"ANS TO PROB SET 3"

ECON 6020

MACRO THEORY I

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WICKENS. PROBLEM 2.4 The central planner's problem is to

$$\text{Maximize } \sum_{s=0}^{\infty} \beta^s U(C_{t+s}, l_{t+s}) \quad (4.1)$$

~~Subject to~~ where $U(C_{t+s}, l_{t+s}) = \ln C_{t+s} + \phi \ln l_{t+s}$, (4.2)

$\beta = \frac{1}{1+\theta}$, $\theta > 0$.

Maximization is subject to

$$y_t = C_t + I_t \quad (4.3)$$

$$\Delta K_{t+1} = I_t - \delta K_t \quad (4.4)$$

$$y_t = F(K_t, n_t) \quad (4.5)$$

$$F(K_t, n_t) = A\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha) n_t^{1-\frac{1}{\gamma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}} \quad (4.6)$$

and $l_t + n_t = 1$ (4.7)

(a) Derive the expressions from which the long-run equilibrium values of optimal consumption, labor, and capital can be obtained.

(b) Derive the long-run equilibrium real interest rate and real wage rate

(c) Comment on the implications for labor of having an elasticity of substitution between capital and labor different from unity. (That is, what are the implications of $\gamma \neq 1$).

(d) Derive the long-run equilibrium capital-labor ratio.

Solution (a) This problem fits the general form of the problem in chapter 2.6 but it adds the particular ~~specifications~~ specifications of the utility function, eqn (4.2) above, and the production function, eqn (4.6) above.

The Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \Big\{ \beta^s U(c_{t+s}, l_{t+s}) + \lambda_{t+s}\big[F(k_{t+s}, n_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}\big] + \mu_{t+s}\big[1 - n_{t+s} - l_{t+s}\big] \Big\} \qquad (4.8)$$

The First-order Conditions Are

$$\frac{\partial \mathcal{L}_t}{\partial C_{t+s}} = \beta^s U_c(t+s) - \lambda_{t+s} = 0 \qquad (4.9a)$$

cf W(2.25)

$$\frac{\partial \mathcal{L}_t}{\partial l_{t+s}} = \beta^s U_l(t+s) - \mu_{t+s} = 0 \qquad (4.9b)$$

cf W(2.26)

$$\frac{\partial \mathcal{L}_t}{\partial n_{t+s}} = \lambda_{t+s} F_n(t+s) - \mu_{t+s} = 0 \qquad (4.9c)$$

cf W(2.27)

$$\frac{\partial \mathcal{L}_t}{\partial K_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}\left[F_K(t+s+1) + 1 - \delta\right] = 0 \qquad (4.9d)$$

cf W(2.28)

From (4.9a) we have that

$$\lambda_{t+s} = \beta^s U_c(t+s) \qquad (4.10)$$

Using (4.10) to Eliminate λ_{t+s} and λ_{t+s+1} in (4.9d) gives

$$-\beta^s U_c(t+s) + \beta^{s+1} U_c(t+s+1)\left[F_K(t+s+1) + 1 - \delta\right] = 0$$

or, for $s=0$,

$$\beta \frac{u_c(t+1)}{u_c(t)}\left[F_K(t+1) + 1 - \delta\right] = 1 \qquad (4.11)$$

Eqn (4.11) is the intertemporal Euler Equation.

Note from (4.2) That

$$u_c(C_{t+s}) = \frac{1}{C_{t+s}} \qquad (4.12)$$

Also, from (4.6) we can obtain

$$F_K(K_t, M_t) = A\left(\frac{1}{1-\frac{1}{\gamma}}\right)\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)-1} \cdot \left(1-\frac{1}{\gamma}\right)\alpha K_t^{\frac{-1}{\gamma}}$$

or $$F_K(K_t, M_t) = \alpha A\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)-1} \cdot K_t^{-\frac{1}{\gamma}}$$

But note that

$$\frac{1}{1-\frac{1}{\gamma}} - 1 = \frac{1 - 1 + \frac{1}{\gamma}}{1-\frac{1}{\gamma}} = \frac{\frac{1}{\gamma}}{1-\frac{1}{\gamma}} = \left(\frac{1}{1-\frac{1}{\gamma}}\right)\frac{1}{\gamma}$$

So we can write

$$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} A^{\frac{1}{\gamma}}\left[\alpha K_t^{1-\frac{1}{\gamma}} + (1-\alpha)M_t^{1-\frac{1}{\gamma}}\right]^{\left(\frac{1}{1-\frac{1}{\gamma}}\right)\frac{1}{\gamma}} K_t^{\frac{-1}{\gamma}}$$

or $$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} Y_t^{\frac{1}{\gamma}} K_t^{\frac{-1}{\gamma}}$$

or

$$F_K(K_t, M_t) = \alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y_t}{K_t}\right)^{\frac{1}{\gamma}} \qquad (4.13)$$

Using (4.12) and (4.13) in (4.11)

$$\beta \frac{C_t}{C_{t+1}} \left[\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y_{t+1}}{K_{t+1}}\right)^{\frac{1}{\gamma}} + 1 - \delta\right] = 1 \qquad (4.14)$$

Let variables with no subscript denote long-run (steady-state) values. Eqn (4.14) then gives

$$\beta \frac{C}{C} \left[\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} + 1 - \delta\right] = 1 \quad \text{or}$$

$$\alpha A^{1-\frac{1}{\gamma}} \left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} = 1 + \theta - 1 + \delta = \theta + \delta \quad \text{or}$$

$$\left(\frac{Y}{K}\right)^{\frac{1}{\gamma}} = A^{\left(\frac{1}{\gamma} - 1\right)} \frac{\theta + \delta}{\alpha} \quad \text{or}$$

$$\boxed{\left(\frac{Y}{K}\right) = A^{1-\gamma} \left(\frac{\theta + \delta}{\alpha}\right)^{\gamma} \qquad (4.15)}$$

Eqn (4.15) Gives the long-run (steady-state) value of (y/k) but it does not seperately determine y or k.

For c/l begin from (4.6) which gives

$$u_{t+s} = \beta^s U_l(t+s)$$

Since from (4.2) $U_l(t+s) = \dfrac{\varphi}{l_{t+s}}$ we have that

$$u_{t+s} = \beta^s \frac{\varphi}{l_{t+s}} \qquad (4.16)$$

Using (4.16) with (4.10) in (4.9c) and setting $s=0$ gives

$$\frac{1}{c_t} F_n(t) = \frac{\varphi}{l_t} \qquad (4.17)$$

From (4.6) and using a derivation that parallels the derivation of (4.13) we obtain that

$$F_n(t) = (1-\alpha) A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}} \qquad (4.18)$$

Use (4.18) in (4.17) to get

$$\frac{\varphi}{l_t} = \frac{(1-\alpha) A^{1-\frac{1}{\gamma}} \left(\frac{y_t}{n_t}\right)^{\frac{1}{\gamma}}}{c_t} \qquad (4.19)\ [w(3)]$$

From this, and using $l = 1-n$, gives that the long-run (steady-state) equilibrium values satisfy

$$\boxed{\frac{c}{l} = \frac{c}{1-n} = \left(\frac{1-\alpha}{\varphi}\right) A^{1-\frac{1}{\gamma}} \left(\frac{y}{n}\right)^{\frac{1}{\gamma}} \qquad (4.20)}$$

(b) The implied equilibrium real wage is equal to the marginal product of labor.

Thus

$$w_t = F_n(k_t, n_t) \qquad (4.21)$$

Using (4.18) and evaluating at the long-run (steady-state) values

$$\boxed{w = (1-\alpha) A^{1-\frac{1}{\gamma}} \left(y/n\right)^{\frac{1}{\gamma}} \qquad (4.22)}$$

The implied Equilibrium Real interest Rate is equal to the Marginal Product of Capital Net of the Depreciation Rate. Thus,

$$r_t = F_K(K_t, M_t) - \delta \qquad (4.23)$$

Using this in (4.11) and evaluating at the steady state gives

$$\beta[r+1] = 1 \quad \text{or} \quad 1+r = 1+\theta$$

or $$\boxed{r = \theta \qquad (4.24)}$$

(d) To Find the long-run Equilibrium Value of $\frac{K}{m}$ Begin from (4.22) which can be written

as $$\left(\frac{y}{M}\right)^{\frac{1}{\gamma}} = \frac{w}{1-\alpha} A^{\frac{1}{\gamma}-1} \quad \text{or}$$

$$\left(\frac{y}{m}\right) = \left(\frac{w}{1-\alpha}\right)^{\gamma} A^{1-\gamma} \qquad (4.25)$$

Note That

$$\frac{K}{m} = \left(\frac{Y}{m}\right)\left(\frac{K}{Y}\right)$$ and use (4.25) and (4.15)

To get

$$\frac{K}{m} = \left(\frac{w}{1-\alpha}\right)^{\gamma} A^{1-\gamma} A^{\gamma-1} \left(\frac{\theta+\delta}{\alpha}\right)^{-\gamma}$$

$$\frac{K}{m} = \left[\frac{\alpha w}{(1-\alpha)(\theta+\delta)}\right]^{\gamma}$$

or, since $r = \theta$, from eqn (4.24),

$$\boxed{\frac{K}{m} = \left[\frac{\alpha w}{(1-\alpha)(r+\delta)}\right]^{\gamma} \qquad (4.26)}$$

WICKENS. PROB 4.1 The Household Budget constraint may be expressed in different ways from eqn (4.2),

$$\Delta a_{t+1} + c_t = x_t + r_t a_t \quad \text{[Text (4.2)]},$$

Derive the Euler eqn for consumption for each of the following ways of writing the budget constraint.

(a) $$\boxed{a_{t+1} = (1+r)\left[a_t + x_t - c_t\right] \qquad (1.1)}$$

Note that this is [Text (4.2)] with constant r.

(b) $$\boxed{\Delta a_t + c_t = x_t + r\, a_{t-1} \qquad (1.2)}$$

(c) $$\boxed{W_t = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s c_{t+s} = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s x_{t+s} + (1+r)a_t \qquad (1.3)}$$

Where W_t is household wealth.

Solution: In all three cases we seek to

$$\max_{\{c_{t+s},\, a_{t+s}\}} \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \quad \text{[Text (4.1)]}$$

Subject to the relevant constraint

(a.) For The Constraint (1.1) The Lagrangian is (1.4)

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \left\{ \beta^s u(c_{t+s}) + \lambda_{t+s}\left[(1+r)(a_{t+s} + x_{t+s} - c_{t+s}) - a_{t+s+1}\right]\right\} \quad (1.4)$$

The FOC are

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_{t+s}(1+r) = 0, \; s \geq 0, \quad (1.5)$$

and

$$\frac{\partial \mathcal{L}_t}{\partial a_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}(1+r) = 0, \; s \geq 0, \quad (1.6)$$

From (1.5): $\lambda_{t+s} = \left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s})$ (1.7)

Use (1.7) to Eliminate λ_{t+s} and λ_{t+s+1} in (1.6)

to get

$$-\left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s}) + \left(\frac{1}{1+r}\right)\beta^{s+1} u'(c_{t+s+1})(1+r) = 0 \quad \text{OR}$$

$$\left(\frac{1}{1+r}\right)\beta^s u'(c_{t+s}) = \beta^{s+1} u'(c_{t+s+1}) \quad \text{or}$$

$$\boxed{\beta \frac{u'(c_{t+s+1})}{u'(c_{t+s})}(1+r) = 1 \quad (1.8)}$$

Eqn (1.8) is The Euler Eqn for Constraint (a).

(b) For constraint (1.2) the Lagrangian is

$$\mathcal{L}_t = \sum_{s=0}^{\infty}\Big\{\beta^s u(c_{t+s}) + \lambda_{t+s}\left[x_{t+s} + (1+r)a_{t+s-1} - c_{t+s} - a_{t+s}\right]\Big\} \qquad (1.9)$$

The FOC are

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = \beta^s u'(c_{t+s}) - \lambda_{t+s} = 0, \quad s \geq 0, \qquad (1.10)$$

and

$$\frac{\partial \mathcal{L}_t}{\partial a_{t+s}} = -\lambda_{t+s} + \lambda_{t+s+1}(1+r) = 0, \quad s \geq 0. \qquad (1.11)$$

From (1.10): $\lambda_{t+s} = \beta^s u'(c_{t+s}) \qquad (1.12)$

Use (1.12) in (1.11) to eliminate λ_{t+s} and λ_{t+s+1} to get

$$-\beta^s u'(c_{t+s}) + \beta^{s+1}u'(c_{t+s+1})(1+r) = 0 \quad \text{or}$$

$$\beta^s u'(c_{t+s}) = \beta^{s+1}u'(c_{t+s+1})(1+r) \quad \text{or}$$

$$\boxed{\beta\frac{u'(c_{t+s+1})}{u'(c_{t+s})}(1+r) = 1 \qquad (1.13)}$$

Eqn (1.13), which is the Euler Eqn for constraint (b) is the same as (1.8).

(c) The Lagrangian for constraint (1.3) is

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) + \lambda_t \left[\sum_{s=0}^{\infty} \frac{x_{t+s} - c_{t+s}}{(1+r)^s} + (1+r) a_t \right] \quad (1.14)$$

Note, a single constraint

As a_t is given we require only the FOC for c_{t+s}.

$$\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} = \beta^s u'(c_{t+s}) - \lambda_t \frac{1}{(1+r)^s} = 0 \quad \text{or}$$

$$\beta^s u'(c_{t+s}) = \lambda_t \frac{1}{(1+r)^s} \quad (1.15)$$

Leading (1.15) one period (i.e. evaluating $\frac{\partial \mathcal{L}_t}{\partial c_{t+s+1}} = 0$)

gives

$$\beta^{s+1} u'(c_{t+s+1}) = \lambda_t \frac{1}{(1+r)^{s+1}} \quad (1.16)$$

Combining (1.15) and (1.16) gives

$$(1+r)^s \beta^s u'(c_{t+s}) = (1+r)^{s+1} \beta^{s+1} u'(c_{t+s+1}) \quad \text{or}$$

$$\boxed{\beta \frac{u'(c_{t+s+1})}{u'(c_{t+s})}(1+r) = 1 \quad (1.17)}$$

Eqn (1.17), which is the Euler Eqn for constraint (c) is the same as (1.13) and (1.8).
For all three note the lack of a time subscript on r, that is, the assumption of a constant real interest rate.

Wickens. Prob 4.2

The representative household chooses $\{C_{t+s}\}_{s=0}^{\infty}$ to maximize $\sum_{s=0}^{\infty} \beta^s U(C_{t+s})$ subject to $\Delta a_{t+1} + C_t = x_t + r_t a_t$, and where $\beta = \frac{1}{1+\theta}$, $\theta > 0$. Assume $r_t = r$, a constant.

(a) Assuming that $r = \theta$ and using the approximation

$$\frac{u'(C_{t+1})}{u'(C_t)} \cong 1 - \sigma \Delta \ln C_{t+1} \qquad (2.1)$$

where $\sigma > 0$, show that optimal consumption is constant.

(b) Does this mean that changes in income have no effect on consumption?

Solution:

(a) Note that this is the same optimization problem as PROB 4.1 part (a). So we can use the Euler eqn ~~from~~ in (1.8) above

$$\beta \frac{u'(C_{t+s+1})}{u'(C_{t+s})}(1+r) = 1 \qquad (2.2)$$

Since $\beta = \frac{1}{1+\theta}$ and $\theta = r$ it follows that

$$\boxed{\beta(1+r) = 1 \qquad (2.3)}$$

and, hence, (2.2) becomes

$$\frac{u'(C_{t+s+1})}{u'(C_{t+s})} = 1 \qquad \text{or for } s = 0$$

$$\frac{u'(C_{t+1})}{u'(C_t)} = 1 \qquad \text{Using (2.1) this gives}$$

$$1 = 1 - \sigma \Delta \ln C_{t+1} \qquad \text{which requires}$$

$$\Delta \ln C_{t+1} = 0 \qquad \text{OR}$$

$$C_{t+1} = C_t \qquad (2.4)$$

Thus, with $\theta = r$, optimal consumption is constant.

(b) In a model without any uncertainty, that is, where all future income is correctly anticipated, then (fully anticipated) changes in income do not affect consumption.

If, however, there were some uncertainty about future income, instead of (2.4) the Euler equation with $\theta = r$ will give

$$\underset{t}{E}\, C_{t+1} = C_t \qquad (2.5)$$

The intertemporal budget constraint gives

$$\boxed{C_t = r \underset{t}{E} \sum_{s=0}^{\infty} \frac{x_{t+s}}{(1+r)^{s+1}} + r\, a_t \qquad (2.6)}$$

<u>DERIVATION of (2.6)</u>: Begin from the Budget Constraint

$$\Delta a_{t+1} + c_t = x_t + r\, a_t$$

Which we can write as

$$a_{t+1} = (1+r)\, a_t + x_t - c_t \qquad (2.7)$$

Let $R \equiv (1+r)$ and Note That (2.7) is a first-order Difference equation with Root $R > 1$. Solve (2.7) Forward. First, write (2.7) as

$(1 - RL)\, a_{t+1} = x_t - c_t$. Thus

$$a_{t+1} = \left(\frac{1}{1-RL}\right)(x_t - c_t) = \left[\frac{-R^{-1}L^{-1}}{1-R^{-1}L^{-1}}\right](x_t - c_t)$$

or $$a_t = (-R^{-1})\left[\sum_{s=0}^{\infty} R^{-s} x_{t+s} - \sum_{s=0}^{\infty} R^{-s} c_{t+s}\right] \quad (2.8)$$

Allow for Uncertainty, Take Expectations of Both Sides, and Note That $E_t a_t = a_t$ to get

$$a_t = R^{-1}\sum_{s=0}^{\infty} R^{-s} \underset{t}{E}\, c_{t+s} - R^{-1}\sum_{s=0}^{\infty} R^{-s} \underset{t}{E}\, x_{t+s} \qquad (2.9)$$

Note from (2.5) that $\underset{t}{E}\, c_{t+s} = \underset{t}{E}\, c_{t+1} = c_t$ so that

$$R^{-1}\sum_{s=0}^{\infty} R^{-s} \underset{t}{E}\, c_{t+s} = \left[R^{-1}\sum_{s=0}^{\infty} R^{-s}\right] c_t =$$

$$\frac{1}{1+r}\left[1 + \left(\frac{1}{1+r}\right) + \left(\frac{1}{1+r}\right)^2 + \cdots\right] c_t = \frac{1}{1+r}\left[\frac{1}{1-\left(\frac{1}{1+r}\right)}\right] c_t$$

$$= \frac{1}{1+r}\left[\frac{1}{\left(\frac{r}{1+r}\right)}\right] = \frac{1}{r}\, c_t .$$

Using this result in (2.9) gives

$$a_t = \frac{1}{r}\, c_t - \sum_{s=0}^{\infty} R^{-(s+1)} \underset{t}{E}\, x_{t+s} \quad \text{or}$$

$$\boxed{c_t = r\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s+1} \underset{t}{E}\, x_{t+s} + r\, a_t \qquad (2.6)}$$

End of Derivation of (2.6)

Lead (2.6) one period

$$C_{t+1} = r \, \underset{t+1}{E} \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} x_{t+s+1} + r a_{t+1} \qquad (2.10)$$

Take Expectations of (2.10)

$$\underset{t}{E}\, C_{t+1} = r \underset{t}{E} \sum_{s=0}^{\infty} \frac{x_{t+s+1}}{(1+r)^{s+1}} + r \underset{t}{E}\, a_{t+1} \qquad (2.11)$$

Subtracting (2.11) from (2.10) gives

$$\boxed{\left(C_{t+1} - \underset{t}{E}\, C_{t+1}\right) = r \sum_{s=0}^{\infty} \frac{\underset{t+1}{E} x_{t+s+1} - \underset{t}{E} x_{t+s+1}}{(1+r)^{s+1}} + r\left(a_{t+1} - \underset{t}{E}\, a_{t+1}\right) \qquad (2.12)}$$

Note from (2.7) that a_{t+1} is entirely ~~deter~~ determined by variables known in period t. Therefore $\underset{t}{E}\, a_{t+1} = a_t$ and (2.12) becomes

$$\left(C_{t+1} - \underset{t}{E}\, C_{t+1}\right) = r \sum_{s=0}^{\infty} \frac{\underset{t+1}{E} x_{t+s+1} - \underset{t}{E} x_{t+s+1}}{(1+r)^{s+1}} \qquad (2.13)$$

Eqn (2.13) shows that any change in the expected present discounted value of lifetime income that occurs between periods t and $t+1$ would have an effect on consumption in period $t+1$.

Consider an unanticipated increase in income solely in period $t+1$. So

$$\underset{t}{E}\, x_{t+1} \neq \underset{t+1}{E}\, x_{t+1} = x_{t+1} \qquad (2.14)$$

But $$\underset{t}{E}\, x_{t+s+1} = \underset{t+1}{E}\, x_{t+s+1} \text{ for } s > 0 \qquad (2.15)$$

Using (2.14) and (2.15) in (2.13) gives

$$c_{t+1} - \underset{t}{E}\, c_{t+1} = \left(\frac{r}{1+r}\right)\left(x_{t+1} - \underset{t}{E}\, x_{t+1}\right) \neq 0 \qquad (2.16)$$

and since $c_t = \underset{t}{E}\, c_{t+1}$ via Eqn (2.5)

$$c_{t+1} - c_t \neq 0 \qquad (2.17)$$

So an unanticipated change in expected future income can alter optimal ~~c~~ consumption.

Wickens - Problem 4.3

(a) Derive the dynamic path of optimal consumption when the utility function exhibits habit persistence.

The utility function is

$$U(C_t) = \frac{(C_t - h_t)^{1-\sigma}}{1-\sigma} \qquad (3.1)$$

Where h_t is exogenous and where the budget ~~Constraint~~ constraint is

$$\Delta a_{t+1} + C_t = x_t + r a_t \qquad (3.2)$$

(b) Obtain the consumption function for this problem assuming that $\beta(1+r)=1$. Consider specifically the case where $h_{t+s} = h_t$ for $s \geq 0$.

(a) In Problem 4.1 part (a) we obtained the Euler Eqn for the general problem with the constraint (3.2), which was (1.8)

$$\beta \frac{u'(C_{t+s+1})}{u'(C_{t+s})}(1+r) = 1 . \qquad (1.8)$$

For the utility function (3.1)

$$u'(C_t) = (C_t - h_t)^{-\sigma}.$$

Using this in (1.8) with $S=0$ gives

$$\beta\left[\frac{C_{t+1} - h_{t+1}}{C_t - h_t}\right]^{-\sigma}(1+r) = 1 \qquad \text{or}$$

$$(C_{t+1} - h_{t+1}) = [\beta(1+r)]^{\frac{-1}{\sigma}} \cdot (C_t - h_t) \qquad (3.3)$$

Eqn (3.3) is a simple first-order difference equation in the single variable $(C_t - h_t)$. Eqn (3.3) gives the dynamic path of optimal consumption

(b) Note that, since we have the same Budget constraint as in prob 4.2 above we can use the forward solution from prob 4.2

$$a_t = (-R^{-1})\left[\sum_{s=0}^{\infty} R^{-s} x_{t+s} - \sum_{s=0}^{\infty} R^{-s} c_{t+s}\right] \quad (2.8)$$

Noting that $R \equiv (1+r)$ this can be written

$$a_t = \sum_{s=0}^{\infty} \frac{c_{t+s} - x_{t+s}}{(1+r)^{s+1}} \quad (3.4)$$

Adding and subtracting h_{t+s} on the RHS of (3.4)

$$a_t = \sum_{s=0}^{\infty} \frac{[c_{t+s} - h_{t+s}] - [x_{t+s} - h_{t+s}]}{(1+r)^{s+1}} \quad (3.5)$$

In Prob 4.2 we showed that if $r = \theta$, which implies $\beta(1+r) = 1$, then optimal consumption will be constant: $c_{t+s} = c_t$ for $s \geq 0$. ~~In the state Eqn (3.5~~ The statement of the problem also gives $h_{t+s} = h_t$ for $s \geq 0$.

Eqn (3.5) can therefore be written as

$$a_t = \sum_{s=0}^{\infty} \frac{c_t - h_t}{(1+r)^{s+1}} - \sum_{s=0}^{\infty} \frac{x_{t+s} - h_{t+s}}{(1+r)^{s+1}} \quad (3.6)$$

But $\sum_{s=0}^{\infty} \frac{c_t - h_t}{(1+r)^{s+1}} = \frac{1}{1+r}(c_t - h_t)\left[\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\right]$

$$= \left(\frac{1}{1+r}\right)(c_t - h_t)\left[\frac{1}{1-\left(\frac{1}{1+r}\right)}\right] = \left(\frac{1}{1+r}\right)(c_t - h_t)\left[\frac{1}{\left(\frac{r}{1+r}\right)}\right]$$

$$= \frac{1}{r}(c_t - h_t).$$

This result in (3.6) gives

$$a_t = \frac{1}{r}(c_t - h_t) - \sum_{s=0}^{\infty} \frac{x_{t+s} - h_{t+s}}{(1+r)^{s+1}} \quad (3.7)$$

[Comparing (3.7) to the corresponding Eqn on p 67 of the Wickens Solution Manual note the typo (sign error) in Wickens]

(3.7) can be re-written as

$$C_t = h_t + r a_t + \left(\frac{r}{1+r}\right)\sum_{s=0}^{\infty}\frac{x_{t+s} - h_{t+s}}{(1+r)^s} \quad (3.8)$$

Note further that with $h_{t+s} = h_t$ for $s \geq 0$

$\left(\frac{r}{1+r}\right)\sum_{s=0}^{\infty}\frac{h_{t+s}}{(1+r)^s} = h_t$ and (3.8) gives

$$C_t = r a_t + \left(\frac{r}{1+r}\right)\sum_{s=0}^{\infty}\frac{x_{t+s}}{(1+r)^s} \quad (3.9)$$

Eqn (3.9) here is, except for uncertainty and the expectations operator, the same as (2.6) from problem 4.2 above.

Thus if $h_t = h_{t+s}$ for $s \geq 0$ the consumption function becomes the standard consumption function.

$$\text{MAX } \underset{t}{E} \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) \quad (1)$$

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \quad (2)$$

$$\text{s.t.} \quad A_{t+1} = (1+r) A_t + y_t - c_t \quad (3)$$

(a)

$$\mathcal{L}_t = \underset{t}{E} \sum_{s=0}^{\infty} \beta^s \left\{ u(c_t) + \lambda_t \left[(1+r) A_t + y_t - c_t - A_{t+1} \right] \right\}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = u'(c_t) - \lambda_t = 0 \quad (4)$$

$$\frac{\partial \mathcal{L}_t}{\partial A_{t+1}} = -\lambda_t + \beta \underset{t}{E} \lambda_{t+1} (1+r) = 0 \quad (5)$$

Eqn (4) gives $\lambda_t = u'(c_t)$ (6)

use (6) in (5) to get

$$u'(c_t) = \beta(1+r) \underset{t}{E} u'(c_{t+1}) \quad (7)$$

on

$$C_t^{-\theta} = \left(\frac{1+r}{1+\rho}\right) E_t\, C_{t+1}^{-\theta} \qquad (8)$$

Eqn (8) is the Euler Eqn.

(b) $$\ln C_{t+1} | t \sim N\left(E_t \ln C_{t+1}, \sigma^2\right)$$

So $$E_t\left[C_{t+1}^{-\theta}\right] = E_t\left[e^{-\theta \ln C_{t+1}}\right]$$

$$= e^{-\theta E_t \ln C_{t+1}}\, e^{\theta^2\sigma^2/2} \qquad (9)$$

(Above b/c $-\theta \ln C_{t+1}|t \sim N(-\theta E_t \ln C_{t+1}, \theta^2\sigma^2)$)

Use (9) in (8) to get

$$\left[e^{-\theta \ln C_t}\right] = \left(\frac{1+r}{1+\rho}\right) e^{-\theta E_t \ln C_{t+1}}\, e^{+\theta^2\sigma^2/2} \qquad (10)$$

or, taking logs,

$$-\theta \ln C_t = \ln\left(\frac{1+r}{1+\rho}\right) - \theta E_t \ln C_{t+1} + \frac{\theta^2\sigma^2}{2}$$

or

$$\boxed{\ln C_t = E_t \ln C_{t+1} + \frac{-1}{\theta}\ln\left(\frac{1+r}{1+\rho}\right) - \frac{\theta\sigma^2}{2} \qquad (11)}$$

Equation (11) or eqn (10) is the answer to part b.

(c) Let $\frac{-\theta\sigma^2}{2} + \frac{-1}{\theta}\ln\left(\frac{1+r}{1+\rho}\right) \equiv -a$, a constant.

Add + Subtract $\ln C_{t+1}$ to RHS (11) to get

$$\ln C_t = \ln C_{t+1} + \left(E_t \ln C_{t+1} - \ln C_{t+1}\right) - a$$

or

$$\boxed{\ln C_{t+1} = a + \ln C_t + u_{t+1} \qquad (13)}$$

where $u_{t+1} = \left(\ln C_{t+1} - E_t \ln C_{t+1}\right)$

u_{t+1} is white noise via Rational Expectations.

~~(d) Re-write (11) as~~

$$\cancel{E_t\left[\ln C_{t+1} - \ln C_t\right] = \frac{\theta\sigma^2}{2}}$$

(d) Re-write (11) as

$$\left[E_t \ln C_{t+1} - \ln C_t\right] = \frac{1}{\theta} \ln\left(\frac{1+r}{1+\rho}\right) + \frac{\theta \sigma^2}{2}$$

$$\uparrow r \Rightarrow \uparrow \ln\left(\frac{1+r}{1+\rho}\right) \Rightarrow \uparrow E_t\left[\ln C_{t+1} - \ln C_t\right]$$

$$\uparrow \sigma^2 \Rightarrow \uparrow \frac{\theta \sigma^2}{2} \Rightarrow \uparrow E_t\left[\ln C_{t+1} - \ln C_t\right]$$

$$\frac{\partial\left[E_t \ln C_{t+1} - \ln C_t\right]}{\partial \theta} = \frac{-1}{\theta^2} \ln\left(\frac{1+r}{1+\rho}\right) + \frac{\sigma^2}{2} \lesseqgtr 0 \; ?$$

So $\uparrow r$ or $\uparrow \sigma^2$ Increases Expected Cons growth.

but $\uparrow \theta$ has an ambiguous effect.