Introduction to Partial Differentiation

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Credit to R Horan & M Lavelle

Table of Contents

1. Partial Differentiation (Introduction)

In the package on **introductory differentiation**, rates of change of functions were shown to be measured by the *derivative*. Many applications require functions with more than one variable: the ideal gas law, for example, is

$$pV = kT$$

where p is the pressure, V the volume, T the absolute temperature of the gas, and k is a constant. Rearranging this equation as

$$p = \frac{kT}{V}$$

shows that p is a function of T and V . If one of the variables, say T, is kept fixed and V changes, then the derivative of p with respect to V measures the *rate of change* of *pressure* with respect to *volume*. In this case, it is called *the partial derivative of p with respect to V* and written as

$$\frac{\partial p}{\partial V}.$$

Example 1 If $p = \frac{kT}{V}$, find the partial derivatives of p:

(a) with respect to T, (b) with respect to V.

Solution

(a) This part of the example proceeds as follows:

$$\begin{aligned} p &= \frac{kT}{V} \\ \therefore \frac{\partial p}{\partial T} &= \frac{k}{V}, \end{aligned}$$

where V is treated as a constant for this calculation. (b) For this part, T is treated as a constant. Thus

$$
\begin{aligned}
p &= kT\frac{1}{V} = kTV^{-1}\,, \\
\therefore \frac{\partial p}{\partial V} &= -kTV^{-2} = -\frac{kT}{V^2}.
\end{aligned}
$$

The symbol ∂ is used whenever a function with more than one variable is being differentiated but the techniques of *partial* differentiation are exactly the same as for (*ordinary*) differentiation.

Example 2 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the function $z = x^2y^3$.

Solution

$$
\begin{aligned}
z &= x^2y^3 \\
\therefore \frac{\partial z}{\partial x} &= 2xy^3 \\
\frac{\partial z}{\partial y} &= x^2 3y^2 \\
, \quad &= 3x^2y^2 .
\end{aligned}
$$

For the first part y^3 is treated as a constant and the derivative of x^2 *with respect to x* is $2x$.

and, For the second part x^2 is treated as a constant and the derivative of y^3 *with respect to y* is $3y^2$.

Exercise 1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions.

(Click on the green letters for solutions.)

(a) $z = x^2y^4$, (b) $z = (x^4 + x^2)y^3$, (c) $z = y^{\frac{1}{2}}\sin(x)$.

2. The Rules of Partial Differentiation

Since *partial differentiation* is essentially the same as *ordinary differentiation*, the *product*, *quotient* and *chain* rules may be applied.

Example 3 Find $\dfrac{\partial z}{\partial x}$ for each of the following functions.

$$\text{(a) } z = xy\cos(xy)\,, \quad \text{(b) } z = \frac{x-y}{x+y}\,, \quad \text{(c) } z = (3x+y)^2\,.$$

Solution

(a) Here $z = uv$, where $u = xy$ and $v = \cos(xy)$ so the *product rule* applies (see the package on the Product and Quotient Rules).

$$u = xy \quad \text{and} \quad v = \cos(xy)$$

$$\therefore \frac{\partial u}{\partial x} = y \quad \text{and} \quad \frac{\partial v}{\partial x} = -y\sin(xy).$$

Thus
$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x} = y\cos(xy) - xy^2\sin(xy).$$

(b) Here $z = u/v$, where $u = x-y$ and $v = x+y$ so the *quotient rule* applies (see the package on the Product and Quotient Rules).

$$u = x - y \quad \text{and} \quad v = x + y$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = 1.$$

Thus

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2} \\
&= \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}.
\end{aligned}$$

(c) In this case $z = (3x + y)^2$ so $z = u^2$ where $u = 3x + y$, and the *chain rule* applies (see the package on the Chain Rule).

$$z = u^2 \quad \text{and} \quad u = 3x + y$$

$$\therefore \frac{\partial z}{\partial u} = 2u \quad \text{and} \quad \frac{\partial u}{\partial x} = 3.$$

Thus
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = 2(3x+y)3 = 6(3x+y).$$

Exercise 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for each of the following functions. (Click on the green letters for solutions.)

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \dfrac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = e^{(x^2+y^2)}$, (f) $z = \sin(x^2 + y)$.

Quiz If $z = \cos(xy)$, which of the following statements is true?

(a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (b) $\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$,

(c) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (d) $\frac{1}{y}\frac{\partial z}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial y}$.

3. Higher Order Partial Derivatives

Derivatives of order two and higher were introduced in the package on **Maxima and Minima**. Finding higher order derivatives of functions of more than one variable is similar to ordinary differentiation.

Example 4 Find $\frac{\partial^2 z}{\partial x^2}$ if $z = e^{(x^3+y^2)}$.

Solution First differentiate z with respect to x, keeping y constant, then differentiate this function with respect to x, again keeping y constant.

$z = \mathrm{e}^{(x^3+y^2)}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2\mathrm{e}^{(x^3+y^2)}$$

using the chain rule

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial(3x^2)}{\partial x}\mathrm{e}^{(x^3+y^2)} + 3x^2\frac{\partial(\mathrm{e}^{(x^3+y^2)})}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = 6x\mathrm{e}^{(x^3+y^2)} + 3x^2(3x^2\mathrm{e}^{(x^3+y^2)})$$

$$= (9x^4 + 6x)\mathrm{e}^{(x^3+y^2)}$$

using the product rule

In addition to both $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, when there are two variables there is also the possibility of a *mixed second order derivative*.

Example 5 Find $\dfrac{\partial^2 z}{\partial x \partial y}$ if $z = \mathrm{e}^{(x^3+y^2)}$.

Solution The symbol $\dfrac{\partial^2 z}{\partial x \partial y}$ is interpreted as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$; in words, first differentiate z with respect to y, keeping x constant, then differentiate this function with respect to x, keeping y constant. (It is this differentiation, first with respect to x and then with respect to y, that leads to the name of *mixed derivative*.)

First with x constant $\quad \dfrac{\partial z}{\partial y} = 2y\mathrm{e}^{(x^3+y^2)}$ (using the chain rule.)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2y\mathrm{e}^{(x^3+y^2)}\right)$$

$$= 3x^2 2y\mathrm{e}^{(x^3+y^2)}$$

Second with y constant $$= 6x^2 y\mathrm{e}^{(x^3+y^2)}.$$

The obvious question now to be answered is: what happens if the order of differentiation is reversed?

Example 6 Find $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ if $z = \mathrm{e}^{(x^3+y^2)}$.

Solution

First with y constant $$\frac{\partial z}{\partial x} = 3x^2\mathrm{e}^{(x^3+y^2)}$$ (using the chain rule).

Second with x constant

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(3x^2 \mathrm{e}^{(x^3+y^2)}\right)\\ &= 2y3x^2\mathrm{e}^{(x^3+y^2)}\\ &= 6x^2 y\mathrm{e}^{(x^3+y^2)} = \frac{\partial^2 z}{\partial x \partial y}\end{aligned}$$

.

As a general rule, when calculating *mixed derivatives* the order of differentiation may be reversed without affecting the final result.

Exercise 3. Confirm the statement on the previous page by finding both $\dfrac{\partial^2 z}{\partial x \partial y}$ and $\dfrac{\partial^2 z}{\partial y \partial x}$ for each of the following functions, whose first order partial derivatives have already been found in exercise 2. (Click on the green letters for solutions.)

(a) $z = (x^2+3x)\sin(y)$, (b) $z = \dfrac{\cos(x)}{y^5}$, (c) $z = \ln(xy)$,

(d) $z = \sin(x)\cos(xy)$, (e) $z = e^{(x^2+y^2)}$, (f) $z = \sin(x^2 + y)$.

Notation For first and second order partial derivatives there is a compact notation. Thus $\dfrac{\partial f}{\partial x}$ can be written as f_x and $\dfrac{\partial f}{\partial y}$ as f_y.

Similarly $\dfrac{\partial^2 f}{\partial x^2}$ is written f_{xx} while $\dfrac{\partial^2 f}{\partial x \partial y}$ is written f_{xy}.

Quiz If $z = e^{-y}\sin(x)$, which of the following is $z_{xx} + z_{yy}$?

(a) $e^{-y}\sin(x)$, (b) 0, (c) $-e^{-y}\sin(x)$, (d) $-e^{-y}\cos(x)$.

4. Quiz on Partial Derivatives

Choose the correct option for each of the following.

Begin Quiz

1. If $z = x^2 + 3xy + y^3$ then $\frac{\partial z}{\partial x}$ is

(a) $2x + 3y + 3y^2$, (b) $2x + 3x + 3y^2$, (c) $2x + 3x$, (d) $2x + 3y$.

2. If $w = 1/r$, where $r^2 = x^2 + y^2 + z^2$, then $xw_x + yw_y + zw_z$ is (a) $-1/r$, (b) $1/r$, (c) $-1/r^2$, (d) $1/r^2$.

3. If $u = \sqrt{\frac{x}{y}}$ then u_{xx} is

(a) $-\frac{1}{4\sqrt{y^3x^3}}$, (b) $-\frac{1}{4\sqrt{yx^3}}$, (c) $-\frac{1}{8\sqrt{y^3x^3}}$, (d) $-\frac{1}{8\sqrt{yx^3}}$.

End Quiz

Solutions to Exercises

Exercise 1(a) To calculate the partial derivative $\frac{\partial z}{\partial x}$ of the function $z = x^2y^4$, the factor y^4 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^4\right) = \frac{\partial}{\partial x}\left(x^2\right) \times y^4 = 2x^{(2-1)} \times y^4 = 2xy^4.$$

Similarly, to find the partial derivative $\frac{\partial z}{\partial y}$, the factor x^2 is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^4\right) = x^2 \times \frac{\partial}{\partial y}\left(y^4\right) = x^2 \times 4y^{(4-1)} = 4x^2y^3 .$$

Click on the green square to return

Exercise 1(b) To calculate $\frac{\partial z}{\partial x}$ for the function $z = (x^4 + x^2)y^3$, the factor y^3 is treated as a constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left((x^4 + x^2)y^3\right) = \frac{\partial}{\partial x}\left(x^4 + x^2\right) \times y^3 = (4x^3 + 2x)y^3 .$$

To find the partial derivative $\frac{\partial z}{\partial y}$ the factor $(x^4 + x^2)$ is treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left((x^4 + x^2)y^3\right) = (x^4 + x^2) \times \frac{\partial}{\partial y} y^3 = 3(x^4 + x^2)y^2 .$$

Click on the green square to return

Exercise 1(c) If $z = y^{\frac{1}{2}} \sin(x)$ then to calculate $\frac{\partial z}{\partial x}$ the $y^{\frac{1}{2}}$ factor is kept constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(y^{\frac{1}{2}} \sin(x)\right) = y^{\frac{1}{2}} \times \frac{\partial}{\partial x}(\sin(x)) = y^{\frac{1}{2}} \cos(x) .$$

Similarly, to evaluate the partial derivative $\frac{\partial z}{}$ the factor $\sin(x)$ is

∂y

treated as a constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(y^{\frac{1}{2}}\sin(x)\right) = \frac{\partial}{\partial y}y^{\frac{1}{2}} \times \sin(x) = \frac{1}{2}y^{-\frac{1}{2}}\sin(x).$$

Click on the green square to return

Exercise 2(a) The function $z = (x^2 + 3x)\sin(y)$ can be written as $z = uv$, where $u = (x^2 + 3x)$ and $v = \sin(y)$. The partial derivatives of u and v with respect to the variable x are

$$\frac{\partial u}{\partial x} = 2x + 3, \qquad \frac{\partial v}{\partial x} = 0,$$

while the partial derivatives with respect to y are

$$\frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial y} = \cos(y).$$

Applying the *product rule*

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x} = (2x+3)\sin(y)\,.$$

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} v + u \frac{\partial v}{\partial y} = (x^2+3x)\cos(y)\,.$$

Click on the green square to return

Exercise 2(b)

The function $z = \frac{\cos(x)}{y^5}$ can be written as $z = \cos(x)y^{-5}$.

Treating the factor y^{-5} as a constant and differentiating with respect to x:

$$\frac{\partial z}{\partial x} = -\sin(x)y^{-5} = -\frac{\sin(x)}{y^5}.$$

Treating $\cos(x)$ as a constant and differentiating with respect to y:

$$\frac{\partial v}{\partial y} = \cos(x)(-5y^{-6}) = -5\frac{\cos(x)}{y^6}.$$

Click on the green square to return

Exercise 2(c) The function $z = \ln(xy)$ can be rewritten as (see the package on logarithms)

$$z = \ln(xy) = \ln(x) + \ln(y).$$

Thus the partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\ln(x) + \ln(y)) = \frac{\partial}{\partial x}\ln(x) = \frac{1}{x}.$$

Similarly the partial derivative of z with respect to y is

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\ln(x) + \ln(y)) = \frac{\partial}{\partial y}\ln(y) = \frac{1}{y}.$$

Click on the green square to return

Exercise 2(d) To calculate the partial derivatives of the function $z = \sin(x)\cos(xy)$ the *product rule* has to be applied

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(xy)\frac{\partial}{\partial x}\sin(x) + \sin(x)\frac{\partial}{\partial x}\cos(xy)\\ \frac{\partial z}{\partial y} &= \cos(xy)\frac{\partial}{\partial y}\sin(x) + \sin(x)\frac{\partial}{\partial y}\cos(xy)\,.\end{aligned}$$

Using the *chain rule* with $u = xy$ for the partial derivatives of $\cos(xy)$

$$\begin{aligned}\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\\ \frac{\partial}{\partial y}\cos(xy) &= \frac{\partial\cos(u)}{\partial u}\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.\end{aligned}$$

Thus the partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(xy)\cos(x) - y\sin(x)\sin(xy)\,, \quad \frac{\partial z}{\partial y} = -x\sin(x)\sin(xy)$$

Click on the green square to return

Exercise 2(e) To calculate the partial derivatives of $z = e^{(x^2+y^2)}$ the *chain rule* has to be applied with $u = (x^2 + y^2)$:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}\left(e^{u}\right)\frac{\partial u}{\partial x} = e^{u}\frac{\partial u}{\partial x}\\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}\left(e^{u}\right)\frac{\partial u}{\partial y} = e^{u}\frac{\partial u}{\partial y}\ .,\end{aligned}$$

The partial derivatives of $u = (x^2 + y^2)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial (y^2)}{\partial y} = 2y\,.$$

Therefore the partial derivatives of the function $z = e^{(x^2+y^2)}$ are

Click on the green square to return

$$\begin{aligned}\frac{\partial z}{\partial x} &= \mathrm{e}^u \frac{\partial u}{\partial x} = 2x\,\mathrm{e}^{(x^2+y^2)}\\ \frac{\partial z}{\partial x} &= \mathrm{e}^u \frac{\partial u}{\partial x} = 2y\,\mathrm{e}^{(x^2+y^2)}\,.\end{aligned}$$

Exercise 2(f) Applying the *chain rule* with $u = x^2 + y$ the partial derivatives of the function $z = \sin(x^2 + y)$ can be written as

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial u}(\sin(u))\,\frac{\partial u}{\partial x} = \cos(u)\,\frac{\partial u}{\partial x}\\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial u}(\sin(u))\,\frac{\partial u}{\partial y} = \cos(u)\frac{\partial u}{\partial y}\,.\end{aligned}$$

The partial derivatives of $u = x^2 + y$ are

$$\frac{\partial u}{\partial x} = \frac{\partial x^2}{\partial x} = 2x\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial y}{\partial y} = 1\,.$$

Thus the partial derivatives of the function $z = \sin(x^2 + y)$ are

Click on the green square to return

$$\begin{aligned}\frac{\partial z}{\partial x} &= \cos(u)\frac{\partial u}{\partial x} = 2x\,\cos(x^2+y)\\ \frac{\partial z}{\partial y} &= \cos(u)\frac{\partial u}{\partial y} = \cos(x^2+y)\,.\end{aligned}$$

Click on the green square to return

Exercise 3(a)

From **exercise 2(a)**, the first order partial derivatives of $z = (x^2 + 3x)\sin(y)$ are

$$\frac{\partial z}{\partial x} = (2x+3)\sin(y), \qquad \frac{\partial z}{\partial y} = (x^2+3x)\cos(y).$$

The *mixed* second order derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left((x^2+3x)\cos(y)\right) = (2x+3)\cos(y) \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left((2x+3)\sin(y)\right) = (2x+3)\cos(y)\,. \end{aligned},$$

Click on the green square to return

Exercise 3(b)

From **exercise 2(b)**, the first order partial derivatives of $z = \dfrac{\cos(x)}{y^5}$ are

$$\frac{\partial z}{\partial x} = -\frac{\sin(x)}{y^5}, \qquad \frac{\partial z}{\partial y} = -5\frac{\cos(x)}{y^6},$$

so the *mixed* second order derivatives are

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-5\frac{\cos(x)}{y^6}\right) = 5\frac{\sin(x)}{y^6} \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{y^5}\right) = 5\frac{\sin(x)}{y^6}. \end{aligned},$$

Click on the green square to return

Exercise 3(c)

From **exercise 2(c)**, the first order partial derivatives of $z = \ln(xy)$ are

$$\frac{\partial z}{\partial x} = \frac{1}{x}, \qquad \frac{\partial z}{\partial y} = \frac{1}{y}.$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) = 0 \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0.
\end{aligned}$$

Click on the green square to return

Exercise 3(d) From **exercise 2(d)**, the first order partial derivatives of $z = \sin(x)\cos(xy)$ are

$$\frac{\partial z}{\partial x} = \cos(x)\cos(xy) - y\sin(x)\sin(xy), \qquad \frac{\partial z}{\partial y} = -x\sin(x)\sin(xy).$$

The *mixed* second order derivatives are

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-x\sin(x)\sin(xy)\right) \\
&= -\sin(x)\sin(xy) - x\cos(x)\sin(xy) - xy\sin(x)\cos(xy) \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos(x)\cos(xy) - y\sin(x)\sin(xy)\right) \\
&= -x\cos(x)\sin(xy) - \sin(x)\sin(xy) - xy\sin(x)\cos(xy)\,,
\end{aligned}$$

N.B. In the solution above a *product of three functions* has been differentiated. This can be done by using two applications of the *product rule*. Click on the green square to return

Exercise 3(e) From **exercise 2(e)**, the first order partial derivatives of $z = \mathrm{e}^{(x^2+y^2)}$ are

$$\frac{\partial z}{\partial x} = 2x\mathrm{e}^{(x^2+y^2)}\,, \qquad \frac{\partial z}{\partial y} = 2y\mathrm{e}^{(x^2+y^2)}\,.$$

The *mixed* second order derivatives are thus

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y\mathrm{e}^{(x^2+y^2)}\right) = 4xy\mathrm{e}^{(x^2+y^2)}\,, \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\mathrm{e}^{(x^2+y^2)}\right) = 4yx\mathrm{e}^{(x^2+y^2)}\,. \end{aligned}$$

Click on the green square to return

Exercise 3(f) From **exercise 2(f)**, the first order partial derivatives of $z = \sin(x^2 + y)$ are

$$\frac{\partial z}{\partial x} = 2x\cos(x^2+y), \qquad \frac{\partial z}{\partial y} = \cos(x^2+y).$$

The *mixed* second order derivatives are thus

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\cos(x^2+y)\right) = -2x\sin(x^2+y)\,, \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\cos(x^2+y)\right) = -2x\sin(x^2+y).
\end{aligned}$$

Click on the green square to return

Solutions to Quizzes

Solution to Quiz:

To determine which of the options is correct, the partial derivatives of $z = \cos(xy)$ must be calculated. From the calculations of **exercise 2(d)** the partial derivatives of $z = \cos(xy)$ are

$$\begin{aligned}\frac{\partial}{\partial x}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\frac{\partial u}{\partial x} = -\sin(u)y = -y\sin(xy)\\ \frac{\partial}{\partial y}\cos(xy) &= \frac{\partial \cos(u)}{\partial u}\frac{\partial u}{\partial y} = -\sin(u)x = -x\sin(xy)\,.\end{aligned}$$

Therefore

$$\frac{1}{y}\frac{\partial}{\partial x}\cos(xy) = -\sin(xy) = \frac{1}{x}\frac{\partial}{\partial y}\cos(xy)$$

The other choices, if checked, will be found to be false. End Quiz

Solutions to Quizzes

Solution to Quiz:

The first order derivatives of $z = e^{-y}\sin(x)$ are

$$z_x = e^{-y}\cos(x), \qquad z_y = -\,e^{-y}\sin(x),$$

where e^{-y} is kept constant for the first differentiation and $\sin(x)$ for the second. Continuing in this way, the second order derivatives z_{xx} and z_{yy} are given by the expressions

$$\begin{aligned} z_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{-y}\cos(x)\right) = -e^{-y}\sin(x) \\ z_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-e^{-y}\sin(x)\right) = e^{-y}\sin(x)\,. \end{aligned}$$

,

Adding these two equations together gives

$$z_{xx} + z_{yy} = 0.$$

End Quiz