Applied Calculus

Integration by Substitution:

1) Magic Substitution Method:

function rational form or trignometric form mai ho..

$$z = \tan\frac{x}{2}$$

$$\frac{dz}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$2dz = \sec^2\frac{x}{2}\,dx$$

$$1+\tan^2\theta = \sec^2\theta$$

$$dx = \frac{2\,dz}{\sec^2 x/2}$$

$$dx = \frac{2\,dz}{1+\tan^2\frac{x}{2}}$$

$$dx = \frac{2\,dz}{1+z^2}$$ (integral operator)

$$\sin 2x = 2 \sin \cancel{x} \cos x$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\sin x = \frac{2 \sin x/2 \cos x/2 \times \cos x/2}{\cos x/2}$$

$$= 2 \tan \frac{x}{2} \cos^2 \frac{x}{2}$$

$$= \frac{2 \tan x/2}{\sec^2 x/2}$$

$$= \frac{2 \tan x/2}{1 + \tan^2 x/2}$$

$\therefore$ $z = \tan x/2$

$z^2 = \tan^2 x/2$

$$\sin x = \frac{2z}{1+z^2}$$ (sin ~~oper~~ operator)

$\cos\theta$:-

$$\cos x = 2 \cos^2 \frac{x}{2} - 1 \quad (\because \cos 2\theta = 2\cos^2\theta - 1)$$

$$\cos x = \frac{2}{\sec^2 x/2} - 1 \qquad \cos^2 \frac{x}{2} = \frac{1}{\sec^2 x/2}$$

$$\cos x = \frac{2}{1 + \tan^2 \frac{x}{2}} - 1 \qquad \cos\theta = \frac{1}{\sec\theta}$$

$(\because \sec^2\theta = 1 + \tan^2\theta)$

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$$\cos x = \frac{2}{1+z^2} - 1$$

$$= \frac{2-1-z^2}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

Problem #1 :-

$$\int \left(\frac{2-\cos x}{2+\cos x} \right) dx$$

Sol

$$\int \left(\frac{2-(1-z^2)/1+z^2}{2+(1-z^2)/1+z^2} \right) \frac{2\,dz}{1+z^2}$$

$$\int \left(\frac{2+2z^2-1+z^2/\cancel{1+z^2}}{2+2z^2+1-z^2/\cancel{1+z^2}} \right) \frac{2\,dz}{1+z^2}$$

$$\int \left(\frac{1+3z^2}{3+z^2} \right) \frac{2\,dz}{1+z^2}$$

$$2\int \frac{(1+3z^2)\,dz}{\cancel{3+3z^2+z^2+z^4}\ (z^2+3)(z^2+1)}$$

$$2\int \frac{(3z^2+1)}{(z^2+3)(z^2+1)}\,dz \quad \text{———} \ 1$$

Let $z^2 = u$

$$\frac{3z^2+1}{(z^2+3)(z^2+1)} = \frac{3U+1}{(U+3)(U+1)}$$

$$\frac{3U+1}{(U+3)(U+1)} = \frac{A}{U+3} + \frac{B}{U+1}$$

$$3U+1 = A(U+1) + B(U+3)$$

Put $U+1 = 0$

$$U = -1$$

$$3(-1)+1 = A(\cancel{-1}+\cancel{1}) + B(-1+3)$$

$$-3+1 = 2B$$

$$-2 = 2B$$

$$-1 = B$$

Put $U+3 = 0$

$$U = -3$$

$$3(-3)+1 = A(-3+1) + B(\cancel{-3}+\cancel{3})$$

$$-9+1 = A(-2)$$

$$-8 = -2A$$

$$4 = A$$

$$\frac{3U+1}{(U+3)(U+1)} = \frac{4}{U+3} - \frac{1}{U+1}$$

$$\frac{3z^2+1}{(z^2+3)(z^2+1)} = \frac{4}{z^2+3} - \frac{1}{z^2+1}$$

$$\Rightarrow 2\int \frac{3z^2+1}{(z^2+3)(z^2+1)}\,dz = 4\int \frac{1}{z^2+3}\,dz - \int \frac{1\,dz}{z^2+1}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$2\int \frac{3z^2+1}{(z^2+3)(z^2+1)} = 4\int \frac{1}{z^2+(\sqrt{3})^2}\,dz - \int \frac{1}{z^2+(1)^2}\,dz$$

$$2\int \frac{3z^2+1}{(z^2+3)(z^2+1)} = 2\left[4\int \frac{1}{z^2+(\sqrt{3})^2}\,dz - \int \frac{1}{z^2+(1)^2}\,dz\right]$$

$$= 2\left[4 \times \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) - \tan^{-1} z\right] + C$$

$$= \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) - 2\tan^{-1} z + C$$

$$= \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C$$

$$=$$

$$\because \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

$f^{-1} f$

$\sin^{-1}(\sin x) = x$

$$= \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - 2\left(\frac{x}{2}\right) + C$$

$$= \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - x + C$$

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Q) $\frac{1+\sin x}{1-\sin x}\, dx$

Integration by Parts :

If $f(x)$ & $g(x)$ are two functions.

$U = f(x)$, $V = g(x)$

$$\int (uv)\, dx = u\int v\, dx - \int \left(u' \int v\, dx\right) dx$$

Trig

Log ln

LIATE

Exp

$U =$

Algebric

jo phly hoga usy U lygy.

Q $\int \ln x\, dx$

$U = \ln x$, $V = 1$

$$\int \ln x\, dx = \ln x \int 1\, dx - \int\left(\frac{1}{x}\int 1\, dx\right) dx$$

$$= x\ln x - \int\left(\frac{1}{\cancel{x}} \cdot \cancel{x}\right) dx$$

$$= x\ln x - \int 1\, dx$$

$$\int \ln x\, dx = x\ln x - x + C$$

Q $\int e^x \sin(e^x)\, dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$= \int \sin u \, du$

$= -\cos u + C$

$= -\cos(e^x) + C$

★ Agr function ka derivative mojood hot integration mai to function ka integral aajaye ga..

e.g $\int e^x \sin x \, dx$

Q $I = \int e^x \sin x \, dx$

$V = e^x$; $U = \sin x$

$\int e^x \sin x \, dx = e^x \int \sin x \, dx - \cancel{\int \left(e^x \int \sin x \, dx \right) dx}$

$= \cancel{e^x \int}$

$\int e^x \sin x \, dx = \sin x \int e^x dx - \int \left(\cos x \int e^x dx \right) dx$

$= \sin x \, e^x - \int \cos x \, e^x \, dx$

$= \sin e^x - \left[\cos x \, e^x - \int \left((-\sin x) \int e^x dx \right) dx \right.$

$$I = \sin x\, e^x - \left[e^x \cos x + \int e^x \sin x\, dx \right]$$

$$I = e^x \sin x - \left[e^x \cos x + I \right]$$

$$I = e^x \sin x - e^x \cos x - I$$

$$2I = e^x \sin x - e^x \cos x$$

$$I = \frac{e^x \sin x - e^x \cos x}{2}$$

$$I = \frac{e^x}{2} (\sin x - \cos x) + c$$

LEIBNITZ'S THEOREM AND Successive differention.

① $y = e^{ax}$ (exponential)

$y_1 = a^1 e^{ax}$

$y_2 = a^2 e^{ax}$

$y_3 = a^3 e^{ax}$

$\vdots$

$y_n = a^n e^{ax}$

$\frac{d}{dx} e^{ax}$

$e^{ax} \cdot \frac{d}{dx}(ax)$

$\frac{d}{dx} e^{ax} \cdot a$

$e^{ax} \cdot a \cdot a$

② $y = \frac{1}{x}$ (Rational)

$y = x^{-1}$

$y_1 = (-1) x^{-2}$

$y_2 = (-1)(-2) x^{-3}$

$y_3 = (-1)(-2)(-3) x^{-4}$

$\vdots$

$y_n = (-1)(-2)(-3) \cdots (-n) x^{-(n+1)}$

$y_n = (-1)^n n! \, x^{-(n+1)}$ → differential co-efficient.

$n = 4$

$y_4 = -4! \, x^{(-5)}$

$y_4 = \frac{-24}{x^5}$

Q. Find the nth differential co-effient of

$f(x) = \cos 2x$

$y' = y_1 = f'(x) = -2 \sin 2x \Rightarrow y_1 = -2 [-\sin 2x]$

$\therefore \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$y_1 = 2 \cos\left(\frac{\pi}{2} + 2x\right)$

$y_2 = -2^2 \sin\left(\frac{\pi}{2} + 2x\right) \cdot \frac{d}{dx} 2x$

$y_2 = -2^2 \sin\left(\frac{\pi}{2} + 2x\right)$

$\theta = \frac{\pi}{2} + 2x$

$y_2 = -2^2 \sin\theta$

$y_2 = 2^2 \cos\left(\frac{\pi}{2} + \theta\right)$

$y_2 = 2^2 \cos\left(\frac{\pi}{2} + \frac{\pi}{2} + 2x\right)$

$$y_2 = 2^2 \cos\left(\frac{2\pi}{2} + 2x\right)$$

$$y_n = 2^n \cos\left(\frac{n\pi}{2} + 2x\right)$$

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Q $y = \sin 6x \cos 2x$

$\therefore \sin U + \sin V = 2 \sin\left(\frac{U+V}{2}\right) \cos\left(\frac{U-V}{2}\right)$

$\frac{U+V}{2} = 6x \quad , \quad \frac{U-V}{2} = 2x$

$U+V = 12x$ — ① $U-V = 4x$ — ②

Add eq ① & eq ②

$2U = 16x$

$U = 8x$

Add U in eq ①

$8x + V = 12x$

$V = 4x$

~~Si~~ eq :

$\sin 8x \sin 4x = 2 \sin 6x \cos 2x$

$\sin 6x \cos 2x = \frac{1}{2} \sin 8x \sin 4x$

$\Rightarrow y^n = \frac{1}{2} (\sin 8x)^n + \frac{1}{2} (\sin 4x)^n$

$\Rightarrow y^n = \frac{1}{2}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right)\right] + \frac{1}{2}\left[4^n \sin\left(4x + \frac{n\pi}{2}\right)\right]$

$y = \sin(ax+b)$

$\therefore y^n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

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$$y^n = \frac{1}{2}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right) + 4^n \sin\left(4x + \frac{n\pi}{2}\right)\right]$$

Q10) If $y = \log\left(x + \sqrt{x^2+1}\right)$ show that $(1+x^2)y'' + xy' = 0$

$$y' = \frac{1}{x + \sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2+1}\right)$$

$$y' = \frac{1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{x + \sqrt{x^2+1}}$$

$$y' = \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$y' = \frac{\cancel{x + \sqrt{x^2+1}}}{\sqrt{x^2+1}} \times \frac{1}{\cancel{x + \sqrt{x^2+1}}}$$

$$y' = \frac{1}{\sqrt{x^2+1}}$$

$$\sqrt{1+x^2}\, y' = 1$$

S.B.S

$$(1+x^2)(y')^2 = 1$$

Diff

$$2x \cdot [y']^2 + (1+x^2)\, 2y'y'' = 0$$

$$2y'\left[xy' + (1+x^2)y''\right] = 0$$

$$xy' + (1+x^2)y'' = 0$$

Q If $y = \tan^{-1} x$

show that

$(1+x^2)y'' + 2xy' = 0$

$$y' = \frac{1}{1+x^2}$$

$y'(1+x^2) = 1$

S. diff

$2xy' + (1+x^2)y'' = 0$

LeiBnITz THEOREM :

$U = f(x)$ and $v = g(x)$

$[f(x)\,g(x)]^n = (U \cdot V)^n$

0th derivative 1st derivative

$$\therefore (U \cdot V)^n = U^n V^0 + nU^{(n-1)} V^1 + \frac{n(n-1)U^{(n-2)} V''}{2!} + \frac{n(n-1)(n-2)\,U^{(n-3)}}{3!} V''' + \cdots + U V^n$$

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