# SOLUTION GUIDE TO MATH GRE FORM 3768

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The questions for this solution guide can be found here.

**Solution 1.** (A) Don't get thrown by the $e$ in the exponent; it's just a number:

$$\int e^{ex}\,dx = \frac{1}{e}e^{ex} + C$$

and since $1/e = e^{-1}$, combining terms gives $e^{ex-1} + C$.

**Solution 2.** (E) Remembering one of the most useful Taylor series: for all $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Recalling also that $3\log 2 = \log(2^3) = \log 8$, we get

$$\sum_{n=0}^{\infty} \frac{(3\log 2)^n}{n!} = e^{3\log 2} = e^{\log 8} = 8$$

**Solution 3.** (D) Since $A = \pi \cdot r^2$,

$$\pi \cdot (r \cdot 1.4)^2 = \pi \cdot 1.96 \cdot r^2 = 1.96A$$

So it's a 96% increase. Nothing complicated here.

**Solution 4.** (A) Using implicit differentiation,

$$\frac{d}{dx}(x + y^4 = 10) \implies 1 + 4y^3 \cdot \frac{dy}{dx} = 0$$

Rearranging the equation, we obtain

$$\frac{dy}{dx} = \frac{-1}{4y^3}$$

**Solution 5.** (C) Recall the most general format of the fundamental theorem of calculus:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = b'(x) \cdot f(b(x)) - a'(x) \cdot f(a(x))$$

So in this case, with a constant as the first limit of integration, we just need to substitute in the top term and multiply by its derivative.

$$h'(x) = \frac{d}{dx}\int_0^{x^2} g(t)\,dt = \frac{d}{dx}(x^2) \cdot g(x^2) = 2xg(x^2)$$

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**Solution 6.** (D) You can brute for this by picking certain points, e.g., does $f(f(0)) = 0$ by visual inspection? But taking a step back, $f(x) = 1/x)$ is a perfectly good example of $f(f(x)) = x$ and we can see that function sitting in answer (D).

Another observation: since $f(x) = f^{-1}(x)$, from a graphical perspective $y = f(x)$ must be symmetric over the line $y = x$, since this is the usual trick to graph the inverse function. That also leaves only (D).

**Solution 7.** (B) We see a lot of $e$ in the answer, so that might remind us that

$$\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t$$

Well that isn't quite what we have here, but we can use the substitution $x = 1/n$ to obtain

$$\lim_{x\to 0}\left(1+\frac{x}{a}\right)^{b/x} = \lim_{n\to\infty}\left(1+\frac{1}{a\cdot n}\right)^{b\cdot n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{a\cdot n}\right)^{a\cdot n}\right)^{b/a} = \left(\lim_{n\to\infty}\left(1+\frac{1}{a\cdot n}\right)^{a\cdot n}\right)^{b/a}$$

where in the last equation we use the fact that $z^{b/a}$ is a continuous function to pull it outside of the limit. The limit evaluates to $e$ and thus we obtain $e^{b/a}$.

Of course we can go directly if we remember that $\lim_{x\to 0}(1+tx)^{1/x} = e^t$ too, and all we need to do is manipulate $x \mapsto x/a$ and move the $(\cdots)^b$ outside the limit.

**Solution 8.** (B) This should be a familiar problem type by now; we need to compute the limit if it exists. "Plugging in" $x = 1$ to $f(x)$ gives $0/0$, so L'Hôpital's rule applies:

$$\lim_{x\to 1}\frac{\log x}{x^2-1} = \lim_{x\to 1}\frac{1/x}{2x} = \lim_{x\to 1}\frac{1}{2x^2} = \frac{1}{2}$$

which completes the problem.

**Solution 9.** (C) Let $A$ denote the event "at least one lightbulb is defective" and let $D$ be the number of defective bulbs. Then $P(A) = P(D = 1) + P(D = 2)$, the sum of two things, when we could compute $P(\neg A) = P(D = 0)$ that zero lightbulbs are defective. All this is to say that solving for "not $A$" is slightly faster.

What is the probability that neither bulb is defective? The odds for the first bulb is $7/10$ and the second is $6/9$. Multiplication gives us $7/15$, but remember that was $P(\neg A)$. That implies $P(A) = 1 - P(\neg A) = 8/15$.

**Solution 10.** (B) For this and similar questions, we want to relate the power series we see to one we've memorized. The one that comes to mind for me is the geometric series power series, which has the same radius of convergence that we're addressing in the problem, namely

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Recall that we can take the derivative of convergent power series without changing the radius, which gives us

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\cdot x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

Since $a_n = n + 1$, the first three terms are $1, 2, 3$.

Instead of deriving, we could also just manipulate the power series directly:

$$\frac{1}{(1-x)^2} = \frac{1}{1-x}\frac{1}{1-x} = (1+x+x^2+\cdots)(1+x+x^2+\cdots)$$

which is enough to determine the first three coefficients; easy enough to find that they are 1, 2, 3 again.

**Solution 11.** (E) The function $f(x)$ is a parabola, so it would be useful to know where its vertex is for this question. Thinking about a general $g(x) = (x-h)^2 + k$ with vertex $(h, k)$, its range is given by $y \geq k$ (since $(x-h)^2$ is nonnegative) and it admits an inverse so long as we restrict our domain to $x \leq h$ or $x \geq h$, i.e., it cannot transgress the $x$ value of the vertex.

So where's our vertex? The implication from the problem is that $k = -8$, but let's recall how to figure this out. We need to complete the square, which goes by the following general method:

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$$

The first equality just adds and subtracts the same quantity so that when we FOIL out $(x - b/2)^2$ we see all the necessary terms. In our case,

$$x^2 + 6x + 1 = x^2 + 6x + 9 - 9 + 1 = (x+3)^2 - 8$$

For the current problem, we indeed have $k = -8$ and $h = -3$ as the problem statement suggested, but it was a good check. This makes $f(x)$ both injective and surjective, so it admits an inverse given by (C), (D), or (E). The easiest way to check this would be by plugging in the vertex value we worked so hard to obtain. We expect $f^{-1}(-8) = -3$.

For (C), we get $6 - \sqrt{36-8} = 6 - \sqrt{24}$ is not going to be an integer. The same problem will apply to (D). This makes (E) the only remaining answer, and checking we do get the expected result.

You could get to (E) more quickly just by plugging in some values and seeing that it works, but without the due diligence that (A) and (B) are false it would be risky.

**Solution 12.** (C) Let's think about the prime factorization of $n$. Recall from either your abstract algebra class or number theory that if $p$ is a prime number such that $p \mid n^5$ (or any power of $n$), then $p \mid n$ as well; this is an if and only if statement. Since $n^5$ only has three prime factors, namely $2, p, q$, we can write

$$n = 2^a \cdot p^b \cdot q^c, \quad a, b, c > 0$$

This relies on remembering that $64 = 2^6$, which will be relevant again shortly.

Now, the statement of the problem is that $p^3$ and $64q^{11}$ both divide $n^5$. These numbers are coprime (since $2 < p < q$) so their least common multiple is just their product. Thinking about this statement in terms of prime factorizations,

$$64p^3q^{11} = 2^6 \cdot p^3 \cdot q^{11} \mid n^5 = 2^{5a} \cdot p^{5b} \cdot q^{5c} \implies 6 \leq 5a \text{ and } 3 \leq 5b \text{ and } 11 \leq 5c$$

as "divisible" is just a fancy way of saying "has more prime factors than". Because $a, b, c$ are positive integers, this implies that $a = 2$, $b = 1$, and $c = 3$ is the minimal choice, which gives (C).

**Solution 13.** (C) Thinking backwards, suppose that $G$ is a complete graph (i.e., every pair of vertices has exactly one edge between them) on $N$ vertices. We know that every vertex is connected to every other, so each vertex has $N-1$ edges giving $N \cdot (N-1)$ edges in all.

But this double-counts each edge, so we have to divide by 2. Put another way, $G$ has $\binom{N}{2}$ ("$N$ choose 2") edges. Hence

$$\frac{N \cdot (N-1)}{2} = 190 \implies N \cdot (N-1) = 380$$

and rather than solve this quadratic polynomial, we can look at the options and notice $20 \cdot 19 = 380$. That makes $N = 20$.

**Solution 14.** (C) We can address (D) and (E) pretty quickly. Concavity is determined by the second derivative, so computing that we have $f''(x) = 2x - 1$. On the interval $(0, 2)$, this function changes signs so are neither concave up nor concave down the whole time.

(C) would be true if $f'(x) > 0$ on $(0, 2)$, which we can sort out quickly as well. We can tell that $f'(1) = 1 - 1 + 1 = 1 > 0$, so we just need to check for roots of this polynomial in $(0, 2)$. Using the quadratic formula, we compute the roots to be

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

which are imaginary, so $f'(x)$ cannot change signs. Thus $f'(x) > 0$ and (C) is confirmed true.

To address (A) and (B), there doesn't seem to be any reason these should be true. If we take an antiderivative of $f'(x)$, we obtain $f(x) = x^3/3 - x^2/2 + x + C$. Note that $f(x)$ is continuous on $[0, 2]$ the closed interval, so we can test the functional equation $f(x) = f(2-x)$ at the endpoints. For us, this means that $f(0) = C$ must equal $f(2) = 8/3 - 4/2 + 2 + C \neq C$ which is impossible for every $C \in \mathbb{R}$. The same argument shows that (B) cannot be true for all $C \in \mathbb{R}$ either.

**Solution 15.** (A) To take a direct approach, set $y = b/x$ for the second equation and substitute into the first:

$$x^2 + (b/x)^2 = a \implies x^4 + b^2 = ax^2 \implies x^4 - ax^2 + b^2 = 0$$

Using the quadratic equation for $t = x^2$, we can solve

$$t = \frac{a \pm \sqrt{a^2 - 4b^2}}{2}$$

For $t$ to have a solution, we need the discriminant $a^2 - 4b^2 \geq 0$. Rearranging and taking square roots, we have $a \geq 2b$. We know that there's no $\pm$ introduced since $a, b > 0$.

But is this sufficient? We know that $x = \pm\sqrt{t}$ are the possible solutions for the original problem we're trying to solve, so we need to make sure one of these options for $t$ is guaranteed to be positive. Luckily this is so: since $a > 0$, $a + \sqrt{a^2 - 4b^2} > 0$ as long as we don't get an imaginary term, which we have ruled out. This makes the answer (A).

As a faster alternative, we can consider the graphical interpretation. We're looking for conditions such that a circle of radius $\sqrt{a}$ (the first equation) intersects with a rectangular hyperbola (the second). The nearest points of the hyperbola to $(0, 0)$ are $(\sqrt{b}, \sqrt{b})$ and $(-\sqrt{b}, -\sqrt{b})$, and these points are at a distance of

$$\sqrt{(\sqrt{b})^2 + (\sqrt{b})^2} = \sqrt{2} \cdot \sqrt{b}$$

from the origin. Since a circle is all points equidistant from the origin, we need the radius of the circle to be this distance or greater for there to be any points of intersection. Thus $\sqrt{a} \geq \sqrt{2} \cdot \sqrt{b}$, giving us the same answer when squaring both sides.

The slickest answer might be the following: consider the inequality $(x-y)^2 \geq 0$. Expanding the quadratic, we have that $x^2 + y^2 \geq 2xy$, a version of the arithmetic-geometric inequality. If we have $xy = b$, then we know that $x^2 + y^2 = a \geq 2b$, which gives us condition (A). That certainly shows that the condition is necessary, and we can also prove it's sufficient by setting $x = y = \sqrt{b}$ for the arithmetic-geometric inequality to actually hold.

**Solution 16.** (D) You might just have the answer to this question in your back pocket, but let's go through the steps to check every statement. By definition, for a subset $D \subset X$, we know that

$$f(D) := \{y \in Y : \exists d \in D \text{ such that } f(d) = y\}$$

Starting with Statement III, it's true essentially by definition. Since every $c \in C$ also satisfies $c \in B$, $f(c) \in f(B)$ too. This reduces the answer choices to (D) or (E), and the correct answer hinges on testing Statement II only.

Using III, as $A \cap B \subset A$, we conclude $f(A \cap B) \subset f(A)$ and similarly $f(A \cap B) \subset f(B)$. We immediately conclude $f(A \cap B) \subset f(A) \cap f(B)$. But the converse has no reason to be true; consider $f(x) = x^2$ defined on $\mathbb{R} \to \mathbb{R}$. Let $A = [0, 1]$ and $B = [-1, 0]$. Then $f(A \cap B) = f(\{0\}) = \{0\}$. But $f(A) = f(B) = [0, 1]$, so the inclusion we proved above is strict. The general phenomenon we are seeing here is that when $f$ is not injective, it can be the case that $f(a) = f(b) = y$ for $a \in A \setminus B$ and $b \in B \setminus A$. This gives a candidate $y \in f(A) \cap f(B)$ that does not come from an $x \in A \cap B$. This lets us conclude the answer is (D).

To double-check Statement I: we know that $A \subset A \cup B$ and $B \subset A \cup B$, so that by III, $f(A) \subset f(A \cup B)$ and $f(B) \subset f(A \cup B)$ and thus $f(A) \cup f(B) \subset f(A \cup B)$. But the reverse inclusion is also true; suppose that $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$, by definition that means $x \in A$ or $x \in B$ so $f(x)$ is in $f(A)$ or $f(B)$.

**Solution 17.** (B) These facts should be in your pocket from your real analysis class, or from a more advanced calculus class. We can use this opportunity to review sequences but this should be a 5-second problem.

By definition, $\lim\limits_{n\to\infty} a_n = a$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n > N$. In particular, this means that the sequence $\{a_n\}$ has to be bounded, as the choice $\varepsilon = 1$ (for example) shows that infinitely many elements of $\{a_n\}$ must be within $(L - 1, L + 1)$. By converse, an unbounded sequence can't possibly converse, which is (A).

We can get (E) pretty quickly as well; rewriting the righthand side of this equivalence tells us that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$||a_n - a| - 0| < \varepsilon \text{ whenever } n > N$$

But that's just the same thing as $|a_n - a| < \varepsilon$, so (E) must be true.

(C) is the definition of a continuous function, at least from the perspective of real analysis as opposed to Calc I, and we won't reiterate the proof here. (D) is the definition of a sequence being Cauchy, and we know in $\mathbb{R}$ a sequence is convergent if and only if it's Cauchy. This does require $\{a_n\}$ to be a sequence of real numbers though, so it's important to read the problem statement carefully.

Thus we conclude the answer is (B), although we should've known this all along. There's no reason a bounded sequence needs to converge, and we can use the counterexample $a_n = (-1)^n$ which bounces between $-1$ and 1 indefinitely. These numbers are not getting any closer together, so the sequence is not Cauchy and therefore not convergent (which is true regardless of metric space).

**Solution 18.** (B) Since Statement II appears in most answers, let's tackle that one first. As we're looking at the graph of $g'(x)$, we know that $g(x)$ has extrema at $g'(x) = 0$, which occur at $x = 2$ and $x = 5$, so that's a good start. The First Derivative Test tells us that if $g'(x)$ goes from positive to negative, we have a local maximum and if $g'(x)$ goes from negative to positive, we have a local minimum. That's exactly what's occurring on the graph and what Statement II says, so that one is true.

We can now look at Statement I, which seems to be making the same conclusions about a different set of points. This is dead wrong; these statements are true about $g'(x)$ but not $g(x)$.

Finally, with Statement III, we have to do some integration. The way we can compare $g(2)$ with $g(5)$ via $g'(x)$ comes from the following:

$$\int_2^5 g'(x)\,dx = g(5) - g(2)$$

So how do we figure out the lefthand side? Since we have the graph, we know this integral is the area under the curve on $[2,5]$. Looking at that region, we see a triangle that's entirely contained below the $x$-axis, so it'll have negative area. This means that $g(5) - g(2) < 0$ (it looks like it's about $-3$ but the specifics don't matter) and they cannot be equal.

Put another way: Rolle's theorem (the specific case of the Mean Value Theorem) tells us that if $g(2) = g(5)$, then there must be $c \in (2,5)$ such that $g'(c) = 0$, which we see doesn't happen. Note that II being true doesn't rule out III being true, but it does given that there are no local extrema in between $x = 2$ and $x = 5$.

**Solution 19.** (C) This is a kind of optimization question, and I'll treat it as such. The distance between the graph of $y = \sqrt{x+3}$ and the origin is $\sqrt{x^2 + y^2}$, which upon substitution becomes a single-variable function of $x$. Minimizing this distance is a question of finding the extrema of that function, but it's much easier to consider minimizing the *square* of the distance.

Specifically, consider $d(x) = x^2 + (\sqrt{x+3})^2 = x^2 + x + 3$. Then if $d(x)$ is at a minimum, so is its square root since $d(x)$ is always positive and $\sqrt{x}$ is a strictly increasing function. To find the minima of $d(x)$, we look at $d'(x) = 2x + 1$. There's one root, namely $x = -1/2$, and indeed $d'(x)$ changes from negative to positive at $x = -1/2$ which means we have a local minimum. It's also "obvious" that this is a minimum rather than a maximum, as it's quite easy to get far away from the origin on the graph of any function. Picking the only answer with $x = -1/2$ completes the problem.

**Solution 20.** (D) There are a few things I would expect out of this answer. First, I expect to see both $f''$ and $g''$ represented. I would also only expect to see $f'$ and $f''$ composed with $g$; no $f' \circ g'$ here, that's not how the chain rule works. So these heuristics rule out everything but (D), which is the right answer.

But there's no way to do be sure without crunching the numbers, in my opinion. Given that the first derivative of the composite function is $f'(g(x)) \cdot g'(x)$,

$$\frac{d}{dx} f'(g(x) \cdot g'(x) = f'(g(x)) \cdot g''(x) + g'(x) \cdot f''(g(x)) \cdot g'(x)$$

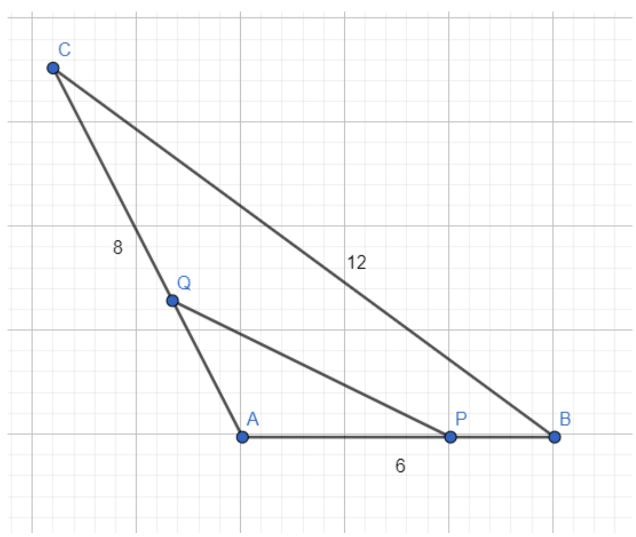
Looking over the answers, that makes it (D).

**Solution 21.** (C) The usual trick here is finding a linear combination of these equations that solves the problem, rather than solving for the actual solutions of $x, y$. That is, we want to find integers $a, b \in \mathbb{Z}$ such that $3a + b = 5 \pmod{13}$ (the $x$-coefficients) and $2a + 7b = 3 \pmod{13}$ (the $y$-coefficient).

My first guess is $a = 1$ and $b = 2$ for the first equation, and in the second equation we have $2 + 14 = 16 \equiv 3$ as required. Lucky? Maybe so, but these problems are usually set up so that the correct solution isn't too hard to find. Therefore we conclude

$$1 \cdot (3x + 2y = 5) + 2 \cdot (x + 7y = 1) \implies 5x + 3x = 5 + 2 = 7$$

**Solution 22.** (D) After drawing out the picture (see below[1]), we see two similar triangles emerge. These are (note the order) $\triangle ACB$ and $\triangle APQ$; the angles $\angle ACB = \angle APQ$ are equal by construction, and $\angle CAB = \angle PAQ$ because points $P, Q$ lie on the legs of the original triangle.




We're looking for the length of $\overline{PQ}$, which corresponds to $\overline{CB} = 12$ in the original triangle. The other leg we know is that $\overline{AP} = 4$ and its corresponding leg $\overline{AC} = 8$. Therefore $\overline{PQ} = 12/2 = 6$.

**Solution 23.** (E) This question looks a lot more complicated than it is. Recall that the system is consistent if it admits at least one solution $(x_i) \in \mathbb{R}^4$. But we can see that $x_1 = a, x_2 = b, x_3 = c, x_4 = 0$ is always a solution; there are more variables than equations here, so we'll always be able to find (in fact) infinitely many solutions for any choice of $(a, b, c) \in \mathbb{R}^3$. This makes the answer (E).

**Solution 24.** (B) A good first determination would be if the null space is 2-dimensional or 1-dimensional, which we can get at by row-reducing the matrix to determine its rank. We see that the bottom row is a multiple of the middle row, and the first two rows are

[1]Thank you GeoGebra for the graphical assistance!

linearly independent, so we've got a rank 2 matrix and thus expect a rank 1 null space by the rank-nullity theorem.

At this point, we can just take (A) and (B) and determine which is correct by doing matrix multiplication.

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ -4 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+9-2 \\ ... \\ ... \end{pmatrix}$$

and we can immediately stop here, since that number isn't zero. That makes (B) the answer.

**Solution 25.** (D) Let's go one constant at a time. The amplitude $A$ is pretty clearly 3 since that's the maximum height of the curve; the only other option is $-3$ which is ruled out because the constants are assumed non-negative. We don't actually need this for the answer, but we have to notice that the graph being "flipped over" compared to $\cos(x)$ cannot be due to the amplitude, but has to be a phase shift.

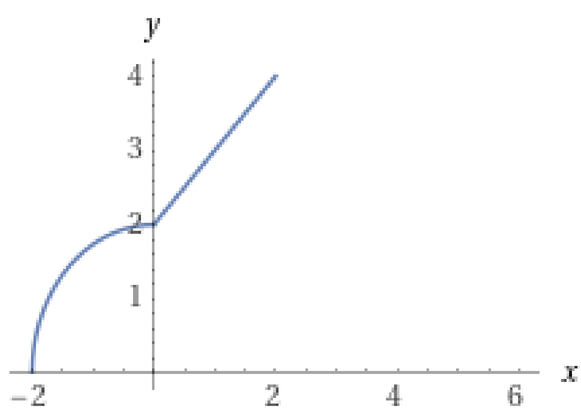
We can look next at the frequency $\omega$. We know that the period $T$ of a trigonometric function is given by $T = 2\pi/\omega$. By looking at the graph, we see the period is $T = \pi$ so $\omega = 2$. At this point, we can solve for $\varphi$ via guess and check, i.e.,

$$f(x) = 3\cos(2x - \varphi), \quad f(0) = 3\cos(-\varphi) = -3 \implies \cos(-\varphi) = -1$$

This means that $-\varphi = n \cdot \pi$ for some $n \in \mathbb{Z}$. That makes (D) the only valid answer.

If you wanted to go all the way to computing the phase shift, this problem is actually tricking us slightly. The standard equation for a trigonometric function should read $A\cos(\omega(x - \theta))$, using $\theta$ instead of $\varphi$. Since $A\cos(\omega x)$ starts at its maximum value (when $A > 0$), we see that we have a shift of $\pi/2$ to the right on the graph, so setting $\theta = \pi/2$ yields $\varphi = \omega\theta = \pi$ as predicted.

**Solution 26.** (C) A classic example of integral = area under the curve for us. The first expression in the "max" is the top half of a circle of radius 2 and the second expression is just a straight line. As we start to draw the graph out, we see that our first intersection point is $(-2, 0)$, where the semicircle is on top, and the second point of intersection is $(0, 2)$ where the line starts to dominate.




Thus the area under the curve is a quarter of the circle of radius 2 plus a trapezoidal area with width 2, height 2 on the left, and height 4 on the right. This explains why the answers all involve a $\pi$ term and a non-$\pi$ term. The quarter circle gives us $4\pi/4 = \pi$ and the trapezoid gives us $2 \cdot (2 + 4)/2 = 6$, yielding (C).

**Solution 27.** (A) When you negate an AND statement, you get an OR statement. So $\neg(Q \text{ AND } R)$ translates to $\neg Q$ OR $\neg R$. The negation of an "if" standment isn't going to be an if statement anymore; it's going to just be the logical propositions. So to negate "if $P$ is true", we'll still have $P$ is true, but the conclusion will be negated. That makes the answer (A); $P$ is true, but it is not implied that $Q$ is true and $R$ is true; one of them is false.

To be honest, I had (E) written down for the first draft and was confused why it wasn't correct. But the negation of a conditional is just a statement of facts to the contrary.[2]

**Solution 28.** (A) We might recognise this first integral as (nearly) the defining PDF for a normal distribution. That's more of a fun fact than useful for the solution to this problem; the super relevant fun fact is that this is not an integral that can be computed by elementary means, i.e., there's no antiderivative function made up of the "usual" functions of everyday life. The $\pi$ might make you suspicious of how to actually get this answer; this actually can be solved via multivariable calculus and polar coordinates[3].

But what we do have here *is* an elementary integral. We can't $u$-substitute, as $u = x^2$ doesn't make our life better when we have to cope with $du = 2x\,dx$, but we can integrate by parts. Despite that, $u = x^2$ isn't going to win us any prizes there either (and I got tricked into trying that while writing this solution).

Instead, set $u = x$ and $dv = x \cdot e^{-x^2}\,dx$, splitting that $x^2$ term in half. Then $du = dx$ and $dv$ is now integrable using a substitution $w = x^2$, $dw = 2x\,dx$:

$$\int x \cdot e^{-x^2}\,dx = \int \frac{1}{2} e^{-w}\,dw = -\frac{e^{-w}}{2} = \frac{-e^{-x^2}}{2}$$

So in total, our integration by parts becomes

$$\int_0^\infty x^2 e^{-x^2}\,dx = x \cdot \frac{-e^{-x^2}}{2}\Big|_0^\infty - \int_0^\infty \frac{-e^{-x^2}}{2}\,dx$$

Now we have some number crunching to do. The middle term we just evaluate: plugging in $x = 0$ gives zero, and on the other end

$$\lim_{x\to\infty} x \cdot \frac{-e^{-x^2}}{2} = \lim_{x\to\infty} \frac{-x}{2e^{x^2}} = 0$$

as the denominator is growing way faster than the numerator; use L'Hôpital's rule if you don't believe me. The other term is addressed by the hint at the beginning: since $e^{-x^2}$ is symmetric around the origin, we can conclude

$$\int_{-\infty}^\infty e^{-x^2} = 2 \cdot \int_0^\infty e^{-x^2} = \sqrt{\pi} \implies \int_0^\infty \frac{e^{-x^2}}{2}\,dx = \frac{\sqrt{\pi}}{4}$$

This is definitely a problem I'd skip and come back to.

**Solution 29.** (B) The function $f(x)$ is increasing when $f'(x) > 0$. By the Fundamental Theorem of Calculus, $f'(x) = (\cos^{23}(x))(2 + \sin^{23}(x))$. As $\sin(x)$ only takes values in $[-1, 1]$, so does $\sin^{23}(x)$; therefore the second term is always positive. The first term is positive so long as $\cos(x)$ is positive, which happens on $(-\pi/2, \pi/2)$ and all $2\pi$-shifts of that. This makes the answer (B).

[2]See, e.g., https://users.math.utoronto.ca/preparing-for-calculus/3_logic/we_3_negation.html

[3]See, e.g., https://math.stackexchange.com/questions/154968/is-there-really-no-way-to-integrate-e-x2

**Solution 30.** (C) Let's start by ruling out things we know are wrong. Remember that $r(t) = (\cos(t), \sin(t))$ would give us the graph of a circle. It seems likely that the graph won't still be circle given that we've messed with it, but it should be some kind of warped circle. That points me to (C), which will turn out to be the right answer.

(A) and (D) don't seem very correct because they've lost all their curviness. To completely validate that: how do we get the point $(1, 1)$? Well $\cos^3(t) = 1$ if and only if $\cos(t) = 1$, and in that case $\sin(t) = 0$ so (D) is impossible. This also rules out (E).

For (A), the listed points are all okay, but how would we get the midpoint $(1/2, 1/2)$ in the first quadrant? If $\cos^3(t) = 1/2$, then $\cos(t) = 1/\sqrt[3]{2}$ which makes it very difficult for $\sin(t) = 1/\sqrt[3]{2}$ too. The Pythagorean identify $\sin^2(t)+\cos^2(t) = 1$ is our friend here; plugging in $1/\sqrt[3]{2}$ in both spots will not work. The same reasoning shows that the graph cannot still be a circle; $\sin^6(t) + \cos^6(t)$ has no reason to always equal 1. That rules out (B) and we can be done.

**Solution 31.** (B) A cube numbered 1-6 on its faces with equal likelihood of any upon rolling is better known as a die, but points to ETS for making this problem less Western-centric and describing the situation in great detail.

If at least 2 of our cubes need to be showing the same number, then our options are 2, 3, or 4 cubes being equal. Conversely, we can solve for the situation where all cubes are different; like we did in Problem 9, we compute the complementary probability and take 1 minus that number.

Imagine we roll the cubes in sequence. The first cube can show anything; the second cube has 5 options, the third 4, and the last one 3. This gives us a probability of

$$1 \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{60}{216} = \frac{5}{18}$$

Its complement is answer (B).

**Solution 32.** (A) The red flag for me is the word "positive" in the first answer. We know that the product of two positive numbers is positive only under non-complex circumstances, and we're explicitly working in the complex numbers. Indeed, in the first set we know $i \in \{a + bi : 0 < a, b \in \mathbb{Q}\}$ but $i^2 = -1$ is not. That solves the problem.

But let's characterize our other sets here for completeness. (B) is almost all of $\mathbb{C}$ except for $a^2 + b^2 = 0$, which only happens at $0 \in \mathbb{C}$; that is, (B) is $\mathbb{C}^\times$ the multiplicative group of the complex numbers (which is, indeed, a group).

(C) doesn't have a lot of options. We can completely characterize the entries: $\{1, -1, i, -i\}$. This is isomorphic to the cyclic group on 4 elements, i.e., the group of fourth roots of unity.

(D) and (E) both have entries that have to satisfy the Pythagorean identity, so they form the unit circle in the complex plane, although (D) is only the rational points of that circle. These are both fine groups; specifically, we know that if $a + bi, c + di$ are two elements, then we have

$$|(a + bi)(c + di)|^2 = |a + bi|^2 \cdot |c + di|^2 = 1 \cdot 1 = 1$$

where $|-|$ denotes the complex norm $|a + bi|^2 = a^2 + b^2$. In the condition that $a, b, c, d \in \mathbb{Q}$, since $\mathbb{Q}$ is closed under multplication (as a field), we know the product will also have rational entries.

**Solution 33.** (A) A little related rates question with a twist. We begin with the formula for the volume of a sphere, $V = 4\pi/3 \cdot r^3$. We then quickly stop, because that formula isn't going to help us for the particular problem. We need to write an equation that incorporates the *depth* of water in the tank, so let's take a step back.

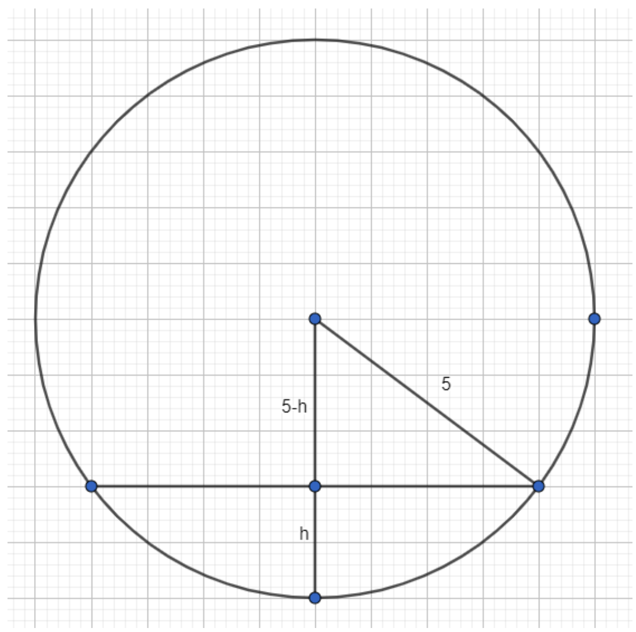
According to various internet sources, it's "fairly easy" to prove that the volume of water in a spherical rank of radius $R$ at height $h$ is given by

$$V(h) = \pi R h^2 - \frac{\pi}{3} h^3$$

Maybe you have this formula in your pocket, maybe not. The only way I can see about this one is actually doing an integral. To derive this formula, we can use

$$V(h) = \int_0^h A(z)\, dz$$

where $A(z)$ is the cross-sectional area of the sphere at height $z$. Luckily all the $A(z)$ are circles, so we just need to find the radius at height $z$. Drawing out what this looks like, in 2D, we obtain the following triangle[4]:




The radius is the last leg of the triangle with radius 5 (our case) and height $5 - h$. We therefore get $r^2 = 25 - (5-h)^2 = 10h - h^2$. The integral is therefore

$$V(h) = \int_0^h A(z)\, dz = \pi \int_0^h 10z - z^2\, dz = \pi(5h^2 - h^3/3)$$

as promised above.

Now that we've got a formula, this problem shouldn't be too difficult. We can take the derivative (and since we've just integrated, that won't be hard) to obtain

$$\frac{dV}{dt} = \pi(10h - h^2) \cdot \frac{dh}{dt}$$

At $h = 2$, we can just plug everything in to obtain $\pi(20-4)(1/3) = 16\pi/3$, which is (A).

There is another way to think about this problem; the maximum volume drain from the tank (that is, the smallest $dV/dt$ can get) is going to occur when the cross-sectional area is at

[4]GeoGebra again

its widest, because infinitesimally we have $dV = (-1/3) \cdot dA$, where $A$ is the cross-sectional area. That area is maximized at the middle of the tank, at $R = 5$ so that $dV/dt$ can never be less than $25\pi/3$. Only (A) is within that bound; all the other rates of decrease are far too large. This observation can save you a lot of time, if you assume there's got to be a trick to it.

**Solution 34.** (A) Let's see if we can find some kind of pattern. In order to compose this function with itself, it'll be more convenient to write it as a single fraction:

$$f(x) = 1 - \frac{1}{x} = \frac{x-1}{x} \implies f\left(\frac{x-1}{x}\right) = 1 - \frac{x}{x-1} = \frac{x-1}{x-1} - \frac{x}{x-1} = \frac{-1}{x-1}$$

Let's look at one more iteration...

$$f^{\circ 3}(x) = f\left(\frac{-1}{x-1}\right) = 1 - (-(x-1)) = x$$

That's fortunate for our computation. This implies that $f^{\circ 100}(x) = f(f^{\circ 99}(x)) = f(x)$, making the answer (A).

This is the first problem on the exam with an under 50% success rate and no wonder: it takes forever.

**Solution 35.** (B) This problem requires some reinterpretation. We are invited by various internet sources to think of this as a Riemann sum, as the idea of a partial sum like this with $n \to \infty$ does recall that. But how is this a Riemann sum?

If we factor out the $n^2$ everywhere, we equivalently have

$$a_n = \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2} \cdot \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2}$$

I think this is quite the leap, but I look forward to readers' comments on the subject. Having established this, set $f(x) = 1/(1 + x^2)$ and we see that this is the $n$th right Riemann sum of

$$\int_0^1 \frac{1}{1 + x^2}\, dx = \arctan(x)\Big|_0^1 = \arctan(1) - \arctan(0) = \pi/4$$

As $\arctan(x)$ is integrable, we know the Riemann sum will converge to the true value, making the answer (B).

Unsurprisingly, this question only had a 31% success rate.

An alternative solution proposed to me by a reader goes as follows: we are going to narrow in on the solution by bounding $a_n$ above and below. An upper bound immediately presents itself: by setting $k = 0$ in all of the terms in the sum of $a_n$, we obtain

$$a_n \leq \sum_{k=1}^{n} \frac{n}{n^2} = 1$$

This inequality will hold in the limit as well, which rules out (C) and (E) as answers. We know that $2 \log 2 > 1$ because $2 \log 2 = \log(2^2) = \log 4$, and $4 > e \approx 2.7$.

The lower bound is a bit more difficult. Recall that for any set of positive numbers $x_1, \ldots, x_m$, there is a series of four inequalities:

$$0 < \frac{m}{\frac{1}{x_1} + \cdots + \frac{1}{x_m}} \leq \sqrt[m]{x_1 \cdots x_m} \leq \frac{x_1 + \cdots + x_m}{m} \leq \sqrt{\frac{x_1^2 + \cdots + x_m^2}{m}}$$

The middle inequality expresses that the arithmetic mean of a set of positive numbers is always at least its geometric mean. On the left of the inequality is the *harmonic mean* and on the right is the *quadratic mean*, which I hadn't thought about in a long time (if I ever did) until receiving this suggestion.

We are going to focus on the harmonic mean and the arithmetic mean for this purpose. If we invert both terms, we get the inequality

$$\frac{m}{x_1 + \cdots + x_m} \le \frac{\frac{1}{x_1} + \cdots + \frac{1}{x_m}}{m}$$

Multiplying through by $m^2$ on both sides will not change this inequality, which yields

$$\frac{m^3}{x_1 + \cdots + x_m} \le \frac{m}{x_1} + \cdots + \frac{m}{x_m}$$

Now, perhaps, we can see where this is taking us. Let $x_k = n^2 + k^2$. Then choosing $n = m$, the righthand side of the above inequality is exactly $a_n$. The denominator of the lefthand side is actually computable now:

$$\sum_{k=1}^{n}(n^2 + k^2) = \sum_{k=1}^{n} n^2 + \sum_{k=1}^{n} k^2 = n^3 + \frac{n(n+1)(2n+1)}{6} = \frac{4n^3}{3} + \text{lower order terms}$$

where we use the "sum of the first $n$ squares" formula that probably was memorized in advance of the test.

Putting this together, the lefthand side of the inequality (the inverted arithmetic mean) has the form

$$\frac{n^3}{\frac{4n^3}{3} + \text{lower order terms}} \implies \lim_{n\to\infty} \frac{n^3}{\frac{4n^3}{3} + \text{lower order terms}} = \frac{1}{4/3} = \frac{3}{4}$$

This inequality will hold in the limit as well, so $\lim_{n\to\infty} a_n \ge 3/4$. That rules out (A) and (D), as $\log 2$ is about 0.7. If you don't have the decimal expansion of $\log 2$ in your head, you can verify it as follows:

$$\log 2 < \frac{3}{4} \iff 2 < e^{3/4} \iff 2^4 = 16 < e^3$$

This again will require you to have $e \approx 2.7$ in your head, so that $e^3 \approx 20$ is computable by hand.

The only answer left is (B).

As yet *another* solution to this problem: notice that $a_n$ are bounded below by setting $k = n$ and computing $n \cdot (1/2n) = 1/2$ and above by setting $k = 0$ and computing $n \cdot (1/n) = 1$. Therefore our answer has to be (A), (B), or (D). You can convince yourself that the series is increasing, so that (A) is also going to be out as $a_2 > 1/2$. Combining this with an alternative version of the integral solution, we have another boundary:

$$\frac{1}{1+x^2} > \frac{1}{1+x} \text{ on } [0,1] \implies \int_0^1 \frac{1}{1+x^2}\,dx > \int_0^1 \frac{1}{1+x}\,dx = \log(1+x)\Big|_0^1 = \log 2$$

So our answer has to be (B) as it's strictly larger than (D).

**Solution 36.** (A) First, let's diagnose what $S$ could be. The only connected subsets of $\mathbb{R}$ are intervals and singletons, and as $S$ has more than one point we're looking at an interval. Since $S$ is bounded, it must be of the form $(a, b)$, $[a, b)$, $(a, b]$, or $[a, b]$ for $a < b \in \mathbb{R}$.

Let's test our intuition on $f(x) = x$, a perfectly good continuous function. There's no reason the image of $S$ needs to be closed, so that rules out Statement I. In the case that $S = (a, b)$, the image $f(S) = (a, b)$ has no maximum value, since it's an open set. That rules out Statement II.

From general principles, we know that the continuous image of a connected subset is still connected, which in our case translates to $f(S)$ is an interval. But there's no reason it needs to be bounded. Consider $f(x) = 1/x$ and $S = (0, 1)$. Then $f(S) = (1, \infty)$, so there's no least upper bound for this set. That makes the answer (A).

This last paragraph might be a little confusing to your intuition. Here's how the usual argument goes for a continuous $f(x)$ defined on all of $\mathbb{R}$. Let $S = (a, b)$, so that $f(S)$ is an interval of some kind. If we take the closure $\overline{S} = [a, b]$, it's closed and bounded, i.e., compact. The continuous image of a compact set is compact, which makes $\overline{f(S)}$ a closed and bounded interval, say $[c, d]$. Thus $f(S)$ has an upper bound $d$ which implies it has a least upper bound. But since our $f(x)$ didn't have to be defined on the closure of $S$, this line of reasoning doesn't apply.

**Solution 37.** (C) This smells like a Lagrange multipliers problem to me. Let $F(x, y, z) = x^2 + y^2 + z^2$. Then extrema of $f(x, y, z)$ subject to $F(x, y, z) = 9$ occur when $\nabla f = \lambda \nabla F$ for some $\lambda \in \mathbb{R}$. Computing these gradients,

$$\nabla f = \langle 1, -3, 2\rangle, \quad \nabla F = \langle 2x, 2y, 2z\rangle$$

which is going to be pretty easy to sort out. We have that $x = 1/2\lambda$, $y = -3/2\lambda$, and $z = 2/2\lambda$. Plugging back into our constraint function will give us the answer, thus

$$(1/2\lambda)^2 + (-3/2\lambda)^2 + (2/2\lambda)^2 = 9 \implies 1 + 9 + 4 = 36\lambda^2 \implies \lambda = \frac{\sqrt{14}}{6}$$

This is a very good sign given how much $\sqrt{14}$ is present in the answer choices.

This gives us $x = 3/\sqrt{14}$, $y = -9/\sqrt{14}$, $z = 6/\sqrt{14}$, and plugging back in to $f(x, y, z)$ gives us $(3 + 27 + 12)/\sqrt{14}$, which sums to $42/\sqrt{14}$ and gives us (C).

We can also get at this answer using Cauchy-Schwarz: let $\vec{u} = \langle 1, -3, 2\rangle$ and $\vec{v} = \langle x, y, z\rangle$. Then

$$|x - 3y + 2z| = |\vec{u} \cdot \vec{v}| \le \|\vec{u}\| \cdot \|\vec{v}\| = \sqrt{1 + 9 + 4} \cdot 3 = 3\sqrt{14}$$

We can assume that we do hit this maximum value, and rearranging it gives us (C).

**Solution 38.** (D) Let's think about this in terms of kernels and images. If $v \in \ker B$, then $A(B(v)) = 0$, so $\ker B \subset \ker A \circ B$. This means the null space of $A \circ B$ is at least dimension 5, getting us down to (D) and (E). In order to maximize the size of the null space, we need to ensure that $\operatorname{im} B$ intersects maximally with $\ker A$. As $\ker A$ has dimension 3 and we're working in $\mathbb{R}^{12}$, there's enough wiggle room for $\operatorname{im} B \cap \ker A = \ker A$. That gives us a total kernel of dimension $3 + 5 = 8$, giving us (D).

To give a specific example here, let $\{e_i\}$ be a basis for $\mathbb{R}^{12}$. Suppose that $B$ is the orthogonal projection onto $e_1, \ldots, e_7$ and $A$ is the orthogonal projection onto $e_4, \ldots, e_{12}$. Then the total composition is the projection onto $e_4, e_5, e_6, e_7$ with an 8-dimensional null space.

**Solution 39.** (E) Since this matrix is upper-triangular, its eigenvalues are the entries along the diagonal. If any of the eigenvalues are zero, then the matrix cannot be invertible. Therefore we have to avoid the situation where $1 + x = 0$, $1 - x = 0$, or $1 + x^2 = 0$. That gives us $\pm 1, \pm i$ as bad values for $x$. We know that four values for $x$ is the most the problem allows, giving us (E).

**Solution 40.** (E) Maybe we can find some kind of pattern here. One line divides the plane into 2 regions; two lines divides it into 4 regions; three could give us 8 regions, but only if the all three intersect at some point, so it's 7 regions. It's not too difficult to also draw out that four lines give us 11 regions, which hints at a pattern: the fourth line gives us 4 more regions. If we just hope that this pattern holds, we will get $5+6+7+8+9+10$ more regions, for a total of $45+11=56$ and answer (E). For the purposes of the GRE, that's where I would stop thinking about this problem.

For a more robust argument[5], consider if we have $N-1$ lines on the plane with none parallel and no three intersecting as a single point. Adding the $N$th line, it isn't parallel to any of the other ones, so there are $N-1$ points of intersection. This gives $N-2$ segments and 2 rays which split existing regions, so that's $N$ more regions. More details (and pictures) are available in the footnote.

It might be worth pointing out that the closed formula here is given by $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}$ as long as $n \geq 2$. This is something like 1 (starting) region plus $n$ lines plus $\binom{n}{2}$ points of intersection in the lines[6]

**Solution 41.** (C) This should be a quick problem for those with linear algebraic facts in their head, but we'll go over them. Starting from the top, (A) is definitely false since not every matrix is diagonalizable. For instance, consider $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which has only one eigenvalue with eigenvector $\langle 1,0,0\rangle$.

Since $M$ has real entries, if it has any complex eigenvalues, they have to come in complex conjugate pairs, which rules out (B). This follows because the eigenvalues are the roots of the characteristic polynomial, which has real coefficients.

For this same reason, the characteristic polynomial has to admit at least one real root, as $M$ is 3-dimensional. Any odd degree polynomial has to have at least one real root by the Intermediate Value Theorem, and that makes (C) necessarily true.

To rule out the rest: if we take $M = I_3$ the identity matrix, it only has one eigenvalue $\lambda = 1$, ruling out (D). This example still works for (E), as the orthogonal eigenbasis for $M$ is just $e_1, e_2, e_3$ but there's still only one eigenvalue. This completes the problem.

**Solution 42.** (A) We can translate this into a linear algebra problem; let $\vec{u}$ be the vector connecting $(1,3,2)$ to $(3,1,2)$ and $\vec{v}$ the vector connecting $(1,3,2)$ to $(-2,0,4)$. That gives $\vec{u} = \langle 2,-2,0\rangle$ and $\vec{v} = \langle -3,-3,2\rangle$. Then as we probably learned in multivariable calculus or linear algebra, the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ has area $\|\vec{u}\times\vec{v}\|$, the magnitude of the cross product. The triangle defined by $\vec{u}$ and $\vec{v}$ has half the area of the parallelogram, so that would solve the problem.

Computing the cross product gives $\vec{u}\times\vec{v} = \langle -4,-4,-12\rangle$, and its magnitude is $\sqrt{176} = 4\sqrt{11}$. The area of the triangle is therefore $2\sqrt{11}$, giving us (A).

[5]See, e.g., https://mvtrinh.wordpress.com/2012/01/05/regions-in-the-plane/

[6]More information on this line of thinking here

**Solution 43.** (E) Looking at the answers, if we can eliminate Statement I we'll be done with the problem, so let's start there first. $xRx$ is easily verified, as $(x-x)(x^2+2)=0$ for any $x$.

Statement II is also true; we know that $xRy$ if $x-y=0$ or if $xy+2=0$. $x-y=0$ if and only if $y-x=0$ and $xy+2=0$ if and only if $yx+2=0$ (because multiplication commutes).

Statement III is testing transitivity, which isn't as obvious. Suppose that $xRy$. Then in one circumstance, $x=y$, so $yRz$ if and only if $xRz$. In the other circumstance, we have $xy+2=0$. Now when $yRz$, we either have $y=z$ or $yz+2=0$. When $y=z$, then $xRz$ if and only if $xRy$, so there's only one edge case to check. In the case that $xy+2=0$ and $yz+2=0$, then we can solve $y=-2/x$ and $z=-2/y$, which means that $x=z$ and this completes the problem.

**Solution 44.** (A) It feels like something curvy is probably the answer, so let's try to rule out (D). we know that one location for $S$ is on the line connecting $A$ and $B$; if the possible points for $S$ forms a line, then it has to be orthogonal to $\overline{AB}$, otherwise the distance can't possibly be constant. But drawing out a triangle for this situation proves that it can't work; $\overline{AS}=6$ and $\overline{SB}=4$, but if we move $S$ one mile to the right or left we have $\sqrt{37}$ and $\sqrt{25}=5$ which doesn't work.

But it's probably best to start with a systematic approach. Let $A=(10,0)$ and $B=(0,0)$ on the plane, where $S=(x,y)$ is a candidate point. We know the distance to $B$ is $\sqrt{x^2+y^2}$ and the distance to $A$ is $\sqrt{(x-10)^2+y^2}$. Therefore we must solve

$$\sqrt{x^2+y^2}+2=\sqrt{(x-10)^2+y^2} \implies x^2+y^2+4\sqrt{x^2+y^2}+4=(x-10)^2+y^2$$

Sorting this out, we can nix the $y^2$ and FOIL out the $(x-10)^2$ to obtain

$$x^2+4\sqrt{x^2+y^2}+4=x^2-20x+100 \implies \sqrt{x^2+y^2}=-5x+24$$

Squaring both sides again is going to give something of the form $y^2=Ax^2+Bx+C$ with $A=24>0$. This isn't a circle or an ellipse, since $x^2$ and $y^2$ have positive coefficients on the opposite sides of the equality. It's also not a parabola, since $y$ is squared. The last option is (A), a branch of a hyperbola.

Helpfully, if you remember that this is one of the definitions of a hyperbola, which is all points which have a constant difference in the distance to two other points, you can be done immediately. In this case, the foci are $A,B$ and the fixed difference is 2 (miles). See also Practice 1, Solution 10.

**Solution 45.** (E) Only (D) and (E) look complicated enough to be correct, but hard to tell what's really going on here. I'll admit to being flummoxed and going to Math StackExchange for help[7].

Here's the idea: we know that $u=u(x,y)=u(f(u,v),g(u,v))$. Take the derivative with respect to $u$ and follow along the chain rule:

$$\begin{aligned}\frac{\partial u}{\partial u}&=1=\frac{\partial u}{\partial x}\frac{\partial f}{\partial u}+\frac{\partial u}{\partial y}\frac{\partial g}{\partial u}\\ \frac{\partial u}{\partial v}&=0=\frac{\partial u}{\partial x}\frac{\partial f}{\partial v}+\frac{\partial u}{\partial y}\frac{\partial g}{\partial v}\end{aligned}$$

[7]See here

We need to pull out that $\partial u/\partial x$ from this mess. The only term that we know isn't in a correct answer is $\partial u/\partial y$, which the bottom equation lets us solve for:

$$\frac{\partial u}{\partial y} = -\frac{(\partial u/\partial x)(\partial f/\partial v)}{\partial g/\partial v}$$

Plugging that back into the first equation, we can solve for $\partial u/\partial x$.

$$\begin{aligned} 1 &= \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial g}{\partial u} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} - \frac{\partial u}{\partial x}\left(\frac{(\partial g/\partial u)(\partial f/\partial v)}{\partial g/\partial v}\right) \\ &= \frac{\partial u}{\partial x}\left(\frac{(\partial f/\partial u)(\partial g/\partial v) - (\partial g/\partial y)(\partial f/\partial v)}{\partial g/\partial v}\right) \end{aligned}$$

when we include the $\partial f/\partial u$ term into the fraction by mutiplying by the denominator. Flipping this fraction around gives us (E).

The quicker way to do it is the following trick, which summarizes all the terrible stuff above into Jacobians. Let $F(u, v) = (f(u, v), g(u, v))$ and let $G(x, y) = (u(x, y), v(x, y))$. Then $F$ and $G$ are inverse functions, so their Jacobians must also be inverses. Therefore given the relationship $J_F^{-1} = J_G$, that is,

$$\begin{pmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{pmatrix}^{-1} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

we can symbolically invert the lefthand matrix to get the same conclusion via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The quickest way to do it is to pick an easy explicit example and hope it carries the day. Let $u = x + y$ and $v = x - y$. Then it's pretty easy to solve $x = \dfrac{u + v}{2}$ and $y = \dfrac{u - v}{2}$. If we think about $x, y$ as the real and imaginary parts of $u$ (respectively), then $v = \overline{u}$ is the conjugate and the above formulas are telling us that the real and imaginary part of a complex number is a function of the number itself and its conjugate (and vice versa).

We can easily compute all the various partial derivatives here: $\partial_u f = 1/2$, $\partial_v f = 1/2$, $\partial_u g = 1/2$, and $\partial_v g = -1/2$. With the knowledge that $\partial_x u = 1$, we can check explicitly. (A), (B), and (C) don't work, and we suspected they wouldn't anyway. Computing (D) gives us

$$\frac{1/2}{(1/2)(-1/2) - (1/2)(1/2)} = \frac{1/2}{-1/2} = -1$$

which is also wrong. Therefore the answer is (E). We notice that the hinges on $\partial_u f = -\partial_v g$ in this example, and indeed given that this example comes from splitting out the real and imaginary parts of the (holomorphic) identity function, we expect this. Thanks to a reader for pointing out that brute force is an effective method.

**Solution 46.** (E) We're given a nice linear differential equation, particularly nice because it only has real-valued coefficients. The general method to solve this equations goes as follows: first, move all the non-$y$ terms to the righthand side of the equals sign, which ETS has helpfully done for us.

We'll first solve the homogeneous version of this equation, $y'' + 2y' + 3y = 0$. Pretending that derivatives are a variable, we have a quadratic equation $w^2 + 2w + 3 = 0$ (where $y$ is the zeroeth derivative so we get $w^0$). Solving for the roots of this polynomial, they're not super pleasant:

$$w = \frac{-2 \pm \sqrt{4-12}}{2} \implies w = -1 \pm i\sqrt{2}$$

The general solution for this differential equation has the form $Ce^{wt}$ for the roots $w$, so we get

$$C_1 e^{(-1+i\sqrt{2})t} + C_2 e^{(-1-i\sqrt{2})t} = y(t)$$

Before we work on the nonhomogeneous part of this problem, we've got a bit of a problem: none of the answers have an $i$ in them.

Not to worry, though: recall Euler's formula, which tells us $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. If we rearrange the above formula a little bit to group together like terms, we obtain

$$C_1 e^{(-1+i\sqrt{2})t} + C_2 e^{(-1-i\sqrt{2})t} = e^{-t}\left(C_1 e^{i\sqrt{2}t} + C_2 e^{-i\sqrt{2}t}\right)$$

Using Euler's formula, we can simplify the content of the parantheses:

$$\begin{aligned} C_1 e^{i\sqrt{2}t} + C_2 e^{-i\sqrt{2}t} &= C_1(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)) + C_2(\cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t)) \\ &= (C_1 + C_2)\cos(\sqrt{2}t) + (C_1 - C_2)i\sin(\sqrt{2}t) \end{aligned}$$

Since we know our $y(t)$ is real-valued, we must have $C_1 = C_2$ so that the $i$ part disappears, which leaves us (effectively) with a single constant $C\cos(\sqrt{2}t)$. Putting our original term back in, we have $Ce^{-t}\cos(\sqrt{2}t)$.

That actually leaves something out, however. If we assume that $C_i \in \mathbb{R}$, this is the only choice, but what if $C_i \in \mathbb{C}$? Then it could be the case that $(C_1 - C_2)i\sin(\sqrt{t})$ actually has a real coefficient, but that only works if $C_i = b_i \cdot i$ for both constants. If these constants are indeed purely imaginary, then $C_1 + C_2$ is going to be non-real, so we have to have $C_1 = -C_2$. That gives us a second solution of $C\sin(\sqrt{2}t)$ which, when put together, gives us our general solution:

$$C_1 e^{-t}\cos(\sqrt{2}t) + C_2 e^{-t}\sin(\sqrt{2}t) = y(t)$$

If you want a simpler reason why we need both a cosine and a sine solution, consider this: they're both equally good solutions to $y'' = -y$, and the fact that we only ended up with cosine the first time was an $\mathbb{R}$-bias that we tried to explain away in the above paragraph.

We still have to solve the nonhomogeneous part of this problem, but we can see the answer is going to be (D) or (E). Answer (A) exists if you screwed up the roots of the original polynomial and (B) and (C) if you forgot the $e^{-t}$ part.

So how to we incorporate that $t$ part? Well it's not just adding $t$ in there, like (D) implies, so the answer is going to be (E). All we're looking for is any *particular* solution to the differential equation $y'' + 2y' + 3y = t$. If we let $y(t) = t$, then we have $y''(t) = 0$, $y'(t) = 1$. So plugging in we get $0 + 2 + 3t \neq t$.

For $y(t) = t/3 - 2/9$, which I'll admit looks fishy if you don't know what you're looking for, we get $y''(t) = 0$, $y'(t) = 1/3$. Plugging in gives us

$$0 + 2(1/3) + 3(t/3 - 2/9) = 0 + 2/3 + t - 2/3 = t$$

as required. That makes the answer (E).

**Solution 47.** (C) This is a highly suspicious line integral, to be presented like this. Indeed, it almost looks like we're taking the integral of something's gradient, as I notice that $y^3$ is the $x$-partial derivative of $xy^3$, whose $y$-partial derivative is $3xy^2$ in the other term.

Indeed, what we've got here is $\nabla F(x,y) \cdot \langle dx, dy\rangle$ for $F(x,y) = xy^3 + 5x^2/2 + 4y^2 + C$. This means that the vector field along which we're integrating is path-independent, so we don't actually have to do a line integral; it just depends on the endpoints.

$$\int_C (5x + y^3)\,dx + (3xy^2 + 8y)\,dy = F(0,3) - F(2,0) = 36 - 10 = 26$$

That gives us (C).

**Solution 48.** (B) We note that $f(x,y)$ hasn't been assumed to even be continuous before getting into this problem.

For Statement I, let's just take a function $f(x,y)$ constant in $y$ and increasing in $x$, say $f(x,y) = x^3$. Then the level curve at $z = c$ is the line $x = \sqrt[3]{c}$ (and constant in $y$). These are all going to be parallel but the graph itself is clearly not a plane. It's possible this could be satisfied by any $f(x,y)$ that's constant in $y$, but sometimes the level curves will be *unions* of parallel lines (e.g., $f(x,y) = x^2$ at $z = 1$) so we can sidestep that issue with a choice which is strictly increasing in $x$.[8]

For Statement II, since the partial derivatives exist, $f(x,y)$ at least has to be continuous. If they're constant, then they're in particular continuous, so $f(x,y)$ is actually a differentiable function. This means that, not only does $f(x,y)$ look locally like a plane, it actually has to be a plane. This is because the normal vector to any point on the graph of $f(x,y)$ (for a differentiable function) is given by $\langle -\partial_x f, -\partial_y f, 1\rangle$. A plane is defined by having a constant normal vector; we can explicitly write $f(x,y) = \partial_x f \cdot x + \partial_y f \cdot y + C$ for some constant $C$.

So that leaves Statement III. There doesn't seem to be any reason this should be the case. Consider $f(x,y) = x^2 + y^2$, which is definitely not a plane. Then its mixed partials are identically zero, as no term has both $x$ and $y$ in it. This rules out III and makes (B) the answer.

**Solution 49.** (E) This is a clear case for the Residue Theorem. We have our positively-oriented closed curve $C$, and the theorem tells us

$$\oint_C \left(\frac{\sin z}{z-1}\right)^2 dz = 2\pi i \cdot \sum \operatorname{Res}(f(z), a_k)$$

where $a_k$ are the poles of the function $f(z) = (\sin z/(z-1))^2$ on the interior of $C$ and Res denotes the residue at that pole.

So where are the poles? Only when $z = 1$, which is (happily enough) the dead center of the circle $C$. If $z = 1$ is a simple pole, then its residue is given by

$$\operatorname{Res}(f(z), 1) = \lim_{z\to 1}(z-1) \cdot \frac{\sin^2(z)}{(z-1)^2} = \lim_{z\to 1} \frac{\sin^2 z}{z-1}$$

which unfortunately does not exist. This is because $z = 1$ isn't actually a simple pole; it has order 2, which is evidenced by the $(z-1)^2$ in the denominator. The residue computation for a pole of order $n$ at $z = c$ is given by

$$\operatorname{Res}(f(z), c) = \frac{1}{(n-1)!} \lim_{z\to c} \frac{d^{n-1}}{dz^{n-1}} (z-c)^n f(z))$$

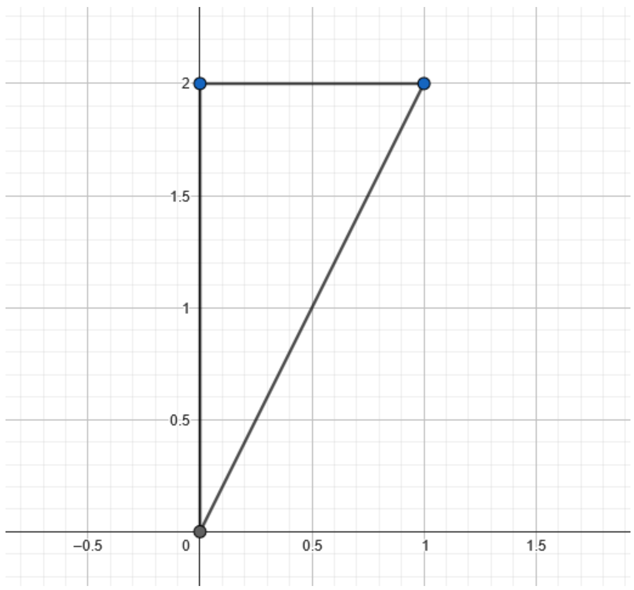
[8]Thanks to a reader for simplifying this solution & correcting a previous error

So for us this becomes

$$\operatorname{Res}(f(z),1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z-1)^2\frac{\sin^2 z}{(z-1)^2}\right) = \lim_{z\to 1}\frac{d}{dz}\sin^2 z = \lim_{z\to 1} 2\sin z\cos z$$

This makes the answer $2\sin(1)\cos(1)$, which isn't an option. Hmm. But when we remember our double angle formulas, we recall $\sin(2\theta) = 2\sin\theta\cos\theta$, which allows us to obtain $\sin 2$ as the final answer, which is (E).

**Solution 50.** (E) Well, I surely don't want to integrate this with respect to $y$ first, as it's not possible (see Problem 28). So let's graph out this region to reverse the order of integration. We have a nice little triangle here, with $0 \le x \le 1$ and $2x \le y \le 2$. This gives $0 \le y \le 2$ and now $x$ is bounded by 0 on the left and $y/2$ on the right.




Hence we can equivalently compute

$$\int_0^2\int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 \left(xe^{y^2}\right)\Big|_0^{y/2}\,dy = \int_0^2 \frac{ye^{y^2}}{2}\,dy$$

Now we can $u$-substitute for $u = y^2$, $du = 2y\,dy$ and compute

$$\int_0^2 \frac{ye^{y^2}}{2}\,dy = \int_0^4 \frac{e^u}{4}\,du = \frac{e^u}{4}\Big|_0^4 = \frac{e^4-1}{4}$$

Giving us (E).

**Solution 51.** (C) Let's run through this a bit to see what's going on. We start with $A = 2$, $B = 1$, $C = 0$. Looking at the main loop, the main step (with the arrows going to the right) is testing whether we've hit $C = A$ or $C > A$; if we hit $C = A$, we print and then increment $A$; otherwise we don't print and then increment $A$. If we're still in $C < A$ then we increase $C$ by $B$ and try again, but this time $B$ is incremented by 2.

So the possible values for $C$ are actually the same every loop; we start at $C = 1$ (within the main testing loop), then $C = 1 + 3$, then $C = 1 + 3 + 5$, as $B$ keeps going up by 2 every loop. $C$ is a sum of successive odd numbers, which means $C$ always has the form $C = n^2$. We only print when $A = C$, so that means we can only print a perfect square, making the answer (C).

**Solution 52.** (E) Recall that in a permutation/symmetric group, the conjugacy classes are exactly the cycle decomposition types. Since we're working with 4 elements, we can have 1-cycles, 2-cycles, 3-cycles, 4-cycles, and the ever mysterious 2-2-cycles, e.g., $(1\,2)(3\,4)$. That's 5 types, so it's (E).

This problem only had a 20% success rate, which I take to mean that people haven't memorized this pretty useful fact about symmetric groups. Commit it to memory and this problem is free; try to work it out on the fly and you'll waste too much time.

**Solution 53.** (D) These are all six possible dot products between the four vectors we've chosen. Let's interpret the statements in terms of directionality; a dot product is negative if the vectors are pointing in opposite directions, zero if they're orthogonal, and positive if they're in the same direction.

Statement I is false; we have four vectors in $\mathbb{R}^2$, and the most "away from each other" these four can be is along the cardinal directions, but then all these dot products would be zero. Two of these vectors have to be going in the same direction no matter what we do.

Now it seems to me that Statements II and III are equivalent. Suppose that II holds, and without loss of generality there are scenarios. We either have $v_1 \cdot v_2, v_1 \cdot v_3 < 0$ or we have $v_1 \cdot v_2, v_3 \cdot v_4 < 0$, that is, the negative dot products either contain a similar vector or they don't. The first case actually is impossible though; under this circumstance we have that $v_2 \cdot v_3 \neq 0$ which would fail the other condition for II.

The transformation $v_1 \mapsto -v_1$ and $v_3 \mapsto -v_3$ gets us between Statement II and Statement III, as it flips the sign of the non-zero dot products and leaves the orthogonal vectors still orthogonal. Since "none of the above" is not an answer choice, we have to conclude that II and III are both true (rather than both false) and the answer is (D).

A particular arrangement that gets us the answer: we can use cardinal directions as we described in I. North/South and East/West give negative dot products, and the other four combinations are zero. To get an arrangement like III, just take North, North, East, East (or something similar).

**Solution 54.** (E) This doesn't look like a fun differential equation to solve; certainly it's not as straightforward as Problem 46 above. If we try to solve the homogeneous equation, we get a solve using separation of variables:

$$y' + 2yt = 0 \implies \frac{dy}{y} = -2t\,dt \implies \log y(t) = -t^2 + C \implies y(t) = Ce^{-t^2}$$

Certainly a good sign, as this looks pretty similar to the nonhomogeneous part of our equation.

If we want to find a particular solution to give us $e^{-t^2}\sin t$, we're going to have to start with something like $y(t) = De^{-t^2}\cos t$ so its derivative involves a $\sin t$. If we just make this guess, we have

$$y'(t) = -2t \cdot De^{-t^2}\cos t - De^{-t^2}\sin t = -2ty(t) - De^{-t^2}\sin t$$

This means that $y' + 2yt = -De^{-t^2}\sin t$, and for us that makes $D = -1$. Therefore the general solution to our differential equation is

$$y(t) = Ce^{-t^2} - e^{-t^2}\cos t = e^{-t^2}(C - \cos t)$$

We can now solve the problem. Since $y(0) = 0$, this means $C = \cos 0 = 1$. Therefore $y(\pi) = e^{-\pi^2}(1+1) = 2e^{-\pi^2}$, which thankfully is (E).

As an alternative: we don't actually need to solve the closed form of the solution, just the value at $\pi$. So if we multiply through our starting equation by $e^{t^2}$, we obtain

$$e^{t^2}(y' + 2yt) = \sin t \implies (e^{t^2} y)' = \sin t$$

as long as we can recognise the derivative of $e^{t^2}$ as hiding on the lefthand side. This is now easy to antiderive: $e^{t^2} y(t) = -\cos(t) + C$, and you can see where we can pick up the above solution at this point using the initial condition of $y(0) = 0$ to finish the problem.

**Solution 55.** (B) Recall how we compute the standard deviation: it's given by taking the sum of the squares of the deviations from the mean, multiplying by $1/N$ for the number of observations $N$, then taking the square root:

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\overline{x} - x_i)^2}$$

If we quadruple the number of observations, then we should $\sqrt{1/4} = 1/2$ the standard deviation. That gives us (B).

**Solution 56.** (E) A classic problem on idempotent rings, i.e., rings where every element is its own square. These can't be very complicated, as you can guess. Let's review:

For Statement I: $(a+1)^2$. Foiling this out gives us $a^2 + a + a + 1$, and this also must equal $a + 1$. This implies that $a^2 + a = 0$, and since $a^2 = a$, $a + a = 0$. Otherwise put, $R$ has characteristic 2, i.e., the canonical homomorphism $\mathbb{Z} \to R$ has kernel $2\mathbb{Z}$.

For Statement II: this asserts that every element is nilpotent, which can't be true since $1 \in R$.

For Statement III: to prove that $R$ is commutative (which it is), consider the product $(a + b)^2$. Foiling this out, we get $a^2 + ab + ba + b^2$. We know that $(a + b)^2 = a + b$ and $a^2 = a$, $b^2 = b$, so putting all this together we obtain $ab + ba = 0$. That lets us conclude that $ab = -ba$; but by Statement I, every element is its own additive inverse, so $-ba = ba$ and we conclude that $ab = ba$. That makes the answer (E).

**Solution 57.** (B) Looks like we're going to have to symbolically integrate by parts here and look for a pattern. If we recall how to integrate $\log x$, we use the choice $u = \log x$ and $dv = dx$. Then

$$\int \log x\, dx = x \log x - \int \frac{x}{x}\, dx = x \log x - x$$

For $I_n$, we should be able to use a similar trick. Let $u = (\log x)^n$ and $dv = dx$. Then $du = n(\log x)^{n-1}/x\, dx$ and $v = x$. We therefore obtain

$$\int (\log x)^n\, dx = x(\log x)^n - \int n(\log x)^{n-1}\, dx = x(\log x)^n - n \cdot I_{n-1}(x)$$

Looking closely at the signs in the answers, this gives us (B).

**Solution 58.** (A) This problem has the form of a "stars and bars" partition. Imagine that we line up all 25 trucks in a row, and we need to divide them between 5 cities. That means putting in 4 bars between trucks so that the first clump goes to City 1, the second to City 2, etc. This means we have 29 "slots" to fill, of which 4 have to be the dividing bars (and the remaining 25 are trucks). That makes the base case of this problem $\binom{29}{4}$.

Now, we've subtracted off $\binom{24}{4}$ for the problem at hand. We can take the situations given and determine which of the ruled out possibilities would give us these combinations. If we continue thinking in stars and bars, however, one might recall that the number of *positive* partitions of 25 trucks into 5 cities would be $\binom{24}{4}$. This follows because, as we imagine lining up the 25 trucks in a row, we now need to place 4 bars strictly between each of the 25 trucks, into 24 slots. We also can't but bars next to each other, because that would give a partition with zero elements.

Looking at the answers, (A) is this situation exactly. We have eliminated all arrangements such that every city receives at least one truck, therefore at least one city must have zero trucks. We could go through and analyze the remainder ((B) is just the complement of (A)), it's just a test of how much time you're willing to waste on incorrect answers.

Perhaps unsurprisingly this problem also has a mere 21% success rate. Stars and bars doesn't come up outside of a discrete math or combinatorics class, and it's worth remembering along with things like the pigeonhole principle for just such an occasion.

**Solution 59.** (C) Recall that the integers invertible modulo $n$ are represented by integers which are coprime to $n$, which in our case gives us $U_{30} = \{1, 7, 11, 13, 17, 19, 23, 29\}$, most of the primes under 30 (since $30 - 2 \cdot 3 \cdot 5$ is the product of the first three primes). It's an abelian group with 8 elements, which doesn't give us a whole lot of choices for its structure, but we don't even need to figure that out to solve the problem.

We're given information about $\varphi(7)$ and $\varphi(11)$. We know that

$$\varphi(7 \cdot 11) = \varphi(7) \cdot \varphi(11) = \varphi(7) \cdot 1 = 7$$

so that gives us $\varphi(77) = 7$. 77 is equivalent to 17 modulo 30, making (C) the answer.

**Solution 60.** (B) When given an $LU$-decomposition of a matrix, we can solve the problem in two steps. First, let $\vec{y} \in \mathbb{R}^4$ such that $L\vec{y} = \vec{b}$. Specifically, that means

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The triangularity of the matrix $L$ is going to make this an easy system of equations. We just work from top to bottom: we have $a + 0 + 0 + 0 = 1$, so $a = 1$. Then $a + b + 0 + 0 = 0$, so $b = -1$. Third, $a + 0 + c + 0 = 1$, so $c = 0$. Finally, $a - b + 0 + d = 0$, so $d = -2$.

Not too bad, we can pretty much turn our brain off and chug when it comes to this sort of solution. Now we have to solve for $U\vec{x} = \vec{y}$, i.e.,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}$$

Unfortunately we can't go straight for $\alpha$, the answer, and need to go from the bottom up instead. $2\delta = -2$, so $\delta = -1$. $\gamma + 2\delta = 0$, so $\gamma = 2$. $\beta + \gamma = -1$, so $\beta = -3$. Finally, $\alpha + \gamma = 1$, so $\alpha = -1$. You could skip computing $\beta$ if you wanted.

**Solution 61.** (E) We can solve this if we can recall/recompute $\Phi_{10}(z)$, the 10th cyclotomic polynomial, which is the minimal polynomial for primitive 10th roots of unity. We know it has to have degree $4 = \varphi(10)$, Euler's totient function of 10, because $1, 3, 7, 9$ are the the only coprime numbers less than 10. We also know that it's a divisor of $z^{10} - 1 = 0$, as the minimal polynomial has to divide any polynomial with the correct roots.

Okay, from this, we can start factoring $z^{10} - 1 = (z^5 - 1)(z^5 + 1)$. The lefthand term is going to take care of all the 5th roots of unity, so we know that $\Phi_{10}(z)$ is a factor of $z^5 + 1$. We can keep going a little further, as $z = -1$ is a root of the righthand term, to obtain

$$z^5 + 1 = (z + 1)(z^4 - z^3 + z^2 - z + 1)$$

This can be accomplished via synthetic division or remembering some of your divisibility tricks. This gives us our required polynomial.

For any polynomial, we know that the sum of the roots is $-1$ times the coefficient of $z$ and the product of the roots is $(-1)^d$ times the constant term, where $d$ is the degree. That makes both $S = 1$ and $P = 1$, giving us (E).

This was the least correctly answered problem on the test, with only 13% correct answers. No surprise; finding these cyclotomic polynomials is pretty annoying.

**Solution 62.** (B) Recall that a metric space is *complete* if every Cauchy sequence of points has a limit that is in that metric space. Recall that a sequence $\{a_n\}$ is Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$. For example, $\mathbb{Q}$ under the standard metric is not complete, as there are plenty of sequences that have no limit, e.g., the truncated decimal expansion of an irrational number, say $1, 1.4, 1.41, 1.414, \dots \to \sqrt{2}$.

(A) is complete, since that's just $\mathbb{R}$ in the standard metric. (E) looks a bit weird, so maybe that's our culprit? What even are the Cauchy sequences in that metric? Actually, a sequence here is Cauchy if and only if it's eventually constant, therefore has a limit in the metric space.

Under what circumstance could we have a Cauchy sequence where the metric is getting closer together but the "numbers" aren't? If we look at (D), suppose we have a Cauchy sequence $\{a_n\}$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be the guaranteed number. Then we have the following manipulation:

$$|\sqrt[3]{a}_n - \sqrt[3]{a}_m| \cdot |\sqrt[3]{a_n^2} + \sqrt[3]{a_n a_m} + \sqrt[3]{a_m^2}| = |a_n - a_m|$$

Because the sequence is Cauchy, it's in particular bounded; this means that complicated term above has some upper bound $M$ for all $n, m \in \mathbb{N}$. Therefore with the replacement of $\varepsilon$ by $\varepsilon/M$, we can prove that the sequence $\{a_n\}$ is also Cauchy in the standard metric on $\mathbb{R}$, and so it has a limit; this must be the same limit as the (B) metric. A similar argument also proves (C) is complete.

This means the answer has to be (B), so what gives? The problem lies in the fact that $\arctan x$ gets very small even as $x$ gets very large, something that's missing in all the other examples. Consider the sequence $\{a_n = n\}$. Then as we know that $\lim_{n\to\infty} \arctan(n) = \pi/2$, there exists $N \in \mathbb{N}$ such that $|\arctan(n) - \pi/2| < \varepsilon$ for all $n > N$. By the triangle inequality, for all $n, m > N$,

$$|\arctan(n) - \arctan(m)| \leq |\arctan(n) - \pi/2| + |\pi/2 - \arctan(m)| < 2\varepsilon$$

So this sequence is indeed Cauchy, but there's no limit; if we had a mysterious point at $\infty$, maybe, but $\mathbb{R}$ certainly doesn't. In fact any "unbounded" sequence in $\mathbb{R}$ that's either positive or negative is going to be Cauchy in the (B) metric, but won't have a limit.

**Solution 63.** (A) There's a devious reduction trick here that will help us out. Set $I_n$ to be the integral we're trying to solve. Consider the difference

$$I_n - I_{n-2} = \int_0^\pi \frac{\sin(nx) - \sin((n-2)x)}{\sin x}\,dx$$

Now there's a trigonometric identity[9] which says that

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

So what's that mean for us? Applying that identity to the numerator,

$$I_n - I_{n-2} = \int_0^\pi \frac{2\sin x\cos((n-1)x)}{\sin x}\,dx = 2\int_0^\pi \cos((n-1)x)\,dx = 0$$

So as long as $n-2 \geq 0$, this identity holds and thus we can reduce $I_{100} = I_2$. Computing that one is easy: recall $\sin(2x) = 2\sin x\cos x$, so that

$$\int_0^\pi \frac{\sin(2x)}{\sin x}\,dx = \int_0^\pi 2\cos x\,dx = 2\sin x\Big|_0^\pi = 0$$

which gives us (A).

Another solution option: let $u = x - \pi/2$ and change variables, letting $I$ be our integral again.

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin(100(u+\pi/2))}{\sin(u+\pi/2)}\,du = \int_{-\pi/2}^{\pi/2} \frac{\sin(100u+50\pi)}{\cos u}\,du = \int_{-\pi/2}^{\pi/2} \frac{\sin(100u)}{\cos u}\,du$$

Now we're on a symmetric domain around $u = 0$. We know that $\sin(100u)$ is an odd function, that is, $\sin(100u) = -\sin(100\cdot -u)$, and $\cos(u) = \cos(-u)$ is an even function. Therefore the quotient $\sin(100u)/\cos u$ is an odd function, and the integral of such a function over a symmetric domain is zero. If you have the suspicion that zero is the answer, a solve like this or trying to prove $I = -I$ is a good method.

**Solution 64.** (C) This smells a lot like L'Hôpital's rule given the setup to the problem, so let's use that as a jumping off point.

For Statement I: if we try to take the limit, we get

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 0} f'(x)}{\lim_{x\to 0} g'(x)}$$

where this last equality follows since $f'(x), g'(x)$ are continuously differentiable. Since the denominator is nonzero, this gives us a value for the limit and we prove I is true.

For Statement II, we're using the same method of proof, but we might hit a wrinkle due to the increased complexity. L'Hôpital's rule still applies since $f^2(0) + f(0) = 2g(0) - g^3(0) = 0$

$$\lim_{x\to 0}\frac{f^2+f}{2g-g^3} = \lim_{x\to 0}\frac{2f\cdot f' + f'}{2g' - 3g^2\cdot g'} = \frac{2f(0)\cdot f'(0) + f'(0)}{2g'(0) - 3g^2(0)\cdot g'(0)} = \frac{f'(0)}{2g'(0)}$$

[9]See https://www.liverpool.ac.uk/ maryrees/homepagemath191/trigid.pdf

Because most of these terms are zero, we can conclude that this limit still exists since $2g'(0) \neq 0$.

For Statement III, we're now asking to extend to a *differentiable* function. If we could rig a situation where $f(x)/g(x) = |x|$, we'd be in great shape. So let's try something straightforward: $f(x) = x \cdot |x|$ and $g(x) = x$. Is $f(x)$ continuously differentiable? It's definitely continuous, and we can write it piecewise as

$$f(x) = \begin{cases} -x^2 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

Then on the branches, we have $f'(x) = -2x$ for $x \leq 0$ and $f'(x) = 2x$ for $x > 0$, which luckily agree at $x = 0$. This gives us exactly the situation of our dreams, and we can conclude III is false and the answer is (C).[10]

**Solution 65.** (E) We're looking for a conditional probability here: the probability that the test is correct *given* that it is positive. Let $A$ denote the event that the man has the disease and let $B$ denote the event that the test is positive; we want $P(A|B)$.

Bayes' theorem tells us that if we can find $P(A)$, $P(B)$, and $P(B|A)$ we can compute $P(A|B)$. The probably that the man has the disease $P(A) = 5\%$ as the problem states. $P(B|A)$ is also given; if a man has the disease, the test is right 24% of the time.

The last piece is $P(B)$, the odds that the test is positive. We can compute $P(B)$ as the sum $P(B \cap A) + P(B \cap \neg A)$. The probability of the man being positive and the test being correct $P(B \cap A) = P(A) \cdot P(B|A) = (0.05)(0.24) = 0.012$. For the other piece, $P(B \cap \neg A) = P(\neg A) \cdot P(B|\neg A) = (0.95)(0.02) = 0.019$.

Putting all this together,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.24 \cdot 0.05}{0.012 + 0.019} = \frac{12}{31}$$

Well that's more than 29% since 12 is more than a third of 31, making the answer 39%. (E) is therefore the answer.

This is why it's important for doctors to study Bayes' theorem!

**Solution 66.** (B) This is probably more of a number theory problem than a topology one, but you need both to get it right. Let's try to figure out if the complement of any of the listed sets is actually open to get started.

Let $U_A$ denote the complement of set (A). Then we know that $n^2 \in U_A$ since it's everything but $n$. But then $n^2 \in U_k$ for some $k$, which means $n^2 \mid k$ and so $n \mid k$ too. This means that $U_A$ cannot be open.

For (B), let $U_B$ denote its complement. Suppose that $m \in U_B$ is some non-multiple of $n$. Then we need to show that $U_m \subset U_B$. But this is immediate; if some divisor $t \mid m$ were a multiple of $n$, then writing $t \cdot d = m$ would imply that $m$ is a multiple of $n$ as well. Therefore we can conclude that $U_B$ is open, and so this set is closed. It may not be minimal, but it's closed.

For (C), this is an open set, so the question must be asked if the open sets are also closed. The complement $V_k$ of a set $U_k$ must contain all the points $\geq k+1$, since they cannot divide $k$. This implies that $2k \in V_k$. But then the open neighbourhood $U_{2k} \subset V_k$, which implies all

[10] Sanity checked at Math StackExchange

the divisors of $2k \in V_k$, including $k$ itself, which means $V_k$ is not the complement. Thus the open sets are not closed and this can't be right.

(D) and (E) might look complicated, but they don't even contain $\{n\}$ for all $n \geq 2$! So we can rule them out immediately and conclude that the answer is (B).