SOLUTION GUIDE TO MATH GRE FORM GR1268

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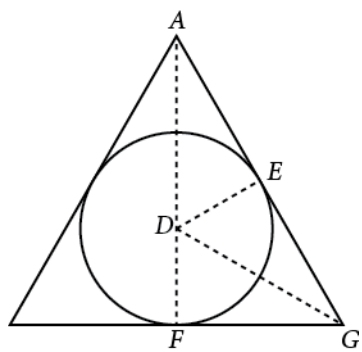
The questions for this solution guide can be found here.

Solution 1. (E) Two applications of L'Hopital's rule:

$$\lim_{x\to 0} \frac{\cos(3x)-1}{x^2} = \lim_{x\to 0} \frac{-3\sin(3x)}{2x} = \lim_{x\to 0} \frac{-9\cos(3x)}{2} = \frac{-9}{2}.$$

You can also use the Taylor expansion of $\cos(3x) = 1 - (3x)^2/2! + (3x)^4/4! \cdots$; after subtracting 1 and dividing, the only term without an x in it is the leading $-9/2$.

Solution 2. (C) Here's how I see it.



We are given $DF = 2$. The triangle ΔDFG is a 30-60-90 triangle, making $FG = 2\sqrt{3}$. Hence one side of the triangle is $4\sqrt{3}$. Using the general formula for the area of an equilateral triangle, $s^2\sqrt{3}/4$, we obtain the area is $12\sqrt{3}$.

It turns out there is an easy formula: for *any* triangle, the radius of the inscribed circle is equal to $2A/P$, where A and P are the area and perimeter of the triangle, respectively. Since A/P is a fixed ratio for an equilateral triangle, we have a special formula $r = s/2\sqrt{3}$ for an equilateral triangle. You can solve the above problem using this (very esoteric) method.

Solution 3. (D) Using u-substitution, let $u = \log x$ so that $du = 1/x\,dx$. This gives

$$\int_{e^{-3}}^{e^{-2}} \frac{1}{x\log x}\,dx = \int_{-3}^{-2} \frac{1}{u}\,du = \log u\Big|_{-3}^{-2} = \log(-2) - \log(-3) = \log(2/3).$$

Solution 4. (A) We know that $\dim(V+W) = \dim V + \dim W - \dim(V\cap W)$. The lefthand side of that equation is at most 7, and we know that $\dim V + \dim W = 8$. Therefore $\dim(V\cap W)$ must be at least 1.

Solution 5. (E) We can quickly count off the pairs that are not permissible. Out of 100 potential pairs, we cannot choose (1,1), (2,4), (3,9), (4,2), and (9,3). That leaves 95/100.

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Solution 6. (C) Just raise everything to the sixth power:

$$(2^{1/2})^6 = 8, \quad (3^{1/3})^6 = 9, \quad (6^{1/6})^6 = 6.$$

Since $f(x) = x^6$ preserves inequalities, this gives us the order.

Solution 7. (C) We have to do some graphical integration to determine the solution. Certainly $f(2) > f(0)$ since we have added positive area on $[0, 2]$. And $f(4) < f(2)$ since we have added negative area on $[2, 4]$. A quick inspection shows that the semicircle of radius one $(f(2) - f(4))$ is far smaller than the quarter-ellipse area $(f(2) - f(0))$. Therefore the proper ordering is $f(0) < f(4) < f(2)$.

Solution 8. (B) Remember that a group necessarily has inverses. Going down the list, $\mathbb{Z} \setminus \{0\}$ definitely does not have all multiplicative inverses. We can stop at this point.

Solution 9. (A) The information given tells us that we are concave up at $x = -1$ and concave down on $(0, 2)$. We also have a maximum, minimum, or saddle point at $x = 0$. Going through our options, (A) is suitable and we can stop there. The other graphs can be checked for their particular issues.

Solution 10. (A) Squaring both sides, we can fiddle around a bit to determine what we're actually dealing with.

$$\begin{aligned}(x+3)^2 + (y-2)^2 = (x-3)^2 + y^2 &\implies (x+3)^2 - (x-3)^2 = y^2 - (y-2)^2 \\ &\implies 12x = 4y - 4.\end{aligned}$$

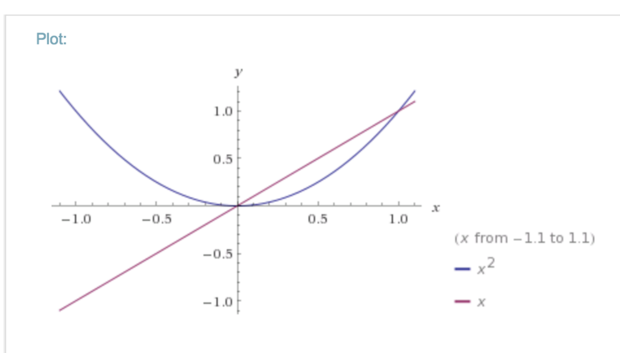
That gives us a line.

Alternatively, we can think symbolically about what the question is asking and take a trip down conic section memory lane. The left-hand side is the distance from (x, y) to $(-3, 2)$ and the right-hand side is the distance to $(3, 0)$. Therefore (x, y) have to be all points equidistant from two points in the plane.

That's definitely not a circle, as those are all (x, y) equidistant from a single point. An ellipse is the set of all (x, y) where the *sum* of the distances to the two foci is a constant, and we have no such constraint here. A parabola is the set of all (x, y) that are equidistant from a single point and the directrix, which is a line rather than a second point. Finally, a hyperbola is the set of (x, y) such that the *difference* of the distances to its two foci is a constant.

Therefore we have to have a line. The set of points we define is the perpendicular bisector to the segment $(-3, 2) \to (3, 0)$.

Solution 11. (B) Since we are rotating around the y-axis, we should phrase things in terms of functions $f(y)$.



It's easy to see that our integral will run from $y = 0$ to $y = 1$. The righthand function is $x = \sqrt{y}$ and the lefthand function is $x = y$. The integral we must perform is

$$\pi \int_0^1 (\sqrt{y})^2 - y^2\,dy = \pi \int_0^1 y - y^2\,dy = \pi\left(y^2/2 - y^3/3\right)\Big|_0^1 = \pi(1/6 - 0) = \pi/6.$$

Alternatively, we can use the method of cylindrical shells. Our integral will run from $x = 0$ to $x = 1$, so the volume is given by

$$2\pi \int_0^1 x(x - x^2)\,dx = 2\pi \int_0^1 x^2 - x^3\,dx = 2\pi(x^3/3 - x^4/4)\Big|_0^1 = 2\pi \cdot (1/12 - 0) = \pi/6$$

giving us the same answer.

Solution 12. (B) Any group of prime order is necessarily cyclic, and hence there is only one up to isomorphism. This limits are choices to (B), (C), and (E). But there are two groups of order 9 (at least): $\mathbb{Z}/3 \times \mathbb{Z}/3$ and $\mathbb{Z}/9$. This makes (B) our only option.

Solution 13. (D) The maximum value of $f(0)$ will be obtained when $f'(x) = -1$ constantly. That would make $f(0) = 5 + 3 = 8$.

Solution 14. (B) Evaluating that integral at $x = c$ gives 0. Therefore we would also have $3c^5 + 96 = 0$. A quick calculation shows $c = -2$.

Solution 15. (A) For a composition of functions, if the first function isn't one-to-one, there's no way the composite is. It's worth mentioning here that the opposite is true for onto: the second function had better be onto.

Solution 16. (B) Put another way, C = A XOR B. If C is false, than either both A and B are true or both A and B are false. (D) and (E) both work, but they're not necessary. (B) is the only one that must be true.

Solution 17. (B) We should see if (D) or (E) has more than 3 solutions to begin with. (E) only has one solution, at $x = 0$, as the righthand side gets arbitrarily close to zero as x gets large but $\sec x$ is bounded away from the x-axis. (D) doesn't have any real solutions at all. (C) also only has one solution.

That leaves (A) and (B). (B) is equivalent to $x^2 + 4x - 15 = 0$, and we can calculate the discriminant is $b^2 - 4ac = 96 > 0$. Therefore it has two real solutions. For (A), we know it has 1 or 3 solutions, so let $f(x) = x^3 + x - 10$. Its derivative $f'(x) = 3x^2 + 1$ is always positive, so $f(x)$ is increasing and can only have 1 solution. This gives us (B).

Note: a previous solution to this problem recommended looking at the discriminant of a cubic polynomal. I no longer recommend this.

Solution 18. (A) We should remember some of our power series. Looking at the solutions, it would be good to recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

So $f(x)$ is the integral of the above power series, so $f'(x)$ is precisely that.

Solution 19. (E) Let us represent $z = a + bi$. Then our limit becomes

$$\lim_{(a,b)\to 0} \frac{(a-bi)^2}{(a+bi)^2} = \lim_{(a,b)\to 0} \frac{a^2 - b^2 - 2abi}{a^2 - b^2 + 2abi}.$$

If we let $a = 0$ (for instance), it is easy to see that the limit is equal to 1. However, if we let $a = b$, then our limit becomes

$$\lim_{a \to 0} \frac{-2a^2 i}{2a^2 i} = -1.$$

Therefore the limit does not exist.

Alternatively, using the polar form of complex numbers, we have $z = re^{i\theta}$ and $\overline{z} = re^{-i\theta}$. Thus

$$\lim_{z \to 0} \frac{(\overline{z})^2}{z^2} = \lim_{r \to 0} \frac{r^2 e^{-2i\theta}}{r^2 e^{2i\theta}} = \lim_{r \to 0} e^{-4i\theta}$$

But this last limit no longer depends on r, so is the value $e^{-4i\theta}$. But θ was a variable, so the limit depends on the angle θ at which one approaches $0 \in \mathbb{C}$. Therefore it does not exist.

Solution 20. (E) This limit looks suspiciously like a derivative. Indeed, noticing that $e = g(0)$, then we could rewrite this as

$$\lim_{x \to 0} \frac{g(g(x)) - g(g(0))}{x - 0}.$$

That is, the derivative of $g(g(x))$ at $x = 0$. By the chain rule, this is $g'(0) * g'(g(0))$. Computing, $g'(x) = 2e^{2x+1}$, so $g'(0) = 2e$ and $g'(e) = 2e^{2e+1}$. That makes our answer $4e^{2e+1+1}$.

Solution 21. (B) We have a symmetric domain here, so for an odd function (e.g. $-f(x) = f(-x)$) its integral is zero. The entire term $\sqrt{1+t^2}\sin^3 t \cos^3 t$ is an odd function because $\sin^3 t$ is odd and the other two functions are even, so its integral amounts to nothing. Therefore the entire integral is equal to

$$\int_{-\pi/4}^{\pi/4} \cos t \, dt = \sin t \Big|_{-\pi/4}^{\pi/4} = \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2}.$$

Solution 22. (C) It looks like our x-coordinates are running over $[-1, 1]$, with y depending on x and z depending on y. To find the volume of the solid, we just need to integrate the constant function 1. We must therefore compute

$$\begin{aligned}
\int_{-1}^{1} \int_{x^2}^{2-x^2} \int_{0}^{y+3} 1 \, dz \, dy \, dx &= \int_{-1}^{1} \int_{x^2}^{2-x^2} y + 3 \, dy \, dx \\
&= \int_{-1}^{1} \big((2-x^2)^2/2 + 3(2-x^2)\big) - \big((x^2)^2/2 + 3(x^2)\big) \, dx \\
&= \int_{-1}^{1} 8 - 8x^2 \, dx \\
&= 8x - 8x^3/3 \Big|_{-1}^{1} = (8 - 8/3) - (-8 + 8/3) = 32/3.
\end{aligned}$$

Solution 23. (D) Examining the choices, we see $S \subset \mathbb{Z}/10$ is a subgroup of an abelian group. Therefore it still have an additive identity and the operation is commutative. It is also closed under addition and multiplication. While S does not contain the multiplicative identity of $\mathbb{Z}/10$, it does have a multiplicative identity. $6 \in S$ is such an identity, as

$$6x = (5+1)x = 5x + x.$$

Since $x \in S$ are all even, $5x = 0$, so $6x = x$.

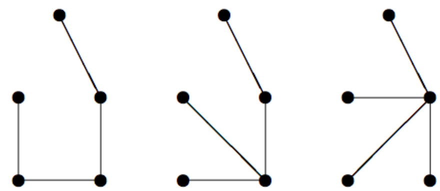


Figure 1. Photo credit Jozef Skokan.

Solution 24. (E) Looking at our answers, we can verify directly that $(-5,1,1,0)$ is a solution. Any multiple of $(-5,1,1,0)$ is also a solution, which shows that (A), (B), (C), and (D) are all true – leaving only (E). Another solution, for example, is $(0,2,-8,5)$

Solution 25. (A) An inflection point of h would be a local maximum or minimum of its derivative h'. One seems to show up around $x = -1.5$.

Solution 26. (D) Fortunately for us, $\mathbb{Z}/11$ is a field. Doing some basic math, we see that $6x \equiv 10$ and $6y \equiv 10$, so $6x + 6y \equiv 9$. Additionally, we see that $6 \cdot 2 \equiv 1$, so $2 \cdot (6x + 6y) = x + y \equiv 7$.

Solution 27. (D) Rewriting this in exponential form, $1 + i = \sqrt{2}e^{i\cdot\pi/4}$. This is much easier to exponentiate. Therefore
$$(1+i)^{10} = 32e^{i\cdot\pi/2} = 32i.$$

Solution 28. (D) Seeing no better option, we can just go down the list. (A) is clearly true, and we can determine if (B) is true if we remember the appropriate formula:
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$
We know $f(1) = 4$ and $f'(1) = 3$. Since f is injective, it is necessary that $f^{-1}(4) = 1$.

Now, for our reference, $g'(x) = \frac{1}{2\sqrt{x}}$, and $g'(1) = 1/2$. To verify (C),
$$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3 \cdot 1 + 4 \cdot 1/2 = 5.$$

To check (D),
$$(g \circ f)'(1) = g'(f(1))f'(1) = g'(4) \cdot 3 = \frac{3}{4} \neq 1/2.$$

Solution 29. (C) It's probably easiest to draw this out for yourself. The maximum degree of any vertex is 2, 3, or 4. If there is a vertex of degree 4, then our tree looks like a star. If the maximum degree of any vertex is 2, then we have a straight line. In the middle case, we obtain a 3-pointed star to which we attach one more vertex – the choice of branch yields isomorphic graphs. See Figure 1.

Solution 30. (A) If we have that $\log x = c \cdot x^4$ at $x = a$ (and only there), then the graphs of the functions are tangent at $x = a$. This is because $\log x$ is concave down and cx^4 is concave up, so there would necessarily be another crossing if we had $\log x > c \cdot x^4$ anywhere. Taking derivatives, we would have that $1/a = c \cdot 4a^3$. From the original equation we also have $\log a = c \cdot a^4$. A little bit of algebra later, we have $c = \frac{1}{4a^4}$, this makes $\log a = 1/4$ so $a = e^{1/4}$. Therefore $c = 1/4e$.

Solution 31. (C) The easiest thing to do is probably to calculate the characteristic polynomial.

$$\begin{aligned}\det\begin{bmatrix}3-\lambda & 5 & 3\\ 1 & 7-\lambda & 3\\ 1 & 2 & 8-\lambda\end{bmatrix} &= (3-\lambda)\det\begin{bmatrix}7-\lambda & 3\\ 2 & 8-\lambda\end{bmatrix} - \det\begin{bmatrix}5 & 3\\ 2 & 8-\lambda\end{bmatrix} + \det\begin{bmatrix}5 & 3\\ 7-\lambda & 3\end{bmatrix}\\ &= (3-\lambda)((7-\lambda)(8-\lambda)-6) - (5(8-\lambda)-6) + (15-3(7-\lambda))\\ &= -(\lambda-2)(\lambda-5)(\lambda-11) \qquad \text{(via some more algebra).}\end{aligned}$$

What works nicely for this problem in particular, though, is examining the matrices $A-\lambda I_3$ for $\lambda \in \{2,3,5\}$. For $\lambda = 2$, we obtain

$$A - 2\cdot I_3 = \begin{bmatrix}1 & 5 & 3\\ 1 & 5 & 3\\ 1 & 2 & 6\end{bmatrix}$$

which has two identical rows so clearly has determinant zero. For $\lambda = 5$,

$$A - 5\cdot I_3 = \begin{bmatrix}-2 & 5 & 3\\ 1 & 2 & 3\\ 1 & 2 & 3\end{bmatrix}$$

we have the same situation. However, for $\lambda = 3$,

$$A - 3\cdot I_3 = \begin{bmatrix}0 & 5 & 3\\ 1 & 4 & 3\\ 1 & 2 & 5\end{bmatrix}$$

so we can't conclude immediately that this matrix has determinant zero. But expansion by minors along the first column is pretty straightforward:

$$\det\begin{bmatrix}0 & 5 & 3\\ 1 & 4 & 3\\ 1 & 2 & 5\end{bmatrix} = 0 - (25-6) + (15-12) = -16 \neq 0$$

Hence $\lambda = 3$ is not an eigenvalue.

As another solution for this nice matrix, note that the sum of each of the rows is 11. This means that $(1,1,1)$ is an eigenvector with eigenvalue 11 (as that's the vector-multiplication way to sum up each of the rows). Recall that the trace of the matrix is the sum of the eigenvalues, which is $3+7+8 = 18$. This means the sum of the other two eigenvalues has to be 7, making (C) look like a good choice. Of course, there's no guarantee that the the other eigenvalues are exactly 2 and 5, so you'd want to make a spot check as above to confirm it.

Solution 32. (E) We can sort this out in two steps and apply the fundamental theorem to each.

$$\frac{d}{dx}\left(\int_{x^3}^{0} e^{t^2}\,dt + \int_0^{x^4} e^{t^2}\,dt\right)$$

For the first,

$$\frac{d}{dx}\int_{x^3}^{0} e^{t^2}\,dt = -\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dt = -3x^2e^{x^6}$$

For the second,

$$\frac{d}{dx}\int_0^{x^4} e^{t^2}\,dt = 4x^3 e^{x^8}$$

All told, our derivative is $x^2e^{x^6}(4xe^{x^8-x^6}-3)$. Of course we can use the more complicated form of the FTC directly (Problem 2.10 in the lecture notes) which allows for variables in both limits of integration.

Solution 33. (C) We might notice a pattern if we start deriving.

$$\begin{aligned} f(x) = \frac{x-1}{e^x} = xe^{-x} - e^{-x} &\implies f'(x) = -xe^{-x} + e^{-x} + e^{-x} = -f(x) + e^{-x} \\ &\implies f''(x) = -(-f(x) + e^{-x}) - e^{-x} = f(x) - 2e^{-x} \\ &\implies f'''(x) = -f(x) + 3e^{-x} \end{aligned}$$

So in general, the nth derivative is going to be $(-1)^n(f(x) - n\cdot e^{-x})$. We want the 19th derivative, so it's going to be $-(xe^{-x} - e^{-x} - 19e^{-x}) = (20-x)e^{-x}$.

Solution 34. (B) An upper triangular matrix is easily verified to be invertible so long as its diagonal entries are all nonzero. Specifically, $\det A$ is still the product of its diagonal entries, so (E) and (D) and (A) are all true. (C) can easily be verified to be true by computing that the bottom-right corner is 25 (the product of upper triangular matrices still being upper triangular). This leaves (B). (B) can be checked directly to be false: if we let $x = (1,0,0,0,0)$, then $Ax = x$.

Solution 35. (B) We can minimise the function $x^2+y^2+z^2 = D(x,y,z)$. We can rewrite this function in terms of two variables since we know that (on our plane) $y = 3-2x-3z$, giving us

$$D(x,z) = x^2 + (3-2x-3z)^2 + z^2.$$

To find the minimum of the function, we should compute its partial derivatives.

$$\begin{aligned} D_x &= 10x + 12z - 12 \\ D_z &= 12x + 20z - 18 \end{aligned}$$

A little bit of algebra gives us a solution at $x = 3/7$ and $z = 9/14$. This gives enough information to conclude the answer is (B).

The alternative way to solve this is probably best. The normal vector to this plane is $(2,1,3)$. The closest point to the origin is along this vector, so is at some point $(2t,t,3t)$. There's only one answer that comes in this form, and that's (B) where $t = 3/14$. This 3/14 actually comes from a formula that you may or may not remember (I didn't, but a reader did): the general solution to this problem for a plane $ax+by+xz = d$ is

$$P = \frac{d}{a^2+b^2+c^2}(a,b,c) \implies P = \frac{3}{4+1+9}(2,1,3) = \left(\frac{6}{14},\frac{3}{14},\frac{9}{14}\right)$$

Solution 36. (C) This is a good time to remind ourselves WHY the false things are false. If (A) were true, it would imply that S is a connected set. (B) is certainly not true if S is a dense set. (C) is the interior of the set S, which is always open. (E) means that S would have to be closed to begin with.

I'm not sure what (D) is supposed to be.

Solution 37. (C) $P^2 = P$ means that P is projection onto some subspace. There is no reason to believe that this should be invertible (consider $P = 0$), but it should definitely be diagonalisable (with eigenbasis some basis of that subspace). III also need not be true if the subspace is anything proper or nontrivial.

Solution 38. (C) The total angle measure of a 10-gon is $180 \cdot 8 = 1440°$. If the polygon is to be convex, all angles must be less than $180°$. If we have 5 acute angles, then the remaining 5 angles would have to make up for $> 1440 - 5 \cdot 90 = 990$ degrees. This is impossible to do and remain convex. If we have 4 acute angles, the remaining 6 angles need to make up for $> 1440 - 4 \cdot 90 = 1080$ degrees. This is our edge case, so the answer must be 3 acute angles.

Solution 39. (D) This problem is nonsense, far as I can tell. The outer while loop iterates i from 2 to n, and the inner while loop prints it out (in a weird way).

Solution 40. (C) There's no reason that $\circ$ should be commutative. We should be a little careful about the distributive laws, however. For II, let $f(x) = x^2$, $g(x) = 1$, and $h(x) = -1$. Then $f(g + h) = 0$, but $f \circ g + f \circ h = 2$. For III, we can verify it directly:

$$((g + h) \circ f)(x) = (g + h)(f(x)) = g(f(x)) + h(f(x)) = (g \circ f)(x) + (h \circ f)(x).$$

By definition, these functions are the same since their values are literally equal. This does not hold for II:

$$(f \circ (g + h)) = f(g(x) + h(x), \quad (f \circ g + f \circ h)(x) = f(g(x)) + f(h(x)).$$

Solution 41. (A) The first plane is determined by the normal vector $(1, 1, 1)$, and the second determined by $(1, -1, 1)$. Therefore the slope of ℓ is determined by a vector perpendicular to those, i.e. the cross product.

$$(1, 1, 1) \times (1, -1, 1) = \det \begin{bmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = (2, 0, -2).$$

So that is the slope of ℓ. We need this to be the normal vector for the plane in question, so it seems that $(1, 0, -1)$ is our best bet (out of the given options).

A more multiple-choice friendly solution is the following: take the slope of each of the solutions and check if it's perpendicular to both $(1, 1, 1)$ and $(1, -1, 1)$. For example, the slope $(1, 1, 1)$ for (B) is definitely not perpendicular to $(1, 1, 1)$ the normal vector of the first plane, so cannot be the right answer. If you start with (A) you get the answer immediately, which is nice, but even checking (B)-(E) does not take very long.

Solution 42. (E) We are taking the discrete metric on $\mathbb{Z}^+$. As such, every singleton set is open. Moreover, since $\mathbb{Z}^+$ is countable, every set is open. Therefore every set is closed as well (having open complement). Finally, suppose $f : \mathbb{Z}^+ \to X$ is any map of topological spaces. Then every $f^{-1}(Y)$ is open for $Y \subset X$ (whether Y is open or not) so f is continuous.

Solution 43. (A) We know that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

To remind you, this follows from the chain rule and rearranging:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Using the same trick, we have[1]

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d\frac{dy}{dx}}{dt} \Big/ \frac{dx}{dt}.$$

Therefore let us go step by step.

$$\frac{dx}{dt} = 2t + 2 = 2(t+1), \quad \frac{dy}{dt} = 12t^3 + 12t^2 = 12t^2(t+1).$$

This makes $dy/dx = 6t^2$. Taking the derivative of that, we get $12t$, and so we finally obtain

$$\frac{d^2y}{dx^2} = \frac{12t}{2t+2} = \frac{6t}{t+1}.$$

Now, the point $(8, 80)$ corresponds to $t = 2$. Plugging that in, we get $12/3 = 4$.

Solution 44. (B) Putting it in simpler terms,

$$\frac{dy}{dx} + xy = x \implies \frac{dy}{dx} = x(1-y) \implies \frac{dy}{1-y} = x\,dx.$$

Integrating both sides, we obtain

$$-\log(1-y) = x^2/2 + C' \implies 1 - y = Ce^{-x^2/2} \implies y = 1 - Ce^{-x^2/2}.$$

Solving the initial value problem gives $C = 2$. Furthermore, as $x \to -\infty$, the second term above vanishes so we get 1 in the limit.

Solution 45. (C) Certainly our solutions are concentrated in $[0, 1]$. We know that every $2\pi/97$ units in x, we get another period of $\cos(97x)$, and each period must meet $y = x$ twice. Therefore there are

$$\frac{1}{2\pi/97} = \frac{97}{2\pi} \approx \frac{97}{6.3} \approx 15$$

periods in $[0, 1]$ and about 30 meetings. There's only one answer in that range, so we'll stick with it.

Solution 46. (C) A very basic related rates problem. Let h be the height of the ladder from the ground and ℓ the distance away from the wall. This gives the relation $h^2 + \ell^2 = 81$. We are also given $d\ell/dt = 2$. At the point when the top of the ladder is 3 metres above the ground, the bottom of the ladder is $\sqrt{81-9} = 6\sqrt{2}$ away. Taking a derivative,

$$2h \cdot \frac{dh}{dt} + 2\ell\frac{d\ell}{dt} = 0 \implies 2 \cdot 3 \cdot \frac{dh}{dt} + 2 \cdot 6\sqrt{2} \cdot 2 = 0.$$

A little bit of math later shows that $dh/dt = -4\sqrt{2}$, giving our answer.

Solution 47. (B) A classic kind of problem. We are clearly continuous and differentiable at 0. Anywhere else, near a rational number there is an irrational number and vice versa. Therefore there can be no continuity anywhere but at 0, and hence no differentiability either.

Solution 48. (B) It would be good to recall the formula for the directional derivative. We take the gradient of the function then take its scalar product with the normalised vector in the direction we want. To begin,

$$\nabla g = (6xy, 3x^2, 1).$$

[1] Notation credit to `https://www.math.hmc.edu/calculus/tutorials/parametric_eq/`.

At the point $(0,0,\pi)$, we have $\nabla g = (0,0,1)$. That works out pretty well for us. The normalised version of the vector $(1,2,3)$ is $(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$. Dotting this with $(0,0,1)$ gives $3/\sqrt{14}$, and since $\sqrt{14} = 3.5$ or so our answer should be closer to 0.8 than 0.2.

Solution 49. (B) The greatest order is given by the product of a 2-cycle and a 3-cycle acting on disjoint elements. That gives order 6.

Solution 50. (D) The sum of the ideals is still an ideal: it is clearly closed under addition (using commutativity of addition), and still under left and right multiplication due to the distributive property. The intersection of ideals is still an ideal, which is not too hard to work out. The product of ideals, however, need not be closed under addition. Consider, for example, $R = \mathbb{Z}[X]$, $U = (2, X)$, and $V = (3, X)$ (the ideals generated by two elements). Then we know that $-2X \in U \cdot V$ and $3X \in U \cdot V$, and hence we should expect $3X - 2X = X \in U \cdot V$. However, there is no way to get X as the product of an element of U and an element of V.

Solution 51. (E) The basis (C) is not orthogonal and (D) is not normal, so we can rule those out. Moving on to the matrix itself, it would be nice to know its rank. We can compute that the first row is the sum of the second and third, making the rank of the row space equal to 2.

That leaves only (A) and (E), but (A) cannot be correct. Our column space contains vectors that have nonzero third entry, so cannot lie in the span of that basis.

Solution 52. (A) Suppose we order the classes to be taught, and know in advance we will give the first two to the first professor, etc. There are 20! ways to order these classes. However, the arrangement is the same if we do pairwise swaps of 1-2, 3-4, etc. There are 10 pairs and each has 2 orientations, so there are 2^{10} essentially the same arrangements.

Solution 53. (A) If we recall the formula for differentiation under the integral, here is the general way:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)\,dy.$$

Looking at our situation, it is simplified greatly since $a(x) = 0$ and $b(x) = x$. Let $h(x,y) = f(y)(y-x)$ for ease of notation. Then

$$\begin{aligned} g'(x) &= h(x,x)\cdot 1 - h(x,0)\cdot 0 + \int_0^x \frac{\partial}{\partial x} h(x,y)\,dy \\ &= 0 - 0 + \int_0^x -f(y)\,dy = \int_0^x -f(x)\,dy \end{aligned}$$

Then $g''(x) = -f(x)$ and $g'''(x) = -f'(x)$. Therefore f needs only one derivative.

Solution 54. (C) We can visualise this as a rectangle in the xy-plane. Consider $[0,3]\times[0,4]$ there. The triangle bounded by the line $y = x$ (and the sides of the rectangle) is all points (x,y) such that $x > y$. This triangle has an area of 4.5, and the rectangle an area of 12. Then the rest of the points (where $x < y$) contribute an area of 7.5. Hence the probability is $7.5/12 = 5/8$.

Solution 55. (E) The best solution is to see how our integral behaves with respect to varying a, b. Let $I(a,b)$ denote the definite integral. Note that $e^{ax} - e^{bx}$ has the same sign as $a - b$

for all $x > 0$ and $a, b > 0$. We should conclude that $I(a, b) > 0$ whenever $a > b > 0$, and this rules out (A). By inspection, if we flip the position of a, b, we can see that $I(b, a) = -I(a, b)$; this rules out (B).

Consider now the integral $I(ta, tb)$ for some $t \in \mathbb{R}$. Then we can recontextualise this integral using u-substitution for $u = tx$, $du = t \cdot dx$:

$$\int_0^\infty \frac{e^{tax} - e^{tbx}}{(1 + e^{tax})(1 + e^{tbx})} \, dx = \int_0^\infty \frac{e^{au} - e^{bu}}{(1 + e^{au})(1 + e^{bu})} \frac{du}{t} = \frac{1}{t} I(a, b)$$

Only (E) satisfies this scaling condition, as (C) and (D) satisfy $I(ta, tb) = t \cdot I(a, b)$ rather than the inverse. Thanks very much to a reader for this elegant solution.

I will include the long, tortured computation I had originally for history's sake. We can use some partial fractions followed up by u-substitution. First,

$$\frac{e^{ax} - e^{bx}}{(1 + e^{ax})(1 + e^{bx})} = \frac{1}{1 + e^{bx}} - \frac{1}{1 + e^{ax}}.$$

Now, consider

$$\int_0^\infty \frac{dx}{1 + e^{bx}}.$$

Let $u = e^{bx}$. Then $du = b \cdot e^{bx} \, dx$, so $\dfrac{du}{b \cdot u} = dx$. Replacing this in, we need to solve

$$\frac{1}{b} \int_1^\infty \frac{du}{u(1 + u)}.$$

This requires some more partial fractions.

$$\frac{1}{u(u + 1)} = \frac{1}{u} - \frac{1}{u + 1}.$$

Putting all that together,

$$\int_0^\infty \frac{dx}{1 + e^{bx}} = \frac{1}{b} \lim_{R \to \infty} (\log u - \log(u + 1)) \Big|_1^R = \frac{1}{b} \lim_{R \to \infty} \log\left(\frac{R}{R + 1}\right) - \log\left(\frac{1}{2}\right) = \frac{\log 2}{b}.$$

That makes our other quantity $\log 2/a$. A little more math gives the answer.

Solution 56. (D) I is certainly true, as it is easy to verify that $\lim_{x \to \infty} \frac{\log x}{\sqrt{x}} = 0$, so the function achieves some maximum on $x \geq 1$. Take C to be that maximum.

If we recall the formula for the landhand side for II,

$$\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \approx n^3.$$

As such, no constant C is going to do the trick for all $n \in \mathbb{N}$.

III brings to mind the power series for $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Then $|\sin x - x|$ is the error of the first order Taylor approximation to $\sin x$, which is given in terms of the next term, which indeed is related to x^3. The specific formula is, for the

Taylor series of order n for $f(x)$ centred at $x = a$,

$$|E_n(x)| \leq \frac{M \cdot |x-a|^{n+1}}{(n+1)!},$$

where M is an upper bound for $|f^{(n+1)}(x)|$ on the interval between a and x. For us, we take $n = 2$ and $a = 0$, and we know that $|f^{(3)}(x)| = \cos x$ attains a maximum of 1. This gives $C = \frac{1}{3!}$, as expected.

Solution 57. (C) I is true, since $\lim_{n\to\infty} x_n$ must be bounded between 0 and $\lim_{n\to\infty} 1/n = 0$. Unfortunately, x_n does not converge inside $(0,1)$. There is no reason therefore that $f(x_n)$ should be a convergent sequence – suppose that $f(x) = 1/x$, so that $f(x_n)$ is certainly not Cauchy. However, if g is uniformly continuous, then g extends to a continuous function on $[0,1]$. Now x_n is a convergent sequence, so $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n) = g(0)$ exists.

Solution 58. (B) For our curve $r(\theta)$, the arc length from θ to $r(0) = (5,0,0)$ is given by

$$\int_0^\theta \|r'(t)\|\, dt.$$

To find what we are actually integrating over,

$$r'(\theta) = (-5\sin\theta, 5\cos\theta, 1) \implies \|r'(\theta)\| = \sqrt{25\sin^2\theta + 25\cos^2\theta + 1} = \sqrt{26}.$$

Then $L(\theta) = \sqrt{26}\cdot\theta$ precisely, so $\theta_0 = \sqrt{26}$. Computing $D(\theta)$ is easy enough:

$$D(\theta) = \sqrt{25\cos^2\theta + 25\sin^2\theta + \theta^2} = \sqrt{25+\theta^2}.$$

Plugging in θ_0 gives $\sqrt{51}$ as our answer.

Solution 59. (E) Out of all these options, (C) is the trickiest one to think about. There is a theorem that if B is a nilpotent matrix (i.e. $B^k = 0$ for some k), then $I - B$ is invertible. In the case of (C), let $B = I - A$, so that $I - (I - A) = A$ is invertible. This makes (C) a valid criterion. (E) is the only one which does not work – suppose we consider A the matrix giving projection onto the xy-plane. Then the vectors $(1,0,0)$, $(0,1,0)$, and $(1,1,1)$ are linearly independent but all map to nonzero vectors.

Solution 60. (D) While it looks like this is the opposite of continuity, that should read 'there exists $\varepsilon > 0$'. What the statement says is that we not only get arbitrarily far away from $f(1)$, but we must for all x sufficiently far away from 1. So as $|x|$ gets very large, so does $|f(x)|$.

Solution 61. (E) We can set this up as a differential equation. Let s denote the amount of salt in the tank, and let t denote time. We have the initial condition of $s(0) = 3$. $s'(t)$ depends on two factors: the salt flowing in and the salt flowing out. The salt flows in constantly at a rate of 0.08 grams per minute, and the salt flows out at a rate of $4\cdot(s/100) = s/25$ grams per minute. Therefore

$$s'(t) = \frac{ds}{dt} = 0.08 - s(t)/25 \implies \frac{ds}{dt} = 0.04(2-s) \implies \frac{ds}{2-s} = 0.04\, dt.$$

Doing the usual calculus,

$$-\log(2-s) = 0.04t + C' \implies 2 - s = Ce^{-0.04t} \implies s(t) = 2 - Ce^{-0.04t}.$$

The initial condition tells us that $C = -1$, so $s(t) = 2 + e^{-0.04t}$. Plugging in $t = 100$ gives our answer.

Solution 62. (C) This question greatly depends on the fact that we are in two dimensions. The complement of S within $[0,1]\times[0,1]$ is given by all points with both coordinates rational. This set is neither closed nor open, and so neither is its complement (and hence neither is S). It is certainly not totally disconnected, and compact would imply closed and bounded (given that we are in Euclidean space).

It is, in fact, connected. We can prove this because it is path connected. Suppose we have two points (x_0, y_0) and (x_1, y_1). Without loss of generality, suppose x_0 is irrational. We can travel straight from (x_0, y_0) to (x_0, y_1). If y_1 is irrational, then we can travel straight to (x_1, y_1). If y_1 is rational, then x_1 must be irrational. If y_0 was irrational to begin with then we can travel (x_0, y_0) to (x_1, y_0) to (x_1, y_1) in a corner.

However, if y_0 is rational, then we should take (x_0, y_0) to (x_0, z) for some irrational z, then (x_0, z) to (x_1, z) and finally to (x_1, y_1). Being path connected, S is connected as well.

Solution 63. (E) If the supremum is positive, it will be the product of the two greatest positive numbers in A and B or the product of the two least negative numbers in A and B. That means we should look for $\sup\cdot\sup$ or $\inf\cdot\inf$. However, it might be the case that the supremum is non-positive: this happens if B contains only negative numbers and A contains only positive numbers. In that case, the greatest value in $A\cdot B$ (i.e. the negative number of smallest magnitude) will be attained by the least (positive) element of A and the greatest (negative) element of B, giving us our third option: $\inf A\cdot\sup B$.

Solution 64. (E) The surface given is the top half of the unit sphere in $\mathbb{R}^3$. Don't do this the hard way – use the divergence theorem. Consider the unit ball $B\subset\mathbb{R}^3$ and its surface ∂B. Then the divergence theorem tells us that

$$\iint_{\partial B}\vec{F}\cdot d\vec{S} = \iiint_B \operatorname{div}\vec{F}\,dV.$$

The righthand side is much more appealing. The divergence of F is just 3, so the righthand integral is the volume of the unit ball times 3, i.e. 4π. Because the vector field $\vec{F}$ is symmetric on the z-axis, we have that

$$\iint_{\partial B}\vec{F}\cdot d\vec{S} = 2\iint_S \vec{F}\cdot d\vec{S} = 4\pi$$

so dividing by 2 gets us the answer.

Solution 65. (E) The real and imaginary parts of an analytic function must satisfy the Cauchy-Riemann equations. That is,

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \qquad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}.$$

Renaming things for our benefit, let our function be $h(x,y) = f(x,y) + i\cdot g(x,y)$. This tells us that

$$\frac{\partial g}{\partial y} = e^x\sin y, \qquad \frac{\partial g}{\partial x} = -e^x\cos y.$$

This gives us the candidate $g(x,y) = -e^x\cos y + C$. Luckily for us, we needn't worry about $+C$ when taking $g(3,2) - g(1,2)$ as it will subtract itself out. Hence

$$g(3,2) - g(1,2) = -e^3\cos 2 + e\cos 2 = (e - e^3)\cos 2.$$

Solution 66. (B) We need to pick elements of order 16 in $\mathbb{Z}/17^\times$. It is easy to rule out $16 \equiv -1$, since -1 has order 2. We see that $5^2 = 25 \equiv 8$, so there's no way that 8 can be a generator. We just need to verify that the order of 5 is more than 8, so we can check 5^8:

$$5^4 = 8^2 = 64 \equiv -4, \quad 5^8 = (-4)^2 = 16 \neq 1.$$

That makes 5 a generator.