CTA200H Project

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1 Introduction

This project develops an analytic model for the capture of migrating planets into a mean motion resonance. The analytic model allows one to predict equilibrium orbital configuration reached by a pair of migrating planets as a function of the planets' relative migration rate and eccentricity damping strength. The analytic model will also allow you to determine the stability of the equilibrium orbital configuration and reveal that, for certain regions of parameter space, capture in resonance is only temporary because the equilibrium orbital configuration is not stable. This makes the analytic model for resonance capture a valuable tool when inferring the potential migration histories of systems observed to host resonant planets today.

1.1 Software

You will use the `celmech` package to develop an analytic model for a mean motion resonance, explore its dynamics, and analytic predictions to direct N-body integrations. This will involve the Python packages `rebound`, `reboundx`, and `celmech`. Each packages can be installed with `pip` from the command line or from each packages respective GitHub repository. The `celmech` documentation here and Jupyter notebook examples found on the GitHub repository of each package will come in handy. You'll probably also want to spend some time familiarizing yourself with some `sympy` basics, which come in handy when working with `celmech`.

1.2 Theory

The equations of motion for the three-body system consisting of a pair of planets of mass m_1 and m_2 orbiting a central star of mass M_* can be written in Hamiltonian form. The Hamiltonian of the system can be written as[1]

$$H(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) = H_{\mathrm{Kep},1}(\mathbf{r}_1, \mathbf{p}_1) + H_{\mathrm{Kep},1}(\mathbf{r}_2, \mathbf{p}_2) + \epsilon H_{\mathrm{int}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) \quad (1)$$

[1] For the exact definition of the coordinates $\mathbf{r}_i$ and $\mathbf{p}_i$ see, e.g., here.

where

$$H_{\mathrm{Kep,i}}(\mathbf{r},\mathbf{p}) = \frac{1}{2\mu_i}|\mathbf{p}_i|^2 - \frac{GM_* m_i}{|\mathbf{r}_i|} \tag{2}$$

$$\epsilon H_{\mathrm{int}} = -\frac{Gm_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_1|} + \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{M_*} \ , \tag{3}$$

and $\mu_i = m_i M_*/(m_i + M_*) \approx m_i$. We included the parameter ϵ when writing the Hamiltonian to indicate that $|\epsilon H_{\mathrm{int}}| \ll |H_{\mathrm{Kep,i}}|$.

The equations of motion can be reformulated by a canonical transformation from the Cartesian position and momentum vectors, $(\mathbf{r}_i, \mathbf{p}_i)$, to the canonical action variables

$$\Lambda_i = \mu_i \sqrt{G(M_* + m_i)a_i} \tag{4}$$

$$\Gamma_i = \mu_i \sqrt{G(M_* + m_i)a_i}(1 - \sqrt{1 - e_i^2}) \tag{5}$$

$$Q_i = \mu_i \sqrt{G(M_* + m_i)a_i}\sqrt{1 - e^2}(1 - \cos(I_i) \tag{6}$$

$$\tag{7}$$

and their conjugate angles $\lambda_i, \gamma_i = -\varpi_i$, and $q_i = -\Omega_i$. In these new variables, the Keplerian Hamiltonian simply becomes

$$H_{\mathrm{Kep,i}} = \frac{1}{2}G^2(M_* + m_i)^2 \mu_3^3 \Lambda_i^{-2} \ .$$

Show that you recover Kepler's law, $n_i^2 a_i^3 = G(M_* + m_i)$ **from this Hamiltonian**, where n_i is the mean motion of the ith planet.

The interaction component of the Hamiltonian does not have a correspondingly simple representation in terms of the canonical variables. However, it should be 2π-periodic in all of the canonical angle variables and we can rewrite it as a cosine[2] series:

$$\epsilon H_{\mathrm{int}} = \sum_{k_1,\ldots,k_6=-\infty}^{\infty} S_{\mathbf{k}}(\Lambda_i, \Gamma_i, Q_i) \cos(k_1\lambda_2 + k_2\lambda_1 + k_3\varpi_1 + k_4\varpi_2 + k_5\Omega_1 + k_6\Omega_2) \ . \tag{8}$$

Rotation and reflection symmetries of the problem imply any non-zero terms in the sum must have $\sum_{i=1}^{6} k_i = 0$ and $|k_5| + |k_6|$ must be even. Also, the amplitdues obey $S_{\mathbf{k}}(\Lambda_i, \Gamma_i, Q_i) \propto e_1^{|k_3|} e_2^{|k_4|} I_1^{|k_5|} I_2^{|k_6|}$ for small eccentricities and inclinations. Writing $\epsilon H_{\mathrm{int}}$ in this way allows us to express our equations of motion in terms of Fourier series as well. For example,

$$\dot{\Lambda}_1 = -\frac{\partial H}{\partial \lambda_1} = \sum_{k_1,\ldots,k_6=-\infty}^{\infty} k_2 S_{\mathbf{k}}(\Lambda_i, \Gamma_i, Q_i) \sin(\mathbf{k} \cdot \boldsymbol{\theta}) \tag{9}$$

[2] A Fourier series should generally contain both sine and cosine terms. Why don't sine terms appear in the Fourier expansion?

where we've written $\mathbf{k}\cdot\boldsymbol{\theta}$ as short-hand for the cosine arguments appearing in Equation (8).

At first glance, this hasn't really simplified anything since we have to contend with the six-fold infinite sum over values the k_i. However, because $\epsilon H_{\rm int}$ is small compared to the Keplerian portion of the Hamiltonian, the planets' orbits will be nearly Keplerian. We can try to obtain an approximate solution to the equations of motion by inserting the ansatz $\lambda_i = n_i(t-T_{i,0})+\epsilon\delta\lambda_i(t)$ along with $\Lambda_i = \Lambda_{i,0}+\epsilon\delta\Lambda_i(t),\ \Gamma_i = \Gamma_{i,0}+\epsilon\delta\Gamma_i(t),$ etc... into the right-hand sides of Equation (9) and the other equations of motion. If we then drop terms that are $\mathcal{O}(\epsilon^2)$ and higher, recalling $\epsilon H_{\rm int}$ is already $\mathcal{O}(\epsilon)$, and integrate with respect to time, we get the approximate solution:

$$\delta\Lambda_1(t) = \sum_{k_1,...,k_6=-\infty}^{\infty} -\frac{k_2 S_{\mathbf{k}}(\Lambda_{i,0},\Gamma_{i,0},Q_{i,0})}{k_1 n_2 + k_2 n_1}\cos(\mathbf{k}\cdot\boldsymbol{\theta}_0)\ . \tag{10}$$

and solutions for the other dynamical variables with a similar form.

There are at least a couple problems with the solution in Equation (10). First, any terms with $k_1, k_2 = 0$ cannot be treated in this manner since they lead to an infinite denominator. Such terms are called *secular* terms and give rise to the exchange of angular momentum between planets while their semi-major axes remain fixed (Why?). Second, any terms for which $k_1 n_2 + k_2 n_1 \approx 0$, will result in a solution that contradicts our assumption that $\delta\Lambda_i \sim \epsilon$ is small. The condition that $k_1 n_2 + k_2 n_1 \approx 0$ means the orbital periods are near an integer ratio, i.e., the planets are near a mean motion resonance (MMR).

Deriving a complete perturbative solution for our dynamical variables in the form of Eq. (10) is apparently not possible due to the appearance of secular and resonant terms. Instead, let's rewrite our Hamiltonian as $H_{\rm int} = H_{\rm slow} + H_{\rm fast}$ where $H_{\rm slow}$ consists the secular and resonant terms and $H_{\rm fast}$ contains the rest of the terms. We'll now try a slightly different perturbative solution of the form $z(t) = \bar{z}(t) + \delta z(t)$ where $z = (\lambda_1, ..., \Lambda_1, ...)$ and satisfies

$$\frac{d}{dt}\bar{z}(t) = [\bar{z}, H_{\rm Kep}(\bar{z}) + H_{\rm slow}(\bar{z})] \tag{11}$$

$$\frac{d}{dt}\delta z(t) = [\bar{z}, H_{\rm fast}(\bar{z})]\ . \tag{12}$$

If we can solve Equation (11) for $\bar{z}$, we can immediately obtain a solution for $\delta z(t)$ by inserting it into Equation (12). The advantage is that now Equation (11) only contains a limited number of important terms from our original 6-fold infinite sum.[3] Furthermore, the amplitudes of the $\delta z(t)$ are generally quite small and we can usually ignore the differences between $z(t)$ and $\bar{z}(t)$.

[3]Actually, $H_{\rm slow}$ should really still contain an infinite number of terms: there are infinitely many terms with $k_1 = k_2 = 0$ and, for planets near a $p:q$ MMR, one should include terms with $(k_1,k_2) = (p,-q),\ (2p,-2q),\ (3p,-3q), ...,$ etc. As we'll see, though, it usually suffices to approximate $H_{\rm slow}(\bar{z})$ with just a few terms with small $|k_i|$ value. Recall that the amplitudes $S_{\mathbf{k}} \propto e_1^{|k_3|}e_2^{|k_4|}$ so that large k terms make small contributions to the equations of motion.

2 Conservative Dynamics

We'll now build up an analytic model for the resonant dynamics of a two-planet system. You should use a set of resonant initial conditions from your RV fits for HD 45364 as a starting configuration.

2.1 (Semi)analytic Equations of Motion

We'll start by setting up a set of semi-analytic equations of motion for the HD 45364 system in the form of Equation (11) using `celmech`. After initializing a `rebound` simulation you can set up a `celmech` model using the `PoincareHamiltonian` class. The documentation details how to add resonant and secular terms up to various order in planets' eccentricities. (Note you can ignore inclination terms since you're modelling a coplanar system.)

1. Compare the time evolution computed using `celmech` to the time evolution computed via direct N-body simulation with `rebound` when including terms up to first, second, and third order in the planets' eccentricities. In particular, plot the time evolution of resonant angles $\theta_i = 3\lambda_2 - 2\lambda_1 - \varpi_i$, the eccentricities, e_i, and the period ratio P_2/P_1 over a few resonant libration periods.

For the rest of the project it is sufficient to work with simple models that only include terms that are first order in eccentricity. You may not get great quantitative agreement with N-body, but these simple models should capture all the relevant qualitative features of the dynamics.

2.2 A simpler model through canonical transformations

The `PoincareHamiltonian` model has 6 degrees of freedom (i.e., there are 6 canonical coordinates and 6 conjugate momenta, so Hamilton's equations give a 12 dimensional ODE.) We can derive a much simpler set of equations through a series of canonical transformations that will reduce the number of degrees of freedom.

1. Show that, if q_i are a set of canonical coordinates with conjugate momenta p_i, then the variables $Q_i = \sum_j A_{ij} q_j$ and $P_i = \sum_j (A^{-1})_{ji} p_j$ form canonical pairs as well (provided A is a non-singular matrix).

2. Show that, if (q, p) are a pair of canonical variables, then so are $(y, x) = (\sqrt{2p}\sin(q), \sqrt{2p}\cos(q))$. Note that the default `celmech` variables $(\eta, \kappa) = \sqrt{2\Gamma} \times (\sin(\gamma), \cos(\gamma))$.

3. Using `celmech` 's `CanonicalTransformation` class, create a new Hamil-

tonian model where the new angle variables are

$$\theta_1 = 3\lambda_2 - 2\lambda_1 - \varpi_1 \tag{13}$$
$$\theta_2 = 3\lambda_2 - 2\lambda_1 - \varpi_2 \tag{14}$$
$$\ell = 3\lambda_2 - 2\lambda_1 \tag{15}$$
$$\psi = \lambda_1 - \lambda_2 \tag{16}$$
$$\phi_1 = \; = -\Omega_1 \tag{17}$$
$$\phi_2 = \; = -\Omega_2 \; . \tag{18}$$

You can use the class method `from_poincare_angles_matrix` to efficiently generate this transfromation. The new Hamiltonian only depends on the angles θ_1 and θ_2 so that the momenta conjugate to the other angles are conserved. (You can even use the keyword argument `new_qp_pairs` to match the notation used here.) Show that the total angular momentum is one of the new conserved quantities. (Hint: the angular momentum per unit mass of a Keplerian orbit is given by $\sqrt{GMa(1-e^2)}$)

4. Use a root-finding method to solve for an equilibrium configuration such that $\dot{\theta}_i = 0$ along with their conjugate momenta which we'll call p_i. In other words, solve for the values $(\theta_1, \theta_2, p_1, p_2)$ that give $\dot{\theta}_i = 0$ and $\dot{p}_i = 0$. Your equations of motion should also depend on the two conserved momenta conjugate to ℓ and ψ, which we'll call L and Ψ. For the values of these quantities, you can simply choose the initial values inherited from the original `rebound` simulation. (Hint: it may be helpful to use the `do_reduction=True` keyword when applying your canonical transformation or otherwise use the function `reduce_hamiltonian` contained in `celmech.hamiltonian`.)

5. Initialize a `rebound` simulation at the equilibrium you found to see how close your analytic solution is to an actual equilibrium by checking the time evolution of $\theta_i(t)$ in the rebound simulation. (This should be relatively straightforward by combining the `new_to_old_array` method of your `CanonicalTransformation` and the `to_Simulation` method of the `Poincare` class.)

3 Dissipative dynamics

Now we'll consider the effects of dissipation on the resonant dynamics. We'll consider the effect of adding migration forces and eccentricity damping forces according to

$$\frac{d \ln e_i}{dt} = -\frac{1}{\tau_{e,i}} \tag{19}$$
$$\frac{d \ln a_i}{dt} = -\frac{1}{\tau_{a,i}} \tag{20}$$

1. Derive equations for $d\Lambda_i/dt$, $d\eta_i/dt$, and $d\kappa_i/dt$ under the effect of Equations (19) and (20).

2. Using the derived equations, include the effects of dissipation in your `celmech` model. You can create a system of ODEs by combining the `flow_func` property of your `PoincareHamiltonian` object with your own user-defined function that implements the effect of dissipation. (I recommend first creating a symbolic representations of the dissipative flow with `sympy`, then using `sympy.lambidfy` to turn it into a function.) Numerically integrate the equations of motion, including dissipation. You can use, e.g., `scipy`'s `solve_ivp` to solve for the evolution of the system. Compare your results to a direct N-body simulation using `rebound` and `reboundx`. You should choose time scales $\tau_{a,i}$ and $\tau_{e,i}$ as you have in previous simulations to roughly reproduce the eccentricities measured from you N-body fits.

3. In a similar fashion, include dissipative dynamics in the reduced Hamiltonian created in Section 2.2. (I suggest making extensive use of `sympy.diff` and the chain rule to do this semi-automatically.) Compare integration results with N-body in a figure.

4 Analytic Determination of Equilibrium

Note: **Due to time constraints, this section won't be required as part of the assignment. However, the true value of deriving our analytic model comes from the results of this section: our model will allow us to determine equilibrium configurations by root-finding rather numerical integration of the equations of motion. This is much more efficient, and will allow you to map expected outcomes over a broad range of parameter space.**

Our reduced Hamiltonian model depends on four dynamical variables, $(\theta_1, \theta_2, p_1, p_2)$, as well as two quantities L and Ψ, which are conserved by the Hamiltonian but evolve under the effects of dissipation. You might expect to solve for the equilibrium configuration reached by the migration planets by solving the equations of motion for

$$\frac{d}{dt}(\theta_1, \theta_2, p_1, p_2, L, \Psi) = 0$$

given a set of $\tau_{a,i}$ and $\tau_{e,i}$. However, these equations will not have a solution in general.

You may have noticed in your migration simulations that the planets can continue migrating together inward or outward together after they settle into resonance and reach fixed eccentricities. In other words, the system remains in a fixed configuration except for an overall scaling factor. This suggests we can find an equilibrium in terms of new variables that account for this overall scale factor. In this section we'll re-write our equations in terms of new variables $p_i' = p_i/L$ and $\Psi' = \Psi/L$ scaled by the total angular momentum of the system.

1. Show that if a set of canonical variables (q_i, p_i) obey Hamilton's equations for a Hamltonian $H(q,p)$, then the variables $(q'_i, p'_i) = (q_i, p_i/\eta)$ obey Hamilton's equations for the Hamiltonian $H'(q', p') = \frac{1}{\eta}H(q', \eta p')$.

2. Derive a new Hamiltonian governing the dynamics of the variables

$$(\theta'_1, \theta'_2, p'_1, p'_2, \Psi') = (\theta_1, \theta_2, p_1/L, p_2/L, \Psi/L)$$

by using the transformation described in Step 1. Show that the new Hamiltonian takes the form $H'(q', p') = L^{-3}h(q', p')$ for some function h.[4]

3. Derive expressions for evolution of the variables $p'_i = p_i/L$ and $\Psi' = \Psi/L$ under the effects of dissipation. The resulting equations can be expressed in terms of p'_i and Ψ' as well as the timescales $\tau_{a,i}$ and $\tau_{e,i}$.

4. Combining the results of Step 2 and 3, show our equations of motion take the form

$$\frac{d}{dt}z = \frac{1}{L^3}[z, h(z)] + f_{\mathrm{dis}}(z; \tau_{e,i}, \tau_{a,i}) \tag{21}$$

where I've used introduce the notation $z = (\theta'_1, \theta'_2, p'_1, p'_2, \Psi')$.

5. Note that these equations still don't have an equilibrium solution since the pre-factor L^{-3} will be evolving with time under the effects of dissipation. Practically, this occurs because we've chosen fixed time scales for the dissipative effects, while the orbital periods of the planets will be increasing or decreasing uniformly. However, provided the dissipative timescales remain much longer than the orbital periods, the practical change in the equilibrium resonant state of the planet pair will be negligble. If we instead allow the time-scales evolve so that to remain constant fractions of the orbital periods we can find and exact equilibrium. Argue that this corresponds to replacing the timescales with $\tau_i \to \tau_i \times (L/L_0)^3$

6. Finally, we can write our equations of motion as

$$\frac{d}{dt}z = \frac{1}{L^3}\left([z, h(z)] + L_0^3 f_{\mathrm{dis}}(z; \tau_{e,i}, \tau_{a,i})\right)$$

and we can solve the expression in parentheses for the value of $z = z_{\mathrm{eq}}$ that gives $\frac{d}{dt}z = 0$.

[4] The disturbing function terms of the Hamiltonian generated by `celmech` include parameters $\Lambda_{i,0}$ and $a_{2,0}$ representing reference values of the original momentum variables Λ_i and the outer planet's semi-major axis, respectively. These parameters should be rescaled too, as $\Lambda_{i,0} \to \Lambda_{i,0}(L/L_0)$ and $a_{2,0} \to a_{2,0} \times \left(\frac{L}{L_0}\right)^2$.