Problem G1. We flip a fair coin ten times, recording a 0 for tails and 1 for heads. In this way we obtain a binary string of length 10.

(a) Find the probability there is exactly one pair of consecutive equal digits.

Solution Given that the coin is flipped 10 times, we can state that there are 2^{10} different binary string patterns. There are 9 potential consecutive pairs of 0 in a binary string of length 10, so there are ${}_9C_1$ possible instances of the presence of only one pair in the string. The pairs are not limited to one side of the coin, so for this problem there are $2 * {}_9C_1$ possible instances. Therefore, we can deduce that the probability is:

$$\frac{2 * {}_9C_1}{2^{10}} = \frac{{}_9C_1}{2^9} = \frac{9}{512}$$

(b) Find the probability there are exactly n pairs of consecutive digits, for every $n = 0, \ldots, 9$.

Solution Continuing with the process used in (a), we can apply the same equation to every $n = 0, \ldots, 9$:

$$\frac{2 * {}_9C_n}{2^{10}} = \frac{{}_9C_n}{2^9}$$

Problem G2. For which positive integers p is there a nonzero real number t such that

$$t + \sqrt{p} \text{ and } \frac{1}{t} + \sqrt{p}$$

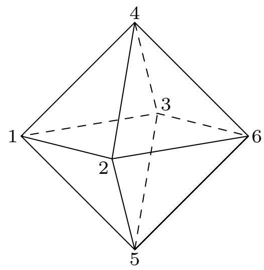
are both rational?

Solution Yeah idk yet man

Problem G3. Points A and B are two opposite vertices of a regular octahedron. An ant starts at point A and, every minute, walks randomly to a neighboring vertex.

(a) Find the expected (i.e. average) amount of time for the ant to reach vertex B.

Solution Let t_i be the expected amount of time for the ant to reach vertex 6 from vertex i on the octahedron.



t_1: $1 + \frac{1}{4}(t_2 + t_3 + t_4 + t_5)$
t_2 and t_3: $1 + \frac{1}{4}(t_1 + t_4 + t_5 + t_6)$
t_4 and t_5: $1 + \frac{1}{4}(t_1 + t_2 + t_3 + t_6)$
t_6: 0

Due to symmetry, we can further state that $t_2 = t_3 = t_4 = t_5$. Therefore:

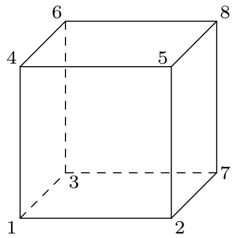
$$t_1 = 1 + \frac{1}{4}(4t_2) = 1 + t_2$$

$$t_2 = 1 + \frac{1}{4}(t_1 + 2t_2 + 0) = 1 + \frac{1}{4}t_1 + \frac{1}{2}t_2 \Rightarrow t_2 = 2 + \frac{1}{2}t_1$$

$$t_1 = 1 + 2 + \frac{1}{2}t_1 \Rightarrow \frac{1}{2}t_1 = 3 \therefore \boxed{t_1 = 6 \text{ minutes}}$$

(b) Compute the same expected value if the octahedron is replaced by a cube (where A and B are still opposite vertices).

Solution Let t_i be the expected amount of time for the ant to reach vertex 8 from vertex i on the cube.



t_1: $1 + \frac{1}{3}(t_2 + t_3 + t_4)$
t_2: $1 + \frac{1}{3}(t_1 + t_5 + t_7)$
t_3: $1 + \frac{1}{3}(t_1 + t_6 + t_7)$
t_4: $1 + \frac{1}{3}(t_1 + t_5 + t_6)$
t_5: $1 + \frac{1}{3}(t_2 + t_4 + t_8)$
t_6: $1 + \frac{1}{3}(t_3 + t_4 + t_8)$
t_7: $1 + \frac{1}{3}(t_2 + t_3 + t_8)$
t_8: 0

Due to symmetry, we can further state that $t_2 = t_3 = t_4$ and $t_5 = t_5 = t_7$. Therefore:

$$t_1 = 1 + \frac{1}{3}(3t_2) = 1 + t_2$$

$$t_2 = 1 + \frac{1}{3}(t_1 + 2t_3)$$

$$t_5 = 1 + \frac{1}{3}(2t_2 + 0) = 1 + \frac{2}{3}(t_2)$$

$$t_2 = 1 + \frac{1}{3}(t_1 + 2(1 + \frac{2}{3}t_2)) = 1 + \frac{1}{3}(t_1 + 2 + \frac{4}{3}t_2) = 1 + \frac{1}{3}t_1 + \frac{2}{3} + \frac{4}{9}t_2 \Rightarrow$$

$$\frac{5}{9}t_2 = 1 + \frac{1}{3}t_1 + \frac{2}{3} \Rightarrow t_2 = \frac{15 + 3t_1}{5}$$

$$t_1 = 1 + \frac{15 + 3t_1}{5} \Rightarrow 5t_1 = 5 + 15 + 3t_1 \Rightarrow 2t_1 = 20 \therefore \boxed{t_1 = 10 \text{ minutes}}$$

Problem G3. For a positive integer n, let $f(n)$ denote the smallest positive integer which neither divides n nor $n+1$.

(a) Find the smallest n for which $f(n) = 9$.

Solution Since $f(n) = 9$, all integers from 1 to 8 must be factors of n or $n+1$. Furthermore, since 8 is a multiple of both 4 and 2, and 6 is a multiple of 2, we can narrow down the necessary factors to 3, 5, 7, and 8. Therefore, either n or $n+1$ must end in a 0 or 5. Combining this limitation with the fact that either n or $n+1$ must be a multiple or one away from a multiple of 3, 7, or 8 further narrows down the possibilities. Using this method, the clear answer reveals itself as $\boxed{n = 104}$, since the factors of 104 and 105 are 3, 5, 8, and 13.

(b) Is there an n for which $f(n) = 2018$?

Solution Similar to (a), all integers from 1 to 2017 must be factors of n or $n+1$. However, since the two factors of 2018 (2 and 1009) both fall into that range, the answer cannot simply be 2017!. Therefore, and equation can be setup to find a number pair where 1009 is only a factor of one of the numbers:

$$\frac{2017!}{1009} = 1009x + 1 \Rightarrow 1009^2 x = 2017! - 1009 \Rightarrow x = \frac{2017! - 1009}{1009^2}$$

Plugging the above expression into SageMath and checking if it is an integer returns true, meaning $\boxed{n = \frac{2017! - 1009}{1009}}$.

(c) Which values can $f(n)$ take as n varies?

Solution $f(n)$ is always the smallest positive integer which cannot be found in the prime factorization of either n or $n+1$.