SOLUTION GUIDE TO MATH GRE FORM GR9367

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The questions for this solution guide can be found here.

Solution 1. (D) We have $f(g(x)) = g(x) + 3 = 5$ for all x, so $g(x) = 2$ for all $x \in \mathbb{R}$.

Solution 2. (C) We have that

$$\frac{\tan x}{\cos x} = \frac{\frac{\sin x}{\cos x}}{\cos x} = \frac{\sin x}{\cos^2 x}.$$

Put this way, we know $\lim_{x\to 0} \cos^2 x = 1$ and $\lim_{x\to 0} \sin x = 0$. Then the limit is $0/1 = 0$.

Solution 3. (A) A straightforward integral.

$$\int_0^{\log 4} e^{2x}\,dx = \frac{e^{2x}}{2}\Big|_0^{\log 4} = \frac{e^{2\log 4} - e^0}{2} = \frac{16-1}{2} = \frac{15}{2}.$$

Solution 4. (C) If A consists precisely of those elements both in B and in A but not in B, then B must lie completely within A. Some of the other things may be true, but are not necessarily true.

Solution 5. (A) We know this function is not differentiable at $x = 0$, but no matter. At $x = -1$, the function $f(x)$ is given by $-x + 3x^2$. As such, $f'(x) = -1 + 6x$, so $f'(-1) = -7$.

Solution 6. (B) We want to minimise the function

$$\begin{aligned} g(b) = \int_b^{b+1} x^2 + x\,dx &= (b+1)^3/3 + (b+1)^2/2 - (b^3/3 + b^2/2) \\ &= \frac{2b^3 + 6b^2 + 6b + 2 + 3b^2 + 6b + 3 - 2b^3 - 3b^2}{6} \\ &= \frac{6b^2 + 12b + 5}{6} = b^2 + 2b + \frac{5}{6}. \end{aligned}$$

Taking the derivative, we get $g'(b) = 2b + 2$, so we choose $b = -1$ to minimise it.

We can also use the fundamental theorem of calculus. Deriving with respect to b, we don't introduce any 'derivative of the limits of integration' terms because everyting is pretty simple:

$$g'(b) = (b+1)^2 + (b+1) - (b^2 + b) = b^2 + 3b + 2 - (b^2 + b) = 2b + 2$$

and obtain the same conclusion.

Solution 7. (D) There is no limit to what x and y can be, but e^α is positive for any $\alpha \in \mathbb{R}$. That means we can appear in any of the octants so long as $z > 0$, which makes 4 of them.

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Solution 8. (B) The main issue is that we cannot take the square root of a negative number. Hence we must have $\tan^2 x - 1 \geq 0$. That means that $1 \leq \tan x$ or $\tan x \leq -1$. Put otherwise, we have $3\pi/4 \geq x \geq \pi/4$ or $-3\pi/4 \leq x \leq -\pi/4$ (except for $\pi/2$ and $-\pi/2$). So while (C) and (E) are tempting, they don't work. That makes (B) our only option.

Another way to think about it:

$$\tan^2(x) - 1 = \frac{\sin^2(x)}{\cos^2(x)} - 1 = \frac{\sin^2(x) - \cos^2(x)}{\cos^2(x)} = \frac{-\cos(2x)}{\cos^2(x)}$$

which you can conclude depending on your recollection of trigonometric identities. The denominator $\cos^2(x)$ is always positive, so we just have to ensure $\cos(x) \neq 0$ and $-\cos(2x) \geq 0$, i.e., $\cos(2x) \leq 0$. This means that $2x$ has to be in the range where cosine is negative, which is $[\pi/2, 3\pi/2]$, so we again require $x \in [\pi/4, 3\pi/4]$. But at $x = \pi/2$ we hit $\cos(x) = 0$ so we once again need to split the range, as above, giving us (B).

Solution 9. (E) This is an easy u-substitution. Let $u = 2 - x^2$, so that $du = -2x\,dx$. Then

$$\int_0^1 \frac{x}{2 - x^2}\,dx = \int_2^1 \frac{du}{-2u} = -\frac{1}{2}\log u\Big|_2^1 = -\frac{1}{2}(0 - \log 2) = \frac{\log 2}{2}.$$

Solution 10. (A) Rather than do some integrals, we can see which of the solutions fits the bill. The solution for $g(x) = g'(x)$ is $g(x) = Ce^x$, so setting $g(x) = f'(x)$ gives us $f(x) = Ce^x + D$. The first initial condition gives us $C = -1$ and the second gives us $D = 1$, so $f(x) = 1 - e^x$.

Solution 11. (C) We might as well just try all these derivatives. We need all three partial derivatives of φ_y, so first,

$$\varphi_y = 2x.$$

We can therefore see that the only nonzero derivative of φ_y is φ_{yx}.

Solution 12. (E) Plugging in 0 certainly doesn't work, giving us $0/0$. Therefore we can use L'Hopital's rule:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 2x}{(1+x)\log(1+x)} &= \lim_{x\to 0} \frac{2\cos 2x}{(1+x)\cdot\frac{1}{1+x} + \log(1+x)} \\ &= \lim_{x\to 0} \frac{2\cos 2x}{1 + \log(1+x)} = \frac{2}{1} = 2.\end{aligned}$$

As we've seen in other problems, taking the first term of the respective Taylor series at $x = 0$ for these functions can also get us to the answer: $\sin(2x) \approx 2x$, $\log(1+x) \approx x$, and $1 + x \approx 1$ (when x is small). That gives us $2x/(1 \cdot x) = 2$. This method never pops into my head first but it's a pretty good one, and quick too.

Solution 13. (A) We can work this out symbolically. Ignoring the fringe case where $n = 0$,

$$\int_1^n x^{-n}\,dx = \frac{x^{-n+1}}{-n+1}\Big|_1^n = \frac{n^{-n+1} - 1}{-n+1}.$$

As n gets very large, $n^{-n+1} \to 0$, and so the whole limit is 0 as well.

You can also prove this using the dominated convergence theorem, where $f_n(x) = \chi_{[1,n]}/x^n$, the characteristic function of the interval $[1, n]$ over x^n. Then $f_n \to 0$ pointwise, and $|f_n(x)| < 1/x^2$ (for $n > 2$), where we can integrate $1/x^2$ on $[1, \infty)$.

As another dominance approach, as x^{-n} is always positive on our domain, we can compute

$$0 \leq \int_1^n x^{-n}\,dx \leq \int_1^\infty x^{-n}\,dx = \frac{x^{-n+1}}{-n+1}\bigg|_1^\infty = \frac{1}{n+1}$$

This upper bound clearly goes to 0 as $n \to \infty$, so we can conclude that our original limit is also 0.

Solution 14. (D) This is a confusingly worded question, but it boils down to this: every 5 years, the value of the dollar is 1/2 what it was before. Therefore the value of the dollar is $(1/2)^{n/5}$ after n years. So what number m gives $2^m \approx 1000000$? We know that $2^{10} = 1024$, so $2^{20} \approx 1000000$. That means we need to wait $5 \cdot 20 = 100$ years.

Solution 15. (D) We need both $f(2)$ and $f'(2)$ to solve this. First,

$$f(2) = \int_1^2 \frac{dt}{1+t^2} = \arctan t\bigg|_1^2 = \arctan 2 - \frac{\pi}{4}.$$

Second,

$$f'(x) = \frac{1}{1+x^2} \implies f'(2) = \frac{1}{5}.$$

Our final answer therefore is

$$y - \left(\arctan 2 - \frac{\pi}{4}\right) = \frac{1}{5}(x-2).$$

Distributing the negative gives the answer.

Solution 16. (A) Let us take the derivative of $f(x)$ to see what we're dealing with.

$$f'(x) = e^{g(x)} \cdot h'(x) + h(x) \cdot g'(x)e^{g(x)} = e^{g(x)}(-g'(x)h(x)) + h(x) \cdot g'(x)e^{g(x)} = 0.$$

This makes f a constant function, so we're done. For the record, there's no reason that g needs to have any special features.

Solution 17. (E) This is a classic type of problem. Suppose that $\arccos \pi/12 = \theta$. Then $\cos\theta = \pi/12$, so that makes $\sin\theta = \sqrt{144-\pi^2}/12$. So:

$$1 - \sin^2(\arccos(\pi/12)) = 1 - \frac{144-\pi^2}{144} = 1 - \left(1 - \frac{\pi^2}{144}\right) = \frac{\pi^2}{144}.$$

But is actually easier than that:

$$1 - \sin^2(\arccos(\pi/12)) = \cos^2(\arccos(\pi/12)) = (\pi/12)^2.$$

Solution 18. (D) Hopefully we can recall this power series. The one that everyone should remember (besides $\sin x$ and $\cos x$) is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We need to modify this by replacing x by $-x^2$, so the given power series corresponds to $\frac{1}{1+x^2}$. Taking the derivative of this gives the answer.

Solution 19. (A) To solve an ODE of this type, we use its characteristic equation. We replace $y^{(n)}$ with r^n and hence get

$$r^3 - 3r^2 + 3r - 1 = 0 \implies (r-1)^3 = 0.$$

The only root of this is $r = 1$ and it appears three times. That gives us three solutions: e^t, te^t, and t^2e^t. There are the particular solutions, so linear combinations of them will be the general solution.

Solution 20. (B) We can see that $z = 6 - x^2 - 2y^2$ is on top and $z = -2 + x^2 + 2y^2$ is on bottom. The height of the solid at (x, y) is $8 - 2x^2 - 4y^2$. That throws (C) out. Looking at our options for the limits of integration, we know that y should depend on x or vice-versa, we can also get rid of (A) and (D). To do the final check, the boundary of our solid is

$$6 - x^2 - 2y^2 = -2 + x^2 + 2y^2 \implies x^2 + 2y^2 = 4.$$

We can see that (E) only goes over a quarter of our domain (which we then multiply by 2), but (B) does the whole thing.

Solution 21. (D) Let us just integrate it and see what happens.

$$\int_0^1 f(x)\,dx = \int_0^a a^2\,dx + \int_a^1 ax\,dx = a^2 \cdot a + \frac{ax^2}{2}\Big|_a^1 = \frac{a + a^3}{2}.$$

That makes $a = 1$ the solution.

Solution 22. (E) We just need to line up the powers of b and c in reverse order, using that $c^{-k} = c^{3-k}$ and $b^{-\ell} = b^{5-\ell}$.

$$b^2cb^4c^2 \text{ gives us } cbc^2b^3.$$

Solution 23. (C) We know that $f(x)$ satisfies the Mean Value Theorem and the Intermediate Value Theorem. II is the MVT. For I, we consider the function $g(x) = f(x) - x$. Then $g(0) = 1$ and $g(1) = -1$, so there exists $x \in (0, 1)$ such that $g(x) = 0$ by the IVT. But $g(x) = f(x) - x = 0$ means $f(x) = x$. However, there's no reason $f(x) > 0$ the whole time.

Solution 24. (A) If A and B are independent events, then $P(A \cap B) = P(A) \cdot P(B)$. But if $P(A \cap B) = P(A) = P(B)$, that would mean that $P(A) = P(A)^2$, which is impossible for values in $(0, 1)$.

As for the other options, it could be the case that A and B differ by a set of measure zero. That could mean that $A = B$ or that one is a proper subset of the other. (E) seems to be true no matter what.

Solution 25. (D) The statement says that $f(x)$ is Lipschitz continuous. It is a useful statement that absolutely continuous functions (such as Lipschitz continuous functions) are differentiable almost everywhere. The proof is somewhat annoying and you don't need to know it particularly.

Solution 26. (C) Looking at the options, there's only one choice. $(1, 1, -1) \cdot (1, 0, 1) = 0$.

Solution 27. (A) We know that $z = f(y)$ is on the surface of revolution, and this occurs when $x = 0$. There's only one option that includes this – which is (A). More specifically, we are drawing circles in the xz-plane with radius $f(y)$.

Solution 28. (B) A subspace has to contain 0 and be closed under addition and scalar multiplication. I is the proper way to take the union of two subspaces, and is indeed a subspace. II is not a subspace, because it needn't be closed under addition. III is the proper way to take the intersection of subspaces, and is a subspace. IV is nothing – it doesn't contain 0!

Solution 29. (E) We should use the limit comparison test with $\sum 1/k$, which we know diverges.

$$\lim_{k\to\infty} \frac{1/k - 1/2^k}{1/k} = \lim_{k\to\infty} 1 - \frac{k}{2^k} = 1.$$

Therefore both series diverge.

Solution 30. (E) The terms in which x_n appears are of the form $x_i x_n$ for $i \in \{1, \ldots, n-1\}$. That means the derivative is just $x_1 + \cdots + x_{n-1}$.

Solution 31. (C) We have

$$\int_0^2 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 x - 1\,dx.$$

The righthand integral is easy enough, just $(x^2/2 - x)\Big|_1^2 = (2-2) - (1/2-1) = 1/2$. For the lefthand integral, we can either be clever or not. If we are not clever, we go through a big trigonometric integration. If we are clever, we remember that $\sqrt{1-x^2}$ is the top half of the unit circle. Therefore the segment in $[0,1]$ is just a quarter-circle with area $\pi/4$. That gives us our answer.

Solution 32. (B) Looking at our answers, it would be good to first verify F_3 is a field. This is the classic example of a quadratic field extension: $\mathbb{Q}[\sqrt{2}]$. F_2 is not a field: we do not have multiplicative inverses on the whole. Suppose, for example, that 3 had an inverse. Then we would have $3a + 3b\sqrt{2} = 1$ for $a, b \in \mathbb{Z}$. Then we must have $b = 0$ and there's no solution to $3a = 1$ in $\mathbb{Z}$. That means that (B) must be our answer.

To be thorough, F_4 is not a field because $\sqrt[4]{2}$ has no inverse, and F_1 is not a field because it too lacks multiplicative inverses. For example, the inverse of $2/3$ should be $3/2$ (seen in $\mathbb{R}$), but there's no way to express $3/2$ with an odd denominator.

Solution 33. (C) There are $n!$ ways to line up the apples from left to right. There is only one correct order, so the probability is $1/n!$.

Solution 34. (E) This is a usual fundamental theorem of calculus problem:

$$\frac{d}{dx}\int_0^{x^2} e^{-t^2}\,dt = 2xe^{-x^4}.$$

Solution 35. (B) Inspired by I, suppose that $f(x) = mx + b$ for $m, b \in \mathbb{Z}$ constants. Then clearly $f(x) = (f(x-1) + f(x+1))/2$, but $f(x)$ need not be strictly increasing or constant. We just need to see if such an $f(x)$ is the only option. Let's define $g(x) = f(x) - f(x-1)$.

Then by rearranging the definition, we have

$$\begin{aligned} g(x) = f(x) - f(x-1) &= \frac{f(x+1) - f(x-1)}{2} \\ &= f(x+1) - \frac{f(x+1) + f(x-1)}{2} \\ &= f(x+1) - f(x) = g(x+1) \end{aligned}$$

so that $g(x)$ is constant, say $g(x) = m$. Hence $f(x) = mx + f(0)$ must be a line, completing the problem.

Solution 36. (D) Working out this complicated description,

$$F(2,2) = F(F(1,2),1) = F(4,1) = 5.$$

Solution 37. (B) We need to remember what elementary row operations do to a matrix. Multiplying the first row of the matrix by 3 multiplies its determinant by 3. Swapping the second and third rows negates the matrix. Finally, subtracting a multiple of one of the rows from another does nothing. That gives us $-1 \cdot 3 \cdot 9 = -27$.

Solution 38. (D) This is the expression of a Riemann sum. Specifically, it is the right Riemann sum representing the integral

$$\int_0^3 x^2 - x\,dx.$$

That means the integral is equal to $x^3/3 - x^2/2\Big|_0^3 = 9 - 9/2 = 9/2$.

Solution 39. (D) $\log(1+y)$ is only defined if $y > -1$. Therefore we only need to solve $\sin(2\pi x) = -1$. This occurs when $2\pi x$ is of the form $-\pi/2 + 2\pi n = \frac{4\pi n - \pi}{2}$. Dividing this by 2π gives $x = \frac{4n-1}{4}$.

Solution 40. (A) I found it easiest to consider this as a volume in $\mathbb{R}^3$. We consider the cube $[0,1]^3$, and we consider the set of points (x, y, z) which satisfy $x \geq yz$. This is given by the integral

$$\begin{aligned} \int_0^1 \int_0^1 \int_{yz}^1 1\,dx\,dy\,dz &= \int_0^1 \int_0^1 1 - yz\,dy\,dz \\ &= \int_0^1 y - \frac{y^2 z}{2}\Big|_0^1 dz = \int_0^1 1 - \frac{z}{2}\,dz \\ &= z - \frac{z^2}{4}\Big|_0^1 = \frac{3}{4}. \end{aligned}$$

Another approach: we know the probability $P(x \geq t) = 1 - t$ for any $t \in [0,1]$. Therefore $P(x \geq yz)$ is just $1 - E(yz)$, the expected value of yz, which in turn is $E(y) \cdot E(z)$ since the variables are independent. Since the expected values are each $1/2$, the probability becomes $1 - (1/2)^2 = 3/4$. This can be formalized by the 'Law of total expectation', which says that $P(x \geq yz)$ is equal to the expected value $E(P(x \geq yz | y, z))$ over all possible values of y, z. That quantity is equal to $E(1 - yz)$ which we distribute through as above.

Solution 41. (B) We are looking for the eigenspace corresponding to $\lambda = 1$. We can compute this:

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+2b \\ -b \end{pmatrix}.$$

This means that $b = -b$ and $a + 2b = a$. We need $b = 0$ and hence a is arbitrary.

Solution 42. (D) If f is infinitely differentiable, then we can use its Taylor series at $x = 0$ to approximate it. We can take the second-order approximation:

$$T_2(x) = f(0) + f'(0) \cdot x + \frac{f''(0) \cdot x^2}{2}.$$

Then we know that the error is given by

$$|f(1) - T_2(1)| < \frac{1^3 \cdot b}{3!}$$

where that b is the same as in the problem. Since $T_2(1) = 1 + 1 + 1 = 3$, we have

$$|f(1) - 3| < \frac{b}{6} \implies \frac{b}{6} - 3 < f(1) < \frac{b}{6} + 3$$

In order to make sure $f(1) < 5$, we see that $b = 12$ is the maximum possible.

Solution 43. (E) We can use the intermediate value theorem to check for this. Let

$$f(x) = x^n - \sum_{i=0}^{n-1} a_i x^i.$$

Then $f(0) = -a_0$ and $f(1) = 1 - \sum_{i=0}^{n-1} a_i$. For I, this implies that $f(0) < 0$ and $f(1) > 0$, so the IVT gives us a root. For III, $f(0) > 0$ and $f(1) < 0$, so the same thing holds. We do not even need to consider II now – it's not even an option.

Solution 44. (D) Let us dissect this a bit. This looks like the limit definition of the double derivative:

$$P''(x) = \lim_{h\to 0} \frac{P(x+2h) + P(x) - 2P(x+h)}{h^2}.$$

The difference for us seems to be some mystery in the h. First, the above is equivalent to

$$P''(x) = \lim_{h\to 0} \frac{P(x+h) + P(x-h) - 2P(x)}{h^2}.$$

under $x \mapsto x - h$. We then need to replace h with $3h$:

$$P''(x) = \lim_{h\to 0} \frac{P(x+3h) + P(x-3h) - 2P(x)}{(3h)^2} = \lim_{h\to 0} \frac{P(x+3h) + P(x-3h) - 2P(x)}{9h^2}.$$

This implies our initial expression is equal to $9P''(x)$.

Alternatively, we can apply L'Hôpital's rule and take derivatives with respect to h:

$$\begin{aligned} \lim_{h\to 0} \frac{P(x+3h) + P(x-3h) - 2P(x)}{h^2} &= \lim_{h\to 0} \frac{3P'(x+3h) - 3P'(x-3h)}{2h} \\ &= \lim_{h\to 0} \frac{9P''(x+3h) + 9P''(x-3h)}{2} \\ &= \frac{18P''(x)}{2} = 9P''(x) \end{aligned}$$

This is a bit faster. There is also a general formula to interpret expressions of this form send to me by a reader, but I'd never heard of it and I don't think it's worth memorizing. Seek it out if you care to add it to your arsenal.

Solution 45. (C) If we go about trying to solve this, assuming that our constants are nonzero,

$$x^2 = \frac{c - by^3}{a} = \frac{f - ey^3}{d} \implies cd - bdy^3 = af - aey^3.$$

We know that $bd \neq ae$, and so

$$cd - af = (bd - ae)y^3$$

gives us a solution for y. Along with a solution for y, we get two solutions for x. That should give us two total solutions.

However, there's a cleaner solution. We can rewrite the problem as a system of of equations, albeit not linear, and obtain

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

The condition $ae \neq bd$ means that the square matrix is invertible, so there is a unique solution

$$\begin{bmatrix} x^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

There is only one option for y, but there are two options for x, leading to two total solutions.

Solution 46. (E) We can work this all out, but we know the following: the coefficient on the righthand side of the x^3 term is exactly a_3, so $a_3 = 1$. Then the coefficient of x^2 is exactly $a_2 - 6a_3$, and must be 0. Therefore $a_2 = 6$. The partial tuple $(-, -, 6, 1)$ uniquely determines an answer.

Solution 47. (C) It is easiest to use Green's theorem for this. We have

$$\oint_{\partial D} L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

For us

$$\oint_C -y\,dx + x\,dy = \iint_E 2\,dx\,dy = 2 \cdot (\pi \cdot a \cdot b).$$

We do have to remember that the area of the ellipse with axes $2a$ and $2b$ is $\pi \cdot a \cdot b$.

Solution 48. (D) The product of cyclic groups is cyclic if and only if all the orders involved are coprime. This doesn't work for (D).

Solution 49. (D) The standard deviation is the square root of the variance, and the variance is defined by

$$\int_{-\infty}^{\infty} (x - E(X))^2 f(x)\,dx,$$

where $E(X)$ is the expected value:

$$\int_{-\infty}^{\infty} x \cdot f(x)\,dx.$$

It might also be convenient to recall that the variance is also calculable as $E(X^2) - E(X)^2$, where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

though in this case the computation is easy in either way.

First,

$$E(X) = \frac{3}{4}\int_{-1}^{1} x - x^3\, dx = 0,$$

because we are integrating an odd function over a symmetric domain. That makes the second computation pretty easy as well.

$$\mathrm{Var}(X) = \frac{3}{4}\int_{-1}^{1} x^2 - x^4\, dx = \frac{3}{4}\left(x^3/3 - x^5/5\right)\Big|_{-1}^{1} = \frac{3}{4}\cdot\frac{4}{15} = \frac{1}{5}.$$

That makes the standard deviation $1/\sqrt{5}$. From a test-taking perspective, one would guess (D) to begin with, as it is the only answer that is the square root of another answer.

Solution 50. (A) The determinant of that matrix is zero if and only if the column space is less than four dimensional. Since it is clear that the last three columns are linearly independent, that would mean that there is a solution to

$$a(x, 1, 0, 0) + b(y, 0, 1, 0) + c(z, 0, 0, 1) = (1, 1, 1, 1).$$

But we can see that $a = b = c = 1$ necessarily, so it means that $x + y + z = 1$. That is the equation to an affine plane, and we can see that $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are all on that plane.

Solution 51. (B) A field automorphism must have $\varphi(0) = 0$ and $\varphi(1) = 1$. By linearity, this means that $\varphi(n) = n$ for all $n \in \mathbb{Z}$. This also means that $\varphi(1/n) = 1/n$ for all $n \in \mathbb{Z} \setminus \{0\}$. By multiplicativity, this means that $\varphi(a/b) = \varphi(a) \cdot \varphi(1/b) = a/b$ for any $a/b \in \mathbb{Q}$. This gives us exactly one field automorphism – the identity.

Solution 52. (D) Reflecting a vector across the x-axis means that the x value stays the same but the y value is reversed. This corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Doubling the vector's length means multiplying this matrix by 2.

Solution 53. (A) The general formula for a complex line integral is, for a parametrisation $Z(t)$, $t \in [a, b]$, of the curve C,

$$\int_C f(z)\, dz = \int_a^b f(Z(t))Z'(t)\, dt.$$

For us, we let $Z(t) = re^{it}$ and $Z'(t) = i \cdot re^{it}$. Let $f(z) = \sum_{k=0}^{n} a_k z^k$. Then

$$\begin{aligned} \int_C P(z)\, dz &= \int_0^{2\pi} i \cdot re^{it} \sum_{k=0}^{n} a_k r^k e^{ikt}\, dt = \int_0^{2\pi} i \sum_{k=0}^{n} a_k r^{k+1} e^{(k+1)it}\, dt \\ &= \sum_{k=0}^{n} a_k r^{k+1} e^{2\pi i(k+1)} - \sum_{k=0}^{n} a_k r^{k+1} e^{0} \end{aligned}$$

$e^{2\pi i(k+1)} = 1$ for any of the values of k we have. Therefore the lefthand and righthand expressions are equal, giving the value zero for the integral.

You can go through this, or you can remember the Cauchy integral formula. The integral of a holomorphic function along a closed curve is zero.

Solution 54. (C) Something relating to the derivative should be our answer, so we will look at (C) or (E). For (E),

$$(f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Depending on how $f(x)$ and $g(x)$ vary, there is no reason that this function should be nondecreasing. However, (C) holds if we take the inequality $f'(x) \geq g'(x)$ and integrating it over the interval $[0,1]$.

Solution 55. (E) The only subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ for some number n. Therefore our three elements should be multiples of the same number. p, pq, and p^q are all multiples of p.

Solution 56. (B) The derived set is also described as the set of limit points of S. For I, the limit points of $A \cup B$ are exactly the limit points of A and the limit points of B. For II, consider two open sets $A = (a,b)$ and $B = (b,c)$. Then $A \cap B = \varnothing$ so it has no limit points. but $A' \cap B' = \{b\}$. For III, if A has no limit points, then trivially $A' \subset A$, which means that A is closed (by definition). For IV, the empty set is open, and it has no limit points.

Solution 57. (C) Certainly this algorithm is better than $O(n)$. We aren't just skimming through the entire list. Therefore the best option is $\log_2 n$.

Solution 58. (C) No problem, we can just take the determinant of $A - \lambda \cdot I$.

$$\det \begin{bmatrix} 2-\lambda & 1-i \\ 1+i & -2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda) - (1-i)(1+i) = \lambda^2 - 4 - 2 = \lambda^2 - 6.$$

That would make $\pm\sqrt{6}$ the eigenvalues.

Solution 59. (A) We know that H is a subgroup of HK. If HK is a finite group, then $|H|$ divides $|HK|$. Therefore $|HK|$ cannot be 30, as $12 \nmid 30$.

Solution 60. (D) We can construct the 'atoms'[1] of this set by the set K:

$$K = \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}.$$

Then the size of the σ-algebra generated by $\{A, B\}$ is equal to $2^{|K|}$. It may be the case that $|K| < 4$, but in the maximal case $|K| = 4$ so our answer is $2^4 = 16$.

Solution 61. (C) These problems can be difficult if not rephrased appropriately. We have to go five blocks east and seven north. This means we need to arrange five Es and seven Ns in a row. This is done in 12! ways, dividing by 7! and 5! to avoid repeating answers.

Solution 62. (B) Recall that open subsets of $[0,1]$ are (unions of) intersections of $[a,b) \cap [0,1]$. This makes a basis of open sets for $[0,1)$ all the half-open sets along with the singleton $\{1\}$. This shows that $[0,1]$ is disconnected by $[0,1)$ and $\{1\}$. For compactness, suppose we take the open cover given by $\{1\}$ and $[0, 1-2^{-n})$ for $n \in \mathbb{N}$. Any finite subcover would miss out on $(1-2^{-N}, 1)$ for some $N \in \mathbb{N}$. Since the lower limit topology is finer than the standard topology on $[0,1]$, it is still Hausdorff.

[1] Terminology and solution from here

Solution 63. (D) This would be a good time for polar coordinates. Let $dx\,dy = r\,dr\,d\theta$. Then our integral becomes

$$\int_0^{2\pi}\int_0^2 e^{-r^2} r\,dr\,d\theta.$$

This has a nice antiderivative now.

$$\int_0^{2\pi}\left(-\frac{e^{-r^2}}{2}\right)\Big|_0^2 d\theta = \int_0^{2\pi}\frac{-e^{-4}+1}{2}\,d\theta = \pi(1-e^{-4}).$$

Solution 64. (C) We can do this using calculus, actually. Suppose that $f'(x) = \lambda \cdot f(x)$. This is a first-order differential equation that is solved by $f(x) = Ce^{\lambda x}$.

Solution 65. (A) It seems like there is only one sensible option. Having a complex power series means that it is analytic.

Solution 66. (B) This looks like a pigeonhole principle problem. We are choosing $n+1$ numbers out of $2n$ numbers. That means that we must choose two numbers that are one apart. However, for n sufficiently large, we needn't pick any primes or squares.