SHORT QUESTIONS: MARKING SCHEME

1. There are several exoplanets observed in the Gliese 876 system $\left(M_G = 0.334 \pm 0.03\, M_{\odot}\right)$ as given in the following table,

Gliese System	Mass	Semi Major Axis (au)
Gliese 876 b	2.2756 ± 0.0045M_J	0.2083 ± 0.000020
Gliese 876 c	0.7142 ± 0.0039 M_J	0.1296 ± 0.000024
Gliese 876 d	6.83 ± 0.40 $M_{\oplus}$	0.0208 ± 0.00000015
Gliese 876 e	14.6 ± 1.7 $M_{\oplus}$	0.3343 ± 0.0013

where $M_{\odot}$ is mass of Sun, M_J is mass of Jupiter ($M_J = 1.89813 \times 10^{27}\ kg$), and $M_{\oplus} is$ mass of Earth.

Find, if any of the exoplanets of Gliese 876 system have resonant orbits (their synodic period is a multiple integer of one of the periods).

Answer and Marking Scheme:

1	**Determine planet periods**	**50**
	Synodic period of planet 1 and 2 (assume $P_1 < P_2$) is $\frac{1}{P_S} = \frac{1}{P_1} - \frac{1}{P_2} \quad \rightarrow \quad P_S = \frac{P_1 P_2}{P_2 - P_1}$ Resonant orbit happens if $P_S = mP_1 \quad \rightarrow \quad \frac{P_1 P_2}{P_2 - P_1} = mP_1 \quad \rightarrow \quad m = \frac{P_2}{P_2 - P_1} > 1, \text{integer}$ Using the data of the Gliesean system we have	

Gliese System	Mass (kg)	Semi Major Axis (m)	Period (Earth, days) $P = 2\pi\sqrt{\frac{a^3}{GM_G}}$
Gliese 876 star	6.64359×10^{29}		
Gliese 876 b	4.31938×10^{27}	3.1161×10^{10}	60.07568
Gliese 876 c	1.35564×10^{27}	1.9388×10^{10}	29.48304
Gliese 876 d	4.079×10^{25}	3.1116×10^{9}	1.8956
Gliese 876 e	8.7194×10^{25}	5.0011×10^{10}	122.1432

2	**Calculate resonant orbit for each planet**	
	From the results above we see that	
	• Gliese 876 c and Gliese 876 b have resonant orbit $$m = \frac{P_b}{P_b - P_c} = 1.96 \approx 2$$	**25**
	• Gliese 876 b and Gliese 876 e have resonant orbit $$m = \frac{P_e}{P_e - P_b} = 1.9679 \approx 2$$	**25**

2. Volcanic activity on Io which has a synchronous orbital period, was proposed to be the result of a tidal heating. Resultant tidal force is the difference in gravitational force experienced by the near and far sides of a satellite orbiting a parent body. Measurements of the surface distortion of Io via satellite radar altimeter mapping indicate that the surface rises and falls by up to 100m during one-half orbit. Only the surface layers will move by this amount. Interior layers within Io will move by a smaller amount, thus we assume that on average, the entire mass of Io is moved through 50m. Mass of Jupiter and Io are $M_J = 1.89813 \times 10^{27}\ kg$ and $m_{Io} = 8.931938 \times 10^{22}\ kg$ respectively. Given that the perijove and apojove distance are r_{peri} = 420000 km and r_{apo} = 423400 km respectively, the orbital period 152853 s, and the radius of Io is R_{Io} = 1821.6 km, calculate the *average power* of this tidal heating of Jupiter's gravitational force on Io. **Hint :** you can use the following approximation $(1+x)^n \approx 1 + nx$ for small *x*.

Answer and Marking Scheme:

1	**Calculate the tidal force** The tidal force, $F_{tidal} = F(r - R_{Io}) - F(r + R_{Io})$ $F_{tidal} = -\frac{GM_J m_{Io}}{(r - R_{Io})^2} + \frac{GM_J m_{Io}}{(r + R_{Io})^2}$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2\left(1 - \frac{R_{Io}}{r}\right)^2} + \frac{GM_J m_{Io}}{r^2\left(1 + \frac{R_{Io}}{r}\right)^2}$	

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\frac{1}{\left(1-\frac{R_{Io}}{r}\right)^2} + \frac{GM_J m_{Io}}{r^2}\frac{1}{\left(1+\frac{R_{Io}}{r}\right)^2}$$

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\left[\frac{1}{\left(1-\frac{R_{Io}}{r}\right)^2} - \frac{1}{\left(1+\frac{R_{Io}}{r}\right)^2}\right]$$

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\left[\frac{1}{\left(1-2\frac{R_{Io}}{r}\right)} - \frac{1}{\left(1+2\frac{R_{Io}}{r}\right)}\right]$$

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\left[\frac{\left(1+2\frac{R_{Io}}{r}\right)-\left(1-2\frac{R_{Io}}{r}\right)}{\left(1-2\frac{R_{Io}}{r}\right)\left(1+2\frac{R_{Io}}{r}\right)}\right]$$ **40**

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\left[\frac{\left(1+2\frac{R_{Io}}{r}-1+2\frac{R_{Io}}{r}\right)}{\left(1-4\left(\frac{R_{Io}}{r}\right)^2\right)}\right]$$ **10**

$$F_{tidal} = -\frac{GM_J m_{Io}}{r^2}\left[\frac{\left(4\frac{R_{Io}}{r}\right)}{\left(1-4\left(\frac{R_{Io}}{r}\right)^2\right)}\right]$$

The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored

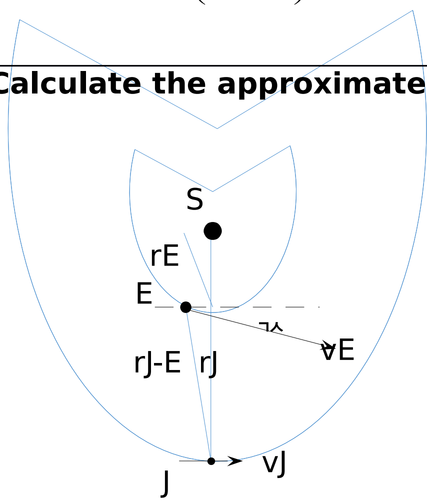
, so that

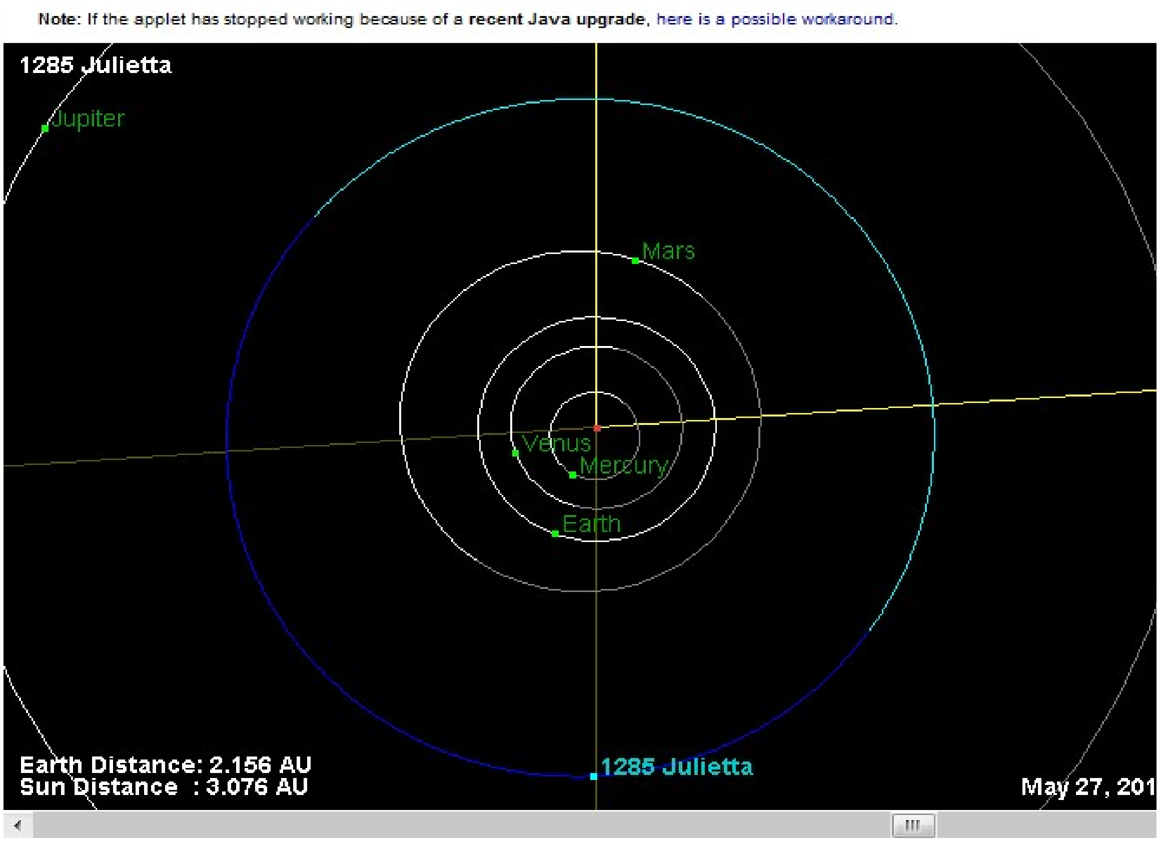
	$$F_{tidal} = -4\frac{GM_J m_{Io} R_{Io}}{r^3}$$	
2	**Calculate the difference of the tidal force** The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove (furthest from Jupiter) $\Delta F = \left\|F_{tidal}(r_{peri}) - F_{tidal}(r_{apo})\right\|$ $= 4GM_J m_{Io} R_{Io}\left[\frac{1}{r_{peri}^3} - \frac{1}{r_{apo}^3}\right] = 2.66\times10^{19}\ N$	**15** **15**
3	**Calculate the average power of work done** Hence, the average power of the work done on the rock during one-half orbit is $\overline{P} = \frac{\Delta F \cdot d}{t_{Io}/2}$ $= 1.74\times10^{16}\ W$	**10** **10**

3. On 27 May 2015 at 02:18:49, the occultation of the star HIP 89931 ($\delta -¿$ Sgr) by the asteroid 1285 Julietta was observed from Borobudur temple, it lasted for only 6.2 s. Assuming that Earth's orbit is circular and the orbit of Julietta is on the ecliptic plane, find the approximate size of asteroid Julietta, if the semi major axis of Julietta $a=2.9914$ au, and at occultation, the distance of Julietta to the Sun and the Earth is 3.076 au and 2.156 au, respectively.

Answer and Marking Scheme:

1	**Calculate the Earth revolution speed** Earth revolution speed is $v_E = \frac{2\pi\bar{r}_E}{T_E} = \frac{2\pi a_E}{T_E} = 29805\ m/s$	**20**
2	**Calculate Julietta revolution speed**	**25**

	Julietta revolution is (since the given data is semimajor axis, the orbit of Julietta is ellipse) $v_J = \sqrt{GM\left(\frac{2}{r_J} - \frac{1}{a}\right)} = 16742{,}9\ m/s$	
3	**Calculate the approximate dimension of Julietta**  From the geometry $\cos\alpha = \frac{r_E^2 + r_J^2 - r_{J-E}^2}{2 r_E r_J} = 0.995$ Thus, Julietta's relative projected speed to Earth is $v_{J,rel} = v_E - v_J \cos\alpha = 13984.16\ m/s$ Hence, the approximate dimension of Julietta is $l_J = v_{J,rel} t_{occil} = 86701.8\ m$	**40** **15**



http://ssd.jpl.nasa.gov/sbdb.cgi?sstr=1285%20julietta;old=0;orb=1;cov=0;log=0;cad=0#orb

4. Recently, an observer is using a hypothetical far-infrared Earth-sized telescope with wavelength in the range of 20 to 640 μm. He finds a static and neutral supermassive black hole with a mass of 21 billions (2.1 x 10^{10}) $M_\odot$. Determine the maximum distance from the observer to this black hole that can be measured with his telescope.

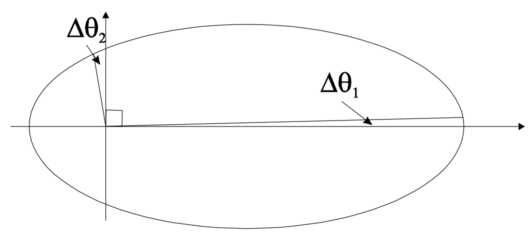
Answer and Marking Scheme:

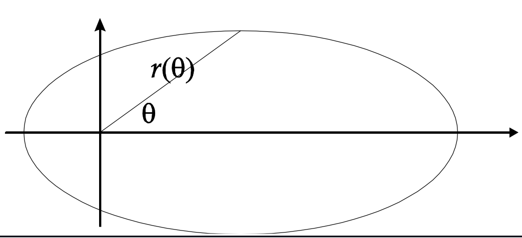
1	**Identify the relationship between the telescope's resolution and the angular size of black hole** This problem can be solved by using relationship between the telescope's resolution and the angular size of black holes which can be seen in this equation:	**25**

	$\phi_{telescope} \le \phi_{blackhole} \quad \Rightarrow \quad 1.2\dfrac{\lambda}{D_{telescope}} \le \dfrac{\text{Diameter of event horizon}}{\text{Observer Distance to galatic' s the center}}$ $1.2\dfrac{\lambda}{D_{telescope}} \le \dfrac{2R_{blackhole}}{d}$	
2	**Identify the relation for determining the distance** Then, it is assumed that the radius of black hole is the Schwarszchild radius; and the telescope is also Earth-sized telescope. Therefore: $1.2\dfrac{\lambda}{2R_{Earth}} \le \dfrac{2R_{Schwarzchild}}{d}$ $d \le \dfrac{4\times R_{Schwarzchild}\times R_{Earth}}{1.2\times\lambda} \quad \Rightarrow \quad d \le \dfrac{4\times M_{Black\ Hole}\times R_{Earth}}{1.2\times M_{Sun}\times\lambda}$ $d \le \dfrac{4\times 2.1x10^{10} M_{Sune}\times R_{Earth}}{1.2\times M_{Sun}\times\lambda} \Rightarrow \quad d \le \dfrac{8.4x10^{10}\times R_{Earth}}{1.2\times\lambda}$	5 0
3	**Calculating the maximum-observer distance** To find the maximum distance, then we should use the minimum λ. Therefore: $d_{\max} = \dfrac{8.4x10^{10}\times R_{Earth}}{1.2\times\lambda_{\min}} = \dfrac{8.4x10^{10}\times 6.3708x10^{6}}{1.2\times 20x10^{-6}} m = 2.230x10^{22} m$	2 5

5. An observer is trying to determine an approximate value of the orbital eccentricity of a man-made satellite. When it is at apogee, in a short time the satellite moved by $\Delta\theta_1 = 2'44$. Meanwhile, when the position of the satellite is perpendicular to its major axis (true anomaly is equal to 90°), within the same duration of time, it is observed that the satellite moved by $\Delta\theta_2 = 21'17$. Find an approximate value of the eccentricity of the satellite's orbit.

Answer and Marking Scheme:

<table>
<tr><td>1</td><td>Identify expression for $r(\theta)$.
We know that $r(\theta) = \frac{a(1-e^2)}{1-e\cos\theta}$. Hence


$$r(0) = \frac{a(1-e^2)}{1-e} = a(1+e)$$
$$r\left(\frac{\pi}{2}\right) = \frac{a(1-e^2)}{1-e\times 0} = a(1-e^2)$$

</td><td>20</td></tr>
<tr><td>2</td><td>Estimate area of sectors using area of triangles.
Estimate the area of swept out sectors as the area of sectors of circles
$A(S_1) \approx 0.5 \times \Delta\theta_1 \times (r(0))^2 = \Delta\theta_1 (a(1+e))^2$</td><td>25</td></tr>
</table>

	$A(S_2) \approx 0.5 \times \Delta\theta_2 \times \left(r\left(\frac{\pi}{2}\right)\right)^2 = \Delta\theta_2 \left(a\left(1-e^2\right)\right)^2$	
3	**Use Kepler's second law to find a relation between the ratio** $\frac{\Delta\theta_1}{\Delta\theta_2}$ **and the eccentricity** e **.** Use Kepler's second law to obtain that $A(S_1) = A(S_2)$ $\quad \Delta\theta_1 \times (a(1+e))^2 = \Delta\theta_2 \times \left(a\left(1-e^2\right)\right)^2$ Thus, $\frac{\Delta\theta_1}{\Delta\theta_2} = \left(\frac{1-e^2}{1+e}\right)^2 = (1-e)^2 = \frac{2'44"}{21'17"} \approx 0.12843$	**40**
4	**Obtain an estimate value of the eccentricity.** Thus, the eccentricity is $e \approx 1 - \sqrt{0.12843} = 0.64163$	**15**

6. Before running an observation, a radio telescope is pointed to a point-source calibrator that has a flux density of 21.86 Jy. However, the measured flux density is 14.27 Jy. If the observation was being made at an elevation of 35 degrees above the horizon, estimate the zenith atmospheric opacity, τ_z .

Answer and Marking Scheme:

1	Atmospheric opacity τ_A can be determined from the measured flux, that is: $S_{meas} = \exp(-\tau_A)\, S_{real}$ $14.27 = \exp(-\tau_A) \times 21.86 \Rightarrow \exp(-\tau_A) = 14.27/21.86 = 0.65$ Then $\tau_A = -\ln(0.65) = 0.43$	**50**
2	Now we have: $\tau_A = \tau_Z \sec z$ $\tau_z = \tau_A \cos z$ $= 0.43 \times \cos(90-35)° = 0.25$ where z is zenith angle.	**50**

7. A cluster of galaxies has a total mass M. A galaxy with mass m is expected to escape from this cluster. Determine the density of the cluster.

Answer and Marking Scheme:

1	This problem will consider as two-body problem with the masses *M* (globular cluster mass) and *m* (galaxy mass).	**10**
2	We have the total energy of this system: $E_T = E_K + E_P = \frac{mV^2}{2} - \frac{GMm}{R}$	**20**
3	Where *V* is velocity of galaxy and *R* radius of globular cluster Galaxy will escape from globular cluster if total energy, $E_T = 0$	**10**
4	$E_T = 0 \rightarrow \frac{mV^2}{2} - \frac{GMm}{R} = 0$ ▭ $V^2 = \frac{2GM}{R}$	**20**
5	By using Hubble's law : $V = HR$	**5**
6	We have $H^2R^2 = \frac{2GM}{R} = \frac{2G\left(\frac{4\pi}{3}R^3\rho\right)}{R}$ ▭ $\rho = \frac{3H^2}{8\pi G}$	**20**
7	$G = 6{,}6738 \times 10^{-8}\, N\, m^2 kg^{-2}$ $H = 70\, km\, s^{-1}\, Mpc^{-1}$ Then $\rho = \frac{3H^2}{8\pi G} = 9.1989 \times 10^{-27}\, kg/m^3$	**15**

8. A strong radio signal from a celestial body has been observed as a burst with very short duration of 700 μs. The observed flux density at a frequency of 1660 MHz is measured to be 0.35 kJy. If the distance of the source is known to be 2.3 kpc, estimate the brightness temperature of this source.

Answer and Marking Scheme:

1	During 700 μs, the radio wave travels about $r = c\,t = 3 \times 10^{10} \times 700 \times 10^{-6} = 2.1 \times 10^{7}$ cm. The region from where the burst originates must be no larger than the distance that light can travel during the duration of the burst. So we estimate that r is the size of the source. ($r = 2.1 \times 10^{7}$ cm, $R = 2.3$ kpc) Flux density observed at 1660 MHz is $S_{1660MHz} = 0.35$ kJy $= 0.35 \times 10^{3} \times 10^{-23}$ ergs s^{-1} cm^{-2} Hz^{-1} $= 3.5 \times 10^{-21}$ ergs s^{-1} cm^{-2} Hz^{-1}	**20**
2	The solid angle subtended by this source of radiation is $\Omega = \pi\,(r/R)^2 = 3.14\,(2.1 \times 10^{7}/2.3 \times 10^{3} \times 3.086 \times 10^{18})^2$ $= 2.75 \times 10^{-29}$ sr	**30**
3	The flux density is related to the total brightness by the relation: $S_{1660MHz} = B_{1660MHz}\,\Omega$ while at this frequency, the total brightness can be approximated from the Rayleigh-Jeans formula: $B_{1660MHz} = 2kT_b\nu^2/c^2$ where T_b is the brightness temperature. Then we have: $T_b = S_{1660MHz}c^2/2k\nu^2\Omega$ $T_b = [3.5 \times 10^{-21} \times (3 \times 10^{10})^2]/[2 \times (1.38 \times 10^{-16}) \times (1660 \times 10^{6})^2 \times (2.75 \times 10^{-29})]$ $= 1.5 \times 10^{26}$ K	**20** **30**

9. Assume that the Sun is a perfect blackbody. Venus is also assumed to be a blackbody, with temperature T_V, radiating about as much energy as it receives from the Sun at its orbital distance of 0.72 au. Suppose that at closest approach to Earth, Venus has an angular diameter of about 66 arcsec. What is the flux density of Venus at closest approach to Earth as observed by a radio telescope at an observing frequency of 5 GHz.

Answer and Marking Scheme:

1	Solar luminosity is $L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$ $= 4 \times 3.14 \times (6.96 \times 10^{10})^2\ (5.67 \times 10^{-5})(5780)^4$ ergs/sec $= 3.9 \times 10^{33}$ ergs/sec	**10**
2	Flux of the Sun intercepted by Venus is $L_V = (A_{V(projected)}/A_{(sphere,0.72au)}) \times L_\odot$ Also, $L_V = 4\pi R_V^2 \sigma T_V^4$ So, $4\pi R_V^2 \sigma T_V^4 = (\pi R_V^2/4\pi d^2) \times L_\odot = (\pi R_V^2/4\pi d^2) \times 4\pi R_\odot^2 \sigma T_\odot^4$	**30**
3	Therefore, $T_V^4 = (R_\odot^2/4d^2) \times T_\odot^4 = [(6.957 \times 10^5 \text{ km})^2/4 \times (0.72 \times 149.6 \times 10^6)^2] \times (5780)^4$ $= (483998490000/46407499776000000) \times (5780)^4 = (4.84 \times 10^{11}/4.64 \times 10^{16}) \times (5780)^4$ $= 1.0429316216908190111241385226915e\text{-}5 \times 1114577187760656$ $= 1.04 \times 10^{-5} \times 1.11 \times 10^{15}$ $= 11624277939.308134327737156481455 = 1.1624 \times 10^{10}$ $T_V = 328.5$ K	**20**
4	The flux density of Venus at closest approach to Earth at an observing frequency of 5 GHz: (1″ = 4.84×10^{-6} rad) $S_{5GHz} = B_{5GHz}\,\Omega = (2kT_b\nu^2/c^2)\,\Omega \qquad (\because T_b = T_V)$ $= (2 \times 1.38 \times 10^{-16} \times 328.5 \times (5 \times 10^9)^2 \times \pi\,(66 \times 4.84 \times 10^{-6})^2)/\,(3 \times 10^{10})^2$ $= 8.07 \times 10^{-22}$ ergs s^{-1} cm^{-2} Hz^{-1} $= 80.7$ Jy	**40**

10. A molecular cloud is known to have a temperature T = 115 K. If interstellar molecules (assumed spherical) have masses of about 5 × 10^{-23} g and radii of 3.5 Å and are in thermal equilibrium with the surrounding gas, can you estimate the frequency they will radiate? Describe your answer mathematically.

Answer and Marking Scheme:

1	The molecules rotate with angular velocity ω and for spherical molecules, their moment of inertia is $I = ⅖ mr^2$, i.e., $I = ⅖ mr^2 = ⅖ (5 \times 10^{-23}\ g)(3.5 \times 10^{-8}\ cm)^2 = 2.45 \times 10^{-38}\ g\ cm^{-2}$	**30**
2	In thermal equilibrium, we have: $½ I \omega^2 = 3/2\ kT$	**30**
3	So we can estimate: $\omega = (3kT/I)^{1/2} = (3 \times 1.38 \times 10^{-16}\ 115/2.45 \times 10^{-38})^{1/2} = 1.39 \times 10^{12}\ s^{-1}$ $\nu = \omega/2\pi = 2.218 \times 10^{11}$ Hz = 221.8 GHz that is, in millimeter waves	**40**

11. Consider a star with its mass density is inversely proportional to the radial distance from the center of the star with the factor of proportionality ¿$5.0\times 10^{13}\,kg/m^2$. If the escape velocity of any objects to leave the star is $v_0 = 1.5\times 10^4\,m/s$, calculate the mass of the star.		
	Answer	
	$\frac{1}{2}mv_0^{\ 2} - \frac{GmM}{R} = 0 \rightarrow R = \frac{2GM}{v_0^{\ 2}}$	30
	$\Delta m(r) = \rho 4\pi r^2 \Delta r = 4\pi\alpha r\Delta r \rightarrow M \ = 2\pi\alpha R^2$ $(dm(r) = \rho 4\pi r^2 dr = 4\pi\alpha r dr \rightarrow M \ = 2\pi\alpha R^2)$	20
	$M = \alpha 2\pi R^2 = \alpha 2\pi \frac{4G^2M^2}{v_0^{\ 4}} \rightarrow M = \frac{v_0^{\ 4}}{8\pi\alpha G^2}$	20
	$M = \frac{v_0^{\ 4}}{8\pi\alpha G^2} = \frac{(1.5\times 10^4\,ms^{-1})^4}{8\pi(5\times 10^{13}\,kgm^{-2})(6.67\times 10^{-11}\mathrm{kg^{-1}m^3s^{-2}})^2} = 9.1\times 10^{21}\,kg$	30

12.A $1 GeV$ energetic proton propagates out from the Sun toward the Earth. What is the travel time of the proton as seen from the Earth?		
	Answer	
	$E_0 = m_0 c^2 = 941\, MeV$	20
	$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{K}{E_0} + 1 = \frac{1000\, MeV}{941\, MeV} + 1 = 2.062$	20
	$\Rightarrow 1-\frac{v^2}{c^2} \approx \frac{1}{4}$ ▭ $v \approx 0.87\, c$	30
	$Elapse\, time = \frac{Earth - Sun\, distance}{speed} = \frac{1.50 \times 10^{11}\, m}{0.87 \times (3 \times 10^8)\, m/s} = 9.6\, m$	30

13. Two protons collide to produce deuteron, positron, and neutrino in the center of a star. Write down the reaction equation and calculate the minimum temperature needed for this reaction. Assume that the neutrino mass is negligible.		
	Answer	
	$+¿+v_e$ $p+p+\Delta E \rightarrow D+e^{¿}$	20
	m $(¿¿\, p+m❑_n)-2m_p$ $m_e+¿$ $¿$ $\Delta E=(m_e+m_D-2m_p)c^2=¿$ $¿(m_e+m❑_n-m_p)c^2$	20
	$\Delta E=(9.109\,10^{31}+1.675\,10^{27}\,1.673\,10^{27})\,kg\,(2.99792\,10^{8}\,ms^{-1})^2=2.6198\times$	20
	$\Delta E \quad k_B T \rightarrow T \quad \frac{\Delta E}{k_B}$	20
	$T=\frac{2.6198\times10^{-13}J}{8.315\frac{J}{mol\,K}}=\frac{2.6198\times10^{-13}mol\,K}{8.315}\times6.02\times10^{23}\frac{1}{mol}$ $¿1.894\times10^{10}\,K$	20

14. Suppose we live in an infinitely large and infinitely old universe where the average density of stars is $n=10^9\,Mpc^{-3}$ and the average stellar radius is equal to the Solar radius. How far, on average, could you see in any direction before your line of sight struck a star? (Assume standard Euclidean geometry holds true in this universe).

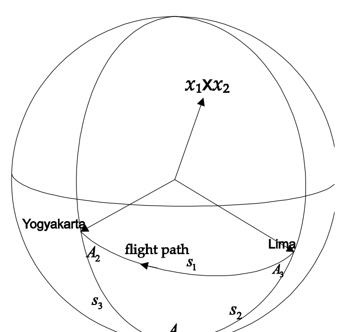
Answer and Marking Scheme:

1.	To decide how far one can see on average in a universe filled with spherical objects of radius *R*, it is simplest to think of a long cylinder along the line of sight. If an object is closer than *R* to the line of sight, then the line of sight intersects its surface. For a distance l, the cylindrical volume which would contain such objects is $\pi R^2 l$. If the density of objects is *n*, then the volume that will on average contain one object is defined by $\pi R^2 l n=1$ and the average distance to which we see before our vision is blocked is $$l=\frac{1}{\pi R^2 n}$$	**50**
2.	The average radius *R* in Mpc $$R=\frac{6.96\times10^8 m}{3.086\times10^{22} m\,Mpc^{-1}}=2.3\times10^{-14}\,Mpc$$	**25**
3.	Then $$l=\frac{1}{3.14\left(2.3\times10^{-14}\,Mpc\right)^2\left(10^9\,Mpc^{-3}\right)}=6.2\times10^{17}\,Mpc$$	**25**

15. An airplane was flying from Lima, capital of Peru ($12^0 2' S$ and $77°1'W$) to Yogyakarta ($7°47'S$ and $110^0 26'E$). The airplane choose the shortest flight path from Lima to Yogyakarta. Neglecting Earth's revolution, find the latitude of the southernmost point of the flight path.

Answer and Marking Scheme:

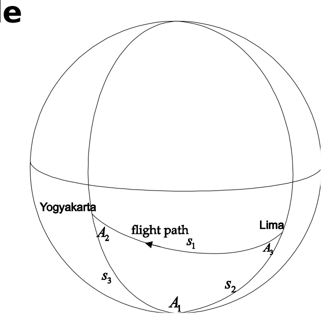
1st version

1	**Determine coordinates of Lima and Yogyakarta.** Assume that the equator is the $z=0$ plane and the Earth is a unit sphere. The Cartesian coordinates of the Lima is $x_1=(x_1,y_1,z_1)=(\cos(-12^0 2')\cos(282°58'48),\cos(-12°2')\sin(282°58'48),\sin(-1$ $¿(0.2197308726,-0.9530236794,-0.2084807188)$ The Cartesian coordinates of the Yogyakarta is $x_2=(x_2,y_2,z_2)=(\cos(-7^0 47')\cos(110^0 26'),\cos(-7°47')\sin(110^0 26'),\sin(-7^0 47'))$	**15**
2	**Determine the normal of the plane defined by Lima, Yogyakarta, and center of the Earth.** The normal of the plane containing the two places and the center of the Earth is $x_1\times x_2=(a,b,c)=(0.3226285776,0.1018712545,-0.12564355210)$	**20**
3	**Find the equation of the (shortest flight) path from Lima to Yogyakarta.** Let the equation of the place be $ax+by+cz=0$ or equivalently $y=-a'x-c'z$ where $a'=-3.16702272082062$ and $c'=1.23335628599724$. For the sake of simplicity, from now let us denote a' and c' as a and c, respectively.  Intersection of the Earth and the plane is the large circle	**25**

	$x^2+(-ax-cz)^2+z^2=1$ or equivalently $(1+a^2)x^2+2acxz+(1+c^2)z^2-1=0$ It is also the equation for the shortest flight path from Lima to Yogyakarta.	
4	**Determine the z-value of the southernmost point of the path (at the point, the discriminant of the equation** $(1+a^2)x^2+2acmx+(1+c^2)m^2-1=0$ **equals zero)**. The plane $z=m$, it intersects the large circle at at either at one or two points. The number of intersection point at the highest or at the lowest point is one, i.e. when the discriminant is zero. $(1+a^2)x^2+2acmx+(1+c^2)m^2-1=0$ The discriminant is $D=(2acm)^2-4(1+a^2)\left[(1+c^2)m^2-1\right]$ $\text{¿}4+4a^2-(4+4a^2+4c^2)m^2$ Therefore $D=0$, if $m=\pm\sqrt{\frac{1+a^2}{1+a^2+c^2}}=\text{¿}$ 0.937444937497383 . Then choose $z=-0.937444937497383$	**30**
5	**Determine the latitude of the southernmost point.** Thus, the latitude of the point is $\sin^{-1}z=1.21521692653748=69.6268010651290°=69°37'36''S.$	**10**

2nd version (using the Law of Cosines for sides)

Answer and Marking Scheme:

1	**Determine length of the sides of the triangle** 	**20**

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$s_2=90°-12°2'=77°58'=1.3608\,rad\,, s_3=90°-7°47'=82°13'=1.4350\,rad\,; A_1=(1$

Using Law of Cosines for sides
$\cos s_1=(\cos s_2)(\cos s_3)+(\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1=2.7724$ .rad

</td><td></td></tr>
<tr><td>2</td><td>

Determine the angles of the triangle
Then use
$\cos s_2=(\cos s_1)(\cos s_3)+(\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2=0.35921$ rad.

Similarly, use Law of Cosines
$\cos s_3=(\cos s_1)(\cos s_2)+(\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3=0.3642$ rad.

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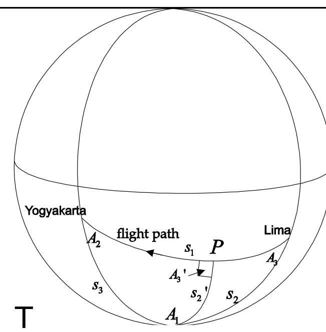
<tr><td>3</td><td>

Use Law of Cosines for sides to construct an equation in $\cos s_2'$.



If P is the southernmost point, then the angle A_3 is a right angle, $A_3=\frac{\pi}{2}$, the Law of

$\cos s_3=\cos s_1'\cos s_2'+\sin s_1'\sin s_2'\cos A_3=\cos s_1'\cos s_2'$ T

hus,

$\cos s_1'=\frac{\cos s_3}{\cos s_2'}=\frac{0.13538}{\cos s_2'}.$

Then substitute it to get

$\cos s_2'=\cos s_1'\cos s_3+\sin s_1'\sin s_3\cos A_2=\frac{\cos s_3}{\cos s_2'}\cos s_3+\sqrt{1-\left(\frac{\cos s_3}{\cos s_2'}\right)^2}\sin s_3\cos A_2$

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<tr><td>4</td><td>

Solve the equation
Let $x=\cos s_2'$. Then

$x=\frac{0.018328}{x}+\sqrt{\frac{x^2-0.018328}{x^2}}\times 0.92756$ Multiply both sides by x to get

$x^2=0.018328+x\sqrt{\frac{x^2-0.018328}{x^2}}\times 0.92756$

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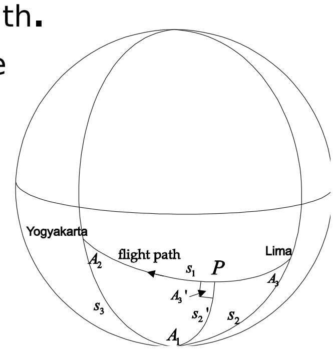
</table>

	$x^2-0.018328=x\sqrt{\frac{x^2-0.018328}{x^2}}\times 0.92756$ $\left(x^2-0.018328\right)^2=\left(x^2-0.018328\right)\times\left(0.92756\right)^2$ $x^4-0.89702x^2+0.016105=0$ which is a quadratic equation of x^2. The roots are ± 0.13538 and ± 0.93739 .	
5	**Determine the Latitude** Thus, $s_2^{'}=\cos^{-1}0.93739=0.35574\,rad=20°23'$ or $s_2^{'}=\cos^{-1}0.13538=1.4350\,rad=82°13'$ The latitude is $69°37'S$ or $7°47'S$. The second one is impossible because it higher then the latitude of Lima. Thus the answer is $69°37^{'}S$	**10**

3rd version (using Napier's rule)

Answer and Marking Scheme:

1	**Determine length of the sides of the triangle** $s_2=90°-12°2^{'}=77°58^{'}=1.3608\,rad\,,s_3=90°-7°47^{'}=82°13^{'}=1.4350\,rad\,;A_1=(1$ Using Law of Cosines for sides $\cos s_1=(\cos s_2)(\cos s_3)+(\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1=2.7724$ .rad	**20**
2	**Determine the angles of the triangle** Then use $\cos s_2=(\cos s_1)(\cos s_3)+(\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2=0.35921$ rad. Similarly, use Law of Cosines $\cos s_3=(\cos s_1)(\cos s_2)+(\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3=0.3642$ rad.	**30**

3	**Find the latitude of the point** Let P be the southernmost point on the flight path. Since the triangle has a right triangle at P, then use one of Napier's rules $\sin s_2' = \sin s_3 \sin A_2$ to get $s_2' = 0.35576\, rad = 20°23'$. Therefore, the latitude of P is $90° - 20°23' = 69°37' S$  	**50**