MATH 31B: BONUS PROBLEMS

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1. Homework 1

§7.1: 28, 38, 45.

§7.2: 31, 33, 40.

§7.3: 44, 52, 61, 71. Also, compute the derivative of x^{x^x}.

2. Homework 2

First, let me say a word about what I call 'log-L'Hôpital' problems. I don't know if these are going to be on the midterm, but I'll give you a few bonus questions on the topic.

You already know that the indeterminate forms

$$\frac{0}{0} \quad \text{and} \quad \frac{\pm\infty}{\infty}$$

are the only ones where you actually apply L'Hôpital's rule. In discussion, I pointed out that if you have

$$\infty - \infty \quad \text{or} \quad 0 \cdot \infty$$

you can get to one of the original indeterminate forms pretty easily. But there's one other class of problems where it's not quite so easy.

Consider something like

$$\lim_{x\to\infty} x^{1/x}.$$

Plugging in gives us ∞^0. On the one hand, infinity to any power should be infinity. But raising anything to the zero should give us 1, so we are a little stuck. But we can get to something solvable in the following way:

$$\lim_{x\to\infty} x^{1/x} = L \implies \log\left(\lim_{x\to\infty} x^{1/x}\right) = \log L$$

Because log is a continuous function, it passes through the limit:

$$\lim_{x\to\infty} \log\left(x^{1/x}\right) = \log L.$$

Now using log rules, we can pull down the exponent.

$$\lim_{x\to\infty} \frac{1}{x} \cdot \log x = \log L$$

This gives us an $0 \cdot \infty$ type situation, which is easily rearranged into

$$\lim_{x\to\infty} \frac{\log x}{x} = \log L.$$

Now we have ∞/∞, so we can apply L'Hôpital's rule and obtain the answer $\log L = 0$. That lets us solve the original problem: $L = e^0 = 1$.

The problems below which rely on this trick will be marked with a $*$.

§7.7: 28, 35, 47*, 48*, 60, 65. Read 61 after doing 60, and try to do it if you're ambitious.

§7.8: 19, 42, 43, 61, 71. Anything in 73-110 is great general practice. In particular, try out 79, 83, 104, and 107. Let me also reiterate what I've said in class: if you only remember one of these inverse trig derivatives, remember $\frac{d}{dx}\tan^{-1} x$!

§7.9: In my experience, hyperbolic trig functions don't show up a lot later in the quarter. It also seems like you've been assigned a sufficient number of problems. If you are particularly interested, try out 33 and 34.

As an added bonus, you can look at the Chapter Review Exercises (p.386) and try any of the problems but 93-108. In particular, 121 and onward be quite the challenge. Based on what I've seen, those will probably be harder than the midterm.

3. Homework 3

§8.1: You've been assigned a large number of integration by parts problems, which is great. After doing 60, you can try 61, 62, and 86. 67-74 look good as well.

4. Homework 4

§8.5: There are a couple sub-concepts to master for the method of partial fractions. For a problem involving long division, try 7. I don't think you'll be doing much factoring in the denominator, but 37 and 38 incorporate this. For the easy sort of irreducible quadratic, look at 13, 40, 41. For the hard sort of irreducible quadratic, 43-46. Finally, if you want a further challenge, try 53 and 54 (using a strategy like you used for 52).

Also, in the realm of §8.5, it looks like Professor Ou has been using trigonometric substitution in class. We skipped the official section of the book. It might be worth reading through 8.3 (for trig sub) and 8.6 (for general knowledge) to get some more perspective. If I get a request for this in person, I'll post some bonus problems from those sections that might prove useful when approaching weird problems that you've not seen before.

§8.9: According to some of you, this is now an optional section. As such, let's ignore it!

5. Homework 5

§9.1: As long as you know the formulas, you'll be fine for arc length and surface area. The problem is with integration. You're going to need to know trigonometric substitution to do some of these integrals. For example, let's consider the following problem: compute the arc length of $f(x) = x^2/2$ on the interval $[0, \sqrt{3}]$. Step 1, set up the integral:

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_0^{\sqrt{3}} \sqrt{1 + x^2}\, dx$$

But you don't know how to do this integral. The idea is that you should do a substitution so that what ends up under the square root is a perfect square. For instance, if you had something like $\sqrt{1 - x^2}$, you could use $x = \cos\theta$ so that

$$1 - \cos^2\theta = \sin^2\theta$$

gives us a perfect square. But we can't do that – we have to use the identity

$$\sin^2\theta + \cos^2\theta = 1 \implies \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies \tan^2\theta + 1 = \sec^2\theta$$

Therefore let $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$. Therefore

$$\int_0^{\sqrt{3}} \sqrt{1 + x^2}\, dx = \int \sqrt{1 + \tan^2\theta} \cdot \sec^2\theta\, d\theta = \int \sec\theta \cdot \sec^2\theta\, d\theta$$

We leave the limits of the integral blank since it'd be a weird computation, and this is a weird enough computation already. But how do we do this integral? I left it to imply we're doing an integration by parts: let $u = \sec\theta$ and $dv = \sec^2\theta\, d\theta$, so that $du = \sec\theta\tan\theta\, d\theta$ and $v = \tan\theta$. Therefore

$$\int \sec\theta \cdot \sec^2\theta\, d\theta = \sec\theta \cdot \tan\theta - \int \tan\theta \cdot \sec\theta \tan\theta\, d\theta$$

To integrate $\sec\theta \cdot \tan^2\theta$, we need to use the identity above: $\tan^2\theta = \sec^2\theta - 1$. That gives us

$$\int \tan^2\theta \sec\theta\, d\theta = \int (\sec^2\theta - 1)\sec\theta\, d\theta = \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

Putting that all together,

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

which when rearranged gives us

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

To finish the problem, look up in the book that the integral of $\sec\theta$ is $\ln|\sec\theta + \tan\theta|$ or ask me how to do it. Hence:

$$\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)$$

Okay, that's all well and good, but we were supposed to be integrating with respect to x, and in particular on $[0, \sqrt{3}]$. Well we know that $\tan\theta = x$, so we can un-subsrtitute some of our answer, but we also need to compute $\sec\theta$ in terms of x. Well, this is where we have our old trick of drawing the triangle. Draw it out for yourself, but if $\tan\theta = x$, then we must have $\sec\theta = \sqrt{1+x^2}$. (We actually already know that from the beginning, since the substitution for $\sqrt{1+x^2}$ gave us $\sec\theta$.) That means we need to evaluate

$$\begin{aligned}\frac{1}{2}\left(\sqrt{1+x^2}\cdot x + \ln|\sqrt{1+x^2}+x|\right)\Big|_0^{\sqrt{3}} &= \frac{1}{2}\left(\sqrt{1+3}\cdot\sqrt{3} + \ln|\sqrt{1+3}+2|\right) - 0\\ &= \sqrt{3} + \ln 2.\end{aligned}$$

A long problem to be sure, but every problem in this sort is just like that. Memorise how it's solved and you'll do fine if it comes up again.

6. Homework 6

§8.7: 27, 33, 41, 48, 60, 76. Also, 84 is a really fun problem that requires some concepts from last quarter.

7. Homework 7

§11.1: You were asked a bunch of questions from this section, so I think the homework stands for itself.

8. Homework 8

§11.2: No problems on the divergence test have been assigned for some reason, nor does it seem we've covered telescoping series. For divergence test problems, do 17-22. 37-42 are a fun series of problems that will make you think a bit. Highly recommended is 47; be aware that these statements are FALSE.
§11.3: I will say this in class, and I will emphasise it again here: do not worry overmuch about the direct comparison test. It is crap and it will lead you to accidents, just like the analogous method for improper integration. The limit comparison test is a strictly better choice.

You should probably be able to do all of 49-78. It's great practice and a portent of problems to come. 79 and 80 are also great conceptual problems and very useful to memorise once you solve them.

9. Homework 9

§11.4: 34 and 35 are great problems that test the hypotheses of the alternating series test. I recommend them. Beyond this, 17-32 is another set of 'see if they converge any way you can'. You could also finish 3-10 to figure out the whole 'absolutely, conditionally, or diverges' issue. If you're feeling bold, do 37.

§11.5: Use the root test whenever you see everything happens to be to the nth power, e.g. 11, 23, 30, and 36-42. Use the ratio test otherwise. 21-27 are good questions that are a bit theoretical. Again I am forced to recommend 43-62 to test your 'any method covered so far' skills.

10. Power Series

§11.6: Without knowing more about what happened during class, I'll do my best to give a brief explanation.

Suppose we take a general power series centered at c:

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Then we can ask whether $F(x)$ exists for various values of x. We know that it works at $x = c$ because $F(c) = a_0$, but we don't know much else. So we should use the ratio (or root) test to see:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n} \right| = \lim_{n\to\infty} |x| \cdot \left| \frac{a_{n+1}}{a_n} \right| = \rho(x).$$

We get some number that depends on x. For whichever values of x gives us $\rho(x) < 1$, we know that the series converges absolutely – that's the ratio test. If $\rho(x) > 1$, the series diverges. If $\rho(x) = 1$, the test is inconclusive.
Okay, let's see this in an example.

$$F(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{9^n}$$

We can actually use the root test for this one, since everything is to the nth power. Do that in general.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-5)^n}{9^n}\right|} = \lim_{n\to\infty} \frac{|x-5|}{9} = \frac{|x-5|}{9}$$

So we want to solve $\frac{|x-5|}{9} < 1$, which means $|x-5| < 9$. That means that 9 is our *radius of convergence*, and as long as $|x-5| < 9$ the series converges absolutely. There is a separate process for checking the case $|x-5| = 9$ (the case where $\rho = 1$) that I don't need to get into if you don't need to know it.

Look through 9-34 to do some radius of convergence problems. The general rules I advise are the following: whenever everything is to the nth power, use the root test. Otherwise use the ratio test. I can't recommend any of the other problems because I think they'll involve material you haven't covered yet.

§11.7: You've been told to memorise Table 2 on page 587, so do so. After you've done that, 3-18 are good problems for practicing turning Taylor series into different Taylor series. 40,-43, and part of 46 and 47 are good to practice the interaction of derivatives/integrals with Taylor series. You can add on 66 and 68 if you'd like more practice with that sort of thing, and 69-70 would be a good challenge.

Review: The chapter review exercises are a great source of problems. You should *not* do: 8, 9, 11, 17-20, 26, 30-31, 36-37. Avoid any of the error bound/CAS questions. Finally, skip 109-112. There might be others that look fishy, but that's what I can see at first glance.