

Problem G1. We flip a fair coin ten times, recording a 0 for tails and 1 for heads. In this way we obtain a binary string of length 10.

- (a) Find the probability there is exactly one pair of consecutive equal digits.

Solution Given that the coin is flipped 10 times, we can state that there are 2^{10} different binary string patterns. There are 9 potential consecutive pairs of 0 in a binary string of length 10, so there are ${}_9C_1$ possible instances of the presence of only one pair in the string. The pairs are not limited to one side of the coin, so for this problem there are $2 \cdot {}_9C_1$ possible instances. Therefore, we can deduce that the probability is:

$$\frac{2 \cdot {}_9C_1}{2^{10}} = \frac{{}_9C_1}{2^9} = \frac{9}{512}$$

- (b) Find the probability there are exactly n pairs of consecutive digits, for every $n = 0, \dots, 9$.

Solution Continuing with the process used in (a), we can apply the same equation to every $n = 0, \dots, 9$:

$$\frac{2 \cdot {}_9C_n}{2^{10}} = \frac{{}_9C_n}{2^9}$$

Problem G2. For which positive integers p is there a nonzero real number t such that

$$t + \sqrt{p} \text{ and } \frac{1}{t} + \sqrt{p}$$

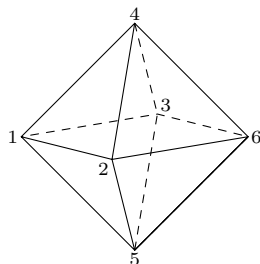
are both rational?

Solution Yeah idk yet man

Problem G3. Points A and B are two opposite vertices of a regular octahedron. An ant starts at point A and, every minute, walks randomly to a neighboring vertex.

- (a) Find the expected (i.e. average) amount of time for the ant to reach vertex B .

Solution Let t_i be the expected amount of time for the ant to reach vertex 6 from vertex i on the octahedron.



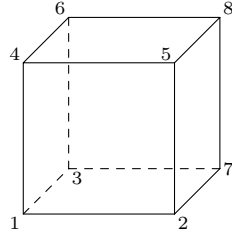
$$\begin{aligned}
t_1: & 1 + \frac{1}{4}(t_2 + t_3 + t_4 + t_5) \\
t_2 \text{ and } t_3: & 1 + \frac{1}{4}(t_1 + t_4 + t_5 + t_6) \\
t_4 \text{ and } t_5: & 1 + \frac{1}{4}(t_1 + t_2 + t_3 + t_6) \\
t_6: & 0
\end{aligned}$$

Due to symmetry, we can further state that $t_2 = t_3 = t_4 = t_5$.
Therefore:

$$\begin{aligned}
t_1 &= 1 + \frac{1}{4}(4t_2) = 1 + t_2 \\
t_2 &= 1 + \frac{1}{4}(t_1 + 2t_2 + 0) = 1 + \frac{1}{4}t_1 + \frac{1}{2}t_2 \Rightarrow t_2 = 2 + \frac{1}{2}t_1 \\
t_1 &= 1 + 2 + \frac{1}{2}t_1 \Rightarrow \frac{1}{2}t_1 = 3 \therefore \boxed{t_1 = 6 \text{ minutes}}
\end{aligned}$$

- (b) Compute the same expected value if the octahedron is replaced by a cube (where A and B are still opposite vertices).

Solution Let t_i be the expected amount of time for the ant to reach vertex 8 from vertex i on the cube.



$$\begin{aligned}
t_1: & 1 + \frac{1}{3}(t_2 + t_3 + t_4) \\
t_2: & 1 + \frac{1}{3}(t_1 + t_5 + t_7) \\
t_3: & 1 + \frac{1}{3}(t_1 + t_6 + t_7) \\
t_4: & 1 + \frac{1}{3}(t_1 + t_5 + t_6) \\
t_5: & 1 + \frac{1}{3}(t_2 + t_4 + t_8) \\
t_6: & 1 + \frac{1}{3}(t_3 + t_4 + t_8) \\
t_7: & 1 + \frac{1}{3}(t_2 + t_3 + t_8) \\
t_8: & 0
\end{aligned}$$

Due to symmetry, we can further state that $t_2 = t_3 = t_4$ and $t_5 = t_6 = t_7$. Therefore:

$$\begin{aligned}
t_1 &= 1 + \frac{1}{3}(3t_2) = 1 + t_2 \\
t_2 &= 1 + \frac{1}{3}(t_1 + 2t_3) \\
t_5 &= 1 + \frac{1}{3}(2t_2 + 0) = 1 + \frac{2}{3}t_2 \\
t_2 &= 1 + \frac{1}{3}(t_1 + 2(1 + \frac{2}{3}t_2)) = 1 + \frac{1}{3}(t_1 + 2 + \frac{4}{3}t_2) = 1 + \frac{1}{3}t_1 + \frac{2}{3} + \frac{4}{9}t_2 \Rightarrow \\
\frac{5}{9}t_2 &= 1 + \frac{1}{3}t_1 + \frac{2}{3} \Rightarrow t_2 = \frac{15 + 3t_1}{5} \\
t_1 &= 1 + \frac{15 + 3t_1}{5} \Rightarrow 5t_1 = 5 + 15 + 3t_1 \Rightarrow 2t_1 = 20 \therefore \boxed{t_1 = 10 \text{ minutes}}
\end{aligned}$$

Problem G4. For a positive integer n , let $f(n)$ denote the smallest positive integer which neither divides n nor $n + 1$.

- (a) Find the smallest n for which $f(n) = 9$.

Solution Since $f(n) = 9$, all integers from 1 to 8 must be factors of n or $n + 1$. Furthermore, since 8 is a multiple of both 4 and 2, and 6 is a multiple of 2, we can narrow down the necessary factors to 3, 5, 7, and 8. Therefore, either n or $n + 1$ must end in a 0 or 5. Combining this limitation with the fact that either n or $n + 1$ must be a multiple or one away from a multiple of 3, 7, or 8 further narrows down the possibilities. Using this method, the clear answer reveals itself as $\boxed{n = 104}$, since the factors of 104 and 105 are 3, 5, 8, and 13.

- (b) Is there an n for which $f(n) = 2018$?

Solution Similar to (a), all integers from 1 to 2017 must be factors of n or $n + 1$. However, since the two factors of 2018 (2 and 1009) both fall into that range, the answer cannot simply be 2017!. Therefore, an equation can be setup to find a number pair where 1009 is only a factor of one of the numbers:

$$\frac{2017!}{1009} = 1009x + 1 \Rightarrow 1009^2x = 2017! - 1009 \Rightarrow x = \frac{2017! - 1009}{1009^2}$$

Plugging the above expression into SageMath and checking if it is an integer returns true, meaning $\boxed{n = \frac{2017! - 1009}{1009}}$.

- (c) Which values can $f(n)$ take as n varies?

Solution $f(n)$ is always the smallest positive integer which cannot be found in the prime factorization of either n or $n + 1$.

Problem G5. A pile with $n \geq 3$ stones is given. Two players Alice and Bob alternate taking stones, with Alice moving first. On a turn, if there are m stones left, a player loses if m is prime; otherwise he/she may pick a divisor $d|m$ such that $1 < d < m$ and remove d stones from the pile.

- (a) Which player wins for $n = 6$, $n = 8$, $n = 10$?
 (b) Determine the winning player of all n .

Problem G6. A perfect power is an integer of the form b^n , where $b, n \geq 2$ are integers. Consider matrices 2×2 whose entries are perfect powers; we call such matrices *good*.

- (a) Find an example for a good matrix with determinant 2019.
 (b) Do there exist any such matrices with determinant 1? If so, comment on how many there could be. (Possible hint: use the theory of Pell equations.)

Problem G7. We consider a fixed triangle ABC with side lengths $a = BC, b = CA, c = AB$, and a variable point X in the interior. The lines through X parallel to \overline{AB} and \overline{AC} , together with line \overline{BC} , determine a triangle T_a . The triangles T_b and T_c are defined in a similar way, as shown in the figure. Let S and p denote the average area and perimeter, respectively, of the three triangles T_a, T_b, T_c .

- (a) Determine all possible values of S as X varies, in terms of a, b, c .
- (b) Determine all possible values of p as X varies, in terms of a, b, c .