MIT PRIMES GENERAL MATH SOLUTIONS

Problem G1. We flip a fair coin ten times, recording a 0 for tails and 1 for heads. In this way we obtain a binary string of length 10.

(a) Find the probability there is exactly one pair of consecutive equal digits.

Solution. Given that the coin is flipped 10 times, we can state that there are 2^{10} different binary string patterns. There are 9 potential consecutive pairs of 0 in a binary string of length 10, so there are ${}_9C_1$ possible instances of the presence of only one pair in the string. The pairs are not limited to one side of the coin, so for this problem there are $2 * {}_9C_1$ possible instances. Therefore, we can deduce that the probability is:

$$\frac{2 \cdot {}_9C_1}{2^{10}} = \frac{{}_9C_1}{2^9} = \frac{9}{512}$$

(b) Find the probability there are exactly n pairs of consecutive digits, for every $n = 0, \ldots, 9$.

Solution. Continuing with the process used in (a), we can apply the same equation to every $n = 0, \ldots, 9$:

$$\frac{2 \cdot {}_9C_n}{2^{10}} = \frac{{}_9C_n}{2^9}$$

Problem G2. For which positive integers p is there a nonzero real number t such that

$$t + \sqrt{p} \text{ and } \frac{1}{t} + \sqrt{p}$$

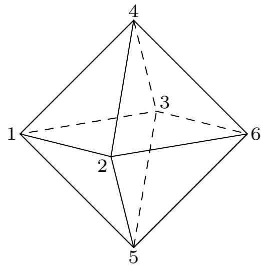
are both rational?

Solution. Yeah idk yet man

Problem G3. Points A and B are two opposite vertices of a regular octahedron. An ant starts at point A and, every minute, walks randomly to a neighboring vertex.

(a) Find the expected (i.e. average) amount of time for the ant to reach vertex B.

Solution. Let t_i be the expected amount of time for the ant to reach vertex 6 from vertex i on the octahedron.



$$t_1\colon\ 1 + \tfrac{1}{4}(t_2 + t_3 + t_4 + t_5)$$

$$t_2 \text{ and } t_3\colon\ 1 + \tfrac{1}{4}(t_1 + t_4 + t_5 + t_6)$$

$$t_4 \text{ and } t_5\colon\ 1 + \tfrac{1}{4}(t_1 + t_2 + t_3 + t_6)$$

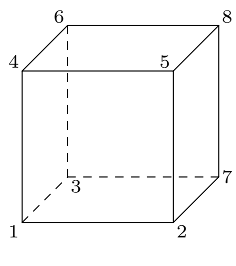
$$t_6\colon\ 0$$

Due to symmetry, we can further state that $t_2 = t_3 = t_4 = t_5$. Therefore:

$$t_1 = 1 + \frac{1}{4}(4t_2) = 1 + t_2$$
$$t_2 = 1 + \frac{1}{4}(t_1 + 2t_2 + 0) = 1 + \frac{1}{4}t_1 + \frac{1}{2}t_2 \Rightarrow t_2 = 2 + \frac{1}{2}t_1$$
$$t_1 = 1 + 2 + \frac{1}{2}t_1 \Rightarrow \frac{1}{2}t_1 = 3 \therefore t_1 = 6 \text{ minutes}$$

(b) Compute the same expected value if the octahedron is replaced by a cube (where A and B are still opposite vertices).

Solution. Let t_i be the expected amount of time for the ant to reach vertex 8 from vertex i on the cube.



$$t_1\colon\ 1 + \tfrac{1}{3}(t_2 + t_3 + t_4)$$

$$t_2\colon\ 1 + \tfrac{1}{3}(t_1 + t_5 + t_7)$$

$$t_3\colon\ 1 + \tfrac{1}{3}(t_1 + t_6 + t_7)$$

$$t_4\colon\ 1 + \tfrac{1}{3}(t_1 + t_5 + t_6)$$

$$t_5\colon\ 1 + \tfrac{1}{3}(t_2 + t_4 + t_8)$$

$$t_6\colon\ 1+\tfrac{1}{3}(t_3+t_4+t_8)$$

$$t_7\colon\ 1+\tfrac{1}{3}(t_2+t_3+t_8)$$

$$t_8\colon\ 0$$

Due to symmetry, we can further state that $t_2 = t_3 = t_4$ and $t_5 = t_5 = t_7$. Therefore:

$$t_1 = 1 + \frac{1}{3}(3t_2) = 1 + t_2$$

$$t_2 = 1 + \frac{1}{3}(t_1 + 2t_3)$$

$$t_5 = 1 + \frac{1}{3}(2t_2 + 0) = 1 + \frac{2}{3}(t_2)$$

$$t_2 = 1 + \frac{1}{3}(t_1 + 2(1 + \frac{2}{3}t_2)) = 1 + \frac{1}{3}(t_1 + 2 + \frac{4}{3}t_2) = 1 + \frac{1}{3}t_1 + \frac{2}{3} + \frac{4}{9}t_2 \Rightarrow$$

$$\frac{5}{9}t_2 = 1 + \frac{1}{3}t_1 + \frac{2}{3} \Rightarrow t_2 = \frac{15+3t_1}{5}$$

$$t_1 = 1 + \frac{15+3t_1}{5} \Rightarrow 5t_1 = 5 + 15 + 3t_1 \Rightarrow 2t_1 = 20 \therefore t_1 = 10 \text{ minutes}$$

Problem G4. For a positive integer n, let $f(n)$ denote the smallest positive integer which neither divides n nor $n+1$.

(a) Find the smallest n for which $f(n) = 9$.

Solution. Since $f(n) = 9$, all integers from 1 to 8 must be factors of n or $n+1$. Furthermore, since 8 is a multiple of both 4 and 2, and 6 is a multiple of 2, we can narrow down the necessary factors to 3, 5, 7, and 8. Therefore, either n or $n+1$ must end in a 0 or 5. Combining this limitation with the fact that either n or $n+1$ must be a multiple or one away from a multiple of 3, 7, or 8 further narrows down the possibilities. Using this method, the answer reveals itself as $n = 104$, since the factors of 104 and 105 are 3, 5, 8, and 13.

(b) Is there an n for which $f(n) = 2018$?

Solution. Similar to (a), all integers from 1 to 2017 must be factors of n or $n+1$. However, since the two factors of 2018 (2 and 1009) both fall into that range, the answer cannot simply be 2017!. Therefore, and equation can be setup to find a number pair where 1009 is only a factor of one of the numbers:

$$\frac{2017!}{1009} = 1009x + 1 \Rightarrow 1009^2x = 2017! - 1009 \Rightarrow x = \frac{2017! - 1009}{1009^2}$$

Plugging the above expression into Sage Math and checking if it is an integer returns true, meaning:

$$n = \frac{2017! - 1009}{1009}$$

(c) Which values can $f(n)$ take as n varies?

Solution. $f(n)$ is always the smallest positive integer which cannot be found in the prime factorization of either n or $n+1$.

Problem G5. A pile with $n >= 3$ stones is given. Two players Alice and Bob alternate taking stones, with Alice moving first. On a turn, if there are m stones left, a player loses if m is prime; otherwise he/she may pick a divisor $d|m$ such that $1 < d < m$ and remove d stones from the pile.

(a) Which player wins for $n = 6$, $n = 8$, $n = 10$?

Solution. If $n = 6$, d must be either 3 or 2. Therefore, in the subsequent round, Alice wins, since she can take 3 stones, leaving Bob with 3, a prime number. If $n = 8$, d must be either 4 or 2. The resulting m would be either 4 or 6, which are both numbers Bob can win from, since 4 immediately leads to a prime pile, and we have already shown that whoever moves first when $n = 6$ will win. If $n = 10$, d must be either 5 or 2. The resulting m would be either 8 or 5. 5 is a prime number, so Alice wins.

$$n = 6\text{: Alice; } n = 8\text{: Bob; } n = 10\text{: Alice}$$

(b) Determine the winning player of all n.

Solution. As seen in (a), the winner of the game when there are n stones is dependent on the winner of the game for potential m values. By recognizing this fact, a list of the first 40 values can be determined:

		11	Bob	21	Bob	31	Bob
3	Bob	12	Alice	22	Alice	32	Bob
4	Alice	13	Bob	23	Bob	33	Bob
5	Bob	14	Alice	24	Alice	34	Alice
6	Alice	15	Bob	25	Bob	35	Bob
7	Bob	16	Alice	26	Alice	36	Alice
8	Bob	17	Bob	27	Bob	37	Bob
9	Bob	18	Alice	28	Alice	38	Alice
10	Alice	19	Bob	29	Bob	39	Bob
		20	Alice	30	Alice	40	Alice

Evidently, the main pattern is that Bob wins every game beginning with an odd numbered n, whereas Alice wins the evens. However, one anomaly to the pattern can be seen; if $n = 2^a$, and $a \geq 2$ is an even integer, Alice wins. If a is odd, Bob wins. This alternation can be explained by the fact that every $m = 2^{a_{i-1}}$ is attainable when $n = 2^{a_i}$ due to the nature of the game.

$$\text{Alice: } 0 = n \pmod 2 \text{ or } n = 2^a, \text{ when } a >= 2 \text{ and } 0 = a \pmod 2$$
$$\text{Bob: } 0 = n \pmod 2 \text{ or } n = 2^a, \text{ when } a >= 2 \text{ and } 0 = a \pmod 2$$

Problem G6. A perfect power is an integer of the form b^n, where b, $n \geq 2$ are integers. Consider matrices 2×2 whose entries are perfect powers; we call such matrices *good.*

(a) Find an example for a good matrix with determinant 2019.

Solution. When put through a program I wrote (included in my CS .zip file), I discovered several small number solutions.

$$\begin{bmatrix} 13^4 & 5^2 \\ 67^2 & 2^2 \end{bmatrix} \text{ and } \begin{bmatrix} 13^2 & 5^2 \\ 67^2 & 26^2 \end{bmatrix}$$

(b) Do there exists any such matrices with determinant 1? If so, comment on how many there could be. (Possible hint: use the theory of Pell equations.)

Solution. Yes. Given:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot d - b \cdot c \tag{1}$$

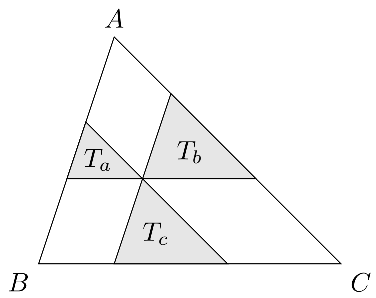
$$x^2 - ny^2 = 1 \tag{2}$$

To apply Pell's equation, x must be a composite number. Assuming x is such, an equation can be created referencing (1) and (2) to satisfy the problem:

$$\det \begin{bmatrix} a^2 & b^p \\ c^2 & d^2 \end{bmatrix} = (a \cdot d)^2 - b^p \cdot c^2 = 1$$

Where $p >= 3$, since there are no integer solutions for $x^2 - ny^2 = 1$ when n is a perfect square. According to Pell's equation, the number of integer solutions for x and y is infinite. Therefore there will be an endless number of composite x solutions, meaning the number of possible matrices is infinite.

Problem G7. We consider a fixed triangle ABC with side lengths $a = BC, b = CA, c = AB$, and a variable point X in the interior. The lines through X parallel to $\overline{AB}$ and $\overline{AC}$, together with line $\overline{BC}$, determine a triangle T_a. The triangles T_b and T_c are defined in a similar way, as shown in the figure.



Let S and p denote the average area and perimeter, respectively, of the three triangles T_a, T_b, T_c.

(a) Determine all possible values of S as X varies, in terms of a, b, c.

Solution. Heron's formula can be used to represent $[\triangle ABC]$ as T:

$$T = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

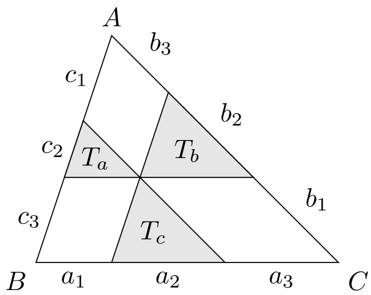
The greatest value T could be occurs when X lies directly on one of the three vertices, since $T = \frac{1}{3}A$. On the other hand, the smallest potential T value occurs when X is at the median of the triangle, since that is when $T_a = T_b = T_c = \frac{1}{9}A = T$. Combining these two concepts provides us with a range in terms of a, b, c:

$$T = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$

$$\frac{1}{9}A \leq S \leq \frac{1}{3}A$$

(b) Determine all possible values of p as X varies, in terms of a, b, c.

Solution. As described by the problem, the inner lines are parallel to their respective outer lines. Therefore, the intersections of the parallel lines create three parallelograms. Given that an aspect of a parallelogram is having two pairs of congruent opposite sides, the line segments $c_1, c_3, a_1, a_3, b_1, b_3$ are all congruent to sides of the the three triangles.



As demonstrated by the diagram, if all labeled line segments relate to a respective side of the three triangles, and since they also combine to create the perimeter of the large triangle, we can deduce that:

$$p = \frac{a+b+c}{3}$$