An embedding of quasicategories in prederivators

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Abstract

We show that the theory of quasicategories embeds in that of prederivators, in that there exists a simplicial functor from quasicategories to prederivators which is an equivalence onto its image. Thus no information is lost in the passage from quasicategories to prederivators, in contrast to the apparent lesson of previous work on the subject, and a certain class of prederivators can serve as a complete axiomatization of homotopy theories or $(\infty, 1)$-categories.

The past five decades have seen many proposed axiomatizations of homotopy theory, beginning with Quillen model categories [?]. Dwyer and Kan found, via their notion of simplicial localization [?], the perspective that a model category should be viewed as a *presentation* of a homotopy theory, rather than as a homotopy theory proper. Via simplicially enriched categories [?, ?], Rezk spaces (complete Segal spaces) [?], or quasicategories (∞-categories, weak Kan complexes) [?, ?], a homotopy theory is now understood to be the same thing as an $(\infty, 1)$-category, a term which we apply equally to objects of any of the three theories just listed. The salient commonality in any model of $(\infty, 1)$-categories is that any two objects have a space of morphisms between them, whose homotopy type is well defined. One can construct an $(\infty, 1)$-category of $(\infty, 1)$-categories, often referred to as the homotopy theory of homotopy theories [?]. In particular there is a well defined (up to homotopy equivalence) space of functors between any two $(\infty, 1)$-categories.

Both $(\infty, 1)$ categories and model categories share a common underlying structure: their homotopy categories. Indeed, for every small category J and for any homotopy theory $\mathcal{C}$, there exists an ordinary category $\mathrm{Ho}(\mathcal{C}^J)$, the homotopy category of J-shaped diagrams in $\mathcal{C}$. That is, each homotopy theory $\mathcal{C}$ gives rise to a 2-functor $\mathrm{Ho}(\mathcal{C}^{(-)})$ sending small categories to possibly large categories. This is known as the "prederivator" of $\mathcal{C}$, which is called a derivator when $\mathcal{C}$ has all (homotopy) limits and colimits.

The concept of derivator has been known longer than most notions of $(\infty, 1)$-category, arising with Heller [?] and Grothendieck [?] as much as thirty years ago. In a derivator, such constructions as the cone on a morphism or the Postnikov tower of a space become functorial, and one motivation of derivators was to find a minimal enhancement of triangulated categories permitting such functoriality. This minimality is reflected in the fact that a prederivator underlies every homotopy theory, no matter the model; but it had generally been thought that the mapping of homotopy theories to prederivators was a kind of truncation, rather than an embedding.

There is an obvious analogy with the homotopy category itself: every homotopy theory $\mathcal{C}$ has an underlying homotopy category $\mathrm{Ho}(\mathcal{C})$, but by no means does the homotopy category determine the homotopy theory. For instance, there is the well known example [**?**] of the stable categories of modules over $\mathbf{F}_{p^2}$ and $\mathbf{F}_p[\varepsilon]$. These are combinatorial model categories [**?**, Section 2.2] with the same homotopy category, but which are not Quillen equivalent. The work of Renaudin [**?**] suggested that the 2-category of prederivators should be thought of as a truncation of homotopy theories one dimension higher, and specifically as the result of strictly inverting the homotopy invertible natural transformations (for model categories, the natural transformations between left Quillen functors giving weak equivalences at the cofibrant objects.) This truncation process would destroy the higher-dimensional information in the spaces of morphisms between homotopy theories, such as the homotopy groups of dimension at least three. We show here that this loss is unnecessary: the prederivator of a homotopy theory is in fact a complete invariant, and indeed the homotopy theory of homotopy theories embeds in a simplicially enriched category of prederivators. In [**?**], Muro and Raptis describe a simplicial enrichment $\mathbf{PDer}_\bullet$ of the category of prederivators. We will show that $\mathbf{PDer}_\bullet$ is the target of a simplicially fully faithful functor HO from the simplicially enriched category of quasicategories, $\mathbf{QCAT}_\bullet$. The latter simplicial category contains a Kan complex-enriched category $\mathbf{QCAT}_{\bullet,\iota}$, which also embeds fully faithfully in an analogous restriction $\mathbf{PDer}_\bullet^{\mathrm{eq}}$ of the full simplicial category of prederivators. In short, all homotopy-theoretic properties of quasicategories, such as their algebraic K-theory, may be recovered from their associated prederivators.

For comparison, the related result proved by Renaudin is that the 2-category of combinatorial model categories, left Quillen functors, and natural transformations, localized at those natural transformations giving weak equivalences on cofibrant objects, embeds 2-fully faithfully in the 2-category $\underline{\mathbf{PDer}}$ of prederivators.

In this work we describe homotopy theories via quasicategories [**?**] rather than model categories. This fresh start permits us not only to avoid the false appearance of a truncation, but to embed the full range of homotopy theories into prederivators. Up to Quillen equivalence, the combinatorial model categories implicated in Reanudin's results are exactly the locally presentable model categories, that is, the left Bousfield localizations of model categories of simplicial presheaves. Thus Renaudin's result applies to a special class of homotopy theories, in particular ones which are (homotopically) complete and cocomplete. Using quasicategories, no such assumptions are required, as we do not require the universal property of presentable model categories [**?**] exploited by Renaudin.

The contents of the paper are as follows. Section 1, on conventions, introduces the notion of a smothering prederivator, which is a technically useful refinement of a strong prederivator. Section 2 constructs the prederivator associated to a quasicategory, and furthermore the ordinary functor $\mathrm{HO} : \mathbf{QCAT} \to \mathbf{PDer}$ associating a prederivator to a quasicategory. Section 3 recalls the construction of the simplicially enriched category $\mathbf{PDer}_\bullet$ of prederivators and the restricted simplicial category $\mathbf{PDer}_\bullet^{\mathrm{eq}}$ of [?], which will be seen to contain the Kan complex-enriched category $\mathbf{QCAT}_{\bullet,\iota}$ of quasicategories. In Section 4 we state the main result and deduce the claim on $\mathbf{QCAT}_\bullet^{\mathrm{eq}}$ as a corollary.

Section 5 is dedicated to the proof of the main theorem embedding $\mathbf{QCAT}_\bullet$ in $\mathbf{PDer}_\bullet$. The result follows quickly from the fact that the ordinary functor $\mathrm{HO} : \mathbf{QCAT} \to \mathbf{PDer}$ sending quasicategories to prederivators is fully faithful. There are essentially two pieces to this latter proposition: Lemma 5.2 and Lemma 5.5. The first says that a certain forgetful functor to "**Cat**-sets" (see Definition 5.1) is faithful for the smothering prederivators defined in Section 1: in essence, morphisms between smothering prederivators are determined by their actions on objects. Lemma 5.5 says that a further forgetful functor from **Cat**-sets to simplicial sets is fully faithful when restricted to the prederivators underlying quasicategories. A finaly brief Section 6 discusses the question of computing the image of quasicategories in prederivators.

1 Conventions and definitions

We begin with notational conventions and the axioms of derivators.

If $\mathcal{C}$ is a category (or a 2-category, simplicially enriched category, etc) with objects c_1 and c_2, we denote the set (or category, simplicial set, etc) of morphisms by $\mathcal{C}(c_1, c_2)$.

We will frequently alternate between viewing the same collection of objects as a category, 2-category, or a simplicially enriched category, which we will generally call simply a simplical category.

Convention 1.1. *We will denote the category, the 2-category, and the simplicial category of foos respectively by*

$$\mathbf{foo}, \underline{\mathbf{foo}}, \mathbf{foo}_\bullet$$

Furthermore, when applicable, the above will designate the category of small *foos while*

$$\mathbf{FOO}, \underline{\mathbf{FOO}}, \mathbf{FOO}_\bullet$$

will refer to large *ones.*

The reason for the capitalized forms is that the theory of prederivators includes reference to such objects as the "category of all large categories." Such apparent nonsense can be readily tamed by appeal to two Grothendieck universes, or by the reader's preferred foundations.

We denote the category $0 < 1 < \cdots < n$ by $[n]$, so that $[0]$ is the terminal category. The simplex category Δ is the full subcategory of **Cat** on the categories $[n]$, as n runs over $\mathbf{N}$.

If S is a simplicial set, that is, a contravariant functor $S : \Delta^{\mathrm{op}} \to \mathbf{Set}$, then we denote its set of n-simplices by $S([n]) = S_n$. The face map $S_n \to S_{n-1}$ which forgets the i^{th} vertex will be denoted d_i^n or just d_i.

Definition 1.2. The nerve $N(J)$ of a category J is the simplicial set defined by the formula $N(J)_n = \mathbf{Cat}([n], J)$.

Recall that the natural extension of N to a functor is a fully faithful embedding of categories in simplicial sets. See [**?**, Proposition B.0.13].

We denote by Δ^n the simplicial set represented by $[n] \in \Delta$; equivalently, $\Delta^n = N([n])$.

For us 2-categories are strict: they have strictly associative composition and strict units preserved on the nose by 2-functors, and our morphisms between 2-functors will be strictly 2-natural transformations. In other words, the 2-category of 2-categories is identical to the **Cat**-enriched category of **Cat**-enriched categories.

We now recall the definitions relevant to the theory of derivators.

Definition 1.3. A *prederivator* is simply a 2-functor $\mathscr{D} : \underline{\mathbf{Dia}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$ from a 2-category $\underline{\mathbf{Dia}}$ such as the 2-category of finite direct categories or that of all small categories to the 2-category $\underline{\mathbf{CAT}}$ of locally small categories. See [**?**, Definition 1.12], for the exact conditions $\underline{\mathbf{Dia}}$ must satisfy.

For categories $J, K \in \underline{\mathbf{Dia}}$, we have a functor $\mathrm{dia}_J^K : \mathscr{D}(J \times K) \to \mathscr{D}(J)^K$ induced by the action of $\mathscr{D}$ on the functors and natural transformations from $[0]$ to K. We refer to dia_J^K as a "partial underlying diagram functor," and when $J = [0]$ simply as the "underlying diagram functor," denoted dia^K.

We generally denote $\mathscr{D}(u)$ by u^*, for $u : J \to K$ a functor in $\underline{\mathbf{Dia}}$.

The 2-functor $\mathscr{D}$ is a *derivator* if it satisfies the first four of the following axioms, a *semiderivator* if it satisfies the first two, and *strong* if it satisfies the fifth. We introduce here a variant (Der5') of the fifth axiom, prederivators satisfying which will be called *smothering*, à la [**?**].

(Der1) Let $(J_i)_{i \in I}$ be a family of objects of $\underline{\mathbf{Dia}}$ such that $\coprod_I J_i \in \underline{\mathbf{Dia}}$ as well. Then the canonical map

$$\mathscr{D}\left(\coprod_I J_i\right) \to \prod_I \mathscr{D}(J_i)$$

is an isomorphism.

(Der2) For every $J \in \underline{\mathbf{Dia}}$, the underlying diagram functor

$$\mathrm{dia}^J : \mathscr{D}(J) \to \mathscr{D}([0])^J$$

is conservative.

(Der3) For every functor $u : J \to K$ in $\underline{\mathbf{Dia}}$, $u^* = \mathscr{D}(u) : \mathscr{D}(K) \to \mathscr{D}(J)$ has both a left and a right adjoint, denoted by $u_!$ and u_* respectively.

(Der4) For every object k of $K \in \underline{\mathbf{Dia}}$, functor $u : J \to K$, and object $X \in \mathscr{D}(J)$, the canonical map $p_! q^* X \to k^* u_! X$ is an isomorphism, where $q : u/k \to J$ is the canonical projection and $p : u/k \to [0]$ is the unique functor to the terminal category. Dually, for u_*.

(Der5) For every $J \in \underline{\mathbf{Dia}}$, the partial underlying diagram functor $\mathrm{dia}_J^{[1]} : \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ is full and essentially surjective.

(Der5') For every $J \in \underline{\mathbf{Dia}}$, the partial underlying diagram functor $\mathrm{dia}_J^{[1]} : \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ is full and surjective on objects.

Remark 1.4. Axiom (Der2) implies that all the partial underlying diagram functors are conservative, so that (Der5') requires exactly that $\mathrm{dia}_J^{[1]}$ be smothering in the sense of [**?**].

To lessen the sense of artificiality in asking in (Der5') for strict surjectivity of a functor, we might note that asking a full conservative functor to strictly surjective onto its essential image is equivalent to asking that it be a fibration in the trivial model structure on the category of categories, that is, the unique model structure whose weak equivalences are equivalences of categories, see [**?**].

2 The prederivator associated to a quasicategory

We recall that a *quasicategory* [**?**], called an ∞-category in [**?**], is a simplicial set Q in which every inner horn has a filler. That is, every map $\Lambda_i^n \to Q$ extends to an n-simplex $\Delta^n \to Q$ when $0 < i < n$, where Λ_i^n is the simplicial subset of the standard simplex Δ^n generated by the faces $d_j\Delta^n$ for $j \neq i$. For instance when $n = 2$, the only inner horn is Λ_1^2, and then the filler condition simply says we may compose "arrows" (that is, 1-simplices) in Q, though not uniquely. Morphisms of quasicategories are simply morphisms of simplicial sets. The quasicategories in which every inner horn has a *unique* filler are exactly the nerves of categories; in particular the nerve functor $N : \mathbf{Cat} \to \mathbf{SSet}$ factors through the subcategory of quasicategories, $\mathbf{QCat}$.

Every quasicategory Q has a homotopy category $\mathrm{Ho}(Q)$, the ordinary category defined as follows. The objects of $\mathrm{Ho}(Q)$ are simply the 0-simplices of Q. For two 0-simplices q_1, q_2, temporarily define $Q(q_1, q_2) \subseteq Q_1$ to be the set of 1-simplices f with initial vertex q_1 and final vertex q_2. Then the hom-set $\mathrm{Ho}(Q)(q_1, q_2)$ is the quotient of $Q(q_1, q_2)$ which identifies homotopic 1-simplices, where two 1-simplices $f_1, f_2 \in Q(q_1, q_2)$ are said to be homotopic if f_1, f_2 are two faces of a 2-simplex in which one of the outer faces is degenerate.

In fact, we have a functor $\mathrm{Ho} : \mathbf{QCat} \to \mathbf{Cat}$ from quasicategories to categories, left adjoint to the nerve $N : \mathbf{Cat} \to \mathbf{QCat}$. Indeed, a morphism $f : Q \to R$ of quasicategories preserves the homotopy relation between 1-simplices, so that it descends to a well defined functor $\mathrm{Ho}(\varphi) : \mathrm{Ho}(Q) \to \mathrm{Ho}(R)$.

The fact that quasicategories are the fibrant objects for a Cartesian model structure on $\mathbf{SSet}$ (see [**?**, 2.2.8]) implies that Q^S is a quasicategory for every simplicial set S and quasicateogry Q. That is, quasicategories are enriched over themselves via the standard simplicial enrichment $(R^Q)_n = \mathbf{SSet}(Q \times \Delta^n, R)$. It is immediately checked that Ho preserves finite products, so that we get finally *the 2-category of quasicategories*, $\underline{\mathbf{QCat}}_{\mathrm{Ho}}$. Its objects are quasicategories, and for quasicategories Q, R, the hom-category $\underline{\mathbf{QCat}}_{\mathrm{Ho}}(Q, R)$ is simply the homotopy category $\mathrm{Ho}(R^Q)$ of the hom-quasicategory R^Q. This permits a simple definition of equivalence of quasicategories, as follows.

Definition 2.1. A map $F : Q \to R$ of quasicategories is said to be an equivalence if and only if it is an equivalence in the 2-category $\underline{\mathbf{QCat}}_{\mathrm{Ho}}$.

Remark 2.2. We recall that an equivalence in a 2-category consists of 1-morphisms $F : C \to D$ and $G : D \to C$ and 2-morphisms $a : \mathrm{id}_C \to GF$ and $b : FG \to \mathrm{id}_D$ such that a and b are isomorphisms. Thus an equivalence of quasicategories is a pair of maps $F : Q \leftrightarrows R : G$ together with two homotopy classes $a = [\alpha], b = [\beta]$ of morphisms $\alpha : Q \to Q^{[1]}, \beta : R \to R^{[1]}$ such that $[\alpha]$ is an isomorphism in $\mathrm{Ho}(Q^Q)$ and so is $[\beta]$, in $\mathrm{Ho}(R^R)$. We can clarify the definition yet further by noting that, for each $q \in Q_0$, the map α sends q to some $\alpha(q) \in Q_1$, and recalling that the invertibility of $[\alpha]$ is equivalent to that of each $[\alpha(q)]$, as recorded for instance by Riehl and Verity:

Lemma 2.3 ([**?**], 2.3.10)**.** *The equivalence class $[f]$ of a map $f : Q \to R^{[1]}$ is an isomorphism in the homotopy category $\mathrm{Ho}(R^Q)$ if and only if, for every vertex $q \in Q_0$ of Q, the equivalence class $[f(q)]$ is an isomorphism in $\mathrm{Ho}(R)$.*

For every quasicategory Q, we now construct a prederivator $\mathrm{HO}(Q) : \underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$. We first extend Ho to a 2-functor of the same name, $\mathrm{Ho} : \underline{\mathbf{QCAT}}_{\mathrm{Ho}} \to \underline{\mathbf{CAT}}$. This still sends a quasicategory to its homotopy category; we must define the action on morphism categories, which will be for each R and Q a functor

$$\mathrm{Ho} : \underline{\mathbf{QCAT}}_{\mathrm{Ho}}(Q, R) = \mathrm{Ho}(R^Q) \to \mathrm{Ho}(R)^{\mathrm{Ho}(Q)} = \underline{\mathbf{CAT}}(\mathrm{Ho}(Q), \mathrm{Ho}(R))$$

This functor is defined as the transpose of the following composition across the product-hom adjunction in the 1-category **Cat**.

$$\mathrm{Ho}(R^Q) \times \mathrm{Ho}(Q) \cong \mathrm{Ho}(R^Q \times Q) \xrightarrow{\mathrm{Ho(ev)}} \mathrm{Ho}(R)$$

For the isomorphism we have used again the preservation of finite products by Ho. The morphism $\mathrm{ev} : R^Q \times Q \to R$ is evaluation, the counit of the adjunction $(-) \times Q \dashv (-)^Q$ between endofunctors of **QCat**.

We also need a 2-functor $N : \underline{\mathbf{Cat}} \to \underline{\mathbf{QCat}}_{\mathrm{Ho}}$ sending a category $J \in \underline{\mathbf{Cat}}$ to $N(J)$. The map on hom-categories is the composition $J^K \cong \mathrm{Ho}(N(J^K)) \cong \mathrm{Ho}(N(J)^{N(K)})$. The first isomorphism is the inverse of the counit of the adjunction $\mathrm{Ho} \dashv N$, which is an isomorphism by full faithfulness of the nerve. The second uses the fact that N preserves exponentials, see [**?**, Proposition B.0.16]. Now we define the associated prederivator.

Definition 2.4. Let Q be a quasicategory. Then the prederivator $\mathrm{HO}(Q) : \underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$ is given as the composition

$$\underline{\mathbf{Cat}}^{\mathrm{op}} \xrightarrow{N^{\mathrm{op}}} \underline{\mathbf{QCat}}_{\mathrm{Ho}}^{\mathrm{op}} \xrightarrow{Q^{(-)}} \underline{\mathbf{QCAT}}_{\mathrm{Ho}} \xrightarrow{\mathrm{Ho}} \underline{\mathbf{CAT}}$$

In particular, $\mathrm{HO}(Q)$ maps a category J to the homotopy category of J-shaped diagrams in Q, that is, to $\mathrm{Ho}(Q^{N(J)})$.

Given a morphism of quasicategories $f : Q \to R$, we have a morphism of prederivators $\mathrm{HO}(f) : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ given as the analogous composition $\mathrm{HO}(f) = \mathrm{Ho} \circ f^{(-)} \circ N$, so that for each category J the functor $\mathrm{HO}(f)_J$ is given by post-composition with f, that is, by $\mathrm{Ho}(f^{NJ}) : \mathrm{Ho}(Q^{NJ}) \to \mathrm{Ho}(R^{NJ})$.

Remark 2.5. The implicit claims made above, for instance, that any quasicategory map $f : Q \to R$ induces a 2- natural transformation $f^{(-)} : Q^{(-)} \to R^{(-)} : \underline{\mathbf{QCat}}_{\mathrm{Ho}}^{\mathrm{op}} \to \underline{\mathbf{QCAT}}_{\mathrm{Ho}}$, follow from the following fact: a monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a 2-functor $(-)_F : \mathcal{V} - \underline{\mathbf{Cat}} \to \mathcal{W} - \underline{\mathbf{Cat}}$ between 2-categories of $\mathcal{V}$- and $\mathcal{W}$-enriched categories. While this claim is precisely what one would expect, this fully general version was not proven until recently; it comprises Chapter 4 of [**?**]. In our case, the functor Ho is monoidal insofar as it preserves products and thus it induces the 2-functor $(-)_{\mathrm{Ho}}$ sending simplicially enriched categories, simplicial functors, and simplicial natural transformations to 2-categories, 2-functors, and 2-natural transformations.

We record the correspondence between completeness properties of quasicategories and of their underlying derivators. It is difficult at this stage to give a completely satisfactory reference for the following result, in large part due to the intricacy of the passage from Kan extensions in the language of [**?**] to a 2-categorical definition as in [**?**]. Riehl and Verity sketch the result in the last pages of the latter paper, where they announce a full proof for future work.

Proposition 2.6. *The prederivator* $\mathrm{HO}(Q)$ *satisfies the axioms (Der1), (Der2), and (Der5'). It has whatever Kan extensions Q has, and these are pointwise. In particular, a complete and cocomplete quasicategory gives rise to a strong (even smothering) derivator.*

We note here that the validity of (Der1) depends on the preservation of arbitrary products by Ho. The nerve has a left adjoint τ_1, the fundamental category functor, defined on arbitrary simplicial sets as well as on quasicategories, but τ_1 fails to preserve infinite products, so general simplicial sets fail to give rise to semiderivators defined on all of $\underline{\mathbf{Cat}}$. To give an example, we recall the description of $\tau_1 S$ as the free category on the 1-skeleton of S modulo the equivalence relation generated by 2-simplices, as in [**?**, Proposition B.0.14].

Example 2.7. Let Sp_n be the *spine* of Δ^n, which is to say the simplicial subset generated by the 1-simplices with adjacent vertices. Then $\tau_1 \mathrm{Sp}_n$ is the free category on the 1-skeleton, that is, $\tau_1 \mathrm{Sp}_n = [n]$. The product category $\prod_{\mathbf{N}} [n]$ has the poset structure in which $(x_0, x_1, ...) \leq (y_0, y_1, ...)$ if and only if $x_i \leq y_i$ for each i. In contrast, the fundamental category of the simplicial set $H = \prod \mathrm{Sp}_n$ has a different order on the same set, in which the differences $y_i - x_i$ must be not only nonnegative but uniformly bounded.

3 The simplicial enrichment of prederivators

Recall that we take (strict) 2-natural transformations as morphisms of prederivators. The default 2-morphisms are the modifications, which give rise to the standard 2-category of prederivators, $\underline{\mathbf{PDer}}$.

Muro and Raptis showed, as we recall now, that prederivators can actually be collected into a simplicially enriched category. First, note that for any prederivator $\mathscr{D}$ and each category $J \in \underline{\mathbf{Dia}}$ we have a shifted prederivator $\mathscr{D}^J = \mathscr{D} \circ (J \times -)$. This shift is a special case of the cartesian closed structure on $\underline{\mathbf{PDer}}$ discussed in [**?**, Section 4]. Explicitly, given two prederivators $\mathscr{D}_1, \mathscr{D}_2$, and denoting by $\hat{J}$ the prederivator represented by a small category J, the exponential is defined by $\mathscr{D}_2^{\mathscr{D}_1}(J) = \mathbf{PDer}(\hat{J} \times \mathscr{D}_1, \mathscr{D}_2)$. Then the 2-categorical Yoneda Lemma implies that the shifted prederivator $\mathscr{D}^J$ is canonically isomorphic to the prederivator exponential $\mathscr{D}^{\hat{J}}$. This makes it easy to interpret expressions such as $\mathscr{D}^\alpha : \mathscr{D}^u \to \mathscr{D}^v : \mathscr{D}^K \to \mathscr{D}^J$, when $\alpha : u \to v : J \to K$ is a natural transformation, by using the internal hom 2-functor.

Remark 3.1. For a natural transformation $\alpha : u \to v : J \to K$ between functors in **Cat**, the preceding definition of $\mathscr{D}^\alpha$ gives only a shadow of the full action of α on $\mathscr{D}$. The natural transformation α itself may be seen as a functor $\bar{\alpha} : J \times [1] \to K$, associated to which we have a prederivator morphism $\mathscr{D}^{\bar{\alpha}} : \mathscr{D}^K \to \mathscr{D}^{J\times[1]}$, that is, a family of functors $\mathscr{D}(K \times I) \to \mathscr{D}(J \times I \times [1])$. This is strictly more information, as composing with the underlying diagram functor $\mathrm{dia}^{[1]}_{J\times I} : \mathscr{D}(J\times I\times[1]) \to \mathscr{D}(J\times I)^{[1]}$ recovers our original $\mathscr{D}^\alpha$. What is happening here is that the entity $\mathscr{D}^{(-)}$ is more than a 2-functor $\underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{PDer}}$: it is a simplical functor $(\mathbf{Cat}^{\mathrm{op}}_\bullet)_N \to \mathbf{PDer}_\bullet$ from the simplicial category of nerves of categories to the simplicial category of prederivators, which we must now define.

For each category J let $\mathrm{diag}_J : J \to J \times J$ be the diagonal functor.

Definition 3.2. We define $\mathbf{PDer}_\bullet$ as a simplicially enriched category whose objects are the prederivators. The mapping simplicial sets have $n-$simplices as follows: $\mathbf{PDer}_n(\mathscr{D}_1, \mathscr{D}_2) = \mathbf{PDer}(\mathscr{D}_1, \mathscr{D}_2^{[n]})$. For $(f, g) \in \mathbf{PDer}_n(\mathscr{D}_2, \mathscr{D}_3) \times \mathbf{PDer}_n(\mathscr{D}_1, \mathscr{D}_2)$, the composition $f * g : \mathscr{D}_1 \to \mathscr{D}_3^{[n]}$ is given by the formula below, in which we repeatedly apply the internal hom 2-functor discussed above Remark 3.1.

$$\mathscr{D}_1 \xrightarrow{g} \mathscr{D}_2^{[n]} \xrightarrow{f^{[n]}} \left(\mathscr{D}_3^{[n]}\right)^{[n]} \cong \mathscr{D}_3^{[n]\times[n]} \xrightarrow{\mathscr{D}_3^{\mathrm{diag}_{[n]}}} \mathscr{D}_3^{[n]}$$

In [?] a restriction of this enrichment, which we now recall, was of primary interest. Each prederivator $\mathscr{D}$ has an "essentially constant" shift by a small category J denoted $\mathscr{D}^J_{\mathrm{eq}}$. This is defined as follows: $\mathscr{D}^J_{\mathrm{eq}}(K) \subseteq \mathscr{D}(J \times K)$ is the full subcategory on those objects $X \in \mathscr{D}(J \times K)$ such that in the partial underlying diagram $\mathrm{dia}^J_K(X) \in \mathscr{D}(K)^J$, the image of every morphism of J is an isomorphism in $\mathscr{D}(K)$. We shall only need $J = [n]$, when an object of $\mathscr{D}^{[n]}_{\mathrm{eq}}(K)$ has as its partial underlying diagram a chain of n isomorphisms in $\mathscr{D}(K)$.

Then we get another simplicial enrichment:

Definition 3.3. The simplicial category $\mathbf{PDer}^{\mathrm{eq}}_\bullet$ is the sub-simplicial category of $\mathbf{PDer}_\bullet$ with $\mathbf{PDer}^{\mathrm{eq}}_n(\mathscr{D}_1, \mathscr{D}_2) = \mathbf{PDer}(\mathscr{D}_1, \mathscr{D}^{[n]}_{2,\mathrm{eq}})$.

As discussed in the introduction, it will be shown in Section 4 that $\mathbf{PDer}^{\mathrm{eq}}_\bullet$ contains the Kan-enriched category of quasicategories.

Above Example 2.7 we briefly described the extension $\tau_1 : \mathbf{SSet} \to \mathbf{Cat}$ of the homotopy category functor Ho to the entirety of **SSet**. By [?, Proposition B.0.15], τ_1 also preserves finite products. Thus the simplicial categories $\mathbf{PDer}_\bullet$ and $\mathbf{PDer}^{\mathrm{eq}}_\bullet$ give rise to 2-categories $\underline{\mathbf{PDer}}_{\tau_1}$ and $\underline{\mathbf{PDer}}^{\mathrm{eq}}_{\tau_1}$ by applying τ_1 to each hom-simplicial set. This leads to the notion of equivalence of prederivators under which Muro and Raptis showed Waldhausen K-theory is invariant.

Definition 3.4. A coherent equivalence of prederivators is a quadruple (F, G, α, β) of prederivator morphisms $F : \mathscr{D}_1 \to \mathscr{D}_2, G : \mathscr{D}_2 \to \mathscr{D}_1, \alpha_1 : \mathscr{D}_1 \to \mathscr{D}_{1,\mathrm{eq}}^{[1]}$, and $\alpha_2 : \mathscr{D}_2 \to \mathscr{D}_{2,\mathrm{eq}}^{[1]}$ such that the vertices of α_1 are GF and $\mathrm{id}_{\mathscr{D}_1}$, and similarly for α_2.

In general, the existence of a coherent equivalence fails to be a symmetric transitive relation on prederivators, because the hom-simplicial sets are not necessarily quasicategories. In particular, two maps $\alpha, \beta : \mathscr{D}_1 \to \mathscr{D}_2^{[1]}$ cannot necessarily be composed, even if their vertices correspond. Up to this caveat, we would expect 1-simplices of $\mathbf{PDer}_\bullet^{\mathrm{eq}}(\mathscr{D}_1, \mathscr{D}_2)$ and 2-morphisms in $\underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$ to correspond. Thus instead the 2-morphisms in $\underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$ arise from zigzags of such 1-simplices.

Every 2-morphism in $\underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$ is an isomorphism, but not every isomorphism in $\underline{\mathbf{PDer}}_{\tau_1}$ arises from $\underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$. A counterexample arises from any simplicial set S in which some 1-simplex f of S represents an isomorphism in $\tau_1 S$ whose inverse is not representable by any 1-simplex. For instance, the simplicial set S generated by three vertices x, y, z, edges $f : x \to y, g : y \to z, h : z \to x, g' : z \to y, h' : x \to z$, and 2-simplices a, b, c, d with ordered sets of edges (f, h', g), $(h, g', f), (g, sy, g')$, and (h', sx, h) respectively, where sy and sx are degenerate.

From the existence of more 2-isomorphisms in $\underline{\mathbf{PDer}}_{\tau_1}$ than in $\underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$, one may infer that more prederivators are equivalent in the former than in the latter 2-category, and indeed that each implication below is generally irreversible, for two prederivators $\mathscr{D}_1, \mathscr{D}_2$:

$$\text{(Existence of a coherent equivalence } \mathscr{D}_1 \to \mathscr{D}_2) \implies (\mathscr{D}_1 \cong \mathscr{D}_2 \in \underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}) \\ \implies (\mathscr{D}_1 \cong \mathscr{D}_2 \in \underline{\mathbf{PDer}}_{\tau_1}) \implies (\mathscr{D}_1 \cong \mathscr{D}_2 \in \underline{\mathbf{PDer}}) \tag{3.5}$$

However, coherent equivalence has much better properties on the prederivators of primary interest in this paper.

Proposition 3.6. *Let Q and R be quasicategories with associated prederivators $\mathrm{HO}(Q), \mathrm{HO}(R)$, as constructed in Definition 2.4. Then the first two implications in Equation 3.5 are reversible. That is, a map $F : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ is a coherent equivalence if and only if F is an equivalence in either of the 2-categories $\underline{\mathbf{PDer}}_{\tau_1}, \underline{\mathbf{PDer}}_{\tau_1}^{\mathrm{eq}}$.*

We will give the proof later, in Remark 4.8.

Remark 3.7. The last implication in Equation 3.5 is virtually never reversible, as the failure of Waldhausen K-theory to be invariant under equivalence in $\underline{\mathbf{PDer}}$ suggests, see [**?**]. To illustrate this, observe that a 2-morphism in $\underline{\mathbf{PDer}}$ is a modification $\Xi : \alpha \to \beta : \mathscr{D}_1 \to \mathscr{D}_2$, that is, a family of natural transformations $\Xi_J : \alpha_J \to \beta_J$ such that given $u : J \to K$, we have

$$\mathscr{D}_2(u)\Xi_K = \Xi_J \mathscr{D}_1(u) \tag{3.8}$$

The components of Ξ_J are morphisms in $\mathscr{D}_2(J)$, heuristically, homotopy classes of morphisms in some background model. Then Ξ_J may be thought of as a transformation between functors, only natural up to homotopy.

In contrast, a 1-simplex F in the mapping simplicial set from $\mathscr{D}_1$ to $\mathscr{D}_2$ is more rigid: F sends each object $X \in \mathscr{D}_1(J)$ to an object of $\mathscr{D}_2(J \times [1])$, so that, roughly, in passing from $\underline{\mathbf{PDer}}$ to $\underline{\mathbf{PDer}}_{\tau_1}$ we have refined a natural tranformation up-to-homotopy to a homotopy coherent natural transformation.

More light is shed on the incoherent versus coherent structures by introducing a second inclusion of **Cat** into **PDer**. Where the Yoneda embedding sends J to the represented presheaf $\hat{J} = J^{(-)}$ on **Cat**, the constant embedding Δ sends J to the constant 2-functor ΔJ.

Proposition 3.9. *We have the following isomorphism of categories, where ΔJ is the constant prederivator valued at J:*

$$\underline{\mathbf{PDer}}(\mathscr{D}_1, \mathscr{D}_2^{\Delta J}) \cong \underline{\mathbf{CAT}}(J, \underline{\mathbf{PDer}}(\mathscr{D}_1, \mathscr{D}_2))$$

Proof. The shifted prederivator $\mathscr{D}_2^{\Delta J}$ is simply the levelwise hom from J, the 2-functor $K \mapsto \mathscr{D}_2(K)^J$. Now for each K this category $\mathscr{D}_2(K)^J$ can be described as the weighted limit $\{J, \mathscr{D}_2(K)\}$. (See [**?**, Chapter 3] for the theory of weighted limits.) The claim we are to prove is that $\mathscr{D}_2^{\Delta J}$ is the same weighted limit $\{J, \mathscr{D}_2\}$ in $\underline{\mathbf{PDer}}$, which is exactly the **Cat**-enriched category (2-category) of **Cat**-functors $[\mathbf{Cat}^{\mathrm{op}}, \mathbf{CAT}]$. By [**?**, (3.16)], weighted limits in enriched functor categories can be constructed levelwise, just like ordinary limits in ordinary functor categories. Thus the weighted limit $\{J, \mathscr{D}_2\}$ exists and we have

$$\{J, \mathscr{D}_2\}(K) = \{J, \mathscr{D}_2(K)\} = \mathscr{D}_2(K)^J = \mathscr{D}_2^{\Delta J}(K)$$

as desired. ∎

Remark 3.10. This says that $\underline{\mathbf{PDer}}$ is cotensored over categories. In particular, when $J = [1]$, we see that modifications between prederivator morphisms $\mathscr{D}_1 \to \mathscr{D}_2$ are identified with single prederivator morphisms $\mathscr{D}_1 \to \mathscr{D}_2^{\Delta([1])}$. Thus the incoherence of the 2-morphisms in $\underline{\mathbf{PDer}}$ compared to those in $\underline{\mathbf{PDer}}_{\tau_1}$ is reflected in missing information in the constant prederivator $\Delta[1]$ relative to the represented prederivator $\widehat{[1]}$. Note that $\widehat{[1]}(J)$ has objects in natural bijection with the sieves in J.

It would be desirable to make a cotensoring claim for $\underline{\mathbf{PDer}}_{\tau_1}$ as well, for instance that the canonical map $\lambda : \underline{\mathbf{PDer}}_{\tau_1}(\mathscr{D}_1, \mathscr{D}_2^{yJ}) \to \underline{\mathbf{CAT}}(J, \underline{\mathbf{PDer}}_{\tau_1}(\mathscr{D}_1, \mathscr{D}_2))$ is an isomorphism. But this is simply false, even in case $J = [1]$; if $\mathscr{D}_1 = \Delta([0])$, the map we would make an isomorphism is the underlying diagram map $\mathscr{D}_2([1]) \to \mathscr{D}_2([0])^{[1]}$. Thus the best we can ask is that λ be full and essentially surjective, or perhaps smothering-and certainly not for an arbitrary prederivator. This case, with $J = [1]$, is one of the weak cotensors studied in [**?**], where it is proved that the analogue of λ is smothering in the 2-category $\underline{\mathbf{QCat}}_{\mathrm{Ho}}$, which we are to show is a full sub-2-category of $\underline{\mathbf{PDer}}_{\tau_1}$. We could also seek an isomorphism between $\mathbf{SSET}_\bullet(NJ, \mathbf{PDer}_\bullet(\mathscr{D}_1, \mathscr{D}_2))$ and $\mathbf{PDer}_\bullet(\mathscr{D}_1, \mathscr{D}_2^{yJ})$. This is more likely, and indeed is another implication of the main result below, when $\mathscr{D}_2$ is the prederivator underlying a quasicategory. In general, it requires that maps into $\mathscr{D}_2$ be determined by their restriction to the categories $[n]$, for instance in the manner of

Lemma 5.5, so that to show for which prederivators this coherent cotensoring exists may be tantamount to computing the image of quasicategories in prederivators.

4 The main result and its corollaries

We shall prove the following in Section 5:

Theorem 4.1. *There is a simplicially enriched extension* $\mathrm{HO} : \mathbf{QCAT}_\bullet \to \mathbf{PDer}_\bullet$ *of the ordinary functor* $\mathrm{HO} : \mathbf{QCAT} \to \mathbf{PDer}$ *which is simplicially fully faithful. That is, for every pair* Q, R *of quasicategories the map* $\mathrm{HO} : \mathbf{QCAT}_\bullet(Q, R) \to \mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))$ *is an isomorphism of simplicial sets.*

First, we give some corollaries. Define $\mathbf{QPDer}_\bullet \subseteq \mathbf{PDer}_\bullet$ to be the image of quasicategories in prederivators, so that the theorem gives an isomorphism of simplicial categories $\mathbf{QCAT}_\bullet \cong \mathbf{QPDer}_\bullet$. In particular, $\mathbf{QPDer}_\bullet$ is not only a simplicial category, but actually a category enriched in quasicategories.

Recall that the inclusion of Kan complexes into quasicategories has a right adjoint ι, which we will call the Kan core. For a quasicategory Q, the core ιQ is the sub-simplicial set such that an n-simplex $x \in Q_n$ is in $(\iota Q)_n$ if and only if every 1-simplex of x is an isomorphism in $\mathrm{Ho}Q$. See of [**?**, Section 1].

As a right adjoint, ι preserves products, so that for any quasicategory-enriched category $\mathcal{C}$ we have an associated Kan complex-enriched category $\mathcal{C}_\iota$ given by taking the core homwise. Recall that $\tau_1 : \mathbf{SSet} \to \mathbf{Cat}$ is the left adjoint of the nerve functor N, so that the restriction of τ_1 to quasicategories agrees up to canonical isomorphism with the homotopy category functor Ho.

Corollary 4.2. *The associated prederivator functor* $\mathrm{HO} : \mathbf{QCAT}_\bullet \to \mathbf{PDer}_\bullet$ *induces an isomorphism of Kan-enriched categories* $\mathrm{HO}_\iota : \mathbf{QCAT}_{\bullet,\iota} \to \mathbf{QPDer}_{\bullet,\iota}$ *and a fully faithful embedding of 2-categories* $\mathrm{HO}_{\tau_1} : \underline{\mathbf{QCAT}}_{\mathrm{Ho}} \to \underline{\mathbf{PDer}}_{\tau_1}$.

Proof. The given Kan-enriched functor and 2-functor exist via the base change construction of enriched category theory; see Remark 2.5. They are defined predictably, in the manner of Equation 4.3 below. For full faithfulness, we just have to show that HO_ι and HO_{τ_1} induce isomorphisms on hom-objects. Given the isomorphism $\mathrm{HO}_{Q,R} : \mathbf{QCAT}_\bullet(Q, R) \to \mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))$ claimed by the theorem, we get isomorphisms

$$(\mathrm{HO}_{Q,R})_\iota : \iota(\mathbf{QCAT}_\bullet(Q, R)) \cong \iota(\mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))) \tag{4.3}$$

This shows that HO_ι is fully faithful, and the argument for τ_1 is the same, up to the isomorphism $\mathrm{Ho} \cong \tau_1 : \mathbf{QCat} \to \mathbf{Cat}$ discussed above the statement. □

Remark 4.4. As discussed in the introduction, the Kan-enriched category $\mathbf{QCAT}_{\bullet,\iota}$ is a model of the homotopy theory of homotopy theories, which thus embeds into prederivators.

An embedding of Kan-enriched categories gives rise to an embedding of homotopy categories, which in a Kan-enriched category are computed by taking connected components of hom-complexes. The second claim of Corollary 4.2 shows that this embedding extends to *the homotopy 2-category* in the sense of [**?**]. The word *the* is partially justified by work of Low [**?**] indicating that the 2-category $\underline{\mathbf{QCAT}}_{\mathrm{Ho}}$ has a central role analogous to that of "*the* homotopy category" $\mathrm{Ho}(\mathbf{Top}) \cong \mathrm{Ho}(\mathbf{SSet})$.

We have not yet established that HO sends equivalent quasicategories to coherently equivalent prederivators. Rather, since equivalences of quasicategories are equivalences in $\underline{\mathbf{QCat}}_{\mathrm{Ho}}$, we have shown that equivalences of quasicategories are sent to equivalences in $\underline{\mathbf{PDer}}_{\tau_1}$, which we saw around Example 3.5 do not generally correspond to coherent equivalences. We will show the two notions of equivalence coincide in this case via a reformulation of equivalences of quasicategories in terms of $\mathbf{QCAT}_{\bullet,\iota}$.

Definition 4.5. Let $\mathcal{S}$ be any simplicial category. Two objects $x, y \in \mathcal{S}$ are said to be homotopy equivalent if there exist maps $f : x \leftrightarrows y : g$ with homotopies-that is, 1- simplices in mapping spaces $\mathcal{S}(x,x)$ and $\mathcal{S}(y,y)$, giving rise to isomorphisms after applying τ_1-between gf, fg, and the respective identities.

Proposition 4.6. *Let $\mathcal{C}$ be a quasicategory-enriched category. Two objects $x, y \in \mathcal{C}$ are equivalent in the 2-category $\mathcal{C}_{\mathrm{Ho}}$ if and only if they are homotopy equivalent in the Kan-enriched category $\mathcal{C}_\iota$.*

Proof. Let $\jmath : \mathbf{Cat} \to \mathbf{Gpd}$ be the right adjoint of the inclusion of groupoids into categories, which simply sends a category to its wide subcategory of isomorphisms. As a product-preserving functor, $\jmath$ gives rise to a groupoid-enriched category $\mathcal{C}_{\jmath \circ \mathrm{Ho}}$. We get another groupoid-enriched category $\mathcal{C}_{\mathrm{Ho} \circ \iota}$ using the homotopy category functor $\mathrm{Ho} : \mathbf{Kan} \to \mathbf{Cat}$, which factors through groupoids. In fact $\mathrm{Ho} \circ \iota \cong \jmath \circ \mathrm{Ho}$, as is shown in of [**?**, Corollary 1.5].

Now let $\mathcal{A}$ be any 2-category; we claim that equivalences in $\mathcal{A}$ correspond to equivalences in the groupoid-enriched category $\mathcal{A}_\jmath$. This is because $\mathcal{A}_\jmath$ has the same objects and arrows as $\mathcal{A}$, and exactly the 2-isomorphisms of $\mathcal{A}$ for its 2-morphisms. The data of an equivalence involves only 1-morphisms and 2-isomorphisms (see Remark 2.2), so it is the same to ask for an equivalence in $\mathcal{A}$ as in $\mathcal{A}_\jmath$.

On the other hand, if $\mathcal{K}$ is any Kan-enriched category, then the homotopy equivalences in $\mathcal{K}$ are exactly the equivalences in the groupoid-enriched category $\mathcal{K}_{\mathrm{Ho}}$. Indeed, an equivalence between x and y in $\mathcal{K}_{\mathrm{Ho}}$ consists of maps $f : x \leftrightarrows y : g$ and 2-morphisms $a : \mathrm{id}_x \to gf, b : fg \to \mathrm{id}_y$. The 2-morphisms a and b are then the equivalence classes of 1-simplices $\alpha \in \mathcal{K}(x,x)_1, \beta \in \mathcal{K}(y,y)_1$ (see the description of the homotopy category here,) so that (f, g, α, β) gives a homotopy equivalence between x, y in $\mathcal{K}$.

Now setting $\mathcal{K} = \mathcal{C}_\iota$ and $\mathcal{A} = \mathcal{C}_{\mathrm{Ho}}$, the first paragraph shows $\mathcal{K}_{\mathrm{Ho}} \cong \mathcal{A}_\jmath$. Thus homotopy equivalences in $\mathcal{K}$ and equivalences in $\mathcal{A}$ both correspond to equivalences in $\mathcal{K}_{\mathrm{Ho}} \cong \mathcal{A}_\jmath$, and the proof is complete. □

Thus, since Corollary 4.2 shows that homotopy equivalences in $\mathbf{QCAT}_{\bullet,\iota}$ are sent to homotopy equivalences in the image, $\mathbf{QPDer}_{\bullet,\iota}$, to show that equivalences of quasicategories are identified with coherent equivalences of their associated prederivators it suffices to show that $\mathbf{QPDer}_{\bullet,\iota}$ includes into $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}$ via an isomorphism onto its image. That is, we shall show that $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$ is a Kan complex, equal to the Kan complex $\iota(\mathbf{QCAT}_{\bullet}(Q, R))$ of maps from Q to R. Then the result will follow since coherent equivalences are just homotopy equivalences in $\mathbf{PDer}_{\bullet}$.

Recall that $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathscr{D}_1, \mathscr{D}_2)$ restricts $\mathbf{PDer}_{\bullet}(\mathscr{D}_1, \mathscr{D}_2)$ to those simplices in which every edge is an isomorphism.

Corollary 4.7. *For any quasicategories Q and R, the simplicial set $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$ is a Kan complex, isomorphic to the Kan complex $\iota(\mathbf{QCAT}_{\bullet}(Q, R))$ of maps from Q to R.*

Proof. As was briefly recalled above, the Kan core of a quasicategory T has the same 0-simplices as T and, as n-simplices, those $x \in T_n$ such that, for every $\ell : [1] \to [n]$, $\ell^* x \in T_1$ represents an isomorphism in the homotopy category $\mathrm{Ho}(T)$.

Thus, applying Theorem 4.1, we can see that $\iota(\mathbf{PDer}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R)) \cong \iota(R^Q)$ has as 0-simplices the maps $f : Q \to R$ and as n-simplices, all those maps $g : Q \to R^{\Delta^n}$ whose restrictions to R^{Δ^1} represent isomorphisms in $\mathrm{Ho}(R^Q)$. By [**?**, Lemma 2.3.10], this holds if and only if for every vertex $q \in Q_0$, each edge of $g(q) \in R_n$ represents an isomorphism in $\mathrm{Ho}(R)$.

Then consider a 1-simplex $x \in \iota(\mathbf{PDer}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))_1$. It has vertices $x_0, x_1 : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ and, for every $q \in \mathrm{HO}(Q)([0])$, the image $x(q) \in \mathrm{HO}(R)([1])$ represents an isomorphism in $\mathrm{Ho}(R) = \mathrm{HO}(R)([0])$. That is, $\mathrm{dia}^{[1]}(x(q)) : x_0(q) \to x_1(q)$ is an isomorphism. We claim that the satisfaction, for each q, of this condition is equivalent to the invertibility of the modification $\Xi : x_0 \to x_1 : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ underlying x. This modification is given, for any $Y \in \mathrm{HO}(Q)(J)$, by $\Xi_Y = \mathrm{dia}^{[1]}_J(x(Y))$. By Definition 3.3, this is the condition determining whether x lies in $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$.

The modification Ξ is an isomorphism if and only if each morphism Ξ_Y is. By (Der2), Ξ_Y is iso if and only if, for each $j \in J$, the pullback $j^*\Xi_Y$ is an isomorphism in $\mathrm{HO}(R)([0]) = \mathrm{Ho}(R)$.

Now, since Ξ is a modification, we have the equality

$$j^*\Xi_Y = \Xi_{j^*X} = \mathrm{dia}^{[1]}_J x(j^*Y)$$

The first equality is an instance of Equation 3.8, and the second of the definition of Ξ. So we see that $j^*\Xi_Y$ is the morphism in $\mathrm{Ho}(R)$ underlying the 1-simplex $x(j^*Y) \in R_1$. Thus $x \in \mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$ if and only if every 1-simplex in the image of x is an isomorphism in $\mathrm{Ho}(R)$, as claimed.

This has shown that the 1-simplices of $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$ coincide with those of $\iota(\mathbf{PDer}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$. Since both of these are generated as simplicial subsets of $\mathbf{PDer}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$ by their 1-simplices, we conclude that the desired isomorphism holds: $\mathbf{PDer}^{\mathrm{eq}}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R)) \cong \iota(\mathbf{PDer}_{\bullet}(\mathrm{HO}(Q), \mathrm{HO}(R))$. □

Remark 4.8. Thus the simplicial category $\mathbf{QPDer}^{\mathrm{eq}}_{\bullet}$ given by applying Muro-Raptis' construction to the image of quasicategories is simply the hom-wise Kan core $\mathbf{QPDer}_{\bullet,\iota}$, and the correspondence identifies coherent equivalences with homotopy equivalences. Thus

Proposition 4.6 implies that coherent equivalences of prederivators underlying quasicategories are the same as equivalences in the 2-category $\underline{\mathbf{QPDer}}_{\mathrm{Ho}}$. Thus the first two implications in Equation 3.1 are reversible in this case, as was promised in Proposition 3.6.

5 The proof

We will now prove Theorem 4.1. The main goal is to prove that the functor HO gives an isomorphism between the sets $\mathbf{QCat}(Q, R)$ and $\mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R))$. This is Proposition 5.6, whose proof has the following outline:

(1) Eliminate most of the data of a prederivator map by showing maps $\mathrm{HO}(Q) \to \mathrm{HO}(R)$ are determined by their restriction to natural transformations between ordinary functors $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Set}$. This is Lemma 5.2.

(2) Show that $\mathrm{HO}(Q)$ and $\mathrm{HO}(R)$ recover Q and R upon restricting the domain to Δ^{op} and the codomain to $\mathbf{Set}$, and that natural transformations as in the previous step are in bijection with maps $Q \to R$. This is Lemma 5.5.

(3) Show that $\mathrm{HO}(f)$ restricts back to f for a map $f : Q \to R$, which implies that HO is faithful, and that a map $F : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ is exactly HO applied to its restriction, which implies that HO is full. This constitutes the proof of Proposition 5.6.

Let us begin with step (1).

Definition 5.1. A **Cat**-set is a presheaf on the category of small categories, that is, an ordinary functor $\mathbf{Cat}^{\mathrm{op}} \to \mathbf{Set}$.

Given a prederivator $\mathscr{D}$, let $\mathscr{D}^{\mathrm{ob}} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{SET}$ be its underlying **Cat**-set, so that $\mathscr{D}^{\mathrm{ob}}$ sends a small category J to the set of objects $\mathrm{ob}(\mathscr{D}(J))$ and a functor $u : I \to J$ to the action of $\mathscr{D}(u)$ on objects.

Recall that where (Der5) requires that $\mathrm{dia} : \mathscr{D}(J \times [1]) \to \mathscr{D}(J)^{[1]}$ be (full and) essentially surjective, (Der5′) insists on actual surjectivity on objects. The following lemma shows that under this assumption most of the apparent structure of a prederivator map is redundant.

Lemma 5.2. *A strict morphism $F : \mathscr{D}_1 \to \mathscr{D}_2$ between prederivators satisfying (Der5′) is determined by its restriction to the underlying* **Cat***-sets $\mathscr{D}_1^{\mathrm{ob}}, \mathscr{D}_2^{\mathrm{ob}}$. That is, the restriction functor from prederivators satisfying (Der5′) to* **Cat***-sets is faithful.*

Proof. The data of a morphism $F : \mathscr{D}_1 \to \mathscr{D}_2$ is that of a functor $F_J : \mathscr{D}_1(J) \to \mathscr{D}_2(J)$ for every J. (Note the simplification here over pseudonatural transformations, which require also a natural transformation associated to every functor and do not induce maps of **Cat**-sets.) The induced map $F^{\mathrm{ob}} : \mathscr{D}_1^{\mathrm{ob}} \to \mathscr{D}_2^{\mathrm{ob}}$ is given by the action of F on objects. So to show faithfulness it is enough to show that, given a family of functions $r_J : \mathrm{ob}(\mathscr{D}_1(J)) \to \mathrm{ob}(\mathscr{D}_2(J))$, that is, the data of a natural transformation of **Cat**-sets, there is at most one 2-natural transformation with components $F_J : \mathscr{D}_1(J) \to \mathscr{D}_2(J)$ and object parts $\mathrm{ob}(F_J) = r_J$.

Indeed, suppose F is given with object parts $r_J = \mathrm{ob}(F_J)$ and let $f : X \to Y$ be a morphism in $\mathscr{D}_1(J)$. Then by Axiom (Der5'), f is the underlying diagram of some $\hat{f} \in \mathscr{D}_1(J \times [1])$. By 2-naturality, the following square must commute:

$$\begin{tikzcd} \mathscr{D}_1(J \times [1]) \arrow[r, "F_{J\times[1]}"] \arrow[d, "\mathrm{dia}_J^{[1]}"'] & \mathscr{D}_2(J \times [1]) \arrow[d, "\mathrm{dia}_J^{[1]}"'] \\ \mathscr{D}_1(J)^{[1]} \arrow[r, "F_J"] & \mathscr{D}_2(J)^{[1]} \end{tikzcd}$$

Indeed, $\mathrm{dia}_J^{[1]}$ is the action of a prederivator on the unique natural transformation between the two functors $0, 1 : [0] \to [1]$ from the terminal category to the arrow category, as is described in full detail below [**?**, Proposition 1.7]. Thus the square above is an instance of the axiom of 2-naturality. It follows that we must have $F_J(f) = F_J(\mathrm{dia}_J^{[1]}\hat{f}) = \mathrm{dia}_J^{[1]}(r_{J\times[1]}(\hat{f}))$.

Thus if F, G are two strict morphisms $\mathscr{D}_1 \to \mathscr{D}_2$ with the same restrictions to the underlying **Cat**-sets, they must coincide, as claimed. □

Remark 5.3. Note the above does not claim that the restriction functor is full: the structure of a prederivator map is determined by the action on objects of each $\mathscr{D}_1(J), \mathscr{D}_2(J)$, but it is not generally true that an arbitrary map of **Cat**−sets, under the unique admissible extension, will induce functors from $\mathscr{D}_1(J)$ to $\mathscr{D}_2(J)$. To conclude this, we would seem to need the simplicial set underlying $\mathscr{D}_1^J$ to satisfy at least the 2-dimensional inner horn filling condition, in order to get sufficiently functorial choices of lifts. There also appears to be no reason that the extension should be 2-natural, as opposed to just natural.

We proceed to step (2) of the proof.

Let us recall the theory of pointwise Kan extensions for ordinary categories, a special case of the theory axiomatized by (Der3) and (Der4). Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors. At least if $\mathcal{E}$ is complete, then we always have a right Kan extension $F_*G : \mathcal{D} \to \mathcal{E}$ characterized by the adjunction formula $\mathcal{E}^{\mathcal{D}}(H, F_*G) \cong \mathcal{E}^{\mathcal{C}}(H \circ F, G)$ and computed on objects by

$$F_*G(d) = \lim_{d/F} G \circ q \tag{5.4}$$

Here d/F is the comma category with objects $(c, f : d \to F(c))$ and morphisms the maps in $\mathcal{C}$ making the appropriate triangle commute, and $q : d/F \to \mathcal{C}$ is the projection.

Lemma 5.5. *Let $j : \Delta^{\mathrm{op}} \to \mathbf{Cat}^{\mathrm{op}}$ be the inclusion. Then for any quasicategory R, the* **Cat***-set* $\mathrm{HO}(R)^{\mathrm{ob}}$ *underlying* $\mathrm{HO}(R)$ *is the right Kan extension of R along j.*

Proof. For any small category J, the **Cat**-set $\mathrm{HO}(R)^{\mathrm{ob}}$ takes J to the set of simplicial set maps from J to R:

$$\mathrm{HO}(R)^{\mathrm{ob}}(J) = \mathrm{ob}(\mathrm{Ho}(R^{N(J)})) = \mathbf{SSET}(N(J), R)$$

We shall show that the latter is the value required of j_*R, which exists and is calculated via Equation 5.1 since **Cat** is complete.

First, one of the basic properties of presheaf categories, sometimes called the co-Yoneda lemma, implies that $N(J)$ is a colimit over its category of simplices. That is, $N(J) = \mathrm{colim}_{\Delta/N(J)} y \circ q$, where $q : \Delta/N(J) \to \Delta$ is the projection and $y : \Delta \to \mathbf{SSet}$ is the Yoneda embedding.

Then we can rewrite the values of $\mathrm{HO}(R)^{\mathrm{ob}}$ as follows:

$$\mathrm{HO}(R)^{\mathrm{ob}}(J) = \mathbf{SSET}(N(J), R) = \mathbf{SSET}(\mathrm{colim}_{\Delta/N(J)} y \circ q, R) \cong$$

$$\lim_{(\Delta/N(J))^{\mathrm{op}}} \mathbf{SSET}(y \circ q, R) \cong \lim_{(\Delta/N(J))^{\mathrm{op}}} R \circ q^{\mathrm{op}}$$

The last isomorphism follows from the Yoneda lemma.

The indexing category $(\Delta/N(J))^{\mathrm{op}}$ has as objects pairs $(n, f : \Delta^n \to N(J))$ and as morphisms $\bar{a} : (n, f) \to (m, g)$, the maps $a : \Delta^m \to \Delta^n$ such that $f \circ a = g$. That is, $(\Delta/N(J))^{\mathrm{op}} \cong N(J)/\Delta^{\mathrm{op}}$, where on the right-hand side $N(J)$ is viewed as an object of $\mathbf{SSET}^{\mathrm{op}}$. Using the full faithfulness of the nerve functor N, we see $(\Delta/N(J))^{\mathrm{op}} \cong J/\Delta^{\mathrm{op}}$, where again $J \in \mathbf{Cat}^{\mathrm{op}}$.

Thus, if q^{op} serves also to name the projection $J/\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, we may continue the computation above with

$$\mathrm{HO}(R)^{\mathrm{ob}}(J) \cong \lim_{J/\Delta^{\mathrm{op}}} R \circ q^{\mathrm{op}}$$

This is exactly the formula for $j_*R(J)$ recalled above. The isomorphism thus constructed is certainly natural with respect to the action on maps of the Kan extension, so the lemma is established. $\square$

We arrive at step (3).

Proposition 5.6. *The homotopy category functor* $\mathrm{HO} : \mathbf{QCAT} \to \mathbf{PDer}$ *is a fully faithful embedding of ordinary categories.*

Proof. Note that, by Lemma 5.5, the restriction of $\mathrm{HO}(Q)^{\mathrm{ob}}$ to a functor $\Delta^{\mathrm{op}} \to \mathbf{SET}$ is canonically isomorphic to Q, since Kan extensions along fully faithful functors are split by restriction. Thus a map $F : \mathrm{HO}(Q) \to \mathrm{HO}(R)$ restricts to a map $\rho(F) : Q \to R$. In fact, we have a natural isomorphism $\rho \circ \mathrm{HO} \cong \mathrm{id}_{\mathbf{QCAT}}$, so that $\rho \circ \mathrm{HO}(f)$ is again f, up to this isomorphism. Indeed, given $f : Q \to R$, we already know how to compute $\mathrm{HO}(f)$ as $\mathrm{Ho} \circ \left(f^{N(-)}\right)$. Then the restriction $\rho(\mathrm{HO}(f)) : Q \to R$, which we are to show coincides with f, is given by $\rho(\mathrm{HO}(f))_n = \mathrm{ob} \circ \mathrm{Ho} \circ f^{\Delta^n}$. That is, $\rho(\mathrm{HO}(f))$ acts by the action of f on the objects of the homotopy categories of Q^{Δ^n} and R^{Δ^n}. In other words, it acts by the action of f on the sets $\mathbf{SSet}(\Delta^n, Q)$ and $\mathbf{SSet}(\Delta^n, R)$; via Yoneda, $\rho(\mathrm{HO}(f))$ acts by f itself.

It remains to show that $\mathrm{HO}(\rho(F)) = F$ for any $F : \mathrm{HO}(Q) \to \mathrm{HO}(R)$. By Lemma 5.2 it suffices to show that the restrictions of $\mathrm{HO}(\rho(F))$ and F to the underlying $\mathbf{Cat}-$sets coincide. Using Lemma 5.5 and the adjunction characterizing the Kan extension, we have

$$\mathbf{Set}^{\mathbf{Cat}^{\mathrm{op}}}(Q^{\mathrm{ob}}, R^{\mathrm{ob}}) = \mathbf{Set}^{\mathbf{Cat}^{\mathrm{op}}}(j_*Q, j_*R) \cong \mathbf{SSet}(j^*j_*Q, R) \cong \mathbf{SSet}(Q, R)$$

In particular, maps between Q^{ob} and R^{ob} agree when their restrictions to Q and R do. Thus we are left to show that $\rho(\mathrm{HO}(\rho(F))) = \rho(F)$. But as we showed above, $\rho \circ \mathrm{HO}$ is the identity map on $\mathbf{SSET}(Q, R)$, so the proof is complete. $\square$

We have just one loose end to tie up to finish the theorem: we must extend HO to a simplicially enriched functor.

Proof of Theorem 4.1. We first set up simplicial set maps defined via Ho.

An n-simplex $f \in \mathbf{QCAT}_n(Q, R)$ is a morphism $f : Q \to R^{\Delta^n}$, which gives rise to a morphism $\mathrm{HO}(f) : \mathrm{HO}(Q) \to \mathrm{HO}(R^{\Delta^n})$. And $\mathrm{HO}(R^{\Delta^n})$ is naturally isomorphic to the shifted prederivator $\mathrm{HO}(R)^{[n]}$. Indeed, HO preserves cotensors by small categories insofar as, for any small categories K and J,

$$\mathrm{HO}(R^{N(K)})(J) = \mathrm{Ho}((R^{N(K)})^{N(J)}) \cong \mathrm{Ho}(R^{N(K\times J)}) = \mathrm{HO}(R)^K(J) \tag{5.7}$$

Recall that on the right-hand side the expression $\mathrm{HO}(R)^K$ indicates the shift of the prederivator $\mathrm{HO}(R)$ by K.

Thus for each n, the functor HO yields a map of sets $\mathbf{QCAT}_n(Q, R) \to \mathbf{PDer}_n(\mathrm{HO}(Q), \mathrm{HO}(R))$. We must assemble the functions just described to a simplicial set map. For any map $a : [n] \to [m]$ in Δ, the corresponding operation $\mathbf{QCAT}_m(Q, R) \to \mathbf{QCAT}_n(Q, R)$ is simply postcomposition with $R^a : R^{\Delta^m} \to R^{\Delta^n}$. Similarly, postcomposition with $\mathrm{HO}(R)^a$ gives the operation $\mathbf{PDer}_m(\mathrm{HO}(Q), \mathrm{HO}(R)) \to \mathbf{PDer}_n(\mathrm{HO}(Q), \mathrm{HO}(R))$ associated to a. We thus wish to show the outer rectangle of the following diagram commutes, where the horizontal arrows of the right-hand square are the canonical isomorphisms of Equation 5.7.

$$\begin{tikzcd}
\mathbf{QCAT}(Q, R^{\Delta^m}) \arrow[r, "\mathrm{HO}"] \arrow[d, "R^a\circ"'] & \mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R^{\Delta^m})) \arrow[r, "\cong"] \arrow[d, "\mathrm{HO}(R^a)\circ"'] & \mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R)^{[m]}) \arrow[d, "\mathrm{HO}(R)^a\circ"'] \\
\mathbf{QCAT}(Q, R^{\Delta^n}) \arrow[r, "\mathrm{HO}"] & \mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R^{\Delta^n})) \arrow[r, "\cong"] & \mathbf{PDer}(\mathrm{HO}(Q), \mathrm{HO}(R)^{[n]})
\end{tikzcd}$$

The left square commutes by functoriality: for every $f : Q \to R^{\Delta^m}$, $\mathrm{HO}(R^a \circ f) = \mathrm{HO}(R^a)\mathrm{HO}(f)$. The right-hand square commutes because $\mathrm{HO}(R^{\Delta^\bullet}) \cong \mathrm{HO}(R)^\bullet$ is a natural isomorphism of simplicial objects in derivators, as follows from the naturality in n of the isomorphism $\mathrm{Ho}((R^{\Delta^n})^{NJ}) \cong \mathrm{Ho}(R^{\Delta^n \times NJ})$.

Now we have simplicial set maps $\mathrm{HO} : \mathbf{QCAT}_\bullet(Q, R) \to \mathbf{PDer}_\bullet(\mathrm{HO}(Q), \mathrm{HO}(R))$, which we must show are compatible with the simplicially enriched compositions on either side. But the composition in $\mathbf{PDer}_\bullet$ is visibly defined to be consistent with that in $\mathbf{QCAT}_\bullet$. Indeed, for n-simplices $f \in \mathbf{QCAT}_n(R, S)$ and $g \in \mathbf{QCAT}_n(Q, R)$, the composition $f * g \in \mathbf{QCAT}_n(Q, S)$ is defined as

$$Q \xrightarrow{g} R^{\Delta^n} \xrightarrow{f^{\Delta^n}} S^{\Delta^n \times \Delta^n} \xrightarrow{S^{\mathrm{diag}_{\Delta^n}}} S^{\Delta^n}$$

Applying HO to the above line gives

$$\mathrm{HO}(Q) \xrightarrow{\mathrm{HO}(g)} \mathrm{HO}(R^{\Delta^n}) \xrightarrow{\mathrm{HO}(f^{\Delta^n})} \mathrm{HO}(S^{\Delta^n \times \Delta^n}) \xrightarrow{\mathrm{HO}(S^{\mathrm{diag}_{\Delta^n}})} \mathrm{HO}(S^{\Delta^n})$$

Now under the natural isomorphism of Equation (5.7), the above composition becomes the following:

$$\mathrm{HO}(Q) \stackrel{\mathrm{HO}(g)}{\longrightarrow} \mathrm{HO}(R^{[n]}) \stackrel{\mathrm{HO}(f)^{[n]}}{\longrightarrow} \mathrm{HO}(S)^{[n]\times[n]} \stackrel{\mathrm{HO}(S)^{\mathrm{diag}_{[n]}}}{\longrightarrow} \mathrm{HO}(S)^{[n]}$$

This is exactly the composition $f * g$ as defined in [**?**].

The identities in the mapping spaces $\mathbf{QCAT}_\bullet(Q,Q)$ and $\mathbf{PDer}_\bullet(\mathrm{HO}(Q),\mathrm{HO}(Q))$ are just the identity maps, so HO is indeed a simplicial functor.

Finally, HO is simplicially fully faithful: this follows immediately from Proposition 5.6, since an isomorphism of simplicial sets is constituted by an isomorphism between sets of n-simplices for every n. □

6 Concluding comments

The most important remaining question is to characterize the essential image of HO. Note that "essential" can be interpreted here in as many ways as there are notions of equivalence of prederivator: most notably, levelwise equivalence, coherent equivalence, and equivalence in $\underline{\mathbf{PDer}}_{\tau_1}$ These three arise before we even mention the possibility of relaxing our prederivator maps to be pseudo.

All that is clear so far is that only semiderivators, that is, prederivators satisfying the axioms (Der1),(Der2), (Der5), can even be levelwise equivalent to the prederivator underlying a quasicategory, since these three axioms are invariant under such equivalences.

It is unlikely that these three axioms constitute a characterization of the image: in particular, it seems implausible that semiderivators should be necessarily determined by simplices in the sense of Lemma 5.5. On the other hand, it appears we have to avoid making our assumptions too strict. Cordier and Porter's work in [**?**] implies that the derivator underlying a simplicial model category $\mathcal{M}$ is incoherently equivalent to the derivator underlying a quasicategory, even though $\mathrm{HO}(\mathcal{M})$ does not have (Der5'), cannot lift homotopy commutative squares to squares without changing an edge, and has an underlying simplicial set which forgets the homotopy theory of $\mathcal{M}$ completely.

Let us recall their main theorem:

Theorem 6.1 (Cordier-Porter)**.** *Let $\mathcal{M}'$ be a locally Kan simplicial category, such as the subcategory of bifibrant objects in a simplicial model category $\mathcal{M}$. Write $N^{\mathrm{hc}}(\mathcal{M}')$ for the quasicategory associated to $\mathcal{M}'$. Then for any small category J, the inclusion of commutative diagrams into homotopy coherent diagrams sends levelwise weak equivalences to homotopy equivalences and induces an equivalence of categories* $\mathrm{Ho}(\mathcal{M}'^J) \cong \mathrm{Ho}(\mathbf{SSet}(N(J), N^{\mathrm{hc}}(\mathcal{M}')))$.

The inclusion functor is certainly 2-natural in J, so we get a levelwise equivalence between the derivators $J \mapsto \mathrm{Ho}(\mathcal{M}^J) \cong \mathrm{Ho}(\mathcal{M}'^J)$ and $J \mapsto \mathrm{Ho}(N^{\mathrm{hc}}(\mathcal{M}')^J)$ associated respectively to a model category and to a quasicategory. Such a levelwise equivalence is guaranteed to have a pseudo-natural quasi-inverse, by taking the mate of each 2-naturality square.