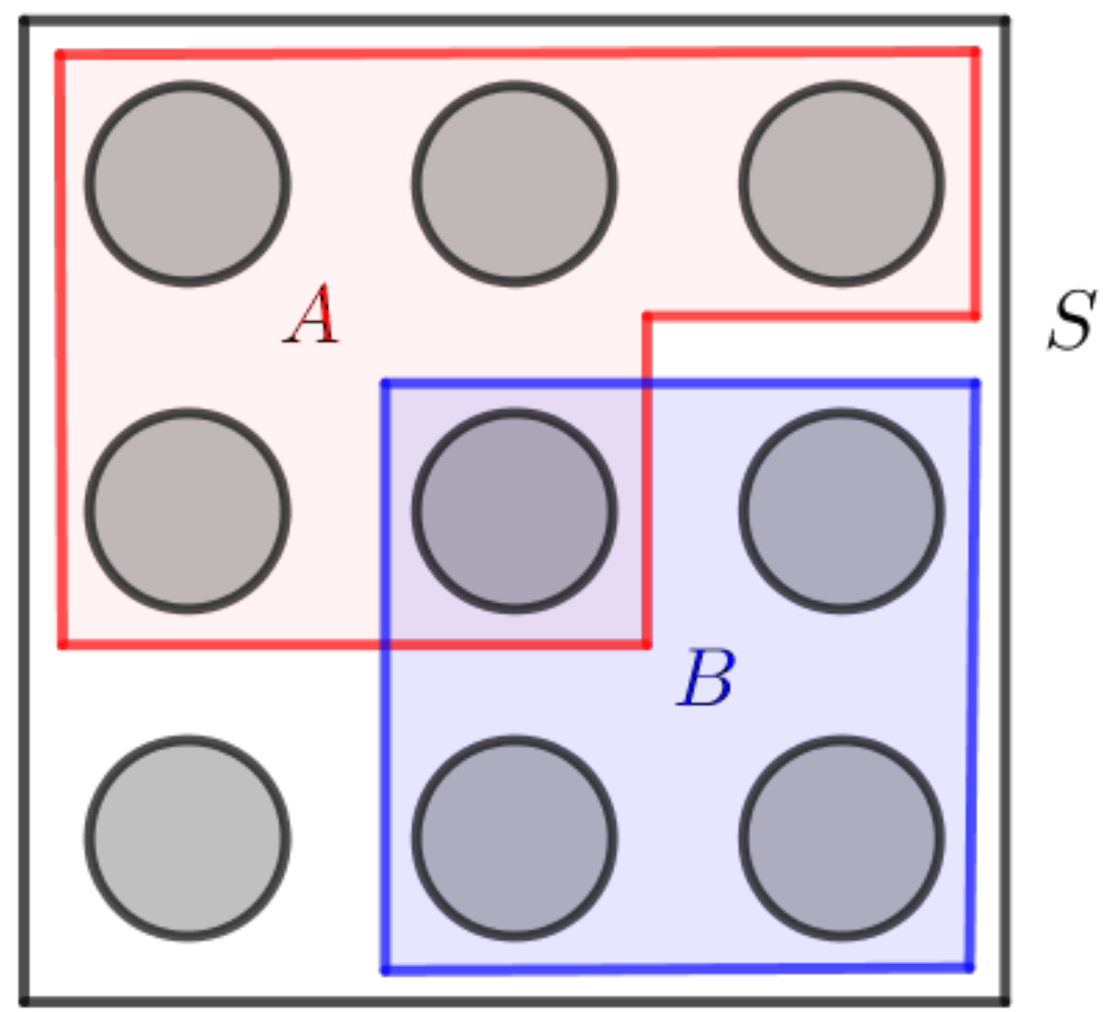
Probability and counting

Sample Spaces

Sample Spaces

Definition 1.2.1 The **sample space** S of an experiment is the set of all possible outcomes of the experiment. An **event** A is a subset of the sample space S, and we say that A **occurred** if the actual outcome is in A.

Fig. 1.1: A sample space with two events, A and B, spotlighted:

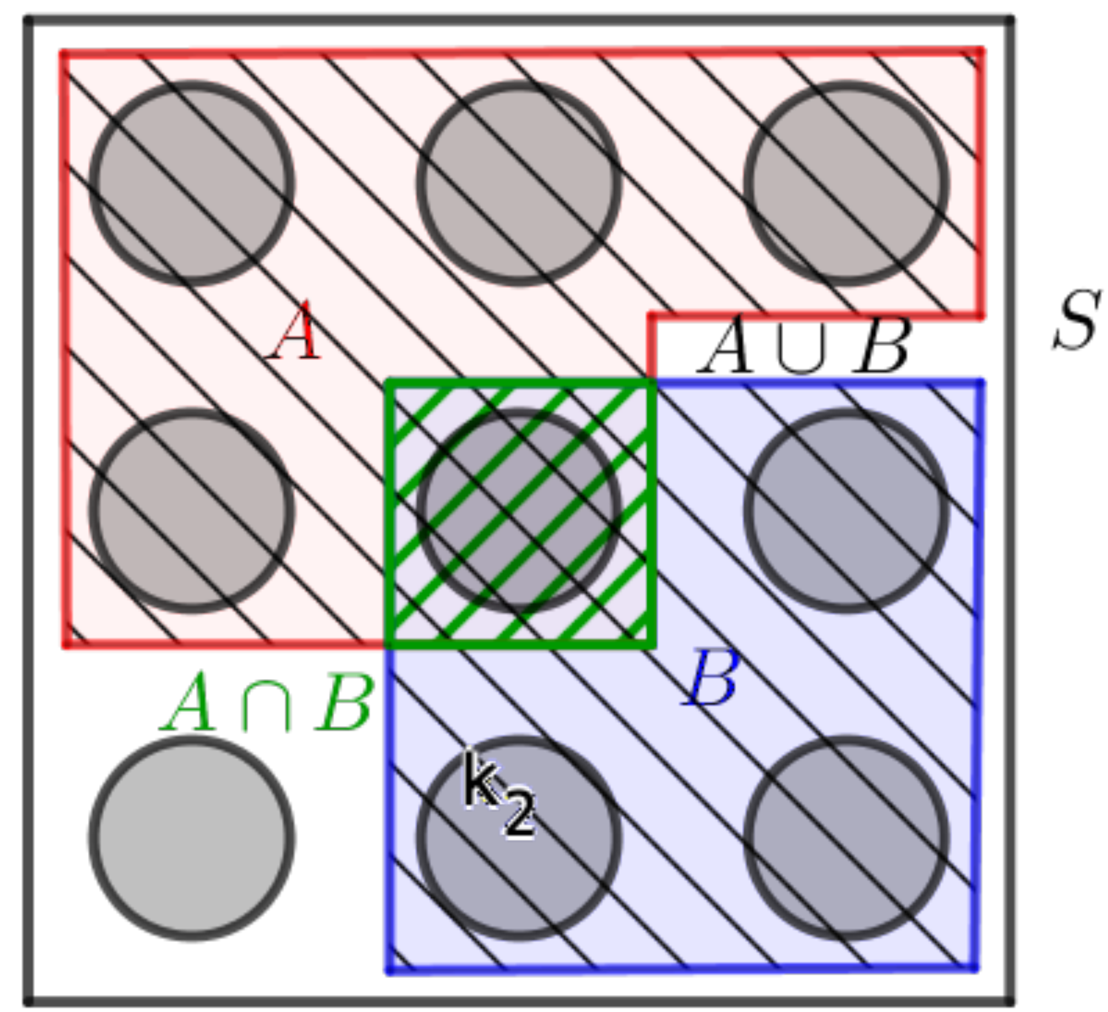


When the sample space is finite, we can visualise it with pebbles. If pebbles are of equal size, they are equally likely to be chosen

Sample Spaces

The union $A \cup B$ is the event that occurs iff (if and only if) **at least one** of A, B occurs. The **intersection** $A \cap B$ is the event that occurs iff **both** A and B occur.

The complement A^c occurs iff A does **not** occur.



De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Sample Spaces

Example 1.2.2 (Coin flips). A coin is flipped 10 times. Heads is H and Tails is T, a possible outcome (pebble) is $HHHTHHTTHT$, the sample space = set of all possible strings of length 10 of H-s and T-s.

Encode H as 1, T as 0, so the outcome is a sequence $(s_1, \ldots, s_{10})$ with $s_j \in \{0,1\}$, and the sample space = space of such sequences.

1. Let A_j = event that the j-th flip is Heads. For example,
$A_1 = \left\{ \left(1, s_2, \ldots, s_{10}\right) : s_j \in \{0,1\} \text{ for } 2 \leq j \leq 10 \right\}$.

2. Let B = event that **at least** one flip was Heads. Then $B = \bigcup_{j=1}^{10} A_j$

Sample Spaces

3. Let C = event that **all** one flips were Heads. Then $C = \bigcap_{j=1}^{10} A_j$

4. Let D = event that there were at least two consecutive Heads.

Then $D = \bigcup_{j=1}^{9} \left(A_j \cap A_{j+1} \right)$

Sample Spaces

Some other relationships between events:

A **implies** B = $A \subset B$ (A is a subset of B)

A and B are **mutually exclusive** = $A \cap B = \varnothing$

$A_1, \ldots, A_n$ are a **partition** of S =

$$A_1 \cup \ldots \cup A_n = S, \quad A_i \cap A_j = \varnothing, \ (i \neq j)$$

Naive definition of probability

Naive definition of probability

Definition 1.3.1 (Naive definition of probability). Let A be an event for an experiment with a finite sample space S. **Naive probability** of A is

$$P_{naive}(A) = \frac{|A|}{|S|} = \frac{\text{number of outcomes favorable to A}}{\text{total number of outcomes in S}}$$

In general, we have

$$P_{naive}(A^c) = \frac{|A^c|}{|S|} = \frac{|S| - |A|}{|S|} = 1 - \frac{|A|}{|S|} = 1 - P_{naive}(A)$$

Actually, this holds even beyond the naive definition.

Naive definition of probability

For our example from above, we have:

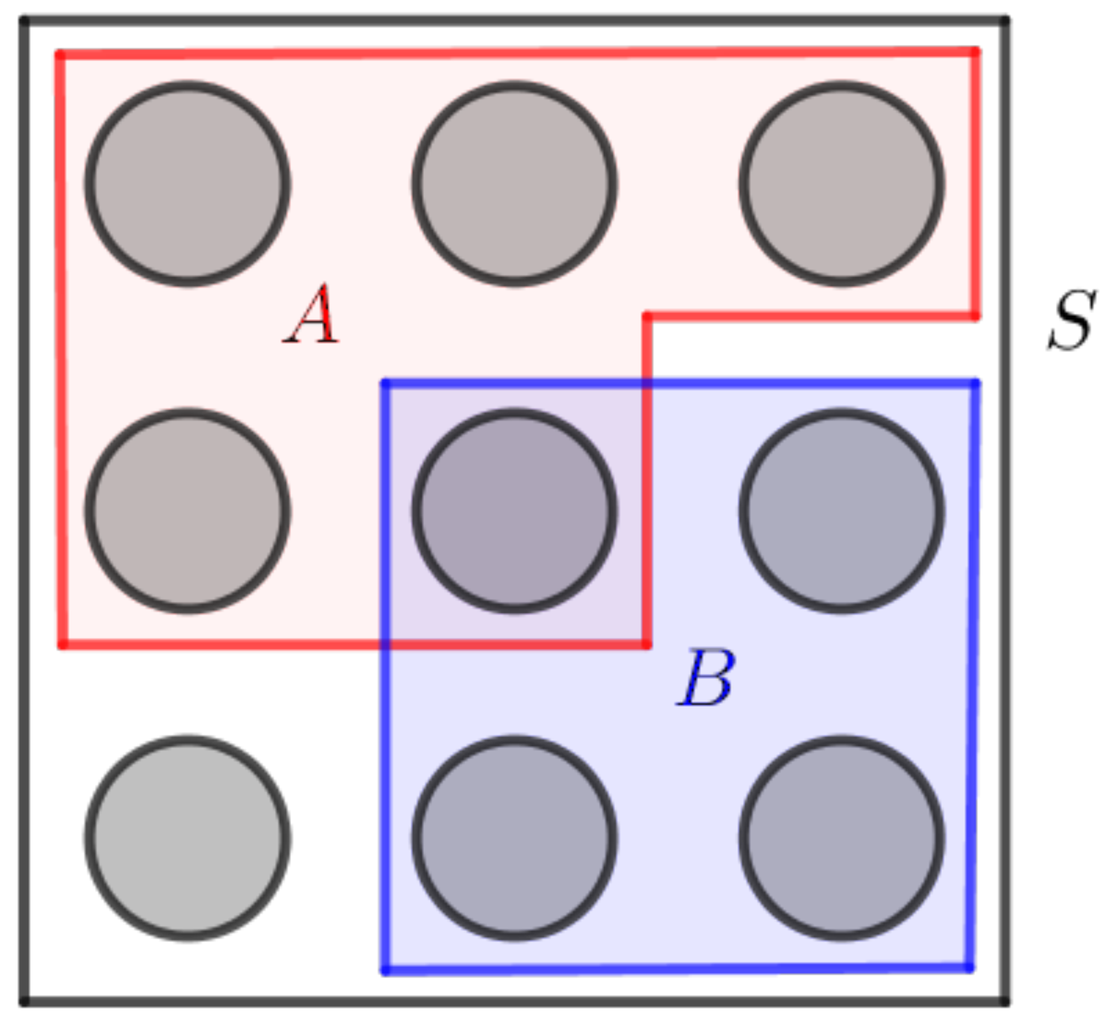
$$P_{naive}(A) = \frac{5}{9}, \quad P_{naive}(B) = \frac{4}{9}, \quad P_{naive}(A \cup B) = \frac{8}{9}, \quad P_{naive}(A \cap B) = \frac{1}{9}$$

And

$$P_{naive}(A^c) = \frac{4}{9}, \; P_{naive}(B^c) = \frac{5}{9}$$

$$P_{naive}((A \cup B)^c) = \frac{1}{9}$$

$$P_{naive}((A \cap B)^c) = \frac{8}{9}$$



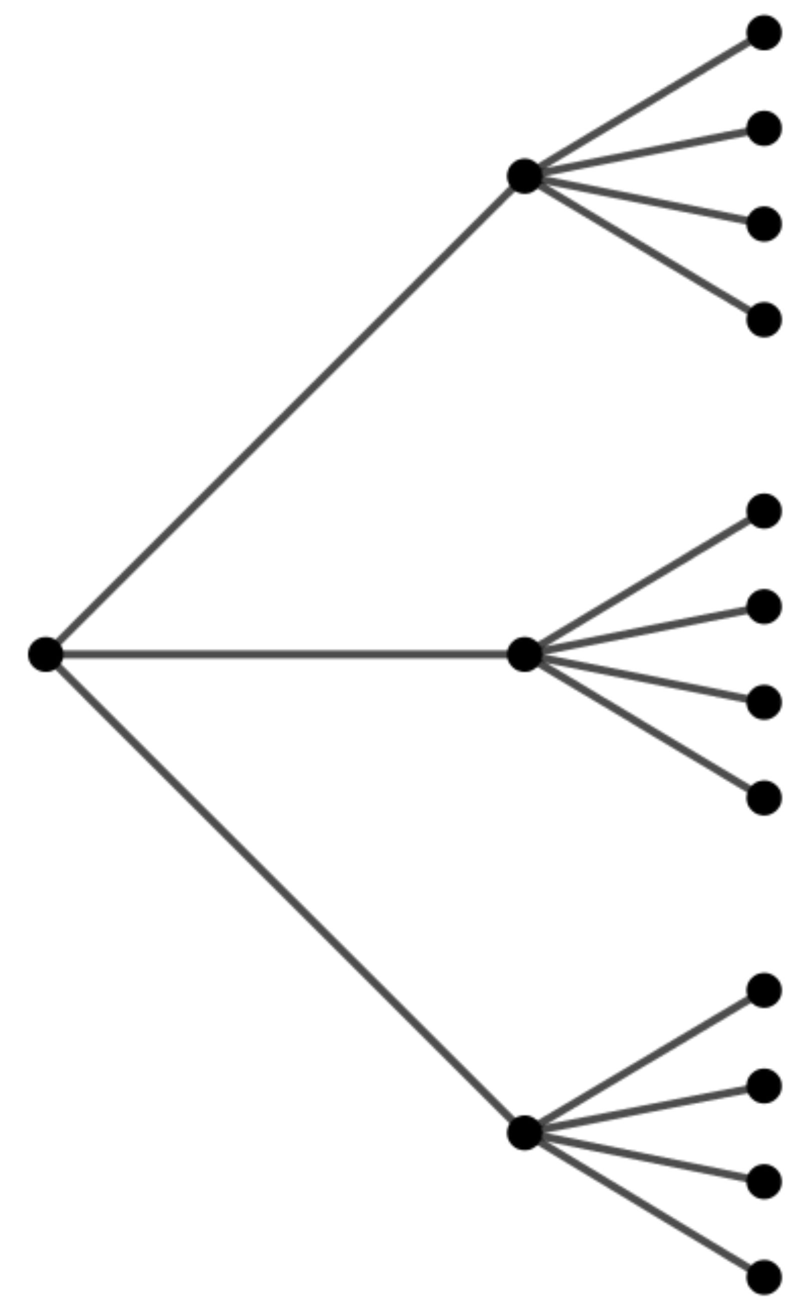
Counting rules

Counting rules

In some problems we can directly count the number of possibilities using **multiplication rile**.

Consider a compound experiment of two sub-experiments, A and B. A has a outcomes, B has b outcomes.

Then the compound experiment has $a \cdot b$ outcomes:



There is no requirement that A is performed before B!

Counting rules

Sampling with replacement. Consider n objects and making k choices of them, one at a time **with replacement** (choosing a certain object and "putting it back").

Theorem 1.4.7. There are n^k outcomes in such an experiment.

Hint: multiplication rule!

Counting rules

Sampling without replacement. Consider n objects and making k choices from them, one at a time **without replacement** (choosing a certain object ant not putting it back, so it can't be chosen again).

Theorem 1.4.8. There are $n(n-1)\ldots(n-k+1)$ outcomes in such an experiment for $1 \leq k \leq n$ and 0 outcomes for $k > n$.

Hint: multiplication rule!

Counting rules

Example: (Permutations and factorials). A **permutation** of 1,2,…,n is an arrangement of them in some order, e.g., 3,5,1,2,4 is a permutation of 1,2,3,4,5.

It follows from the previous counting rule (sampling without replacement) with $k = n$ that there are $n \cdot (n-1) \ \ldots \ 2 \cdot 1$ outcomes here. Denote that number $n!$ – **factorial** of n.

Counting rules

Example: (Birthday problem/paradox). There are k people in a class. Assume each person's birthday is **equally likely** to be any of the 365 days of the year (exclude Feb 29) and that people's birthdays are **independent** (we'll define that formally later). **Question:** What is the probability that **at least one** pair of people have the same birthday?

Counting rules

Example: (Birthday problem/paradox). There are k people in a class. Assume each person's birthday is **equally likely** to be any of the 365 days of the year (exclude Feb 29) and that people's birthdays are **independent** (we'll define that formally later). **Question:** What is the probability that **at least one** pair of people have the same birthday?

Solution: Let's count the complement instead – the number of ways to assign birthdays to k people such that **no** two people share a birthday. This is sampling with replacement, so $365 \cdot 364 \ldots (365 - k + 1)$ such choices, so

$$P(\text{no birthday match}) = \frac{365 \cdot 364 \ldots (365 - k + 1)}{365^k}$$

Counting rules

$$P(\text{at least 1 birthday match}) = 1 - \frac{365 \cdot 364 \ldots (365 - k + 1)}{365^k}$$