Sample Problems

Compute each of the following integrals. Assume that a and b are positive numbers.

1. $\displaystyle\int \sin x\ dx$
2. $\displaystyle\int \cos 5x\ dx$
3. $\displaystyle\int \cos x \sin^4 x\ dx$
4. $\displaystyle\int \csc^2 x\ dx$
5. $\displaystyle\int \tan x\ dx$
6. $\displaystyle\int \cot x\ dx$
7. $\displaystyle\int \sec x\ dx$
8. $\displaystyle\int \csc x\ dx$
9. $\displaystyle\int \sin^2 x\ dx$
10. $\displaystyle\int \sin^3 x\ dx$
11. $\displaystyle\int \sin^4 x\ dx$
12. $\displaystyle\int \sin^5 x\ dx$
13. $\displaystyle\int \frac{1}{a^2 + b^2x^2}\ dx$
14. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\ dx$
15. $\displaystyle\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}}\ dx$
16. $\displaystyle\int_0^{\pi/3} \sqrt{1 + \cos 2x}\ dx$
17. $\displaystyle\int_0^{\pi/2} \sqrt{1 - \cos x}\ dx$
18. $\displaystyle\int \tan^3 x\ dx$
19. $\displaystyle\int \sin 7x \cos 3x\ dx$
20. $\displaystyle\int \sin 10x \sin 4x\ dx$

Practice Problems

1. $\displaystyle\int \cos 3x\ dx$
2. $\displaystyle\int \sin\left(4x - \frac{\pi}{5}\right)\ dx$
3. $\displaystyle\int \sec\theta \tan\theta\ d\theta$
4. $\displaystyle\int \sec^2\theta\ d\theta$
5. $\displaystyle\int x \tan\left(x^2\right)\ dx$
6. $\displaystyle\int \cot(2x - \pi)\ dx$
7. $\displaystyle\int \cos^2 x\ dx$
8. $\displaystyle\int \cos^2(2x)\ dx$
9. $\displaystyle\int \cos^3 x\ dx$
10. $\displaystyle\int \cos^4 x\ dx$
11. $\displaystyle\int \cos^5 x\ dx$
12. $\displaystyle\int \sin x \cos^5 x\ dx$
13. $\displaystyle\int \sin^3 x \cos^5 x\ dx$
14. $\displaystyle\int \tan^2 x\ dx$
15. $\displaystyle\int_0^{\pi/6} \sqrt{1 - \cos 6x}\ dx$
16. $\displaystyle\int \sin 2a \cos 8a\ da$
17. $\displaystyle\int \cos b \cos 11b\ db$
18. $\displaystyle\int \sin 6\theta \sin 14\theta\ d\theta$
19. $\displaystyle\int \cos 11m \sin 3m\ dm$

Sample Problems - Answers

1.) $-\cos x + C$ 2.) $\dfrac{1}{5}\sin 5x + C$ 3.) $\dfrac{1}{5}\sin^5 x + C$ 4.) $-\cot x + C$ 5.) $-\ln|\cos x| + C = \ln|\sec x| + C$

6.) $\ln|\sin x| + C$ 7.) $\ln|\sec x + \tan x| + C$ 8.) $-\ln|\csc x + \cot x| + C$ 9.) $\dfrac{1}{2}x - \dfrac{1}{4}\sin 2x + C$

10.) $\dfrac{1}{3}\cos^3 x - \cos x + C$ 11.) $-\dfrac{1}{4}\sin 2x + \dfrac{1}{32}\sin 4x + \dfrac{3}{8}x + C$ 12.) $-\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{1}{5}\cos^5 x + C$

13.) $\dfrac{1}{ab}\tan^{-1}\left(\dfrac{b}{a}x\right) + C$ 14.) $\sin^{-1}\left(\dfrac{x}{a}\right) + C$ 15.) $2\ln\left|\sec\left(\sqrt{x}\right) + \tan\left(\sqrt{x}\right)\right| + C$ 16.) $\dfrac{\sqrt{6}}{2}$

17.) $2\sqrt{2} - 2$ 18.) $\dfrac{1}{2}\sec^2 x + \ln|\cos x| + C$ 19.) $-\dfrac{1}{20}\cos 10x - \dfrac{1}{8}\cos 4x + C$ 20.) $\dfrac{1}{12}\sin 6x - \dfrac{1}{28}\sin 14x + C$

Practice Problems - Answers

1.) $\dfrac{1}{3}\sin 3x + C$ 2.) $-\dfrac{1}{4}\cos\left(4x - \dfrac{\pi}{5}\right) + C$ 3.) $\sec\theta + C$ 4.) $\tan\theta + C$ 5.) $\dfrac{1}{2}\ln\left|\sec\left(x^2\right)\right| + C$

6.) $\dfrac{1}{2}\ln|\sin(2x - \pi)| + C$ 7.) $\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + C$ 8.) $\dfrac{1}{2}x + \dfrac{1}{8}\sin 4x + C$ 9.) $\sin x - \dfrac{1}{3}\sin^3 x + C$

10.) $\dfrac{3}{8}x + \dfrac{1}{4}\sin 2x + \dfrac{1}{32}\sin 4x + C$ 11.) $\dfrac{1}{5}\sin^5 x - \dfrac{2}{3}\sin^3 x + \sin x + C$ 12.) $-\dfrac{1}{6}\cos^6 x + C$

13.) $-\dfrac{1}{6}\cos^6 x + \dfrac{1}{8}\cos^8 x + C$ 14.) $-x + \tan x + C$ 15.) $\dfrac{\sqrt{2}}{3}$ 16.) $\dfrac{1}{12}\cos 6a - \dfrac{1}{20}\cos 10a + C$

17.) $\dfrac{1}{20}\sin 10b + \dfrac{1}{24}\sin 12b + C$ 18.) $\dfrac{1}{16}\sin 8\theta - \dfrac{1}{40}\sin 20\theta + C$ 19.) $\dfrac{1}{16}\cos 8m - \dfrac{1}{28}\cos 14m + C$

Sample Problems - Solutions

1. $\displaystyle\int \sin x \; dx$

Solution: This is a basic integral we know from differentiating basic trigonometric functions. Since $\dfrac{d}{dx}\cos x = -\sin x$, clearly $\dfrac{d}{dx}(-\cos x) = \sin x$ and so $\displaystyle\int \sin x \; dx = \boxed{-\cos x + C}$.

2. $\displaystyle\int \cos 5x \; dx$

Solution: We know that $\dfrac{d}{dx}\cos x = -\sin x + C$. We will use substitution. Let $u = 5x$ and then $du = 5dx$ and so $\dfrac{du}{5} = dx$.

$$\int \cos 5x \; dx = \int \cos u \; \left(\frac{du}{5}\right) = \frac{1}{5}\int \cos u \; du = \boxed{\frac{1}{5}\sin 5x + C}$$

Note: Once we have enough practice, there is no need to perform this substitution in writing. We can just simply write $\displaystyle\int \cos 5x \; dx = \frac{1}{5}\sin 5x + C$.

3. $\displaystyle\int \cos x \sin^4 x \; dx$

Solution: Let $u = \sin x$. Then $du = \cos x dx$.

$$\int \cos x \sin^4 x \; dx = \int \sin^4 x \;\; (\cos x dx) = \int u^4 \;\; u = \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\sin^5 x + C}$$

4. $\displaystyle\int \csc^2 x \; dx$

Solution: We need to remember that $\dfrac{d}{dx}\cot x = -\csc^2 x$.

$$\int \csc^2 x \; dx = -\int -\csc^2 x \; dx = \boxed{-\cot x + C}$$

5. $\displaystyle\int \tan x \; dx$

Solution: Let $u = \cos x$. Then $du = -\sin x \; dx$.

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x}\; dx = \int \frac{1}{u}\;\; (\sin x du) = \int \frac{1}{u}\;\; (-du) = -\int \frac{1}{u}\; du = -\ln|u| + C = -\ln|\cos x| + C \\ &= \ln\left|(\cos x)^{-1}\right| + C = \boxed{\ln|\sec x| + C}\end{aligned}$$

6. $\displaystyle\int \cot x \; dx$

Solution: Let $u = \sin x$. Then $du = \cos x \; dx$.

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x}\; dx = \int \frac{1}{u}\;\; (\cos x du) = \int \frac{1}{u}\; du = \ln|u| + C = \boxed{\ln|\sin x| + C}$$

7. $\displaystyle\int \sec x \; dx$

Solution: $\displaystyle\int \sec x \; dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx$

From here we will use substitution. Recall that $\dfrac{d}{dx} \sec x = \sec x \tan x$ and $\dfrac{d}{dx} \tan x = \sec^2 x$. Let $u = \sec x + \tan x$. Then $du = \left(\sec x \tan x + \sec^2 x\right) dx$.

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx = \int \frac{1}{u} \; \left(\sec^2 x + \sec x \tan x\right) dx = \int \frac{1}{u} \; du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

8. $\displaystyle\int \csc x \; dx$

Solution: $\displaystyle\int \csc x \; dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \; dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \; dx$

From here we will use substitution. Recall that $\dfrac{d}{dx} \csc x = -\csc x \cot x$ and $\dfrac{d}{dx} \cot x = -\csc^2 x$. Let $u = \csc x + \cot x$. Then $du = \left(-\csc^2 x - \csc x \cot x\right) dx$.

$$\begin{aligned} \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \; dx &= \int \frac{1}{u} \; \left(\csc^2 x + \csc x \cot x\right) dx = \int \frac{1}{u} \; (-du) = -\int \frac{1}{u} \; du = -\ln|u| + C \\ &= \boxed{-\ln|\csc x + \cot x| + C} \end{aligned}$$

9. $\displaystyle\int \sin^2 x \; dx$

Solution: Recall the double angle formula for cosine, $\cos 2x = 1 - 2\sin^2 x$. We solve this for $\sin^2 x$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\begin{aligned} \int \sin^2 x \; dx &= \int \frac{1}{2}(1 - \cos 2x) \; dx = \frac{1}{2}\left(\int 1 \; dx - \int \cos 2x \; dx\right) = \frac{1}{2}\left(x + C_1 - \frac{1}{2}\sin 2x + C_2\right) \\ &= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C} \end{aligned}$$

10. $\displaystyle\int \sin^3 x \; dx$

Solution:

$$\int \sin^3 x \; dx = \int \sin x \sin^2 x \; dx = \int \sin x \left(1 - \cos^2 x\right) \; dx$$

Let $u = \cos x$. Then $du = -\sin x dx$

$$\begin{aligned} \int \sin^3 x \; dx &= \int \sin x \left(1 - \cos^2 x\right) \; dx = \int \left(1 - \cos^2 x\right)(\sin x dx) = \int \left(1 - u^2\right)(-du) = \int \left(u^2 - 1\right) du \\ &= \frac{1}{3}u^3 - u + C = \boxed{\frac{1}{3}\cos^3 x - \cos x + C} \end{aligned}$$

11. $\displaystyle\int \sin^4 x \, dx$

Solution: We use the double angle formula for cosine to express $\sin^2 x$.

$$\cos 2x = 1 - 2\sin^2 x \quad \Longrightarrow \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 \, dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \, dx = \frac{1}{4}\int (1-\cos 2x)^2 \, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) \, dx$$

We use the double angle formula for cosine again to express $\cos^2 2x$.

$$\cos 4x = 2\cos^2 2x - 1 \quad \Longrightarrow \quad \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$

$$\begin{aligned}
\int \sin^4 x \, dx &= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) \, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right) \, dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8}\right) \, dx = \int \left(-\frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{3}{8}\right) \, dx \\
&= -\frac{1}{2}\left(\frac{1}{2}\right)\sin 2x + \frac{1}{8}\left(\frac{1}{4}\right)\sin 4x + \frac{3}{8}x + C = \boxed{-\frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \frac{3}{8}x + C}
\end{aligned}$$

12. $\displaystyle\int \sin^5 x \, dx$

Solution: This method works with odd powers of $\sin x$ or $\cos x$. We will separate one factor of $\sin x$ from the rest which will be expressed in terms of $\cos x$.

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \sin^4 x \, dx = \int \sin x \sin^4 x \, dx = \int \sin x \left(\sin^2 x\right)^2 \, dx = \int \sin x \left(1 - \cos^2 x\right)^2 \, dx \\
&= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \, dx
\end{aligned}$$

We proceed with substitution. Let $u = \cos x$. Then $du = -\sin x dx$.

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \, dx = \int \left(1 - 2\cos^2 x + \cos^4 x\right) \, (\sin x dx) \\
&= \int \left(1 - 2u^2 + u^4\right) \, (-du) = \int \left(-1 + 2u^2 - u^4\right) \, du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\
&= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}
\end{aligned}$$

13. $\displaystyle\int \frac{1}{a^2 + b^2x^2} \, dx$

Solution: The basic integral here is $\displaystyle\int \frac{1}{x^2+1} \, dx = \tan^{-1} x + C$. We need a substitution under which $a^2x^2 = b^2u^2$. This would be convenient because then

$$\frac{1}{a^2x^2 + b^2} = \frac{1}{b^2u^2 + b^2} = \frac{1}{b^2} \cdot \frac{1}{u^2+1}$$

So we will pursue this substitution. We solve $a^2x^2 = b^2u^2$ for a possible value of u and obtain $u = \dfrac{a}{b}x$. Then $du = \dfrac{a}{b}dx$ and so $\dfrac{b}{a}du = dx$.

$$\begin{aligned}\int \frac{1}{a^2x^2+b^2}\,dx &= \int \frac{1}{b^2u^2+b^2}\left(\frac{b}{a}du\right) = \int \frac{1}{b^2}\cdot\frac{1}{u^2+1}\cdot\frac{b}{a}\,du = \frac{b}{ab^2}\int \frac{1}{u^2+1}\,du = \frac{1}{ab}\tan^{-1}u + C \\ &= \boxed{\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right) + C}\end{aligned}$$

14. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Solution: The basic integral here is $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$. We need a substitution under which $x^2 = a^2u^2$. This would be useful because then

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-a^2u^2}} = \frac{1}{\sqrt{a^2(1-u^2)}} = \frac{1}{a\sqrt{1-u^2}}$$

So we will pursue this substitution. We solve $x^2 = a^2u^2$ for a possible value of u and obtain $x = au$ and $dx = adu$.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-a^2u^2}}\,(adu) = \int \frac{a}{a\sqrt{1-u^2}}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du- = \sin^{-1}u + C = \boxed{\sin^{-1}\left(\frac{x}{a}\right) + C}$$

15. $\displaystyle\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}dx$.

$$\begin{aligned}\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int \frac{\sec\left(\sqrt{x}\right)}{2\sqrt{x}}\,dx = 2\int \sec\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}dx\right) = 2\int \sec u\;du = 2\ln\left|\sec u + \tan u\right| + C \\ &= \boxed{2\ln\left|\sec\left(\sqrt{x}\right) + \tan\left(\sqrt{x}\right)\right| + C}\end{aligned}$$

16. $\displaystyle\int_0^{\pi/3} \sqrt{1+\cos 2x}\,dx$

Solution: We will yet again use the double angle formula for cosine, this time to eliminate the square root.

$$\cos 2x = 2\cos^2 x - 1 \quad\Longrightarrow\quad 2\cos^2 x = \cos 2x + 1$$

$$\int_0^{\pi/3} \sqrt{1+\cos 2x}\,dx = \int_0^{\pi/3} \sqrt{2\cos^2 x}\,dx = \sqrt{2}\int_0^{\pi/3} \sqrt{\cos^2 x}\,dx = \sqrt{2}\int_0^{\pi/3} \left|\cos x\right|\,dx$$

Since $f(x) = \cos x$ is positive on $\left[0, \dfrac{\pi}{3}\right]$, we can simplify $\left|\cos x\right| = \cos x$

$$\sqrt{2}\int_0^{\pi/3} \left|\cos x\right|\,dx = \sqrt{2}\int_0^{\pi/3} \cos x\,dx = \sqrt{2}\left(\sin x\Big|_0^{\pi/3}\right) = \sqrt{2}\left(\sin\frac{\pi}{3} - \sin 0\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - 0\right) = \boxed{\frac{\sqrt{6}}{2}}$$

17. $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos x}\ dx$

Solution:

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \Longrightarrow \qquad 2\sin^2\theta = 1-\cos 2\theta$$

We substitute $\theta = \dfrac{x}{2}$ into this and obtain

$$2\sin^2\frac{x}{2} = 1-\cos x$$

$$\int_0^{\pi/2} \sqrt{1-\cos x}\ dx = \int_0^{\pi/2} \sqrt{2\sin^2\frac{x}{2}}\ dx = \sqrt{2}\int_0^{\pi/2} \left|\sin\frac{x}{2}\right|\ dx$$

Since $f(x) = \sin\dfrac{x}{2}$ is non-negative on $\left[0, \dfrac{\pi}{2}\right]$, we can simplify $\left|\sin\dfrac{x}{2}\right| = \sin\dfrac{x}{2}$

$$\begin{aligned}\sqrt{2}\int_0^{\pi/2} \sin\frac{x}{2}\ dx &= \sqrt{2}\left(-2\cos\frac{x}{2}\Big|_0^{\pi/2}\right) = -2\sqrt{2}\left(\cos\frac{x}{2}\Big|_0^{\pi/2}\right) = -2\sqrt{2}\left(\cos\frac{\pi}{4}-\cos 0\right)\\ &= -2\sqrt{2}\left(\frac{\sqrt{2}}{2}-1\right) = -2+2\sqrt{2} = \boxed{2\sqrt{2}-2}\end{aligned}$$

18. $\displaystyle\int \tan^3 x\ dx$

Solution: Let $u = \cos x$. Then $du = -\sin x dx$

$$\begin{aligned}\int \tan^3 x\ dx &= \int \frac{\sin^3 x}{\cos^3 x}\ dx = \int \sin x\frac{\sin^2 x}{\cos^3 x}\ dx = \int \frac{1-\cos^2 x}{\cos^3 x}\ \sin x dx = \int \frac{1-u^2}{u^3}\ (-du) = \int \frac{u^2-1}{u^3}\ du\\ &= \int \frac{u^2}{u^3} - \frac{1}{u^3}\ du = \int \frac{1}{u} - u^{-3}\ du = \ln|u| - \frac{u^{-2}}{-2} + C = \ln|u| + \frac{1}{2u^2} + C\\ &= \ln|\cos x| + \frac{1}{2}\sec^2 x + C\end{aligned}$$

19. $\displaystyle\int \sin 7x\cos 3x\ dx$

Solution: We wil use the product-to-sum identities to trasform this product into a sum. We write the sine formula for the sum and the difference of these two angles.

$$\begin{aligned}\sin 10x &= \sin(7x+3x) = \sin 7x\cos 3x + \cos 7x\sin 3x\\ \sin 4x &= \sin(7x-3x) = \sin 7x\cos 3x - \cos 7x\sin 3x\end{aligned}$$

We will add the two equations

$$\begin{aligned}\sin 10x + \sin 4x &= 2\sin 7x\cos 3x\\ \frac{1}{2}(\sin 10x + \sin 4x) &= \sin 7x\cos 3x\end{aligned}$$

We can very easily integrate $\dfrac{1}{2}(\sin 10x + \sin 4x)$

$$\begin{aligned}\int \sin 7x\cos 3x\ dx &= \int \frac{1}{2}(\sin 10x + \sin 4x)\ dx = \frac{1}{2}\int \sin 10x + \sin 4x\ dx\\ &= \frac{1}{2}\left(\frac{1}{10}\right)(-\cos 10x) + \frac{1}{2}\left(\frac{1}{4}(-\cos 4x)\right) + C = \boxed{-\frac{1}{20}\cos 10x - \frac{1}{8}\cos 4x + C}\end{aligned}$$

20. $\displaystyle\int \sin 10x \sin 4x \; dx$

Solution: We wil use the product-to-sum identities to trasform this product into a sum. We write the cosine formula for the sum and the difference of these two angles.

$$\begin{aligned}
\cos 14x &= \cos(10x + 4x) = \cos 10x \cos 4x - \sin 10x \sin 4x \\
\cos 6x &= \cos(10x - 4x) = \cos 10x \cos 4x + \sin 10x \sin 4x
\end{aligned}$$

We will subtract the first equation from the first one

$$\begin{aligned}
\cos 6x - \cos 14x &= 2 \sin 10x \sin 4x \\
\frac{1}{2}(\cos 6x - \cos 14x) &= \sin 10x \sin 4x
\end{aligned}$$

We can very easily integrate $\frac{1}{2}(\cos 6x - \cos 14x)$

$$\begin{aligned}
\int \sin 10x \sin 4x \; dx &= \int \frac{1}{2}(\cos 6x - \cos 14x) \; dx = \frac{1}{2}\int \cos 6x - \cos 14x \; dx \\
&= \frac{1}{2}\left(\frac{1}{6}\right)(\sin 6x) - \frac{1}{2}\left(\frac{1}{14}(\sin 14x)\right) + C = \boxed{\frac{1}{12}\sin 6x - \frac{1}{28}\sin 14x + C}
\end{aligned}$$

