

# **MULTIPLE INTEGRALS**

# TOPICS DISCUSSED

- ❖ INTRODUCTION
- ❖ REGION OF INTEGRATION
- ❖ CHANGING THE ORDER OF INTEGRATION
- ❖ PLANE AREA USING DOUBLE INTEGRATION
  - CARTISEAN FORM
  - POLAR FORM

# INTRODUCTION

- ❖ When a function  $f(x)$  is integrated with respect to  $x$  between the limits  $a$  and  $b$ , we get the double integral  $\int_a^b f(x)dx$ .
- ❖ If the integrand is a function  $f(x, y)$  and if it is integrated with respect to  $x$  and  $y$  repeatedly between the limits  $x_0$  and  $x_1$  (for  $x$ ) and between the limits  $y_0$  and  $y_1$  (for  $y$ ) we get a **double integral** that is denoted by the symbol

$$\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) dx dy.$$

- ❖ Extending the concept of double integral one step further, we get the **triple integral**, denoted by

$$\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z) dx dy dz .$$

# EVALUATION OF DOUBLE AND TRIPLE INTEGRALS

- ❖ To evaluate  $\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) dx dy$  first integrate  $f(x, y)$  with respect to  $x$  partially, treating  $y$  as constant temporarily, between the limits  $x_0$  and  $x_1$ .
- ❖ Then integrate the resulting function of  $y$  with respect to  $y$  between the limits  $y_0$  and  $y_1$  as usual.
- ❖ In notation  $\int_{y_0}^{y_1} \left[ \int_{x_0}^{x_1} f(x, y) dx \right] dy$  ( for double integral)

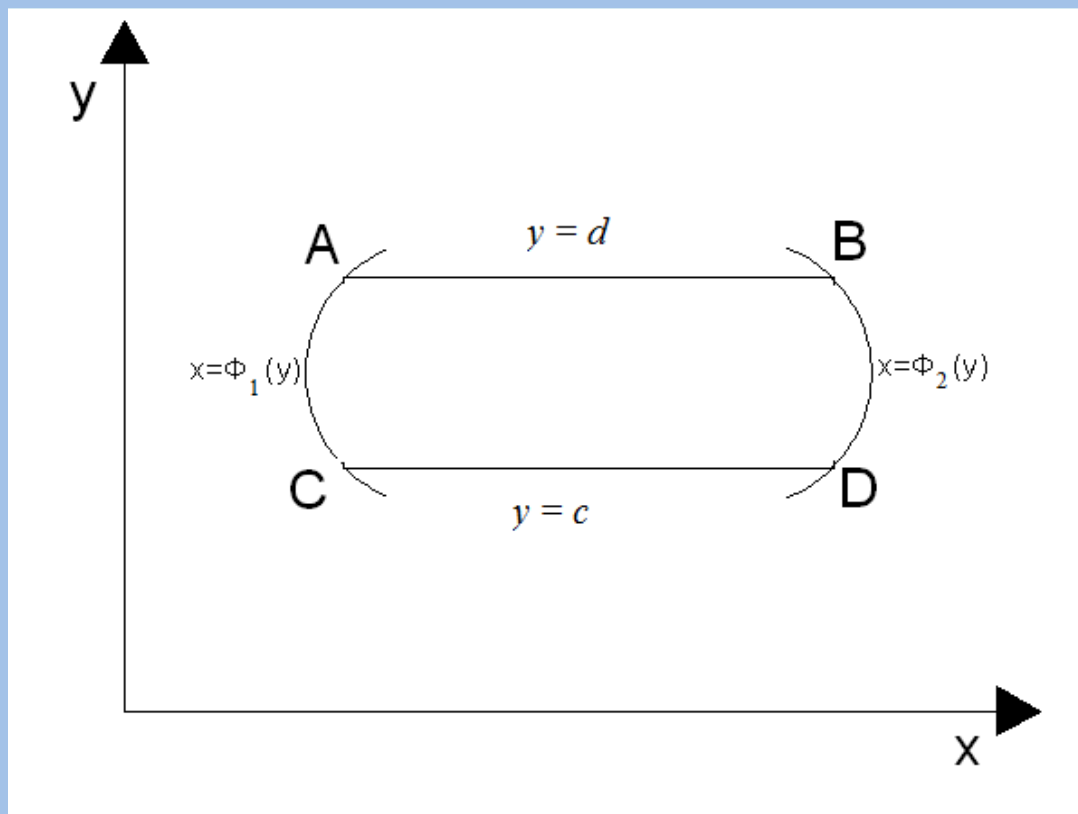
$\int_{z_0}^{z_1} \left\{ \int_{y_0}^{y_1} \left[ \int_{x_0}^{x_1} f(x, y, z) dx \right] dy \right\} dz$  ( for triple integral).

Note:

- ❖ Integral with variable limits should be the innermost integral and it should be integrated first and then the constant limits.

# REGION OF INTEGRATION

Consider the double integral  $\int_c^d \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) dx dy$ ,  $x$  varies from  $\phi_1(y)$  to  $\phi_2(y)$  and  $y$  varies from  $c$  to  $d$ . (i.e)  $\phi_1(y) \leq x \leq \phi_2(y)$  and  $c \leq y \leq d$ . These inequalities determine a region in the  $xy$  - plane, which is shown in the following figure. This region ABCD is known as the region of integration



### EXAMPLE :1

Evaluate  $\int_0^1 \int_0^2 y^2 x \, dy \, dx$

**Solution:**

$$\int_0^1 \int_0^2 y^2 x \, dy \, dx = \int_0^1 x [y/3]_0^2 \, dx$$

$$= \frac{8}{3} \int_0^1 x \, dx$$

$$= \frac{8}{3} \left[ \frac{x^2}{2} \right]_0^1$$

$$= \frac{4}{3}$$

### EXAMPLE :2

Evaluate  $\int_2^3 \int_1^2 \frac{1}{xy} dy dx$

**Solution:**

$$\begin{aligned}\int_2^3 \int_1^2 \frac{1}{xy} dy dx &= \int_2^3 [\log x]_1^2 \frac{1}{y} dy \\ &= (\log 2 - \log 1) \int_2^3 \frac{1}{y} dy \\ &= \log 2 [\log y]_2^3 \\ &= \log 2 (\log 3 - \log 2) \\ &= \log 2 \cdot \log \left( \frac{3}{2} \right)\end{aligned}$$

### EXAMPLE :3

Evaluate  $\int_0^2 \int_1^3 \int_1^2 xy^2 z dz dy dx$

**Solution:**

$$\begin{aligned}\int_0^2 \int_1^3 \int_1^2 xy^2 z dz dy dx &= \int_0^2 \int_1^3 \left[ \frac{z^2}{2} \right]_1^2 xy^2 dy dx \\ &= \int_0^2 \int_1^3 \frac{3}{2} xy^2 dy dx \\ &= \frac{3}{2} \int_0^2 \left[ \frac{y^3}{3} \right]_1^3 x dx \\ &= \frac{26}{2} \left[ \frac{x^2}{2} \right]_0^2 = 26\end{aligned}$$



### EXAMPLE :4

Evaluate  $\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz$

**Solution:**

$$\begin{aligned}\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz &= \int_0^1 dx \int_0^2 dy \left[ \frac{z^2}{2} \right]_1^2 yx^2 \\ &= \frac{3}{2} \int_0^1 \left[ \frac{y^2}{2} \right]_0^2 x^2 dx \\ &= \frac{3}{2} \int_0^1 2 x^2 dx \\ &= \left[ \frac{x^3}{3} \right]_0^2 = 1\end{aligned}$$

### EXAMPLE :5

Evaluate  $\int_0^\pi \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin \theta \, dr d\theta d\phi$

**Solution:**

$$\begin{aligned}\int_0^\pi \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin \theta \, dr d\theta d\phi &= \int_0^\pi \int_0^{\frac{\pi}{2}} \sin \theta \left[ \frac{r^3}{3} \right]_0^1 d\theta d\phi \\&= \frac{1}{3} \int_0^\pi \int_0^{\frac{\pi}{2}} \sin \theta d\theta d\phi \\&= \frac{1}{3} \int_0^\pi [-\cos \theta]_0^{\frac{\pi}{2}} d\phi \\&= \frac{1}{3} \int_0^\pi d\phi \\&= \frac{\pi}{3}\end{aligned}$$

### EXAMPLE :6

Evaluate  $\int_0^1 \int_0^x dx dy$

**Solution:**

$$\begin{aligned}\int_0^1 \int_0^x dy dx &= \int_0^1 [y]_0^x dx \\ &= \int_0^1 x dx \\ &= \left[ \frac{x^2}{2} \right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

### EXAMPLE :7

Evaluate  $\int_0^a \int_0^x \int_0^y xyz dx dy dz$

**Solution:**

$$\begin{aligned} I &= \int_0^a \int_0^x \left[ \int_0^y z dz \right] xy dy dx \\ &= \int_0^a \int_0^x \left[ \frac{z^2}{2} \right]_0^y xy dy dx \\ &= \int_0^a \int_0^x \left[ \frac{y^2}{2} \right] xy dy dx \\ &= \int_0^a \int_0^x \left[ \frac{y^3}{2} \right] dy x dx = \int_0^a \left[ \frac{y^4}{8} \right]_0^x x dx \\ &= \left[ \frac{x^6}{48} \right]_0^a = \frac{a^6}{48} \end{aligned}$$

### EXAMPLE :8

Evaluate  $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx dy dz}{\sqrt{1-x^2-y^2-z^2}}$

**Solution:**

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[ \sin^{-1} \left( \frac{z}{\sqrt{1-x^2-y^2}} \right) \right]_0^{\sqrt{1-x^2-y^2}} dx dy \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dx dy = \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx \\ &= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx \\ &= \frac{\pi}{2} \left[ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right]_0^1 \\ &= \frac{\pi^2}{8} \end{aligned}$$

### EXAMPLE :9

Evaluate  $\int_0^\pi \int_0^{a \sin \theta} r dr d\theta$

**Solution:**

$$\begin{aligned} I &= \int_0^\pi \left[ \frac{r^2}{2} \right]_0^{a \sin \theta} d\theta \\ &= \frac{1}{2} \int_0^\pi a^2 \sin^2 \theta d\theta \\ &= \frac{a^2}{2} \int_0^\pi \left[ \frac{1 - \cos 2\theta}{2} \right] d\theta \\ &= \frac{a^2}{2} \times \frac{1}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^\pi = \frac{\pi a^2}{4} \end{aligned}$$

# PROBLEMS FOR PRACTICE

Evaluate the following

1.  $\int_0^2 \int_0^1 4xy \, dx dy$

Ans: 4

2.  $\int_1^b \int_1^a \frac{1}{xy} \, dx dy$

Ans:  $\log a \cdot \log b$

3.  $\int_0^1 \int_0^x dx dy$

Ans:  $1/2$

4.  $\int_0^\pi \int_0^{\sin \theta} r \, dr d\theta$

Ans:  $\pi/4$

5.  $\int_0^1 \int_0^2 \int_0^3 xyz \, dx dy dz$

Ans:  $9/2$

6.  $\int_0^1 \int_0^z \int_0^{y+z} dz dy dx$

Ans:  $1/2$

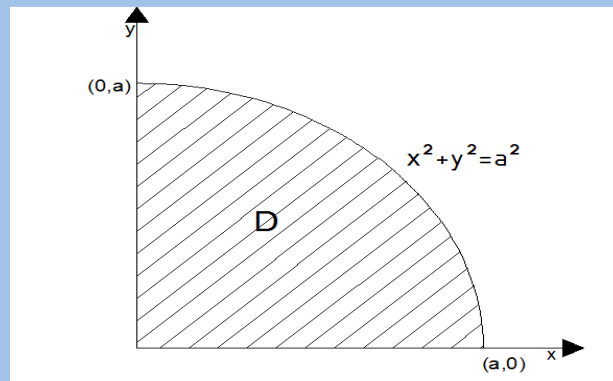
### EXAMPLE :10

Sketch the region of integration for  $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x, y) dy dx$ .

#### Solution:

Given  $x = 0$  and  $x = a$  ;  $y = 0$  and  $y^2 = a^2 - x^2$

$$y = 0 \text{ and } x^2 + y^2 = a^2$$



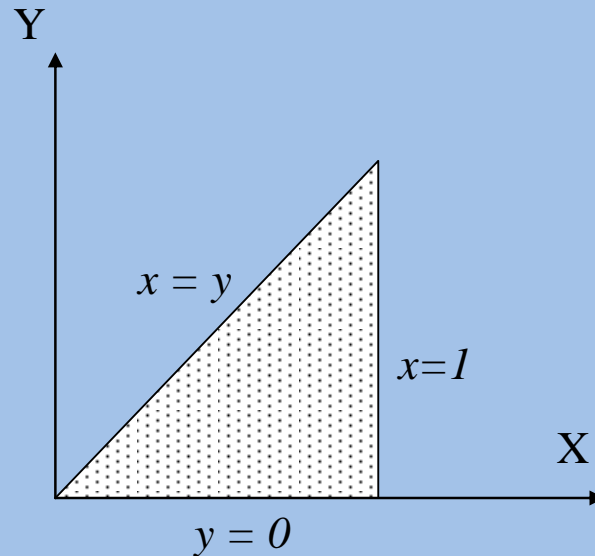


### EXAMPLE :11

Sketch the region of integration for  $\int_0^1 \int_0^x f(x, y) dy dx$ .

### Solution:

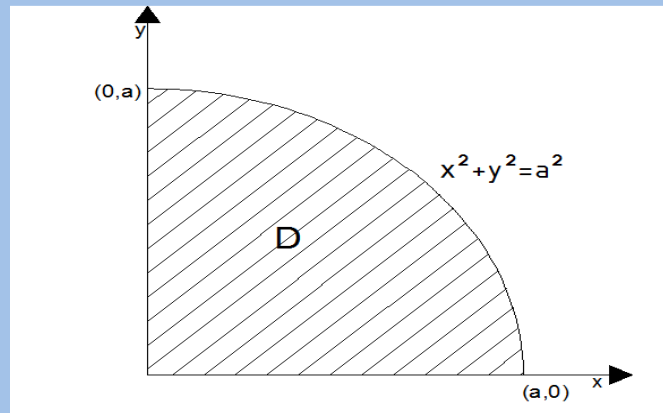
Given  $x = 0$  ;  $x = 1$  and  $y = 0$  ;  $y = x$ .



### EXAMPLE :12

Evaluate  $\iiint_D xyz \, dx dy dz$  where D is the region bounded by the positive octant of the sphere  $x^2 + y^2 + z^2 = a^2$

**Solution:**



$$\begin{aligned} I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz dy dx \\ &= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[ \frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dy dx \end{aligned}$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy (a^2 - x^2 - y^2) dy dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} x (a^2 y - y x^2 - y^3) dy dx$$

$$= \frac{1}{2} \int_0^a \left[ a^2 \frac{y^2}{2} - x^2 \frac{y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} x dx$$

$$= \frac{1}{8} \int_0^a x (a^2 - x^2)^2 dx$$

$$= \frac{1}{8} \int_0^a (a^4 x - 2a^2 x^3 + x^5) dx$$

$$= \frac{1}{8} \left[ a^4 \frac{x^2}{2} - 2a^2 \frac{x^4}{4} - \frac{x^6}{6} \right]_0^a = \frac{a^6}{48}.$$

# PROBLEMS FOR PRACTICE

1. Sketch the region of integration for the following

$$(i) \int_0^4 \int_{\frac{y^2}{4}}^y \frac{y dx dy}{x^2 + y^2}$$

$$(ii) \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y dy dx$$

$$(iii) \int_0^1 \int_x^1 \frac{y dx dy}{x^2 + y^2}$$

2. Evaluate  $\iiint_V (xy + yz + zx) dx dy dz$ , where  $V$  is the region of space bounded by  $x=0, x=1, y=0, y=2, z=0$  and  $z=3$ .

Ans:  $33/2$

3. Evaluate  $\iiint_V \frac{dx dy dz}{(1+x+y+z)^3}$ , where  $V$  is the region of space bounded by  $x=0, y=0, z=0$  and  $x+y+z=1$

Ans:  $\frac{1}{16} (8 \log 2 - 5)$

4. Evaluate  $\iiint_V dx dy dz$ , where  $V$  is the region of space bounded by  $x=0, y=0, z=0$  and  $2x+3y+4z=12$ .

Ans: 12

## CHANGE OF ORDER OF INTEGRATION

- ❖ If the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.
- ❖ When the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e.  $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy$  will take the form

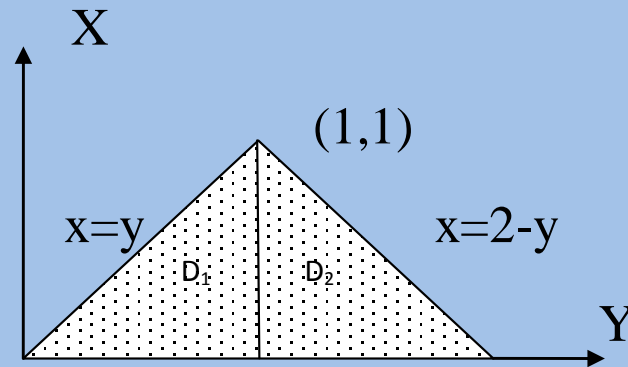
$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x, y) dy dx$$

- ❖ This process of converting a given double integral into its equivalent double integral by changing the order of integration is called the **change of order of integration**.

### EXAMPLE :13

Evaluate  $\int_0^1 \int_y^{2-y} xy dx dy$  by changing the order of integration.

**Solution:**



Given  $y : 0$  to  $1$  and  $x : y$  to  $2-y$

By changing the order of integration,

In Region  $D_1$   $x : 0$  to  $1$  and  $y : 0$  to  $x$ .

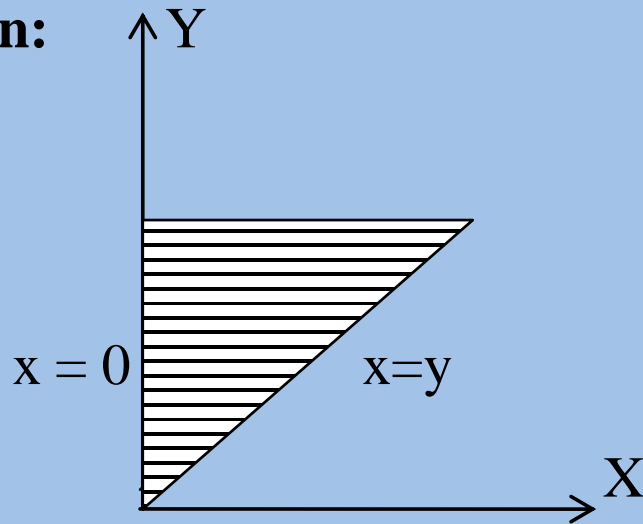
In Region  $D_2$   $x : 1$  to  $2$  and  $y : 0$  to  $2-x$ .

$$\begin{aligned}\int_0^1 \int_y^{2-y} xy dx dy &= \int_0^1 \int_0^x xy dy dx + \int_1^2 \int_0^{2-x} xy dy dx \\&= \int_0^1 x \left[ \frac{y^2}{2} \right]_0^x dx + \int_1^2 x \left[ \frac{y^2}{2} \right]_0^{2-x} dx \\&= \frac{1}{2} \int_0^1 x^3 dx + \frac{1}{2} \int_1^2 [4x - 4x^2 + x^3] dx \\&= \frac{1}{2} \left[ \frac{x^4}{4} \right]_0^1 + \frac{1}{2} \left[ 2x^2 - \frac{4x^3}{3} + \frac{x^4}{4} \right]_1^2 \\&= \frac{1}{8} + \frac{5}{24} = \frac{1}{3}\end{aligned}$$

### EXAMPLE :14

Evaluate  $\int_0^{\infty} \int_0^y y e^{-\frac{y^2}{x}} dx dy$  by changing the order of integration.

**Solution:**



Given  $x=0$ ,  $x = y$ ,  $y = 0$ ,  $y = \infty$  .

By changing the order of integration  $y: x$  to  $\infty$ ,  $x : 0$  to  $\infty$



$$\begin{aligned}
\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}} dx dy &= \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} dy dx \\
&= \int_0^\infty \int_x^\infty ye^{-\frac{y^2}{x}} d\left(\frac{y^2}{2}\right) dx \\
&= \frac{1}{2} \int_0^\infty \left[ \frac{e^{-\frac{y^2}{x}}}{-1/x} \right]_x^\infty dx = \frac{1}{2} \int_0^\infty xe^{-x} dx
\end{aligned}$$

Take  $u = x, dv = e^{-x} dx$  implies  $du = dx, v = -e^{-x}$ ,

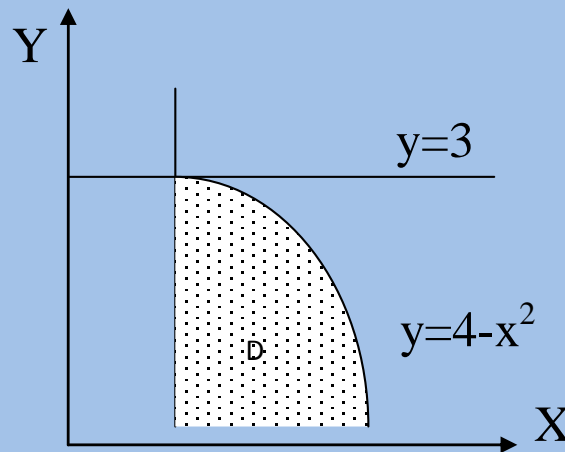
by integration by parts,

$$= \frac{1}{2} \left[ x \left( \frac{e^{-x}}{-1} \right) - e^{-x} \right]_0^\infty = \frac{1}{2}$$

### EXAMPLE :15

Evaluate  $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy$  by changing the order of integration.

**Solution:**



Given  $y=0, y=3$  and  $x=1, x=\sqrt{4-y}$

By changing the order of integration,

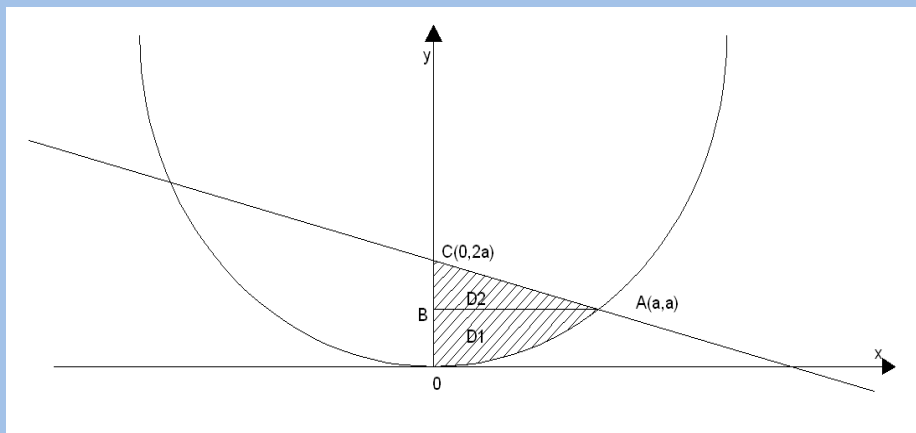
In region D,  $x : 1$  to  $2$  and  $y : 0$  to  $4-x^2$

$$\begin{aligned}
\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy &= \int_1^2 \int_0^{4-x^2} (x+y) dy dx \\
&= \int_1^2 \left[ xy + \frac{y^2}{2} \right]_0^{4-x^2} dx \\
&= \int_1^2 \left[ x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx \\
&= \int_1^2 \left[ \frac{x^4}{4} - x^3 - 4x^2 + 4x + 8 \right] dx \\
&= \left[ \frac{x^5}{10} - \frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2 + 8x \right]_1^2 \\
&= \frac{241}{8}
\end{aligned}$$

### EXAMPLE :16

Evaluate  $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$  by changing the order of integration.

**Solution:**



Given  $y : x^2/a$  to  $2a - x$  and  $x : 0$  to  $a$

By changing the order of integration,

In Region  $D_1$   $x : 0$  to  $\sqrt{ay}$  and  $y : 0$  to  $a$ .

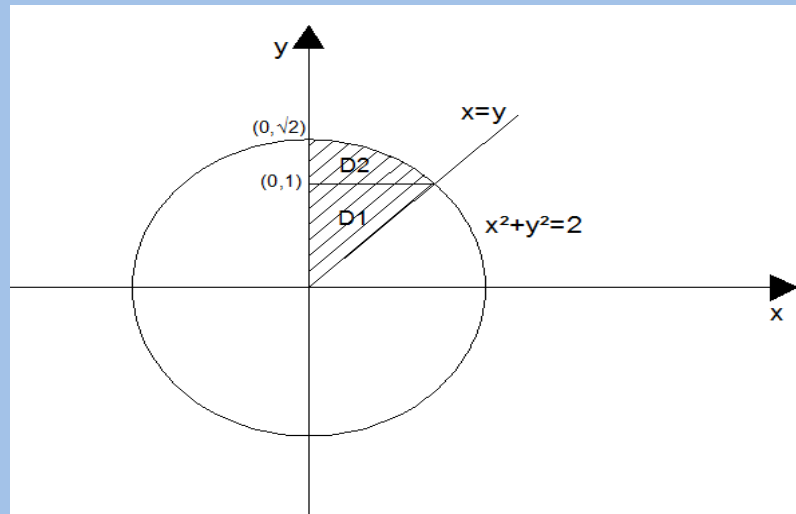
In Region  $D_2$   $x : 0$  to  $2a - y$  and  $y : a$  to  $2a$ .

$$\begin{aligned}
\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx &= \int_0^a \int_0^{\sqrt{ay}} xy \, dy \, dx + \int_a^{2a} \int_0^{2a-y} xy \, dy \, dx \\
&= \int_0^a y \left[ \frac{x^2}{2} \right]_0^{\sqrt{ay}} dy + \int_0^1 y \left[ \frac{x^2}{2} \right]_0^{2a-y} dy \\
&= \frac{a}{2} \int_0^a y^2 \, dy + \frac{1}{2} \int_a^{2a} [4a^2y - 4ay^2 + y^3] dy \\
&= \frac{a}{2} \left[ \frac{y^3}{3} \right]_0^a + \frac{1}{2} \left[ 2a^2y^2 - \frac{4ay^3}{3} + \frac{y^4}{4} \right]_a^{2a} \\
&= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}.
\end{aligned}$$

### EXAMPLE :17

Evaluate  $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy dx$  by changing the order of integration.

**Solution:**



Given  $x = 0$ ,  $x = 1$  and  $y = x$ ,  $y^2 = 2 - x^2$

By changing the order of integration

In Region  $D_1$ ,  $y : 0$  to  $1$ ,  $x : 0$  to  $y$

In Region  $D_2$ ,  $y : 1$  to  $\sqrt{2}$ ,  $x : 0$  to  $\sqrt{2 - y^2}$

$$\begin{aligned}
I &= \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}} dx dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dx dy \\
&= \int_0^1 \left[ \sqrt{x^2+y^2} \right]_0^{\sqrt{2}} dy + \int_1^{\sqrt{2}} \left[ \sqrt{x^2+y^2} \right]_0^{\sqrt{2-y^2}} dy \\
&= \int_0^1 (\sqrt{2}y - y) dy + \int_1^{\sqrt{2}} (\sqrt{2} - y) dy \\
&= \left( (\sqrt{2} - 1) \frac{y^2}{2} \right)_0^1 + \left( \sqrt{2}y - \frac{y^2}{2} \right)_1^{\sqrt{2}} \\
&= 1 - \frac{1}{\sqrt{2}}
\end{aligned}$$

# PROBLEMS FOR PRACTICE

Evaluate the following by changing the order of integration

1.  $\int_0^a \int_x^a (x^2 + y^2) dy dx$  Ans:  $\frac{a^4}{3}$

2.  $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy dy dx$  Ans:  $\frac{3a^4}{8}$

3.  $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y dx dy$  Ans:  $\frac{a^3}{6}$

4.  $\int_0^1 \int_y^{2-y} xy dx dy$  Ans:  $\frac{1}{3}$



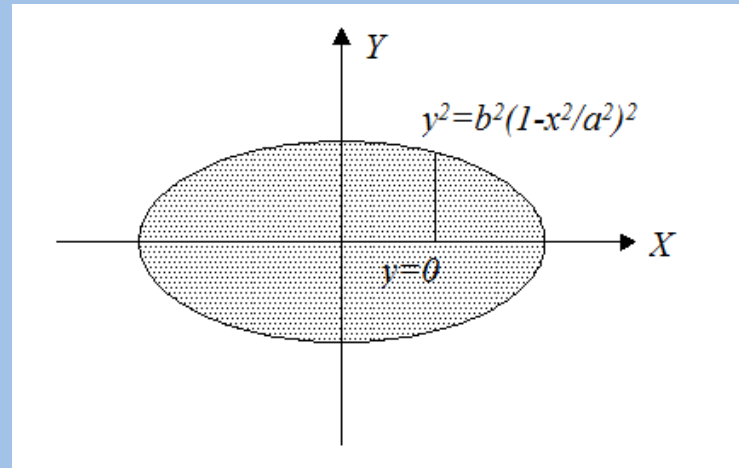
# PLANE AREA USING DOUBLE INTEGRAL

## **CARTESIAN FORM**

**EXAMPLE :18**

Find by double integration, the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Solution:**



$$A = 4 \iint dy dx = 4 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy dx$$

$$= 4 \int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx$$

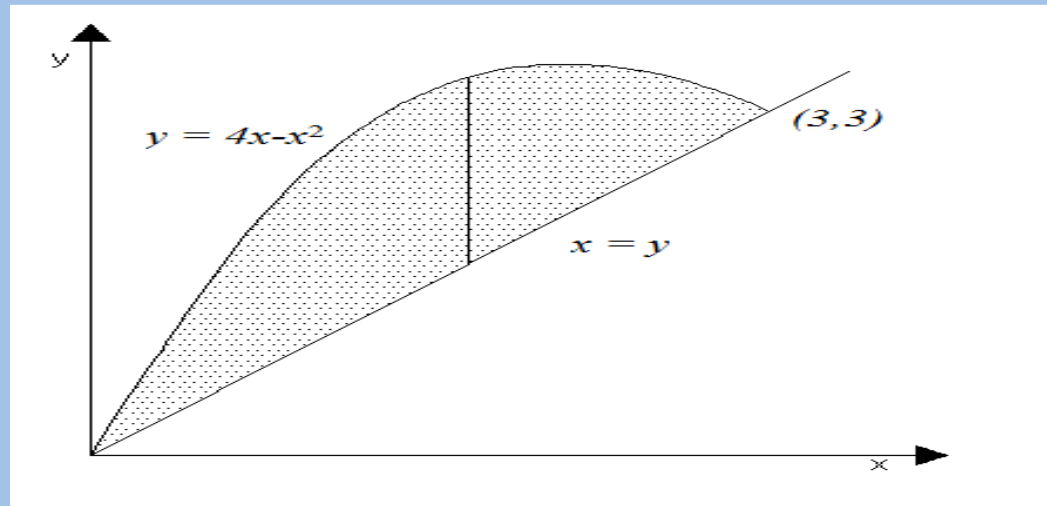
$$= \frac{4b}{a} \left[ \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi ab \text{ sq. units.}$$

**EXAMPLE :19**

Find the area between the parabola  $y = 4x - x^2$  and the line  $y = x$ .

**Solution:**



Given  $y = 4x - x^2$  and  $y = x$ , solving for x,

$$x = 4x - x^2 \Rightarrow 0 = 3x - x^2 \Rightarrow 0 = (3 - x)x \Rightarrow x = 0, 3$$

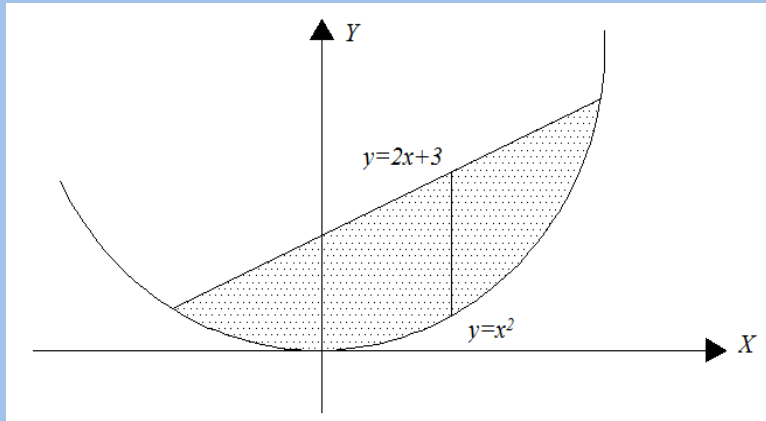
$$A = \int_0^3 \int_x^{4x-x^2} dy dx = \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (3x - x^2) dx$$

$$= \left[ \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{9}{2}$$

### EXAMPLE :20

Find the area between the parabola  $y = x^2$  and the line  $y = 2x + 3$ .



**Solution:**

Given  $y = x^2$  and  $y = 2x + 3$ .

solving for  $x$ ,  $x^2 = 2x + 3 \Rightarrow x = -1, 3$

$$A = \int_{-1}^3 \int_{x^2}^{2x+3} dy dx = \int_{-1}^3 [y]_{x^2}^{2x+3} dx$$

$$= \int_{-1}^3 (2x + 3 - x^2) dx$$

$$= \left[ \frac{2x^2}{2} + 3x - \frac{x^3}{3} \right]_{-1}^3 = \frac{32}{3}$$

# PLANE AREA USING DOUBLE INTEGRAL

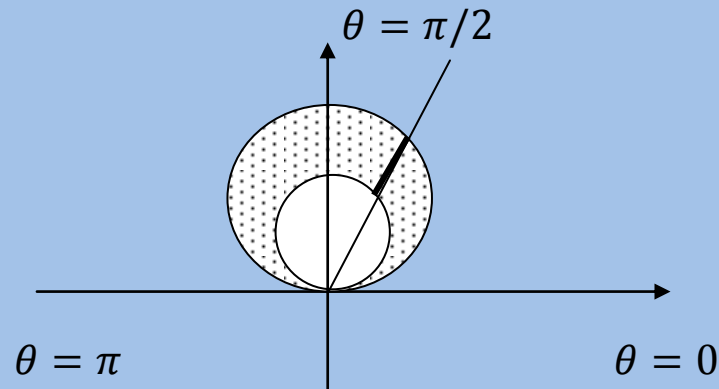
## **POLAR FORM**

### EXAMPLE :21

Find the area bounded by the circle

$$r = 2 \sin \theta \text{ and } r = 4 \sin \theta.$$

**Solution:**



$$A = \int_0^{\pi} \int_{2 \sin \theta}^{4 \sin \theta} r \, dr \, d\theta = \int_0^{\pi} \left[ \frac{r^2}{2} \right]_{2 \sin \theta}^{4 \sin \theta} d\theta$$

$$= 6 \int_0^{\pi} (\sin \theta)^2 d\theta$$

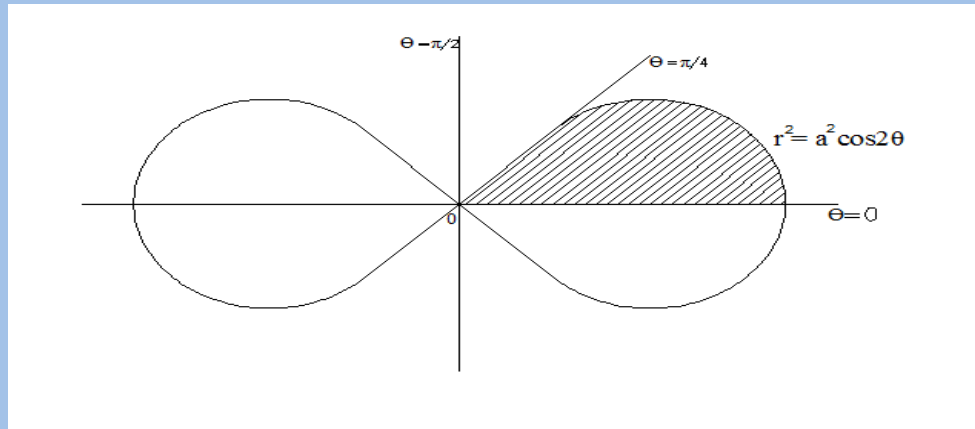
$$= 3 \int_0^{\pi} (1 - \cos 2\theta) d\theta$$

$$= 3 \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} = 3\pi .$$

### EXAMPLE :22

Find the area enclosed by the lemniscate  $r^2 = a^2 \cos 2\theta$  by double integration.

**Solution:**



If  $r = 0$  then  $\cos 2\theta = 0$  implies  $\theta = \frac{\pi}{4}$ .

$$A = 4 \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{a^2 \cos 2\theta}} r \, dr \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} \left[ \frac{r^2}{2} \right]_0^{\sqrt{a^2 \cos 2\theta}} d\theta$$

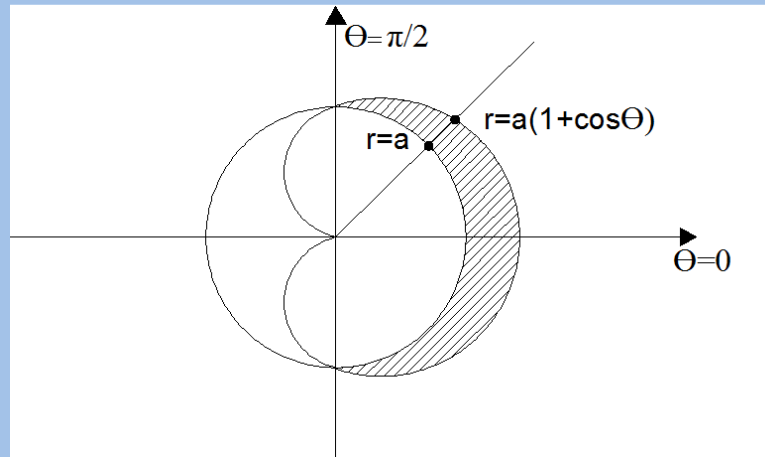
$$= 4a^2 \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta}{2} d\theta$$

$$= 4 \left[ \frac{a^2 \sin 2\theta}{4} \right]_0^{\frac{\pi}{4}} = a^2.$$

### EXAMPLE :23

Find the area that lies inside the cardioids  $r = a(1 + \cos \theta)$  and outside the circle  $r = a$ , by double integration.

**Solution:**



Solving  $r = a(1 + \cos \theta)$  and  $r = a$

$$\Rightarrow a(1 + \cos \theta) = a$$

$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}.$$



$$A = 2 \int_0^{\frac{\pi}{2}} \int_a^{a(1+\cos \theta)} r \, dr \, d\theta = 2 \int_0^{\frac{\pi}{2}} \left[ \frac{r^2}{2} \right]_a^{a(1+\cos \theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}} [(a(1 + \cos \theta))^2 - a^2] d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} [2 \cos \theta + (\cos \theta)^2] d\theta$$

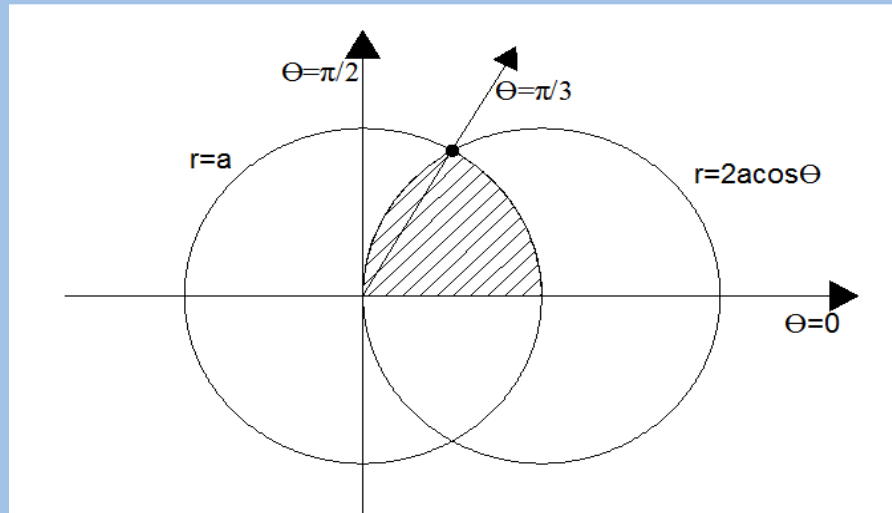
$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} [4 \cos \theta + 1 + \cos 2\theta] d\theta$$

$$= \frac{a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} + 4 \sin \theta \right]_0^{\frac{\pi}{2}} = \frac{a^2}{2} (\pi + 8) .$$

**EXAMPLE :24**

Find the common area to the circles  $r = a$  ,  $r = 2a \cos \theta$ .

**Solution:**



Given  $r = a$  ,  $r = 2a \cos \theta$  , solving

$$\Rightarrow a = 2a \cos \theta$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \pi/3$$

when  $r = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \pi/2$

$$A = 2 \iint r dr d\theta$$

$$= 2 \int_0^{\frac{\pi}{3}} \int_0^a r dr d\theta + 2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2a \cos \theta} r dr d\theta$$

$$= 2 \int_0^{\frac{\pi}{3}} \left[ \frac{r^2}{2} \right]_0^a d\theta + 2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[ \frac{r^2}{2} \right]_0^{2a \cos \theta} d\theta$$

$$= a^2 \int_0^{\frac{\pi}{3}} d\theta + 2a^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos \theta)^2 d\theta$$

$$= a^2 [\theta]_0^{\frac{\pi}{3}} + 2a^2 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= a^2 \frac{\pi}{3} + 2a^2 \left( \frac{\pi}{2} - \frac{\pi}{3} \right) - a^2 \frac{\sqrt{3}}{2}$$

$$= a^2 \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right)$$

# PROBLEMS FOR PRACTICE

1. Find by double integration, the area bounded by the parabolas  $x^2 = 4ay$  and  $y^2 = 4ax$ .

Ans:  $\frac{16a^2}{3} \text{ sq. units.}$

2. Find by double integration, the smallest area bounded by the circle  $x^2 + y^2 = 9$  and the line  $x + y = 3$ .

Ans:  $\frac{9}{4} (\pi - 2) \text{ sq. units.}$

3. Find by double integration, the area common to the parabola  $y^2 = x$  and the circle  $x^2 + y^2 = 2$ .

Ans:  $\left(\frac{1}{3} + \frac{\pi}{2}\right) \text{ sq units.}$

4. Find by double integration, the area lying inside the circle  $r = a \sin \theta$  and outside the coordinate  $r = a(1 - \cos \theta)$ .

Ans:  $a^2 \left(1 - \frac{\pi}{4}\right) \text{ sq. units.}$

**THE END**