Monetary Policy - EC861 - BC: Fall 2009 - Lecture Notes

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Contents

Chapter 1

Linking data and economic models

Very good background reading on the subject is the book "Structural Macroeconometrics" by Dejong and Dave, Princeton University Press.

1.1 An example

A DSGE model is in general an economic model describing how the economy evolves over time. The stochastic nature of the model refers to the fact that the economy is hit, during its evolution, by random shocks such as technological change, fluctuations in oil prices, or errors in policy-making.

An example is the prototypical RBC model. We solve this model using a planner's approach since we know that the competitive equilibrium of the model is Pareto-efficient.

For simplicity, assume that capital does not depreciate, that labor is supplied inelastically (and normalized to 1), that utility is logarithmic in consumption. Then the representative agent problem can be written as:

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right)$$

subject to

$$K_t - K_{t-1} = A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t \tag{1}$$

Optimal consumption implies

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha \left(\frac{A_{t+1}}{K_t} \right)^{1-\alpha} + 1 \right) \right) \tag{2}$$

Assume technology follows an AR(1) process in logs, that is

$$\log A_t = \rho \log A_{t-1} + \log U_t \tag{3}$$

where ρ is the autocorrelation of the shock. Assume that $\log U$ has mean zero, finite variance.

The system made by (1) to (3) is a non-linear system with rational expectations. We usually solve them in the following steps

1. Find the steady state.

   In steady state variables are constant, that is

$$\begin{aligned}
\log A &= 0->A=1 \\
C &= K^{\alpha} \\
1-\beta &= \alpha\beta\left(\frac{1}{K}\right)^{1-\alpha} -> \left(\frac{\alpha\beta}{1-\beta}\right)^{\frac{1}{1-\alpha}} = K
\end{aligned}$$

2. Linearize around the steady state

   (a) equation 1

$$C_t = A_t^{1-\alpha}K_{t-1}^{\alpha} - K_t + K_{t-1}$$

$$\begin{aligned}
\log C_t &= \log\left(A_t^{1-\alpha}K_{t-1}^{\alpha} - K_t + K_{t-1}\right) \\
&\quad \text{take total differential around steady state} \\
\frac{1}{C}dC_t &= \frac{1}{C}\left(\left(1-\alpha\right)A^{-\alpha}K^{\alpha}dA_t + \alpha A^{1-\alpha}K^{\alpha-1}dK_{t-1} - dK_t + dK_{t-1}\right) \\
c_t &= \frac{\left(1-\alpha\right)K^{\alpha}}{C}dA_t + \alpha\frac{K^{\alpha-1}}{C}dK_{t-1} - \frac{1}{C}dK_t + \frac{1}{C}dK_{t-1} \\
c_t &= \left(1-\alpha\right)A_t + \alpha k_{t-1} - \frac{K}{C}k_t + \frac{K}{C}k_{t-1} \\
\frac{K}{C} &= \frac{K}{K^{\alpha}} = \frac{\alpha\beta}{1-\beta} \\
0 &= -c_t + \left(1-\alpha\right)a_t - \frac{\alpha\beta}{1-\beta}k_t + \frac{\alpha}{1-\beta}k_{t-1}
\end{aligned}$$

   (b) equation 2

$$\begin{aligned}
E_t\left(\frac{C_{t+1}}{C_t}\right) &= \beta E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
\log E_t\left(C_{t+1}\right) - \log\left(C_t\right) &= \log\beta + \log E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
E_t\log\left(C_{t+1}\right) - \log\left(C_t\right) &= \log\beta + \log E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
&\quad \text{steady state of both sides is 1} \\
E_tc_{t+1} - c_t &= \left(\alpha\left(1-\alpha\right)\left(\frac{A}{K}\right)^{-\alpha}\left(\frac{1}{K}dE_tA_{t+1} - \frac{A}{K^2}dK_t\right)\right) \\
E_tc_{t+1} - c_t &= \alpha\left(1-\alpha\right)\left(\frac{1}{K}\right)^{1-\alpha}\left(E_ta_{t+1} - k_t\right) \\
E_tc_{t+1} - c_t &= \frac{\left(1-\alpha\right)\left(1-\beta\right)}{\beta}\left(E_ta_{t+1} - k_t\right) \\
0 &= -E_tc_{t+1} + c_t + \frac{\left(1-\alpha\right)\left(1-\beta\right)}{\beta}\left(E_ta_{t+1} - k_t\right)
\end{aligned}$$

   (c) equation 3

$$a_t = \rho a_{t-1} + u_t$$

This is a dynamic system of 3 equations in 3 unknowns. To use a more compact notation, we prefer to write it in the following form

$$
\begin{aligned}
0 &= E_t\left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t\right] && (4) \\
\mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t && (5)
\end{aligned}
$$

where

- $\mathbf{x}_t$ is the column vector collecting all the endogenous variables of the model. This variables can be further divided into state and jump variables.
- $\mathbf{z}_t$ collects all the exogenous stochastic processes. Obviously, one can treat the variables in $\mathbf{z}$ as belonging to $\mathbf{x}$ and treat the shocks are pure iid.

In our above example

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} c \\ k \end{bmatrix} \\
\mathbf{z} &= [a]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{F} &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \\
\mathbf{G} &= \begin{bmatrix} -1 & -\frac{\alpha\beta}{1-\beta} \\ 1 & -\frac{(1-\alpha)(1-\beta)}{\beta} \end{bmatrix} \\
\mathbf{H} &= \begin{bmatrix} 0 & \frac{\alpha}{1-\beta} \\ 0 & 0 \end{bmatrix} \\
\mathbf{L} &= \begin{bmatrix} 0 \\ \frac{(1-\alpha)(1-\beta)}{\beta} \end{bmatrix} \\
\mathbf{M} &= \begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix} \\
\mathbf{N} &= [\rho]
\end{aligned}
$$

1.2 The general form of a linearized DSGE model

To summarize, a linearized DSGE model can be written in the following form

$$
\begin{aligned}
0 &= E_t\left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{u}_{t+1} + \mathbf{M}\mathbf{u}_t\right] \\
\mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t
\end{aligned}
$$

1.3 The solution of a linearized DSGE model

What one is looking for is the recursive equilibrium law of motion that describes the endogenous variables as a function of the STATE, that is:

$$\mathbf{x}_t = \mathbf{P}\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{z}_t \tag{6}$$

i.e., matrices $\mathbf{P}$, $\mathbf{Q}$ such that the equilibrium is described by these rules. Notice that (3) describes the solution, in that, once we know the state described by current shocks $\mathbf{z}_t$ and last period values of our variables $\mathbf{x}_{t-1}$, we also know the values of $\mathbf{x}_t$. Sometimes we find it convenient to write

$$\mathbf{x}_t = \mathbf{P}\left(\boldsymbol{\theta}\right)\mathbf{x}_{t-1} + \mathbf{Q}\left(\boldsymbol{\theta}\right)\mathbf{z}_t$$

to make clear that the matrices $\mathbf{P}$ and $\mathbf{Q}$ are complicated functions of the underlying parameters of the model.

Next, what we do is to plug the matrices in (4) and (5) in a computer, to obtain (6).

In our toy example above, set

$$\begin{aligned} \alpha &= 0.33 \\ \beta &= 0.99 \\ \rho &= 0.98 \end{aligned}$$

Then

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} 0 & 0.6589 \\ 0 & 0.9899 \end{bmatrix} \begin{bmatrix} c_{t-1} \\ k_{t-1} \end{bmatrix} + \begin{bmatrix} 0.1755 \\ 0.0151 \end{bmatrix} [z_t]$$

Notice that both consumption and capital are a distributed lag of the single innovation to productivity z_t. That is, there is just one factor driving both variables. This is important when it comes to comparing models with data.

1.4 The restricted VAR representation of the linearized model

A typical feature of the solution of a linearized DSGE model is that it can be rearranged in a way to admit a VAR representation.

Denote with $\mathbf{x}$ the vector collecting the variables that you take to the data (observables, like output and consumption), with $\mathbf{y}$ those that appear in the model but you do not want in estimation or cannot perfectly observe (like productivity and capital), with $\mathbf{u}$ the vector of iid innovations in the model. In the typical economic model, the shocks are uncorrelated, so $E\mathbf{u}\mathbf{u}'$ is a diagonal matrix.

The decision rules for the model can then be written as:

$$
\begin{aligned}
\dim \mathbf{x} &= n_x \times 1 \\
\dim \mathbf{y} &= n_y \times 1 \\
\dim \mathbf{u} &= n_x \times 1 \\
\mathbf{x}_t &= \mathbf{P}_{n_x \times n_x} \mathbf{x}_{t-1} + \mathbf{Q}_{n_x \times n_y} \mathbf{y}_{t-1} + \mathbf{R}_{n_x \times n_x} \mathbf{u}_t \\
\mathbf{y}_t &= \mathbf{S}_{n_y \times n_x} \mathbf{x}_{t-1} + \mathbf{U}_{n_y \times n_y} \mathbf{y}_{t-1} + \mathbf{V}_{n_y \times n_x} \mathbf{u}_t
\end{aligned}
$$

Given that we do not observe $\mathbf{y}$, we want to solve this system of equation for $\mathbf{x}_t$ only. Denote L the lag operator, and $\mathbf{I}$ a conformable identity matrix. Then:

$$
\begin{aligned}
(\mathbf{I} - \mathbf{P}L)\, \mathbf{x}_t &= \mathbf{Q}L\mathbf{y}_t + \mathbf{R}\mathbf{u}_t \\
(\mathbf{I} - \mathbf{U}L)\, \mathbf{y}_t &= \mathbf{S}L\mathbf{x}_t + \mathbf{V}\mathbf{u}_t
\end{aligned}
$$

With simple algebra, we can easily express our whole system as a traditional VAR:

$$
\begin{aligned}
\left(\mathbf{I} - \mathbf{P}L - \mathbf{Q}L\,(\mathbf{I} - \mathbf{U}L)^{-1}\,\mathbf{S}L\right)\mathbf{x}_t &= \left(\mathbf{Q}L\,(\mathbf{I} - \mathbf{U}L)^{-1}\,\mathbf{V} + \mathbf{R}\right)\mathbf{u}_t \\
(\mathbf{I} - \Theta\,(L))\,\mathbf{x}_t &= (\Psi\,(L) + \mathbf{R})\,\mathbf{u}_t \qquad (7)
\end{aligned}
$$

The representation in (7) is a VARMA. In the most general case, both $(\mathbf{I} - \Theta\,(L))$ and $(\Psi\,(L) + \mathbf{R})$ are infinite order polynomials.

The impact effect on $\mathbf{x}_t$ of a shock to the vector $\mathbf{u}$ will be given by $\mathbf{R}$: on the LHS of the above equation, the only contemporaneous term on $\mathbf{x}$ is given by the $\mathbf{I}$ matrix, whereas on the I the only contemporaneous term on $\mathbf{u}$ is $\mathbf{R}$. That is

$$
\frac{d\mathbf{x}_t}{d\mathbf{u}_t} = \mathbf{R}
$$

See Chari, Kehoe and McGrattan (2004) critique: they have argued that omitting variables makes estimation of (7) with any number of lags impossible because the model is (7) is a VARMA with infinite lags in $\mathbf{x}_t$ and infinite lags in $\mathbf{u}_t$. In other words, the theoretical model implies in general an infinite order VAR unless $\mathbf{Q}$ is a matrix of zeros, which means that we take to the data all the state variables of the model.

1.5 Nonlinear models

Nonlinear approximations of structural models are represented using three equations, written with variables typically expressed in levels. The first equation is

$$
s_t = f\,(s_{t-1}, v_t)
$$

where s_t collects the state variables, v_t the collection of structural shocks incorporated in the model.

The second equation is the policy function, which incorporates the optimal specification of the control variables c_t included in the model as a function of the shocks

$$c_t = c\left(s_t\right)$$

The third equation maps the full collection of model variables into the observables

$$X_t = \widetilde{g}\left(s_t, c_t, v_t, u_t\right)$$

where the presence of u_t reflects the possibility that the observations of X_t are associated with measurement error.

1.6 Addenda to chapter 1

1.6.1 The Equity Premium and a Decentralization of the RBC model

We assume output is produced according to $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$. We introduce depreciation and preferences other than log utility. Hours are fixed and normalized to unity, so $L = 1$. Z_t follows a standard AR(1) process in logs.

We consider a decentralization in which households own firms, firms issue equity shares S, priced at q, and pay dividends d per each unit of shares owned. See for instance Section 4.3.3 (page 221) in Herr and Maussner, or Chapter 10 in Cooley's book "Frontiers of Business Cycle Research" for a similar scheme.

Household solve their maximization problem of

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \frac{C_s^{1-\sigma}}{1-\sigma} \right)$$

subject to:

$$q_t S_t + C_t = (d_t + q_t) S_{t-1} + w_t L_t \tag{1}$$

yielding

$$\frac{q_t}{C_t^{\sigma}} = \beta E_t \left(\frac{q_{t+1} + d_{t+1}}{C_{t+1}^{\sigma}} \right) \tag{2}$$

Firms maximize their discounted profits, by solving:[1]

$$\max E_t \left(C_t^{-\sigma} \left(Y_t + (1-\delta) K_{t-1} - K_t - w_t \right) + \beta C_{t+1}^{-\sigma} \left(Y_{t+1} + (1-\delta) K_t - K_{t+1} - w_{t+1} \right) + ... \right)$$

the objective of the firm is motivated by the fact that managers of the firm maximize the value of the firm for its owners, and this value is equal to the present discounted value of all current and expected future cash flows. Here,

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} \tag{3}$$

yielding

$$\begin{aligned} \frac{1}{C_t^{\sigma}} &= \beta \left(E_t \frac{1}{C_{t+1}^{\sigma}} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) \right) && (4) \\ (1-\alpha) Y_t &= w_t && (5) \end{aligned}$$

In equilibrium, demand and supply of funds must be equal, and total output must equal consumption plus investment

$$\begin{aligned} C_t + K_t &= Z_t K_{t-1}^{\alpha} + (1-\delta) K_{t-1} && (6) \\ S_t &= 1 && (7) \end{aligned}$$

[1] See for instance Jermann, JME, "Asset Pricing in Production Economies", 41, 1998, 257-75.

In (5), we simply normalize the supply of shares of the firm to unity. Given the stochastic process for Z_t, equations (1) to (7) can be solved for $q, C, d, K, w, Y,\ S$ (it is understood tha $L = 1$ at all times).

In steady state, given Y, we have

$$\begin{aligned} d + w &= C \\ w &= (1-\alpha) Y \\ S &= 1 \\ q &= \frac{\beta}{1-\beta} d \\ C + \delta K &= Y \\ K &= \frac{\alpha\beta}{1-\beta(1-\delta)} Y \end{aligned}$$

hence in steady state the value of the firm equals its capital stock. In fact

$$\begin{aligned} qS &= \frac{\beta}{1-\beta} d = \frac{\beta}{1-\beta} (Y - \delta K - (1-\alpha) Y) = \\ &= \frac{\beta}{1-\beta} (\alpha Y - \delta K) = \\ &= \frac{\beta}{1-\beta} \left(\alpha \frac{1-\beta(1-\delta)}{\alpha\beta} - \delta \right) K = K \end{aligned}$$

Introducing a riskless bond in the household problem gives, from $q_t S_t + C_t + p_t B_t = w_t + (d_t + q_t) S_{t-1} + B_{t-1}$, an Euler equation of the kind

$$p_t = \beta E_t \left(\frac{C_t^\sigma}{C_{t+1}^\sigma} \right).$$

Second-order approximations to a model of this kind can be used to study the equity premium puzzle, once the returns on the two assets (bonds and equity) are defined.[2] See e.g. Romer's ("Advanced Macroeconomics" textbook) and the dynare file `rbcequity1.mod`.[3] **Note in particular how the different returns on the various assets are defined.**

In comparing the numbers from this exercise with those from the literature, note that most papers on the equity premium puzzle deal with endowment economies, and try to make sense of the observed equity

[2] See for instance Romer's second edition, page 351. Manipulating the appropriate Euler equations, one can show that the theoretical equity premium is approximately equal to

$$E(req) - rbo = \sigma cov\left(req, g^C\right)$$

where req is the return on equity, rbo is the return on bonds, and g^C is the growth rate of consumption, and

$$cov\left(req, g^C\right) = corr\left(req, g^C\right) stdev\left(g^C\right) stdev(req)$$

[3] In the Dynare file, I use the equilibrium result that

$$C_t = d_t + w_t$$

to eliminate d_t from the model. This follows from

$$q_t S_t + C_t = (d_t + q_t) S_{t-1} + w_t L_t$$

when $L = 1$ and $S = 1$.

premium conditioning on observed values for the standard deviation of equity returns (around 15 percent on annual basis) and the standard deviation of consumption growth (around 3 percent on annual basis).

This example shows how solutions that are accurate up to the second-order allow studying problems that simple linearizations cannot handle. To learn more about this, a useful reading is Schmitt-Grohe and Uribe (2004), JEDC, 28, 755-775, “Solving dynamic general equilibrium models using a second-order approximation to the policy function”.

1.6.2 Adding Leverage

Firms maximize their discounted profits, by solving:

$$\max E_t \begin{pmatrix} C_t^{-\sigma}\left(Y_t + (1-\delta) K_{t-1} - K_t - w_t + p_t B_t - B_{t-1}\right) \\ +\beta C_{t+1}^{-\sigma}\left(Y_{t+1} + (1-\delta) K_t - K_{t+1} - w_{t+1} + p_{t+1} B_{t+1} - B_t\right) + ... \end{pmatrix}$$

and $B_t = \nu K_t$. We need to choose now a value for ν. This happens because in this class of models debt and capital are perfect substitutes as forms of financing for the firm in the nonstochastic steady state of the model, so we need to exogenously set a value of ν in order to go around this indeterminacy.

The difference with the previous case is that dividends are no longer equal to consumption less labor income. Before, we had that:

$$d_t = C_t - w_t$$

whereas now, from the household budget constraint, we have that:

$$d_t = C_t - w_t + \nu\left(p_t K_t - K_{t-1}\right)$$

in practice, dividends rise when the firm increases its debt.

See Jermann (JME, 1995) for more on this.

1.6.3 Adding Banks

Households receive wages and returns on deposits D_t, as well as profits from banks

$$C_t + D_t = w_t + R_t D_{t-1} + \pi_t$$

Firms produce goods, and pay wages and rent on K, thus maximizing

$$Y_t - w_t - R_t^K K_{t-1}$$

Banks accumulate capital that they own, finance themselves with deposits and lend capital to firms. They have the technology to transform deposits into new capital

$$\pi_t + K_t + R_t D_{t-1} = D_t + \left(R_t^K + 1 - \delta_K\right) K_{t-1}$$

The RHS is the source of funds for the banks. Returns on capital (i.e., loans made to firms) and new deposits. The LHS are the uses: banks pay back depositors, create new capital and pay back profits if any. Banks maximize the pdv of π_t, using the households stochastic discount factor.

For a richer model of banks, see Chari, Christiano and Kehoe (JMCB, 1995).

1.6.4 Random Walk in Productivity

If productivity follows a random walk, the variables of the model will inherit the random walk behavior from productivity itself. We express therefore the variables in terms of detrended variables, by scaling them by the level of productivity. We set up this problem from the planner's point of view. The problem is to solve:

$$\max E_t\left(\sum_{s=t}^{\infty}\beta^{s-t}\log C_s\right)$$

The constraints are:

$$\begin{aligned} K_t-(1-\delta)K_{t-1} &= A_t^{1-\alpha}K_{t-1}^{\alpha}-C_t & (1.1)\\ A_t &= A_{t-1}U_t & (1.2)\end{aligned}$$

where $\ln U_t\sim N\left(0,\sigma^2\right)$. The first order condition is

$$\frac{1}{C_t}=\beta E_t\left(\frac{1}{C_{t+1}}\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha}+1-\delta\right)\right). \quad (1.3)$$

The specification of productivity above assumes productivity follows a random walk in logs. As a consequence, consumption and capital will inherit the random walk properties of productivity, and will not be stationary. We need therefore to find a transformation of variables that ensures that they are stationary.

Define the transformed variables:[4]

$$\widetilde{C}_t=\frac{C_t}{A_t},\widetilde{K}_t=\frac{K_t}{A_t},g_t=\frac{A_t}{A_{t-1}}$$

The equilibrium conditions of the model and the optimality conditions are now:

$$\begin{aligned} \frac{K_t}{A_t}-(1-\delta)\frac{K_{t-1}}{A_{t-1}}\frac{A_{t-1}}{A_t} &= \frac{A_t^{1-\alpha}K_{t-1}^{\alpha}}{A_t}-\frac{C_t}{A_t}\\ \frac{1}{\frac{C_t}{A_t}} &= \beta E_t\left(\frac{1}{\frac{C_{t+1}}{A_{t+1}}}\frac{A_t}{A_{t+1}}\left(\alpha\left(\frac{A_{t+1}A_t}{A_tK_t}\right)^{1-\alpha}+1-\delta\right)\right)\\ g_t &= U_t\end{aligned}$$

or

$$\begin{aligned} \widetilde{K}_t-(1-\delta)\frac{\widetilde{K}_{t-1}}{g_t} &= \frac{\widetilde{K}_{t-1}^{\alpha}}{g_t^{\alpha}}-\widetilde{C}_t\\ \frac{1}{\widetilde{C}_t} &= \beta E_t\left(\frac{1}{g_{t+1}\widetilde{C}_{t+1}}\left(\alpha\left(\frac{g_{t+1}}{\widetilde{K}_t}\right)^{1-\alpha}+1-\delta\right)\right)\\ g_t &= U_t\end{aligned}$$

The solution can be written in terms of the following stationary variables $\widetilde{C}_t$ and $\widetilde{K}_t$. Of course, one can go back to the non-transformed variables after the solution is found.

[4] For a paper describing alternative specifications of the productivity process, see Ireland (2001), Journal of Economic Dynamics & Control, 25, 703-719, "Technology shocks and the business cycle: An empirical investigation".

1.6.5 Deterministic Trends

If productivity follows a deterministic trend, the variables of the model will inherit the deterministic trend from productivity itself. We express therefore the variables in terms of detrended variables. We set up this problem from the planner's point of view. The problem is to solve:

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right)$$

The constraints are:

$$\begin{aligned} K_t - (1-\delta) K_{t-1} &= A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t && (1.4) \\ \ln A_t &= t \ln G + \ln Z_t && (1.5) \end{aligned}$$

where $\ln Z_t$ is a serially correlated disturbance, i.e. $\ln Z_t = \rho \ln Z_{t-1} + u_t$, where u_t has zero mean and standard deviation equal to σ. Note that this is equivalent to writing $A_t = G^t Z_t$. Here, $G > 1$ is the gross growth rate of technology, and C and K will all grow at the same rate in the balanced growth path of the model.

The first order condition is:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha \left(\frac{A_{t+1}}{K_t} \right)^{1-\alpha} + 1 - \delta \right) \right). \qquad (1.6)$$

Define the transformed variables:

$$\widetilde{C}_t = \frac{C_t}{G^t}, \widetilde{K}_t = \frac{K_t}{G^t}, Z_t = \frac{A_t}{G^t}$$

Note the slight difference with the random walk case. Here, we scale A_t only by the trend component of G^t.

The equilibrium conditions of the model and the optimality conditions are now:

$$\begin{aligned} \frac{K_t}{G^t} - (1-\delta) \frac{K_{t-1}}{G^{t-1}} \frac{1}{G} &= \frac{A_t^{1-\alpha}}{G^{(1-\alpha)t}} \frac{K_{t-1}^{\alpha}}{G^{\alpha(t-1)}} \frac{1}{G^{\alpha}} - \frac{C_t}{G^t} \\ \frac{1}{\frac{C_t}{G^t}} &= \beta E_t \left(\frac{1}{\frac{C_{t+1}}{G^{t+1}}} \frac{1}{G} \left(\alpha \left(\frac{A_{t+1}}{G^{t+1}} \frac{G^t}{K_t} G \right)^{1-\alpha} + 1 - \delta \right) \right) \\ \frac{A_t}{G^t} &= Z_t \end{aligned}$$

or

$$\begin{aligned} \widetilde{K}_t - \frac{1-\delta}{G} \widetilde{K}_{t-1} &= \frac{\widetilde{K}_{t-1}^{\alpha}}{G^{\alpha}} - \widetilde{C}_t \\ \frac{1}{\widetilde{C}_t} &= \frac{\beta}{G} E_t \left(\frac{1}{\widetilde{C}_{t+1}} \left(\alpha \left(\frac{G}{\widetilde{K}_t} \right)^{1-\alpha} + 1 - \delta \right) \right) \\ \ln Z_t &= \rho \ln Z_{t-1} + u_t \end{aligned}$$

Of course, one can go back to the non-transformed variables after the solution is found.

1.6.6 King Plosser and Rebelo utility

King Plosser and Rebelo (1988) identified the class of utility function compatible with stationary labor. They are

$$u = \ln C_t + v\left(1 - L_t\right) \text{ if } \sigma = 1$$
$$\frac{1}{1-\sigma} C_t^{1-\sigma} v\left(1 - L_t\right) \text{ if } \sigma > 1$$

with $v'\left(1 - L_t\right) > 0$ and $v''\left(1 - L_t\right) < 0$.

King, Plosser, and Rebelo (1988) show that if consumption grows with an exponential trend due to labor augmenting technological progress, then the only class of utility functions that is compatible with the absence of a long-run trend in labor supply is given by the equation above.

1.6.7 Writing down Model Equations using Dynare: Basic RBC Model

- Install dynare onto your computer
- Create a mod file called `rbclog1.mod` (add its folder in matlab path, change matlab path)
- Set path, change current directory and, to launch the file, type in `dynare rbclog1.mod`
- The mod file is below:

1. `// example 1 from EC861`
2. `var k a c ;`
3. `varexo u;`
4. `parameters ALPHA, BETA, RHO, SIGMA;`
5. `ALPHA = 0.33;`

   `BETA = 0.99;`

   `RHO = 0.98;`

   `SIGMA = 0.01;`
6. `model;`
7. `exp(k) - exp(k(-1)) = (exp(a))^(1-ALPHA)*exp(k(-1))^ALPHA - exp(c) ;`
8. `1/exp(c) = BETA/exp(c(+1))*(ALPHA*(exp(a(+1))/exp(k))^(1-ALPHA)+1) ;`
9. `a=RHO*a(-1)+u ;`
10. `end;`
11. `initval;`

    `a = 0;`

    `k = 0;`

    `c = 0;`

    `end;`
12. `steady;`
13. `shocks;`
14. `var u; stderr SIGMA;`
15. `end;`
16. `stoch_simul(periods=1000,order=1,irf=40) c k a ;`

The solution

```
MODEL SUMMARY


 Number of variables:  3
 Number of stochastic shocks:  1
 Number of state variables:  2
 Number of jumpers:  2
 Number of static variables:  0
MATRIX OF COVARIANCE OF EXOGENOUS SHOCKS
Variables u
u 0.000100


POLICY AND TRANSITION FUNCTIONS
             c        k        a
Constant 1.716840 5.202546    0
k(-1)    0.659032 0.989921    0
a(-1)    0.172105 0.014841 0.980000
u        0.175618 0.015144 1.000000


MOMENTS OF SIMULATED VARIABLES
VARIABLE MEAN STD. DEV. VARIANCE SKEWNESS KURTOSIS
c 1.717993 0.033345 0.001112 -0.222992 0.036180
k 5.204072 0.042119 0.001774 -0.240439 0.053512
a 0.000827 0.047849 0.002289 -0.150949 0.104964
CORRELATION OF SIMULATED VARIABLES
VARIABLE c k a
c 1.0000 0.9815 0.7353
k 0.9815 1.0000 0.5919
a 0.7353 0.5919 1.0000
AUTOCORRELATION OF SIMULATED VARIABLES
VARIABLE 1 2 3 4 5
c 0.9991 0.9975 0.9958 0.9940 0.9920
k 1.0004 1.0001 0.9996 0.9990 0.9981
a 0.9773 0.9558 0.9349 0.9146 0.8949
```

1.6.8 The equity premium

```
//% RBC Model solved with second order, to calculate equity premium
  //% example 2 from EC861, FALL 2007
  //% September 4, 2007
  //%
  //% VARIABLES
  //% c :  log of consumption
  //% w :  log of wage
  //% k :  log of capital stock
  //% q :  log of equity price
  //% y :  log output
  //% p :  log of bond price
  //% a :  log of technology process
  //% gc :  growth rate of consumption
  //% req :  return on equity
  //% rbo :  return on bond
  //% ep :  equity premium
  //% Declare endogenous and exogenous variables
  var c w k q y p a req rbo ep gc ;
  varexo u;
  //% Declare model parameters (convention:  use uppercase for parameters,
  //% lowercase for variables)
  parameters ALPHA, BETA, RHO, SIGMA, DELTA, STDERRU ;
  ALPHA = 0.33 ;
  BETA = 0.96 ;
  RHO = 0.80 ;
  STDERRU = 0.025 ;
  SIGMA = 5 ;
  DELTA = 0.10 ;
  //% Write down model equations here.  Remember:  as many equations as variables,
  //% and put ";" at the end of each equation
  model;
  exp(q)/(exp(c)^SIGMA) = BETA/(exp(c(+1))^SIGMA)*(exp(q(+1))+exp(c(+1))-exp(w(+1))) ;
  1/(exp(c)^SIGMA) = BETA/(exp(c(+1))^SIGMA)*(ALPHA*(exp(y(+1))/exp(k))+1-DELTA) ;
  exp(w) = (1-ALPHA)*exp(y) ;
  exp(c) + exp(k) - (1-DELTA)*exp(k(-1)) = exp(y) ;
```

```
exp(y) = exp(a)*exp(k(-1))^ALPHA ;
exp(p) = BETA*(exp(c)^SIGMA)/(exp(c(+1))^SIGMA) ;
req = (exp(q)+exp(c)-exp(w))/exp(q(-1)) - 1 ;
rbo = 1/exp(p) - 1 ;
ep = req - rbo ;
gc = c - c(-1) ;
a=RHO*a(-1)+(1-RHO^2)^0.5*u ;
end;
//% Make guess for initial values of variables
initval;
a = 0;
c = 0;
k = 0;
p = 0;
q = 0;
w = 0;
y = 0;
ep = 0;
gc = 0;
req = 0;
rbo = 0;
end;
steady;
shocks;
var u; stderr STDERRU ;
end;
stoch_simul(order=2,irf=10) gc ep req rbo ;
//% disp(' ') ;
//% disp('theoretical equity premium (cov b/w req and gc, times SIGMA)') ;
//% disp(SIGMA*oo_.var(1,3)) ;
```

1.6.9 Some Notes about Dynare

1. The way Dynare shows the coefficients of the decision rules (for the first order approximation) is as follows

$$y_t = \overline{y} + A\left(y_{t-1} - \overline{y}\right) + Bu_t$$

   See the dynare manual to check where the coefficients are stored

   `http://www.cepremap.cnrs.fr/dynare/download/manual/manual.pdf`

2. Dynare is not immune from bugs. For instance version 3 will not compute theoretical moments unless there is a static variable in the model. Or, it will give an error message if you call one of the variables nh, for instance (there might be some other name conflicts as well, so beware!)

3. The tricky part of solving a DSGE model is how to find the matrices A and B. I expect that you know how to (1) compute impulse responses; (2) calculate moments; (3) do variance decompositions; (4) do transitions from one steady state to another WITHOUT using Dynare built-in commands.

4. All variables that are known at the beginning of the period must be dated $t-1$. This convention differs from most textbooks and published papers, which typically use a different timing convention.

5. Wouter Den Haan has written a very nice set of lecture notes on Matlab that I urge you to read and study, especially if you plan to be a macroeconomist.

   `http://www1.feb.uva.nl/mint/wdenhaan/Juillard.htm`

1.7 Homework 1

The answer must be written in a clear and polished way, and so to ensure replication of the results. Include in the answer the printout of the key Matlab and Rats files. For each data series that you use, provide the exact description and series code and time period. If you use the menu-driven VAR.src routine provided in Rats, explain step by step which menu options you follow to obtain the answer. Part of writing a good paper is ensuring replication of the results.

1. Consider a standard RBC model like the one we studied in class. Assume:

   (a) The household utility function is:

   $$\max E_t \left(\sum_{s=t}^{\infty} b_t \beta^{s-t} \left(\log C_s + \tau \log \left(1 - N_t\right)\right)\right)$$

   (b) Capital depreciation at rate δ

   (c) Technology follows a random walk with drift, so that:

   $$Y_t = \left(A_t N_t\right)^{1-\alpha} K_{t-1}^{\alpha}$$

   $$\log A_t = \log A_{t-1} + \mu + \varepsilon_{At}$$

   (d) There is a discount rate shock, so that the effective discount factor is $b_t \beta$, where b_t:

   $$\log b_t = \rho \log b_{t-1} + \left(1 - \rho^2\right)^{1/2} \varepsilon_{bt}$$

   Write the model in stationary form.

2. Set $\alpha = 0.33$, $\delta = 0.025$, $\rho = 0.9$, $\sigma_A = 0.01$, $\sigma_b = 0.02$, $\tau = 2$, $\beta = 0.99$, $\mu = 0.005$. Obtain the decision rules for this model. Plot the impulse response of output growth and log hours to both discount rate shocks and technology shocks.

   (a) Discuss short and long-run effects of these shocks on the model variable. What happens to the LEVEL of output and hours after a technology shock? After a discount factor shock?

   (b) Are log output and/or log hours a random walk?

3. Drawing the shocks from their distribution, simulate the model variables for 200 periods. Plot log output and log consumption. Is consumption smoother than output?

4. From the simulated variables, construct the variables first difference of logged labor productivity (output per hours) and the logged level of labor supply (hours)

   $$n_t$$

   $$\Delta \left(y_t - n_t\right)$$

Run a VAR on the simulated model variables using the Blanchard-Quah methodology and two lags. Identify the technology shock as the shock that leads to a permanent increase in labor productivity in the long-run. What happens to labor supply in response to a technology shock? Do the responses to the VAR match up with the responses obtained from the model?

5. Run a VAR on real world series for $\Delta(y_t - n_t)$ and n_t (suggestion: $y - n$ is log of output per hours; n is log hours per capita; see e.g. what the Gali March AER 1999 paper does in constructing the variables). Focus on the period 1960-1999, or more or less. Describe carefully how you construct the series. Use the VAR.SRC routine provided in RATS. You can download most macro series from the St.Louis Fed FRED website. Alternatively, use the sample programs provided on the course webpage at:

   `http://www2.bc.edu/~iacoviel/teach/0910/EC861_files/VARS.zip`

   (a) What happens to labor supply after a technology shock? (This is the LSVAR - labor-supply stationary VAR)

   (b) What happens after a technology shock if hours are in first differences rather than in levels? (This is the DSVAR).

1.7.1 Solution to homework 1

See files `homemork1.mod` (Dynare), `homemork1_bivariatevar.m` and `homework1.prg` (Rats) on the course webpage (they are hidden).

1. The original law of motion for productivity is $\log A_t = \log A_{t-1} + \mu + \varepsilon_{At}$. Let $g_t = \frac{A_t}{A_{t-1}}$. Then, written in terms of the transformed variables, the model is

$$\begin{aligned}
\widetilde{K}_t - (1-\delta)\frac{\widetilde{K}_{t-1}}{g_t} &= \widetilde{Y}_t - \widetilde{C}_t \\
b_t \frac{1}{\widetilde{C}_t} &= \beta b_{t+1} E_t\left(\frac{1}{g_{t+1}\widetilde{C}_{t+1}}\left(\alpha\frac{\widetilde{Y}_{t+1}g_{t+1}}{\widetilde{K}_t} + 1 - \delta\right)\right) \\
(1-\alpha)\frac{\widetilde{Y}_t}{N_t\widetilde{C}_t} &= \frac{\tau}{1-N_t} \\
\widetilde{Y}_t &= (N_t)^{1-\alpha}\frac{\widetilde{K}^{\alpha}_{t-1}}{g^{\alpha}_t} \\
\log g_t &= \mu + \varepsilon_{At} \\
\log b_t &= \rho \log b_{t-1} + \varepsilon_{bt}
\end{aligned}$$

where variables with a tilde are scaled by the productivity term A. So, for instance to make output and capital and consumption stationary we divide these variables by the level of productivity, so that

$$\begin{aligned}
\widetilde{K}_t &\equiv \frac{K_t}{A_t} \\
\widetilde{C}_t &\equiv \frac{C_t}{A_t} \\
\widetilde{Y}_t &\equiv \frac{Y_t}{A_t} = \frac{(A_t N_t)^{1-\alpha} K^{\alpha}_{t-1}}{A_t} = N_t^{1-\alpha}\frac{K^{\alpha}_{t-1}}{A^{\alpha}_t} = N_t^{1-\alpha}\frac{K^{\alpha}_{t-1}}{A^{\alpha}_{t-1}}\frac{A^{\alpha}_{t-1}}{A^{\alpha}_t} = N_t^{1-\alpha}\widetilde{K}^{\alpha}_{t-1}\frac{1}{g^{\alpha}_t}
\end{aligned}$$

where

$$g_t \equiv \frac{A_t}{A_{t-1}}$$

The variables are $\left(\widetilde{K}_t, \widetilde{C}_t, \widetilde{Y}_t, N_t, g_t, b_t\right)$. In Dynare, we enter the log of the each variable.

2. Using the notation in the text, the growth rate of productivity is:

$$gyl = \frac{Y_t/N_t}{Y_{t-1}/N_{t-1}} = \frac{Y_t N_{t-1} A_{t-1} A_t}{A_t N_t Y_{t-1} A_{t-1}} = \frac{g_t \widetilde{Y}_t N_{t-1}}{\widetilde{Y}_{t-1} N_t}$$

whereas hours are stationary.

   (a) Following a technology shock, productivity growth rises for one quarter, and hours increase persistently. In the long run, the level of technology (the cumulative sum of gyl) is permanently higher, whereas hours return to the baseline.

   Following a discount factor shock, consumption rises, hours fall, productivity growth gyl temporarily increases, but then falls below the baseline after the shock, and in the long-run its level is unchanged.

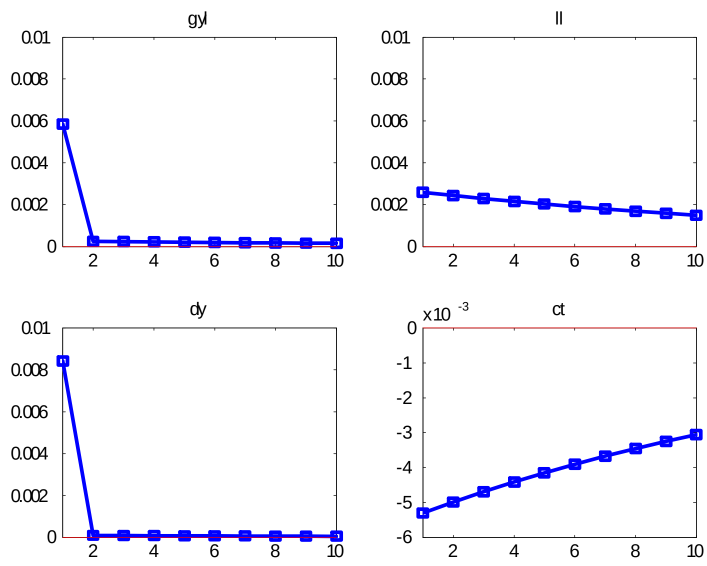


Figure 1.1: Model impulse response to a technology shock

(b) Log output is a random walk, log hours are stationary.

3. Using the model solution, we can obtain simulated series for the variables. For instance, consumption growth is

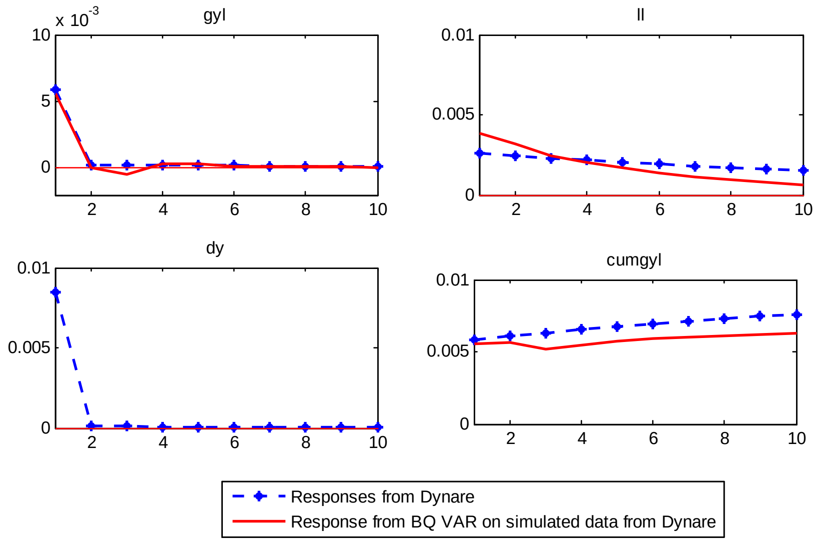
$$g_{C,t} = \frac{C_t}{C_{t-1}} = \frac{\widetilde{C}_t}{\widetilde{C}_{t-1}} \widetilde{g}_t.$$

Similarly, we can plot output growth. I obtain that qoq output growth is more volatile (standard deviation of 1%) than qoq consumption growth (standard deviation of 0.70%). Notice that if one where to assume that productivity growth is serially correlated one could reverse this result: this findings are related to the so-called excess smoothness puzzle. In practice, the reason why consumption is smoother than output is that growth shocks are iid. If these growth shocks were serially correlated, consumption would jump a lot in response to productivity shocks, and could be more volatile than output. One can obtain this result by specifying an equation for productivity growth as

$$\log g_t = (1 - \rho)\,\mu + \rho \log g_{t-1} + \varepsilon_{At}$$

4. Now to the VAR. Note that in the model, if one focuses only on gyl and l as observables, the technology shock will have a permanent effect on the level of both variables in the long-run (that is, the cumulative impulse responses of gyl and l to a tech shock will be nonzero), where the discount rate shock will have a non-zero impact on the cumulative response of l, and a zero impact on the cumulative response of gyl.

I run a VAR on the simulated model variables, and I identify the technology shock as the only shock that leads to a permanent increase in labor producitivity. The two sets of impulse responses are roughly in the same ballpark. The VAR is estimated with two lags on the variables gyl and ll.



5. Finally, I run a VAR on real world series for $\Delta(y_t - n_t)$ and n_t. I use the following transformations in RATS

   SET LL = LOG(HOANBS) - LOG(CNP16OV)

   SET YL = LOG(GDPC1) - LOG(HOANBS)

   DIFF YL / GYL

   DIFF LL / DLL

   (a) In the LSVAR, I run

       SOU(NOECHO) C:\E\BC_TEACH\EC861_F09\VAR\VAR.SRC

       @VAR 1960:1 1997:4

       # GYL LL

       I find that hours fall in response to a technology shock, although not significantly.

   (b) In the DSVAR, the drop in hours is even stronger.

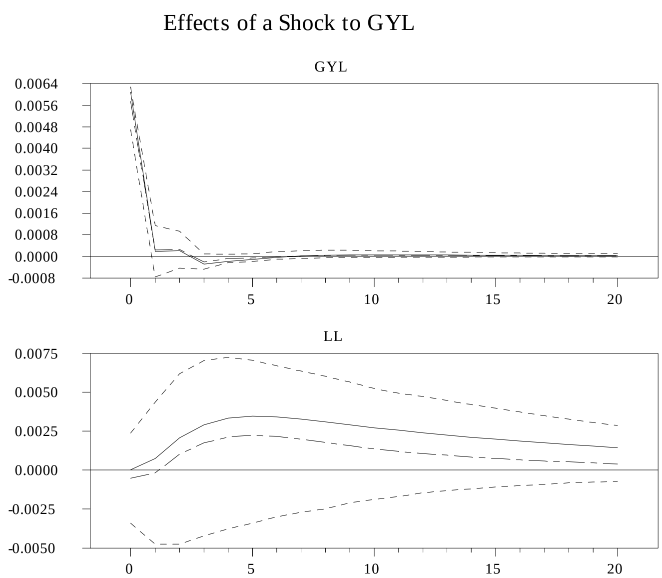


Figure 1.2: BQ VAR on true US data.

Chapter 2

Maximum Likelihood Estimation of DSGE Models

In this chapter, I closely follow Chapter 4 of Dejong and Dave, especially Section 3.

Another good reading is Fernandez-Villaverde (2009), at http://papers.nber.org/papers/w14677

2.1 Introduction

The general problem is as follows. Consider the solution of a DSGE model, again

$$\begin{aligned} x_t &= F(\mu)\, x_{t-1} + G(\mu)\, v_t \\ z_t &= H' x_t \end{aligned}$$

where $E(Gvv'G') = GE(vv')\,G' = Q$. Note that both F and G are non-linear functions of the vector of model parameters μ. Also, $\dim(x) = m \times 1$, $\dim(z) = n \times 1$, $\dim(H) = m \times n$. Let $P_{t|t-1} = E\left(\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)'\right)$

We are interested in obtaining estimates of the unknown parameters in the vector μ based on a sample observations about $z^T \equiv \{z_t\}_{t=1}^T$.

We define as maximum likelihood estimates of the model the values of μ that maximize the likelihood associated with a particular sample of realizations of z over time. To maximize the likelihood, of course, we need to have an algorithm to calculate it.

Below, we will consider two cases.

1. The simple case is when all the variables in x are "observed". In that case (provided that there are more shocks than observables), calculation of the likelihood proceeds as in standard econometric textbooks.

2. The more complicated case is when $n < m$. In this case, one need to find a way to infer the value of x from observations on z. Inferring the values of x is essential to calculate the likelihood of z. To do

so, we need a particular algorithm (the Kalman filter) used to produce assessment of the conditional probability associated with the time-t observation z_t, given the history of past realizations $z^{t-1} \equiv \{z_j\}_{j=1}^{t-1}$. $L\left(z_t|z^{t-1}\right)$.

Regardless of whether case 1 or 2 applies, we write the likelihood associated with a particular realization of z at time t as $L\left(z_t|z^{t-1}\right)$.

The sequence of conditional likelihoods $\left\{L\left(z_t|z^{t-1}\right)\right\}_{t=1}^{T}$ are independent over time, thus

$$L\left(z^T\right) = \Pi_{t=1}^{T} L\left(z_t|z^{t-1}\right)$$

$L\left(z^T\right)$ is the likelihood of our model. But, how do we compute $L\left(z_t|z^{t-1}\right)$?

2.2 Writing Down the Likelihood

2.2.1 Case I: All Variables are Observables

Consider the case in which $z_t = x_t$. Conditional on $\{x_j\}_{j=1}^{t-1}$, we note that the optimal forecast of x_t is given by

$$\widehat{x}_t = F x_{t-1}$$

so that

$$\widehat{e}_t = x_t - F x_{t-1}$$

and the conditional likelihood associated with a particular realization of x_t can be assessed as the likelihood assigned to $\widehat{e}_t$ by its assumed probability distribution

$$L\left(x_t|x^{t-1}\right) = p_e\left(\widehat{e}_t\right)$$

The likelihood evaluation begins by inserting x_1 into its unconditional distribution, which is $N(0, Q)$, hence (using the formula for a multivariate normal distribution with mean 0, realization given by x_1 and variance covariance matrix Q)

$$L\left(x_1|\mu\right) = (2\pi)^{-m/2}\left|Q^{-1}\right|^{1/2}\exp\left[-\frac{1}{2}\left(x_1' Q^{-1} x_1\right)\right]$$

then, for $t = 2, ..., T$, we have

$$L\left(x_t|\mu\right) = (2\pi)^{-m/2}\left|Q^{-1}\right|^{1/2}\exp\left[-\frac{1}{2}\left(x_t - F x_{t-1}\right)' Q^{-1}\left(x_t - F x_{t-1}\right)\right]$$

Finally the sample likelihood is the product of the individual likelihoods.

$$L\left(x|\mu\right) = \Pi_{t=1}^{T} L\left(x_t|\mu\right)$$

2.2.2 Case II: Some Unobservables, and the use of the Kalman Filter

When some of the state variables cannot be observed, the procedure to compute the likelihood is similar, but we need some method to estimate the evolution of the states over time based on information conveyed by the observables. The Kalman filter is an algorithm that enables the computation of the likelihood of a (linearized DSGE) model recursively. Hidden states and observables are described by a state space system that is perturbed at each point by Gaussian shocks with zero mean and known covariances. Here I illustrate a recursive version of the Kalman filtering problem that allows computing recursively the likelihood of a model.

Because the computation is recursive, we start at time 0, where we can calculate the following:

- $z_{1|0}$: Conditional expectation of z_1 (the vector of unobservables that enter the computation of the likelihood) given observations on z_0
- z_1: Actual observations on z_1
- $x_{1|0}$: Conditional expectation of x_1 given observations on z_0

At time zero, we want to derive the best estimate of x_1. KEY QUESTION IS HOW TO OBTAIN x_1 GIVEN $x_{1|0}$ and z_1.

At time 0 :

$$x_{1|0} = 0$$

$$P_{1|0} = FP_{1|0}F' + Q$$

where

$$P_{1|0} = E\left[\left(x_1 - x_{1|0}\right)\left(x_1 - x_{1|0}\right)'\right]$$

This way, one can construct the associated values for the observables z, given by:

$$\begin{aligned} z_{1|0} &= H'x_{1|0} = 0 \\ \Omega_{1|0} &= E\left[\left(z_1 - z_{1|0}\right)\left(z_1 - z_{1|0}\right)'\right] = H'P_{1|0}H \rightarrow \text{innovation covariance} \end{aligned}$$

The two object above are used to compute the corresponding likelihood function of z_1, which is a normal variable with mean $z_{1|0}$ and variance $\Omega_{1|0}$, that is $z_1\tilde{\ } N\left(z_{1|0}, \Omega_{1|0}\right)$

$$L\left(z_1|\mu\right) = (2\pi)^{-m/2}\left|\Omega_{1|0}^{-1}\right|^{1/2}\exp\left[-\frac{1}{2}\left(z_1'\Omega_{1|0}^{-1}z_1\right)\right].$$

Next, the values of $x_{1|0}$ and $P_{1|0}$ are updated to construct new updates of $x_{1|1} \equiv x_1$ and $P_{1|1} \equiv P_1$.

$$\begin{aligned} x_{1|1} &= \underset{\text{old value}}{x_{1|0}} + \underset{\text{Kalman gain}}{P_{1|0}H\Omega_{1|0}^{-1}}\underset{\text{prediction error}}{\left(z_1 - z_{1|0}\right)} \rightarrow \text{updated state estimate} \\ P_{1|1} &= P_{1|0} - P_{1|0}H\Omega_{1|0}^{-1}H'P_{1|0} \rightarrow \text{updated covariance estimate} \end{aligned}$$

Remark 1 *The term $K_1 \equiv P_{1|0} H \Omega_{1|0}^{-1}$ denotes the Kalman gain matrix. It is a minimum-mean square estimator that yields the best prediction of x_1 given estimates of $x_{1|0}$, z_1 and $z_{1|0}$. It can be derived as follows; consider the covariance matrix of x_1, $P_{1|1}$*

$$\begin{aligned}
P_{1|1} &= E\left[\left(\underset{actual}{x_1} - \underset{mean}{x_{1|1}}\right)\left(x_1 - x_{1|1}\right)'\right] \\
\textit{use } x_{1|1} &= x_{1|0} + K_1\left(z_1 - z_{1|0}\right) \textit{ for some } K_1 \textit{ to be determined} \\
P_{1|1} &= cov\left(x_1 - x_{1|0} - K_1\left(z_1 - z_{1|0}\right)\right) \\
P_{1|1} &= cov\left(x_1 - x_{1|0} - K_1 H'\left(x_1 - x_{1|0}\right)\right) \\
P_{1|1} &= E\left[\left(I - K_1 H'\right) \underset{variance\ of\ \ x_1 - x_{1|0}}{P_{1|0}} \left(I - K_1 H'\right)'\right] \\
& \textit{We minimize the expected value of the square of the magnitude of this vector.} \\
& \min_{K_1} trace\left(P_{1|1}\right) \\
\textit{gives } K_1 &= P_{1|0} H \Omega_{1|0}^{-1}
\end{aligned}$$

Next, for every other period $t = 2, ..., T$, we have:

$$\begin{aligned}
x_{t|t-1} &= F x_{t-1} \\
P_{t|t-1} &= F P_{t|t-1} F' + Q \\
z_{t|t-1} &= H' x_{t|t-1} \\
\Omega_{t|t-1} &= H' P_{t|t-1} H \\
L\left(z_t|\mu\right) &= (2\pi)^{-m/2}\left|\Omega_{t|t-1}^{-1}\right|^{1/2} \exp\left[-\frac{1}{2}\left(\left(z_t - z_{t|t-1}\right)' \Omega_{t|t-1}^{-1}\left(z_t - z_{t|t-1}\right)\right)\right] \\
x_{t|t} &= x_{t|t-1} + P_{t|t-1} H \Omega_{t|t-1}^{-1}\left(z_t - z_{t|t-1}\right) \\
P_{t|t} &= P_{t|t-1} - P_{t|t-1} H \Omega_{t|t-1}^{-1} H' P_{t|t-1} \\
L\left(z|\mu\right) &= \Pi_{t=1}^{T} L\left(z_t|\mu\right)
\end{aligned}$$

2.3 Some toy examples on Identification

2.3.1 Example 0: the simplest example ever

We have five independent observations from the normal distribution for the variable y_t, mean zero and unit standard deviation. The oservations turn out to be

$$\{-0.5925, 0.3298, -0.9984, 1.8028, -0.5416\}$$

Using the formula and notation above

$$\begin{aligned}
m &= 1 \\
Q &= 1 \text{ (variable has unit stdev)} \\
T &= 5
\end{aligned}$$

$$\begin{aligned} L &= \Pi_{t=1}^{T}\left((2\pi)^{-m/2}\left|Q^{-1}\right|^{1/2}\exp\left[-\frac{1}{2}\left(x_t - Fx_{t-1}\right)' Q^{-1}\left(x_t - Fx_{t-1}\right)\right]\right) \\ &= (2\pi)^{-5/2}\exp\left(-\frac{\left(0.5925^2 + 0.3298^2 + 0.9984^2 + 1.8028^2 + 0.5416^2\right)}{2}\right) = 8.2948\times 10^{-4} \end{aligned}$$

$$\ln L = \ln 8.2948\times 10^{-4} = -7.0947$$

check file **basic1.mod**

2.3.2 Example 1: the RBC Model

The solution to the RBC model can be written as (in log deviation from the steady state, say)

$$\begin{aligned} k_t &= a_{kk}k_{t-1} + a_{kz}z_t \\ z_t &= \rho z_{t-1} + e_t \end{aligned}$$

Suppose we have observations on k_t over time. Then we can calculate, given k_t, the sequence of values for e_t which are need to compute the likelihood. We can make the dependence of e_t on μ explicit, and then the likelihood will be

$$L\left(e^T\right) = \prod_{t=1}^{T}\frac{1}{\sigma_e\sqrt{(2\pi)}}\exp\left[-\frac{1}{2}\frac{e_t(\mu)^2}{\sigma_e^2}\right]$$

2.3.3 Example 2: toy Phillips curve

See `toy_pc1.mod`

Consider the model

$$\begin{aligned} \pi_t &= \beta E_t\pi_{t+1} + ky_t \\ y_t &= \rho y_{t-1} + e_t \end{aligned}$$

where e_t is iid with zero mean and variance σ^2.

Assume our only obervable is π_t. Written in state space form, the solution to our model takes the form

$$\begin{aligned} x_t &= Fx_{t-1} + Gv_t \\ z_t &= H'x_t \end{aligned}$$

where:

$$
\begin{aligned}
x_t &= \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \\
v_t &= [e_t] \\
F &= \begin{bmatrix} 0 & \frac{\kappa\rho}{1-\beta\rho} \\ 0 & \rho \end{bmatrix} \\
G &= \begin{bmatrix} \frac{\kappa}{1-\beta\rho} \\ 1 \end{bmatrix} \\
z_t &= [\pi_t] \\
H' &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}'
\end{aligned}
$$

Consider the rational expectations solution for π_t

$$\pi_t = \rho\pi_{t-1} + \frac{\kappa}{1-\beta\rho}\varepsilon_t$$

MLE will recover ρ, and only one parameter among β, σ_ε and κ. In this simple model, maximum likelihood will fail to recover separately estimates of κ,β, σ_ε.

2.3.4 Example 3: A simple Dynamic New-Keynesian Model

Consider the following log-linear model for output y, inflation π and the nominal interest rate R:

$$
\begin{aligned}
y_t &= E_t y_{t+1} - R_t + E_t\pi_{t+1} + g_t \\
\pi_t &= k y_t + \beta E_t\pi_{t+1} + u_t \\
R_t &= \phi\pi_t
\end{aligned}
$$

Assume g and u are iid. The solution will take the form:

$$
\begin{aligned}
y_t &= \frac{1}{1+\phi k}g_t - \frac{\phi}{1+\phi k}u_t \\
\pi_t &= \frac{k}{1+\phi k}g_t + \frac{1}{1+\phi k}u_t
\end{aligned}
$$

Hence estimation cannot recover β, and at most three parameters among σ_g, σ_u, ϕ and k. To think about why, the series are iid, and all that enters the likelihood function is their variance and their covariance.

$$
\begin{aligned}
var\left(y\right) &= \frac{1}{\left(1+\phi k\right)^2}\sigma_g^2 + \frac{\phi^2}{\left(1+\phi k\right)^2}\sigma_u^2 \\
var\left(\pi\right) &= \frac{k^2}{\left(1+\phi k\right)^2}\sigma_g^2 + \frac{1}{\left(1+\phi k\right)^2}\sigma_u^2 \\
cov\left(y,\pi\right) &= \frac{k}{\left(1+\phi k\right)^2}\sigma_g^2 - \frac{\phi}{\left(1+\phi k\right)^2}\sigma_u^2
\end{aligned}
$$

There are three equations in four unknowns.

Implementation in Dynare of Example 1

```
var x y ;
varexo e_y ;
parameters beta kappa rho ;
beta = 0.9;
kappa = 0.05;
rho = 0.8;
model;
x = beta*x(+1)+kappa*y ;
y = rho*y(-1)+e_y;
end;
//% Let exy=kappa/(1-beta*rho)
//% The solution for this model is
//% x = rho*exy*y(-1) + exy*e
//% y = rho* y(-1) + e
//% written in terms of x only
//% x = rho*x(-1) + exy*e
//% Hence one can identify only rho; and one parameter between (1) kappa, (2) beta or (3)
stderr(e_y),
//% conditional on making the right guess about the other two parameters
//% maximum two parameters
shocks;
var e_y; stderr 0.01;
end;
//% Simulates model and generates artificial time series
//% The option simul_seed=1 resets Matlab seed
stoch_simul(order=1,periods=500,irf=0,simul_seed=1,nomoments);
//% Saving x in matlab mat file data1, used in estimation later
save data1 x ;
//% BAYESIAN=0 :  classical estimation
//% BAYESIAN=1 :  bayesian estimation
BAYESIAN=0;
if BAYESIAN==0;
 estimated_params;
 rho,0.9,-1,1;
 stderr e_y,0.01,0.0001,0.1;
```

```
 end;
 varobs x ;
 estimation(datafile=data1);
end;
if BAYESIAN==1;
 estimated_params;
 beta,0.95,0,1,beta_pdf,0.5,0.15;
 rho,0.95,0,1,beta_pdf,0.9,0.05;
 stderr e_y,0.01,0,0.2,gamma_pdf,0.02,0.015;
 end;
 varobs x;
estimation(datafile=data1,mode_compute=4,mh_nblocks=1,mode_check,mh_jscale=1,mh_replic=10000);
end;
```

2.4 Bayesian Estimation

2.4.1 Overview

In the classical estimation, parameters are treated as fixed but unknown, and the likelihood function is interpreted as a sampling distribution from the data. The realizations of z are interpreted as one of the many possible realizations from $L(z|\mu)$ that could have been obtained, and inferences regarding μ are statements regarding probabilities associated with particular realizations of z given μ.

The basic idea of Bayesian estimation is to take observations on z as given, and to make inferences about the distribution of μ conditional on z. In this setup, the probabilistic interpretation of μ gives rise to a potential avenue for incorporating judgements about μ through a prior distribution $\pi(\mu)$. From the definition of joint probability, we have that:

$$p(\mu, z) = L(z|\mu)\pi(\mu)$$

reversing the role of μ and z gives

$$p(z, \mu) = P(\mu|z)\, p(z).$$

Solving for $P(\mu|z)$ gives

$$P(\mu|z) = \frac{L(z|\mu)\pi(\mu)}{p(z)} \propto L(z|\mu)\pi(\mu)$$

In this formulation, $P(\mu|z)$ is the posterior distribution of μ, which assign values to alternative values of μ; this definition is the center of Bayesian analysis (it is the Bayes rule).

Remark 2 *If you look at the file DsgeLikelihood.m in Dynare, this is the file that calculates the posterior of the model for given data and values of μ. Dynare first calculates $L(z|\mu)$ using for instance DiffuseLikelihood1.m (or similar files, depending on whether you have trends, unit roots or other special frills in the model); then Dynare evaluates the prior density of μ, using the function priordens.m. Finally, $P(z|\mu)$ is calculated as the sum of the two.*

2.4.2 Additional Issues

A key computational problem involves how to compute the distribution of μ, $P(\mu|z)$: standard Monte Carlo integration techniques cannot be used, because one cannot draw random numbers directly from $P(\mu|z)$. Typically, we use Markov Chain Monte Carlo (MCMC) techniques, in particular the Metropolis-Hastings algorithm which is a particular version of the MCMC algorithm. The idea of the algorithm is to explore the distribution and to weigh to outcomes appropriately.

A summary of the Metropolis algorithm works as follows.

GOAL: Construct Markov chain μ(t) with stationary distribution P(μ). The distribution P is unknown, but a function dominating the density can be calculated at μ.

Steps

1. Use a numerical optimization routine to maximize $\ln P(\mu|z) = \ln L(z|\mu) + \ln \pi(\mu)$. Denote the posterior mode by $\widetilde{\mu}$. Let Σ be the inverse of the Hessian computed at the posterior mode $\widetilde{\mu}$.

2. Draw μ_0 from $N\left(\mu_0, c_0^2\Sigma\right)$, or let $\mu_0 = \widetilde{\mu}$

3. For $s = 1, ..., nsim$, draw μ from the proposal distribution $N\left(\mu_{s-1}, c^2\Sigma\right)$. The jump from μ_{s-1} is accepted ($\mu_s = \mu$) with probability equal to $\min\left(1, \frac{P(\mu|z)}{P\left(\mu_{s-1}|z\right)}\right)$ and rejected ($\mu_s = \mu_{s-1}$) otherwise.

4. Approximate the posterior expected value of a function $h(\mu)$ by $\frac{1}{sim}\sum_{i=1}^{nsim} h(\mu_s)$

See Chapter 9 in Dejong and Dave for more details.

Chapter 3

Vector Autoregressions

3.1 The data and how to analyze them

You can download all the NIPA (National Income and Product Accounts data) from:

`http://www.bea.doc.gov/bea/dn/nipaweb/DownSS2.asp`

For most macroeconomic time series (as well as NIPA data), the best resource is:

the FRED database `http://research.stlouisfed.org/fred2/`

For a good econometric package which deals with time-series data, you need RATS (www.estima.com).

3.2 Money, prices and output in the short run

Tools employed to estimate the impact of monetary policy have changed over time.

- Friedman and Schwartz (1963): a - rather subjective, according to some - look at 100 years of US monetary history shows that income expansions/contracts were always preceded by expansions/contractions in the money supply. (e.g. Great Depression resulted from a fall in the supply of money, the result of a contractionary Federal Reserve monetary policy plus bank failures and a subsequent failure of the Federal Reserve to react appropriately).

- Earliest "econometric" attempt by Friedman and Meiselman (1963): test whether monetary or fiscal policy was more important in affecting nominal income:

$$y_t^n = y_t + p_t = y_0^n + \sum_{i=0} a_i A_{t-i} + \sum_{i=0} b_i m_{t-i} + \sum_{i=0} h_i z_{t-i} + u_t$$

  where y^n is log of nominal income, A is autonomous expenditure, m is a monetary aggregate and z is a vector of control variables. They found b coefficients were significantly different from zero, a coefficients were not. This is example of St.Louis equation. This test is however misspecified if m_t is endogenous, since in this case $E\left(m_t, u_t \neq 0\right)$ and OLS estimates are inconsistent.

- Similar regression have been run with real income on I and using Granger causality tests:

$$y_t = \sum_{i=1} b_i m_{t-i} + \sum_{i=1} b_i y_{t-i} + \sum_{i=1} c_i z_{t-i} + e_t$$

Stock and Watson (1989) do find evidence that money Granger causes output. However results are debated (see Walsh page 19). In particular, adding an interest rate to this equation reduces the apparent predicting power of money (Bernanke and Blinder, 1992).

3.3 The monetary shocks

Before we turn to shocks, let us summarize with Uhlig (2001): "Eyeball econometrics suggests a strong cause-and-effect from Federal Funds Rate movements to real GDP: whenever interest rates rise, growth rates fall shortly afterwards. This is particularly visible for 1968 through 1983. It seems easy to conclude from this picture, that the question about the effects of monetary policy on output is answered clearly: contractionary monetary policy leads to contractions in real GDP".

How does the economy respond to an exogenous monetary policy shock (MPS)? Why do we care for exogenous shocks? Mainly because we want to isolate exogenous monetary actions from the systematic policy response to other developments in the economy.

How do we isolate exogenous MPS? The general and most followed strategy involves estimating the systematic component of monetary policy, i.e. the parameters of the feedback rule. However, there are other strategies too (e.g. Romer dates)

3.4 The VAR approach

If we choose to pursue a VAR strategy: identification is usually accomplished in one of three ways:

(1) Using a *recursive approach*, ordering the variables in a Wold causal chain; this is what CEE call the recursiveness assumption: the idea is that the policy shock is ortogonal to the variables in the feedback rule itself, i.e. time t variables in the Fed reaction function are not affected at time t by MPS.

(2) Adopting a *structural approach* in which theoretical restrictions are imposed on the instantaneous interactions of shocks. This approach involves estimating a broader set of economic relations, which sometimes can be controversial. The literature has made progress in this direction, e.g. Uhlig (2001) assumes "that a contractionary monetary policy shock does not lead to increases in prices, increase in nonborrowed reserves, or decreases in the federal funds rate *for a certain period following a shock",* while Canova and DeNicolo (2000) identify MPS by imposing sign restrictions on the cross-correlations of variables in response to shocks.

(3) Using *long-run restrictions.*

3.4.1 The general approach

Consider a system represented by a $n \times 1$ vector of endogenous variables z_t. If $\mathbf{z}_t$ is covariance stationary then it has Wold moving average representation given by:

$$\mathbf{z}_t = \Gamma(L)\,\mathbf{u}_t \tag{1}$$

where $\Gamma(L) = \Gamma_0 + \Gamma_1 L + \Gamma_2 L^2 + ...$ is a $n \times n$ matrix of polynomials in the lag operator L and $\mathbf{u}_t$ is a vector of white noise disturbance terms. By an appropriate normalization, assume that Γ_0 has 1's along the diagonal and that $E\mathbf{u}\mathbf{u}' = \sum_u$ is a diagonal matrix. In this MA representation, the $\mathbf{u}$'s are the *fundamental structural disturbances* (e.g. a technology shock and a money shock), and we are interested in estimating the response of the elements of z to innovations in $\mathbf{u}$.

However, we cannot estimate (1) directly. The sample information contained in our observations can be estimated with a vector autoregression of the form:

$$\mathbf{z}_t = A_1\mathbf{z}_{t-1} + A_2\mathbf{z}_{t-2} + ... + A_p\mathbf{z}_{t-p} + \boldsymbol{\varepsilon}_t \tag{2}$$

using the lag operator notation this can be written as:

$$\left[I_n - A_1 L - A_2 L^2 - .... - A_p L^p\right]\mathbf{z}_t \equiv A(L)\mathbf{z}_t = \boldsymbol{\varepsilon}_t \tag{3}$$

or

$$\mathbf{z}_t = C(L)\,\boldsymbol{\varepsilon}_t \tag{4}$$

where $C(L) = A(L)^{-1}$. Identifying a VAR implies decomposing the prediction error $\boldsymbol{\varepsilon}_t$ into the economically meaningful or fundamental innovations $\mathbf{u}_t$.

Comparing (4) with (1), they are *observationally equivalent.* Any representation of the process (1) can be formed by taking any nonsingular matrix G with $1'$s along the main diagonal and writing:

$$C(L)\boldsymbol{\varepsilon}_t = \Gamma(L)\,\mathbf{u}_t = \Gamma(L)\,GG^{-1}\mathbf{u}_t \tag{5}$$

so that:

$$C(L) = \Gamma(L)\,G \text{ and } \mathbf{u}_t = G\boldsymbol{\varepsilon}_t \tag{6}$$

Therefore, in order to recover estimates of the unobservable structural disturbances, $\mathbf{u}_t$, from the observable VAR residuals, $\widehat{\boldsymbol{\varepsilon}_t}$, it is necessary to estimate the matrix G.

The covariance matrix of the VAR residuals, $\sum_\varepsilon$, is related to G^{-1} and $\sum_u$ by

$$\Sigma_u = G\Sigma_\varepsilon G^{'} \tag{7}$$

We have therefore $n^2 - n = n(n-1)$ unknown elements to estimate in G and n elements to estimate in Σ_u.

The total is n^2. However the covariance matrix of VAR residuals only contains $n(n+1)/2 < n^2$ bits of sample information; hence, $n(n-1)/2$ additional restrictions are required for identification.[1].

We can consistently (e.g. by OLS) estimate the lag polynomial $C(L)$. A consistent estimate of $\sum_\varepsilon$ can be computed from the correlation matrix of the residuals of the OLS regression. The structural model will be identified so long as we introduce enough restrictions to determine the elements of G uniquely. Once we have G, we can in fact recover both $\Gamma(L)$ and $\mathbf{u}_t$ by exploiting (6). Hence, as shown by (6), *a priori* restrictions on $\Gamma(L)$ and/or G or its inverse can allow us to recover the structural form.

How is this done in practice? By imposing identifying assumptions.

3.4.2 The Recursiveness Assumption

In the recursiveness assumption, we assume that shocks are constructed in such a way that innovations to the first variable affect all other variables contemporaneously, innovation to the second variable contemporaneously affect all other variables but the first, and so on, until the last variable affects all variable with a one period delay. This is tantamount to impose a triangular structure on the matrix G^{-1}. From

$$\boldsymbol{\varepsilon}_t = G^{-1}\mathbf{u}_t$$

$$\begin{bmatrix} \text{obs. effect of shocks on vrb } z_1 \rightarrow \\ \text{obs. effect of shocks on vrb } z_2 \rightarrow \\ ... \\ ... \\ \text{obs. effect of shocks on vrb } z_n \rightarrow \end{bmatrix} \begin{bmatrix} e_{1t} \\ e_{2t} \\ ... \\ ... \\ e_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b_{21} & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \\ \\ \\ u_{nt} \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \\ \\ \\ \end{matrix} \begin{bmatrix} \text{structural shock 1} \\ ... \\ \\ \\ ... \end{bmatrix}$$

$$e_{1t} = u_{1t}, e_{2t} = b_{21}u_{1t} + u_{2t}, ..., e_{nt} = b_{n1}u_{1t} + ... + u_{nt}$$

The idea is simple: in the VAR equation for y_t, the residual e_{nt} is a linear combination of all structural shocks, and the innovation u_{nt} only affects z_{nt} contemporaneously. By restricting all shocks but the last not to affect the last variable in the first period, we are saying that the observed effect on z_n in the first period in only due to the structural shock n. Hence all elements of the last column but the last have to be equal to zero for this to be true.

How do we calculate G^{-1}? Turns out that, given

$$\Sigma_\varepsilon = G^{-1}\Sigma_u \left(G^{-1}\right)' \tag{7}$$

a Choleski decomposition of the matrix Σ_ε returns $G^{-1}\Sigma_u^{1/2}$, thus allowing to read off both the elements of $\Sigma_u^{1/2}$ (on the main diagonal) and the elements of G^{-1} (once the appropriate rescaling is done)

Notice that with the recursiveness assumption you have equivalence between the two observations:

[1] However, these conditions might be neither necessary nor sufficient for identification. See the discussion in Christiano, Eichembaum and Evans (1999, Handbook of Macroeconomics).

1. shocks to x do not contemporaneously affect variable y

2. variable y contemporaneously affects the decision of the agent of choosing a given level of x. (for more on this, see Christiano, Eichembaum and Evans (1999) Handbook of Macroeconomics).

In both cases we say that *x is ordered after y* in the VAR when using the Choleski factorization of the variance covariance matrix.

3.4.3 Long-run restrictions

If $\mathbf{z}_t$ is covariance stationary, then it can be inverted and has a Wold moving average representation given by:

$$C(L)\boldsymbol{\varepsilon}_t = \Gamma(L)\,\mathbf{u}_t = \Gamma(L)\,GG^{-1}\mathbf{u}_t \tag{5}$$

The idea of using long-run restrictions is to impose restrictions on the long-run (or, the cumulative) effects of shocks, which are provided by the matrix $\Gamma(L)$ evaluated at 1.

Example 3 *Consider an example where the two variable are output growth and unemployment. Assume they are both covariance stationary. Then use:*

$$C(1) = \Gamma(1)\,G$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} 1 & g_{12} \\ g_{21} & 1 \end{bmatrix}$$

In the representation above, we have 4 equations in 6 unknowns. In addition, the $\Sigma_\varepsilon = G^{-1}\Sigma_u G^{-1\prime}$ formula provide 3 more equations with 2 more unknowns. The LR restriction idea imposes that $\Gamma(1)$ is lower triangular, that is, one shock affects the level of both variables in the long-run, whereas one shock affects all but one, and so on. If the first variable is u and the second is Δy, this means that only one shock affects has a non-zero effect on Δy in the long-run. It is typical to interpret this shock as a "technology" shock, whereas the shock that has a zero effect on Δy in the long-run can be interpreted as a demand shock: this happens because this shock has a zero effect on the cumulative response of Δy.

Remark 4 *Since an invertible MA representation only exists for $I(0)$ VAR processes, we implement the Blanchard-Quah procedure by checking for stationarity (or differencing if the variables are $I(1)$) and work with $I(0)$ variables.*

Given that $\Sigma_\varepsilon = G^{-1}\Sigma_u G^{-1\prime}$ and $G = \Gamma(1)^{-1} C(1)$, the Blanchard-Quah methodology implies that:

$$\Sigma_\varepsilon = G^{-1}\Sigma_u G^{-1\prime} = C(1)^{-1}\,\Gamma(1)\,\Sigma_u^{1/2}\Sigma_u^{1/2}\Gamma(1)'\,C(1)^{-1\prime}$$

Given that:

$$\Gamma(L)\,u = C(L)G^{-1}u$$

it follows that:

$$\Gamma(1)\Sigma_u\Gamma(1)' = C(1)G^{-1}\Sigma_u G^{-1\prime}C(1)' = C(1)\Sigma_e C(1)'$$

hence a Choleski decomposition of $C(1)\Sigma_e C(1)'$ will give us $\Gamma(1)\Sigma_u^{1/2}$.

After that, we just use $G^{-1}\Sigma^{1/2} = C(1)^{-1}\Gamma(1)\Sigma_u^{1/2}$ to produce the desired decomposition.

3.4.4 The Common Trends Methodology

For an application and easy description of this methodology, as well as references, see Iacoviello (2002), "House Prices and the Business Cycle in Europe: A VAR Analysis".

Common trends and cointegration The specific model can be represented by a $n \times 1$ vector of endogenous variables X_t, which has the following form:

$$GX_t = G_1 X_{t-1} + .. + G_k X_{t-k} + u_t \tag{1}$$

where X_t and u_t are of dimension $(n \times 1)$, u_t is a vector of white noise and mutually orthogonal structural shocks, k is the lag length and G's and μ are unknown coefficients. The reduced form is:

$$X_t = A_1 X_{t-1} + ... + A_k X_{t-k} + \varepsilon_t \tag{2}$$

where $\varepsilon_t = G^{-1}u_t$, $A_i = G^{-1}G_i$, $E\varepsilon\varepsilon' = G^{-1}G^{-1\prime} = \Sigma_e$. This model can be reparametrized (in first differences and with an error correction term) as follows:

$$\begin{aligned} \Delta X_t &= \Pi X_{t-1} - (A_2 + ... + A_k)\Delta X_{t-1} - ... - A_k \Delta X_{t-k+1} + \varepsilon_t && (3) \\ A(L)\Delta X_t &= \Pi X_{t-1} + \varepsilon_t && (4) \end{aligned}$$

where $\Pi = A_1 + ... + A_k - I$.

If the series are non-stationary and cointegrated, then the following holds: $0 < r = \operatorname{rank}\Pi < n$ and the equation (4) above is the VECM form of the model.

From equation 4 we want to obtain the moving average representation:

$$\Delta X_t = C(L)\varepsilon_t \tag{5}$$

This is obtained as follows. Define the transformation matrix

$$M \equiv \begin{bmatrix} P' \\ \beta \end{bmatrix}, \ \beta' P = 0$$

and the matrices

$$D(L) \equiv \begin{bmatrix} I_{n-r} & 0 \\ 0 & (1-L)I_r \end{bmatrix}, \quad D_\perp(L) \equiv \begin{bmatrix} (1-L)I_{n-r} & 0 \\ 0 & I_r \end{bmatrix} \Rightarrow D(L)D_\perp(L) = (1-L)I_n$$

Also, let α^* be an $n \times n$ matrix such that:

$$\alpha^* \equiv \begin{bmatrix} 0_{n\times(n-r)} & \alpha_{n\times r} \end{bmatrix}$$

It can be verified that:

$$\alpha\left(\beta' X_t\right) = \alpha^*\left(D_\perp\left(L\right) M X_t\right)$$

Premultiply both sides of the VECM in (4) by M:

$$MA\left(L\right)\Delta X_t = M\alpha\left(\beta' X_{t-1}\right) + M\varepsilon_t = M\alpha^*\left(D_\perp\left(L\right) M X_{t-1}\right) + M\varepsilon_t \tag{6}$$

This can be rewritten as:

$$MA\left(L\right) M^{-1} D\left(L\right) D_\perp\left(L\right) M X_t - M\alpha^* L\left(D_\perp\left(L\right) M X_t\right) = M\varepsilon_t \tag{7}$$

$$M\left(A\left(L\right) M^{-1} D\left(L\right) - \alpha^* L\right) X_t^* = R\left(L\right) X_t^* = M\varepsilon_t \tag{8}$$

The VAR has now been transformed in a VAR with a new n dimensional variable, X_t^*[2], where:

$$X_t^* \equiv D_\perp\left(L\right) M X_t$$

Inverting (8) yields:

$$X_t^* = R\left(L\right)^{-1} M\varepsilon_t \tag{9}$$

Noting that $\Delta X_t = M^{-1} D\left(L\right) X_t^*$, we have:

$$\Delta X_t = M^{-1} D\left(L\right) R\left(L\right)^{-1} M\varepsilon_t \tag{10}$$

Therefore:

$$C\left(L\right) = M^{-1} D\left(L\right) R\left(L\right)^{-1} M \tag{11}$$

Identification of permanent and transitory shocks. Subject to identification, an observationally equivalent representation for ΔX_t is:

$$\Delta X_t = \Gamma\left(L\right) u_t \tag{12}$$

We know that $\Gamma(1)$ measures the long-run effect of the structural shocks. Engle and Granger (1987) have shown that the columns of $C(1)$ are orthogonal to the cointegrating vectors β, so $\beta' C\left(1\right) = 0$. Thus, any basis for n-dimensional vectors can be divided into a space spanned by the r cointegrating vectors and a

[2] In the example in the Rats file, for instance, given $X_t = [y \quad mp \quad hp \quad r \quad \pi]'$, we have:

$$X_t^* = \begin{bmatrix} \Delta y + b_y \Delta mp + \tau \Delta hp \\ -b_i \Delta mp + \Delta i + \Delta \pi \\ -b_y y + mp + b_i i \\ -\tau y + hp \\ i - \pi \end{bmatrix}$$

space spanned by the $n-r$ linearly independent columns of $C(1)$. Since $\beta' C(1) = 0$, for any β there are $(n-r)r$ independent reduced-form coefficients of $C(1)$. The G matrix contains n^2 parameters; for given β, there are $(n-r)\,n$ independent RF coefficients in $C(1)$, as many as in $\Gamma(1)$; Σ contains information $n(n+1)/2$ parameters Hence we need $n^2 - (n+1)\,n/2 = n(n-1)/2$ parameters to identify the model.

Structural and reduced form are linked at $L = 1$ by:

$$
\begin{aligned}
C(1) &= \Gamma(1)\,G \\
\varepsilon &= G^{-1}u
\end{aligned}
$$

King, Plosser, Stock and Watson suggest the following methodology:

1) Partition $\Gamma(1)$ so that $\Gamma(1) = [P \mid 0]$, where P is a $n \times (n-r)$ matrix whose columns represent the long-run responses of the variables to permanent shocks, whereas the long-run responses to the temporary shocks are assumed to be zero. These are the sources of the common stochastic trends among the variables. For the remaining r shocks, permanent effects are assumed to be zero, so these shocks have only temporary effects. This imposes $(n-r)r$ identifying restrictions.

2) Partition the shocks according to:

$$
u = \begin{bmatrix} u_{n-r} \\ u_r \end{bmatrix}
$$

where u_{n-r} denotes shocks whose permanent effects are nonzero, while u_r denotes shocks whose permanent effects are zero (transitory shocks).

3) Partition G conformably to $\Gamma(1)$ with its first $n-r$ and last r rows as G_{n-r} and G_r respectively. We have that $C(1) = \Gamma(1)\,G = PG_{n-r}$ as well as:

$$
C(1)\Sigma_e C(1)' = PP'
$$

There are $(n-r)(n-r+1)/2$ independent equations on the LHS and $(n-r)^2$ free parameters in P. Hence we need $(n-r)(n-r-1)/2$ additional restrictions on P, which can be dealt with assuming that this matrix is lower triangular.

4) $C(1)$ has rank $n-r$, hence, in order to decompose it, one cannot use standard Choleski decomposition procedure for $C(1)\Sigma C(1)'$. To deal with this it is possible for instance define P by $P = \widetilde{P}\Theta$, where $\widetilde{P}$'s columns are known coefficients specified a priori and $\Theta_{(n-r)(n-r)}$ is a lower triangular matrix of coefficients to be estimated.

5) Conformably with $\widetilde{P}\Theta$, let D be an $(n-r)\times n$ matrix solving $C(1) = \widetilde{P}D$, such as $D = (\widetilde{P}'\widetilde{P})^{-1}\widetilde{P}'C(1)$. Get Θ with a lower triangular Choleski decomposition of $D\Sigma D'$, then use Θ to calculate P.

6) Given that $C(1) = PG_{n-r} = \widetilde{P}\Theta G_{n-r} = \widetilde{P}D$, we have $G_{n-r} = \Theta^{-1}D$, hence we can obtain the structural shocks u with permanent effects by premultiplying the reduced form residuals ε by G_{n-r}.

7) Hence:

$$G_{n-r}\Delta X_t = G_{n-r}\Pi X_{t-1} - ... - G_{n-r}G_k\Delta X_{t-k+1} + G_{n-r}\mu\ Z_t + u_{n-r,t}$$

Once G_{n-r} is estimated, the dynamic effects on X_t of the shocks with permanent effects are obtained using:

$$\Gamma(L)_{n-r} = C(L)\left[G^{-1}\right]_{n-r} = C(L)\Sigma G'_{n-r}$$

where $\left[G^{-1}\right]_{n-r}$ denotes the first $n-r$ columns of G^{-1}.

8) Identification of the structural parameters associated with the shocks with only transitory effects can proceed from: $[0 \quad I_r]_{r\times n} = [G_r\Sigma G_{n-r} \quad G_r\Sigma G'_r]$. One possibility to identify G_r is specify a triangular structure in it.

3.4.5 Non-Choleski restrictions, and back to comparisons between model and data.

Sometimes economic theory suggests more than $n(n-1)/2$ restrictions on the relationship between structural and reduced form.

To give a more concrete example, we consider the simple example of a dynamic model with nominal rigidities which is widely used in monetary policy analysis. To have a feeling on this, see for instance the paper by McCallum (2001) "Should Monetary Policy Respond Strongly to Output Gaps?", available at http://www.nber.org/papers/W8226[3]

The model equations are:

$$\begin{aligned}
y_t &= E_t y_{t+1} - \theta(r_t - E_t\pi_{t+1}) + \sigma_g g_t && \text{(a)}\\
\pi_t &= \alpha y_t + \beta E_t\pi_{t+1} + \sigma_u u_t && \text{(b)}\\
r_t &= \phi_r r_{t-1} + (1-\phi_r)(1+\phi_\pi)\pi_t + (1-\phi_r)\phi_y y_t + \sigma_e e_t && \text{(c)}
\end{aligned}$$

where:

- y is output, π is inflation and r is the nominal interest rate, expressed in deviation from the steady state
- (1) is a dynamic IS curve; (2) is the Phillips curve; (3) is a Taylor rule
- θ is intertemporal elasticity of substitution; α is the elasticity of inflation to output; β is the weight on future inflation in the price setting equation; ϕ_r,ϕ_π and ϕ_y are parameters of the Taylor rule
- g_t represents a preference shocks; e_t is an interest rate shock; u_t is a cost-push shocks that moves inflation up. In the model, we can assume that shocks are autocorrelated or not, and we can set their relative importance to any arbitrary level.

[3] McCallum model also includes a discussion an extra-exogenous variable, the potential output y^*. We assume that this variable is constant in this example.

Assume now that this model is a true representation of the reality. The solution to this model will have the form of equation (7) in the previous chapter. I set autocorrelation of each shock to 0, standard deviations σ_g, σ_u and $\sigma_e = 0$ and set $\theta = 1$, $\alpha = 0.03$, $\beta = 0.99$, $\phi_r = 0.8, \phi_\pi = 1, \phi_y = 1$.

After rounding, the model decision rules are

```
        r    p    y
r(-1) 0.52 -0.1 -1.2
eps_e 0.66 -0.1 -1.5
eps_g 0.14 0.01 0.68
eps_u 0.26 0.96 -0.6
```

In terms of our VAR notation,

$$\begin{bmatrix} e_y \\ e_p \\ e_r \end{bmatrix} = \begin{bmatrix} 0.7\sigma_y & -0.6\sigma_p & -1.5\sigma_r \\ 0.01\sigma_y & 1\sigma_p & -0.1\sigma_r \\ 0.14\sigma_y & 0.26\sigma_p & 0.68\sigma_r \end{bmatrix} \begin{bmatrix} u_y \\ u_\pi \\ u_r \end{bmatrix}$$

Notice that I have kept the variance of each shock as a potential unknown here, and that blowing up the variance of one shock blows the observed effect of that shock on the variables when hit by that shock.

Let $u_y = u_\pi = u_r = 1$. Then $E\varepsilon\varepsilon' = BB'$ or:

$$\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 0.36\sigma_p^2 + 2.25\sigma_r^2 + 0.49\sigma_y^2 \\ \sigma_p^2 + 0.01\sigma_r^2 + 0.0001\sigma_y^2 \\ 0.0676\sigma_p^2 + 0.4624\sigma_r^2 + 0.0196\sigma_y^2 \\ -0.6\sigma_p^2 + 0.15\sigma_r^2 + 0.007\,\sigma_y^2 \\ -0.156\,\sigma_p^2 - 1.02\sigma_r^2 + 0.098\,\sigma_y^2 \\ 0.26\sigma_p^2 - 0.068\,\sigma_r^2 + 0.0014\sigma_y^2 \end{bmatrix}$$

What does the above relationship mean?

When we estimate a VAR, what we typically estimate consistently is Σ_e, the covariance matrix of the errors. The model tells us exactly the numbers that should go into the covariance matrix.

At most, what we are left with is choice of the variance of the shocks to as to be as close as possible to the estimated VCOV matrix.

A cheap way to test our model? In the data, we estimate the unrestricted VCOV matrix. Denote with $|\Sigma_R|$ the determinant of the resulting estimate of the restricted variance-covariance matrix. Then the χ^2 test statistic:

$$\chi^2 = |\Sigma_R| - |\Sigma|$$

with 3 degrees of freedom can tell us whether we reject or not the set of identifying restrictions. See the RATS manual (in particular the command cvmodel) and the Enders book (Chapter 5) for more on this.

3.4.6 Summarizing

You can see that the idea is the following.

Given that:

$$\mathbf{z}_t = \Gamma(L) u_t = C(L)\varepsilon_t, \; Euu' = \Sigma_u \text{ (diagonal)}, \; E\varepsilon\varepsilon' = \Sigma_\varepsilon$$

or

$$C(L)\varepsilon_t = \Gamma(L) u_t = \Gamma(L) A A^{-1} u_t$$

you can exploit two of the following

$$C(L) = \Gamma(L)A \text{ and } \Sigma_\varepsilon = A^{-1}\Sigma_u A^{-1\prime} \text{ and } \varepsilon = A^{-1}u$$

normally when you use long-run restrictions you exploit first and second, when you use short-run ones you exploit the second and the third.

3.4.7 Digression: Obtaining confidence intervals

Suppose we estimate the following AR(1) process

$$y_t = \underset{(s)}{a}\, y_{t-1} + e_t$$

where s is the standard error of the coefficient. The IRF can be written as $\phi(t) = a^t$

For $t > 0$, if we assume that the coefficient a is normally distributed, then in period 1 a 95% confidence interval would be $(a - 2s, a + 2s)$.

In higher order systems, however, the various $a's$ will be correlated, and with nonstationary variables the normality assumption might be wrong.

Therefore, we normally obtain confidence intervals via bootstrapping.

1. Estimate the AR process from the actual data (of length, say, T) through OLS and denote with $\widehat{a}$ the estimate of a and with $\{\widehat{e}_t\}$ the sequence of the estimated residuals.

2. For a sample size of length T, draw with replacement T numbers using randomly selected numbers from the sequence $\{\widehat{e}_t\}$. That way, you will have a simulated series of length T that you can call $\{\widehat{e}_t^s\}$ that has the same properties of the error process

3. Simulate the time series $\{\widehat{y}_t^S\}$ for given $\widehat{a}$ and $\{\widehat{e}_t^s\}$. E.g. $\{\widehat{e}_t^s\} = \{1, -2, 2, -1, 1\}$, $\widehat{a} = 0.5$. Then

$$\begin{aligned} y_0 &= 0 \text{ (initialization)} \\ y_1 &= .5y_0 + 1 = 1 \\ y_2 &= .5y_1 - 2 = .5 - 2 = -1.5 \\ y_3 &= -.75 + 2 = 1.25 \\ y_4 &= .625 - 1 = -.375 \\ y_5 &= -.1875 + 1 = .8225 \end{aligned}$$

So $\left\{\widehat{y}_t^S\right\} = \{1, -1.5, 1.25, -.375, .8225\}$

4. Estimate an AR(1) process for $\left\{\widehat{y}_t^S\right\}$. If you try that, you would get $\widehat{a} = -.86$. Obtain the IRF. Store it.

5. Repeat the process 2 to 4 several hundred of times. You will obtain several hundred $\{\widehat{e}_t^s\}$, $\widehat{a}$ and IRF

6. Construct an interval based on all the IRF that excludes the highest 2.5 and the lowest 2.5 so to obtain a 95% confidence interval.

This way, there is no need to assume anything about the distribution of the AR coefficients. In a VAR, the only difference is that the error sequence must be chosen in a way to keep the appropriate error structure. For instance, in our bivariate VAR with a Choleski ordering of the residuals

$$\begin{bmatrix} \widehat{u}_y \\ \widehat{u}_m \end{bmatrix} = \begin{bmatrix} \widehat{e}_1 \\ \widehat{e}_2 - b\widehat{e}_1 \end{bmatrix}$$

so we draw from the two error sequences (one for which equation), but make sure that we take the correlation into account when generating the $u's$ sequence.

3.5 What should we remember?

1. What are MPS? They can represent (a) stochastic shifts in the preferences of the policymakers, or in the bargaining power of different members of the MPC; (b) desire by the Fed to signal a change in policy; (c) measurement error on data available to Fed

2. There is agreement about the qualitative effects of a MPS; some say there is also agreement about the timing; some say there is also quantitative agreement about the effects of a monetary policy shock.

3. There is agreement that MPS account for little of the volatility of aggregate output

Chapter 4

Incomplete Markets Models

Our models so far assume a representative agent, so they have difficulty in addressing on distributional issues. Thinking about credit forces us to address these issues, for if while money is a publicly supplied asset, credit is not.

To understand the role of credit in business cycle, we need to have some sort of heterogeneity across the population, across income, or age, or preferences. We present three simple models of interaction between credit and the business cycle. Our main emphasis is how to solve computationally these models. We are also interested in presenting background for models where nonconvexities and nonlinearities force us to think about solution methods other than brute force linearization.

4.1 Background and notation

4.1.1 A basic model of consumption and saving under certainty

The basic model. A household must choose a policy for next period savings s' to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c + s' &= Rs + z \\ c &\geq 0 \\ s' &\in S \end{aligned}$$

where S is the range of all admissible values of s, from $s_{\min}$ to $s_{\max}$. R and z are constant.

Associated with this problem is a Bellman equation

$$v(s) = \max_{s'} (u(c) + \beta v(s'))$$

Here, $v(s)$ is the optimal value of the objective function, starting from the asset (s). A solution to this problem is a value function that satisfies the equation above and the associated policy function $s' = g(s)$ mapping the current state into the optimal choice of assets to carry next period.

Some remarks

- Key to write the Bellman equation is that the problem is recursive. That is, v does not depend on t, but only on s today. If v is time-invariant, g is time-invariant too.
- The value function solves the Bellman equation above. It is a functional equation in the unknown function v. (A functional equation is an equation expressed in terms of both independent variables and unknown functions, which are to be solved for). In other words, the solution to the Bellman equation is the value function itself.[1]
- If $u(c)$ is concave and twice continuosly differentiable, there exist a value function v and a policy function g that solve the problem above.
- The value function can be calculated as the limit of the following sequence of steps $i = 0, 1, ...$

$$v^{i+1}(s) = \max_{s'} \left(u(c) + \beta v^{i}(s') \right)$$

  with $v^0 = 0$ for all values of s.

- Assuming differentiability of the value function, we can write the solution to the value function as $0 = u'(c)\frac{dc}{ds'} + \beta v'(s') -> u'(c) = \beta v'(s')$. At the optimum $v'(s) = u'(c)$; because this condition it satisfied for all values of s', we also have $v'(s') = u'(c')\frac{\partial c'}{\partial s'} = u'(c') R$ thus implying $u'(c) = \beta R u'(c')$; hence, for all t, the solution to the household problem will satisfy this condition.

Now I describe a commonly used algorithm for the computation of household decision functions by value function iteration. The algorithm works as follows:

1. Choose a grid of points for assets s
2. Initialize the value function for each value of s. Let it be v^0
3. Given v^0, compute a new value function v^1 and its associated policy function g_1. That is, for each index i_s, find the index for s' that maximizes v^1 given the initial v^0.
4. Check for convergence (that is, check whether the largest absolute value of the difference between v^0 and v^1 is sufficiently small), otherwise set $v^0 = v^1$ and $g_0 = g_1$ and return to step 3

[see file `cakeeating1.m` for an implementation]

[1] Consider for instance solving $f(xy) = f(x) + f(y)$ for x and y greater than zero. The solution to this equation is $f = \log$.

In the example file, we can set for illustrative purposes $R = 1$, $z = 0$, and $u = \log c$. With this notation, the agent's problem becomes the classic cake-eating problem[2] (see e.g. `http://www.ucl.ac.uk/~uctpjea/matlab.pdf`) for which the value function and the consumption function are of the form:

$$\begin{aligned} v &= A + B \log s \\ c &= (1 - \beta)\, s \end{aligned}$$

4.1.2 A Basic model of consumption and saving under uncertainty

A household must choose a policy for next period savings s' to maximize

$$E \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to

$$\begin{aligned} c + s' &= Rs + z \\ c &\geq 0 \\ s' &\in S \end{aligned}$$

where S is the range of all admissible values of s, from $s_{\min}$ to $s_{\max}$. Here we assume that z is random, and can take on a range of values with some probability. For simplicity, we assume that z can be high or low, (z_l, z_h), and their evolution over time is described by the transition matrix

$$P\left(z'|z\right) = \text{Prob}\left(z_{t+1} = z'|z_t = z\right) = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix}$$

In the matrix above, the element π_{ij} describes the probability of moving to state j, given state i, so that $\sum_j \pi_{ij} = 1$

Associated with this problem is a Bellman equation

$$v\left(z, s\right) = \max_{s'} \left(u\left(c\right) + \beta E v\left(z', s'\right)|z\right)$$

where $Ev\left(z', s'\right)|z = P\left(z'|z\right) v\left(z, s\right)$

Here, $v\left(z, s\right)$ is the optimal value of the objective function, starting from the asset income state (z, s). A solution to this problem is a value function that satisfies the equation above and the associated policy function $s' = g\left(z, s\right)$ mapping the current state into the optimal choice of assets to carry next period.

Computation of household decision functions by VFI

1. Choose a grid of equally spaced points for assets

[2] It is called cake-eating problem because it boils down to finding the solution to the following problem: given a cake (with good preservatives!), what is the best way to eat it? All at once? A little each day? or by some other pattern of consumption?

2. Initialize the value function for each value of s and z. Let it be v^0. Given v^0, compute the associated Ev^0, which is just a weighted sum of the form: $Ev^0(z,s)|z = \pi_{zl}V(l,s) + \pi_{zh}V(h,s)$ for each z, s.

3. Compute a new value function v^1 and its associated policy function g_1. That is, for each possible realization of is and iz, find the index for s' that maximizes v^1 given the initial v^0 and its associated Ev^0.

4. Check for convergence, otherwise set $v^0 = v^1$ and $g_0 = g_1$ and return to step 3

[see file `aiyagari1.m` for an implementation; set option `DEATON=1`]

4.2 General Equilibrium models of credit without aggregate shocks

To move from the models above to general equilibrium requires imagining an economy populated with a large number of households each solving a version of the problem above.

If individual z is a random variable that is independently distributed across households, the law of large numbers guarantees that in equilibrium aggregates will be constant and we can study stationary equilibria of these economies. These stationary equilibria are characterized by aggregate variables and distributions of the state variables that are constant, while the income status and the wealth level of individual households change over time.

4.2.1 Huggett: General equilibrium no capital, no aggregate uncertainty (HM, 5.3.1)

In Huggett's model, we compute the solution to the household problem as before:

$$s' = g(z,s)$$

At each point in time, there are households who differ in their current z and their current s. The question is: suppose we start from a arbitrary distribution of wealth and income, will the model endogenously generate some unique wealth distribution? So long as the "monotone mixing condition" is satisfied,[3] it can be proved that there exists a stationary distibution of wealth. This distribution is constructed by aggregating individual statistics.

The next step is to consider equilibrium considerations. Suppose there is no storage, the only thing that households can do is to borrow and lend to each other at the market rate R. The market clearing condition

[3] See for instance `http://www.compmacro.com/makoto/200701econ552/note/note_im_aiyagari.pdf`.

Roughly speaking, the monotone mixing condition states that there exists t and s such that there is a strictly positive probability that an agent who starts from the worst state reaches above s after t periods, and that another agent who starts from the best state reaches below s after t periods.

In other words, the MMC guarantees that there is a sufficient mixing even among the agents in the best and the worst state. Rios-Rull calls the condition as the "American dream and American Nightmare" condition.

can then be written as

$$\sum_z \int_{s_{\min}}^{s_{\max}} s f(z,s)\, ds = 0$$

where

- $f(z,s)$ is the joint density of the state variables (that is, for each "pair", z, s, we measure how often the distribution is there)
- $\int_{s_{\min}}^{s_{\max}} ..ds$ denotes the sum over all the possible asset realizations
- $\sum_z$ denotes the sum over all the possible income realizations

Let us give a formal definition of equilibrium. An equilibrium is an interest rate R, a policy function g, and a stationary distribution f such that

1. g solves the household problem
2. The stationary distribution f is induced by the household policy functions
3. The loan market clears

Computation: this is done as follows:

1. Make a guess for R
2. Compute g
3. Compute stationary asset distribution
4. Check if markets clear
5. Update R and return to step 2 if they do not.

What is the stationary asset equilibrium? In the stationary equilibrium, the distribution of assets is stationary. We refer to it as the stationary/invariant distibution. In particular, we aim a computing the joint distribution functions over earnings and assets $f(z,s)$. The individual states are all the possible realizations of z and s.

How do we compute the stationary asset distribution induced by the policy functions? There are several possibilities. One method uses Monte Carlo simulation. It works as follows

1. Choose a sample size of N individuals
2. Assign to each household an initial wealth s and initial income z
3. Simulate the decision rules for T periods of the individuals, aggregate and compute statistics from the sample
4. Check for convergence of the distribution. Otherwise increase N or T.

[see file `aiyagari1.m` for an implementation; set option `HUGGETT=1`]

4.2.2 Aiyagari: As above, adding capital (HM, Chapter 5)

Here, we keep heterogeneity at the household level, but keep the assumption that all firms are equal and can act as a representative firm. As a consequence, we can formulate the model as a decentralized economy and study households and firms separately.

In Aiyagari, assets take the form of real capital. There is an aggregate production function, $Y = f(K, L)$, and L is the total amount of efficiency units in the economy, that is

$$L = \sum_{z_i} z_i P^* (z_i)$$

where we can reinterpret z as efficiency units and $P^*(z)$ is the stationary distribution of P across individuals, i.e. a vector indicating for each value of z_i the (constant) fraction of individuals that are in the z_i state.[4]

the integral above is the invariant distribution of households over efficiency units.

The individual problem is the same as before. The value of the objective function for an individual characterized by productivity z and wealth s is

$$V(z, s) = \max_{s'} \left(u(c) + \beta E \left(V(z', s') \right) | z \right)$$

subject to budget constraint and the stochastic process for his idiosyncratic productivity.

The individual budget constraint is

$$c + s' = Rs + wz$$

Aiyagari interprets the single asset as homogeneous physical capital, denoted by k. Assuming that capital depreciates at rate δ, the individual budget constraint can be equivalently rewritten as

$$c + k' = (r_k + 1 - \delta) k + wz$$

The two equations coincide so long as we take $s = k$ and $R = r_k + 1 - \delta$. One interpretation is that there are loans and capital are perfect substitutes as a means of saving. From the household point of view, we can thus interpret negative values of s as borrowing, and assume that a centralized institution collects all savings and transforms them into capital at the end of the period.

The aggregate production function has as its arguments average capital and average employment, which is constant and normalized to unity. The production function determines in equilibrium r_k and w.

Computation of equilibrium:

1. Make an initial guess for the equilibrium K

[4] If P is the transition matrix, the associated stationary distribution is the solution to the following equation

$$P^* = P^* P$$

or

$$P^* (I - P) = 0 \text{ or } \left(I - P' \right) P^{*\prime} = 0$$

where P^* is a vector, and I is the identity matrix. It can be shown that P^* is the eigenvector associated with the unit eigenvalue of P. See for instance Ljungqvist and Sargent, second edition, pages 30 and 31.

2. Given the guess for K, Compute the implied w and R. They are respectively $w = f_L$ and $R = f_K + 1 - \delta$

3. Compute household's decision functions.

4. Compute average assets across households $E(s') = \sum_z \int_{s_{\min}}^{s_{\max}} sf(z,s)\, ds$

5. Check if 2 and 5 make an equilibrium, that is $E(s') = K$ otherwise update.

[see file `aiyagari1.m` for an implementation; set option `AIYAGARI=1`]

The results of Aiyagari models:

1. Average interest rate is lower than in economy without idiosyncratic uncertainty: this can be interpreted as aggregate precautionary saving

2. Few agents are credit constrained in equilibrium, and self-insurance works quite well even with one asset only

4.3 Models with aggregate shocks

4.3.1 Krusell and Smith: As above, adding aggregate uncertainty

In the models above, aggregate variables are constant. This makes the models tractable. In particular, because aggregate variables are constant, for given constant wage w and interest rate R, the relevant state variables in the household problem are the idiosyncratic shock z and the asset level s.

Krusell and Smith (1998) augment the Aiyagari model by adding an exogenous aggregate technology shock to the production function. Each household, as before, continues to receive an idiosyncratic labor endowment shock z that is constant on average (and normalized to unity) for each value of the aggregate shock. The production function is:

$$Y = AK^{\alpha}N^{1-\alpha}$$

In addition, there is an aggregate productivity shock A, governed by a Markov process, so that the household Bellman equation becomes

$$V(z,k;\lambda,A) = \max_{s'} \left(u(c) + \beta E\left[V\left(z',k';\lambda',A'\right)|A,z,\lambda\right]\right)$$

where I have now replaced s with k : just to make sure, k is individual assets, and K is the average of k across agents. In addition, $\lambda(z,k)$ is the joint distribution each period of the state variable (that is, a cross-sectional distribution of (capital, employment) pairs. The household constraints are:

$$\begin{aligned}
c + k' &= R(K,A)\,k + w(K,A)\,z + (1-\delta)\,k \\
R &= R(K,A) = A\alpha(K)^{\alpha-1} \\
w &= w(K,A) = A(1-\alpha)K^{\alpha} \\
\lambda' &= H(\lambda,A)
\end{aligned}$$

where we normalize total employment to unity. Total capital is determined by

$$K_t = \sum_z \int_{k_{\min}}^{k_{\max}} k\lambda_t\left(z,k\right)ds$$

this is the same definition that we used for Aiyagari's model, with the added fact that the distribution itself is a random variable perturbed by the aggregate shock A_t.

Krusell and Smith propose a way to approximate the recursive equilibrium of this economy using simulations, in the following steps

1. Choose the number of moments m to approximate the distribution λ_t

2. Guess a parametric law of motion for H mapping today's m into tomorrow's m'

3. Assume initial values for the parameters of H

4. Given H, use dynamic programming to solve the Bellman equation

$$V\left(z,k;m,A\right) = \max_{k'}\left(u\left(c\right) + \beta E\left[V\left(z',k';m',A'\right) \mid \left(A,z,m\right)\right]\right)$$

   where notice we have m instead of λ, subject to the assumed law of motion for m

5. From the solution of this problem, simulate the dynamics of the distribution function to estimate the law of motion for the moments m

6. Iterate until the fit is good and the parameters of H converge.

The main results

- Approximate aggregation holds

- Model does not fit well wealth distribution in the US

[see file `krusell1.m` for a sketch implementation]

A basic code for incomplete markets models

```
% Cakeeating1
% SOLVING BY VALUE FUNCTION ITERATION THE
% SIMPLEST MODEL OF CONSUMPTION AND SAVING WITHOUT UNCERTAINTY
clear
close all
ns=2000;
smin=0.0001;
smax=1;
BETA=0.90;
R=1; % An interest rate that generates going down to the lowest saving
Z=0.0001; % Z is income flow, in the cake eating problem is essentially zero.
S=linspace(smin,smax,ns);
%-----------------------------------------------------
% Initialization of value functions
% Initialize choices and value
%-----------------------------------------------------
v=ones(1,ns);
c=ones(1,ns);
% A decent guess
V = log((1-BETA)*ones(1,ns))/(1-BETA);
% A bad guess
% V = (ones(1,ns))/(1-BETA);
newV = V;
EV = V;
idecS = zeros(1,ns);
SPRIME = S ;
%-----------------------------------------------------
% Iterate on value function until convergence
%-----------------------------------------------------
diffV = 1;
iter = 1;
tic
while (iter <= 500) & (diffV > 1e-6)
 % Calculate expected future value
 EV=V ;
 for is = 1:ns
 c = max( 1e-200, Z - SPRIME + R*S(is) )
 v = log(c) + BETA*EV ;
 % "c", "v", EV are vectors of size [1,ns]
 % We look in each state for the SPRIME that maximizes v
 [newV(1,is), idecS(1,is)] = max ( v ) ;
 end
 diffV = max(abs((newV(:)-V(:))));
 iter = iter + 1;
 V = newV ;
 disp(diffV);
 figure(10)
 plot(S,V);
```

```
 drawnow
end
toc
%-----------------------------------------------------
% Calculate Decision Rules
%-----------------------------------------------------
figure(gcf+1)
Sdec = S(idecS(:))  ;
% Here I set Sdec(1)=Sdec(2) since things are not well defined there
Sdec(1) = Sdec(2) ;
plot(S,Sdec,'r'); hold on;
plot(S,S,'k')
legend('Savings','45 line')
xlabel('S')
ylabel('S prime')
%-----------------------------------------------------
% Compute and plot consumption function
%-----------------------------------------------------
figure(gcf+1)
for is=1:ns
Cdec(is) = Z - Sdec(is) + R*S(is) ;
end
plot(S(2:end),Cdec(2:end),'r'); hold on;
title('Consumption function')
xlabel('S')
ylabel('C')
%-----------------------------------------------------
% Simulate an asset accumulation path
%-----------------------------------------------------
simS(1)=max(S);
for t = 2:50
 simS(t) = interp1(S,Sdec,simS(t-1)) ;
 simC(t) = Z - simS(t) + R*simS(t-1) ;
 simCOH(t) = Z + R*simS(t-1) ;
end
figure(gcf+1)
subplot(3,1,1)
plot(simS(2:end))
title('Simulated savings over time')
subplot(3,1,2)
plot(simC(2:end))
title('Simulated consumption')
subplot(3,1,3)
plot(simCOH(2:end))
title('Simulated beginning of period savings')
```

4.4 Homework 2

Consider the following RBC model. The representative agent problem can be written as:

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right)$$

subject to

$$K_t - (1-\delta) K_{t-1} = A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t$$

Assume technology follows an AR(1) process in logs, that is

$$\log A_t = \rho \log A_{t-1} + U_t$$

where U_t is an iid process with zero mean. Set $\alpha = 0.33$ $\beta = 0.99$ $\rho = 0.98$, $\delta = 0.02$ and $\sigma_U = 0.01$.

1. Use Dynare to obtain the linearized (not loglinearized) decision rules for this model for C and K. Rearrange the decision rules so that C and K are function of K_{-1} and current A only, rather than K_{-1}, A_{-1} and U.

   Now write down a program to solve the same model using nonlinear methods (value function iteration on a grid).

2. First, using the steady state value for K, set the grid for K to be centered around this value plus/minus say 10%. Use 200 points for the grid of K.

3. Approximate the process for technology using a 9 state Markov chain (use the function `markovappr.m` on the course webpage). Discuss the meaning of the output of the function markovappr.

4. Write a program that solves for the non-linear policy functions for C_t and K_t as a function of lagged K and current A. Plot the policy functions for C and K as a function of beginning of period K when technology is in the lowest possible state and in the highest possible state. Explain.

5. Simulate the economy for, say, 1000 periods, starting from the steady state. Compare the properties of C and K in the linearized model (Dynare) and in the non-linear model (for instance, run a regression of simulated C and K on lagged K and current A, and check whether the coefficients are similar or different to those in Dynare. (If they are very different, then you got something wrong).

`Hints:`

- Note the syntax of markovappr: by default, the process that is approximated by markovappr has mean zero!

- In the simulated output, check that simulated capital does not hind the bound of the grid... if that happens, it means that probably either the grid is too narrow, or that there is a coding error somewhere.

Answer to Homework 2

1. See file rbclevel1.mod for the Dynare part. Using Dynare, the linearized decision rules are

$$\begin{aligned}\overline{C} &= 2.53 \\ \overline{K} &= 35.65 \\ \begin{bmatrix} C_t - \overline{C} \\ K_t - \overline{K} \end{bmatrix} &= \begin{bmatrix} 0.042485 \\ 0.967616 \end{bmatrix} \left[K_{t-1} - \overline{K}\right] + \begin{bmatrix} 0.719687 \\ 1.459430 \end{bmatrix} \left[A_t - \overline{A}\right]\end{aligned}$$

   Note that Dynare writes the decision rule as a function of lagged A and current U. Using the decision rule for A, one can replace lagged A and current U with current A only. The coefficient on current A will be equal to the coefficient on current U.

2. This is obvious.

3. The syntax that one should use is something like:

   [P, logZ, probst, alambda, asigmay] = markovappr(RHO,SIGMA,3,nz);

   This will return the transition matrix of logZ in P, and the values of logZ (approximately $A_t - \overline{A}$) in the vector logZ. The number 3 controls the width of the discretized state space. Tauchen uses m=3.

4. See the matlab file http://www2.bc.edu/~iacoviel/teach/0809/EC861_files/rbcvfi1.m

5. My simulated regression output is

$$\begin{bmatrix} C_t - 2.4944 \\ K_t - 34.98 \end{bmatrix} = \begin{bmatrix} 0.0436 \\ 0.9665 \end{bmatrix} \left[K_{t-1} - \overline{K}\right] + \begin{bmatrix} 0.8618 \\ 1.3190 \end{bmatrix} \left[A_t - \overline{A}\right]$$

   Turns out that the elasticity of K to A in smaller than in Dynare. Most likely, it depends on the grid for K not being fine enough. For nk=1000 and nz=13, I get coefficients much closer to the ones of the linearized model.

Chapter 5

Multi-sector Models

Understanding sectoral heterogeneity and how to model it.

5.1 Why do we need multi-sector models? A look at the data

US DATA, logged and normalized to 0 in 1965. Variables are per capita.

5.2 A Brief Look at Multi-Sector Models

There is no prototypical multi-sector model to look at. Sectoral heterogeneity is just one of the many possible ways we can extend the canonical RBC model, so here I am more interested in the modeling structure, rather than the particular insights that come out of these models.

5.2.1 The one sector as a two sector

See `twosecrbc1.mod`

The social planner maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, n_{ct}, n_{kt}\right)$$

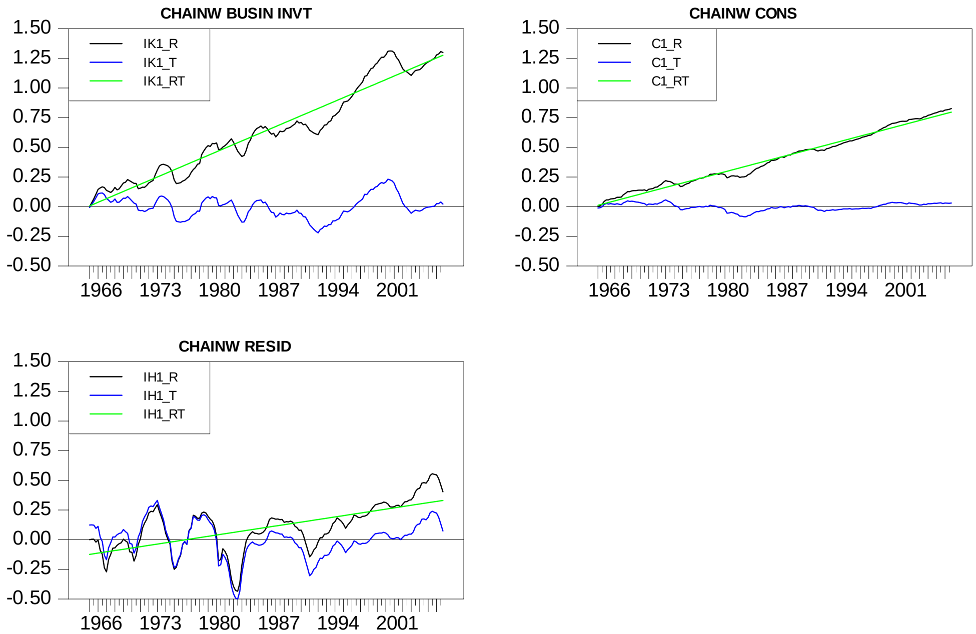
subject to

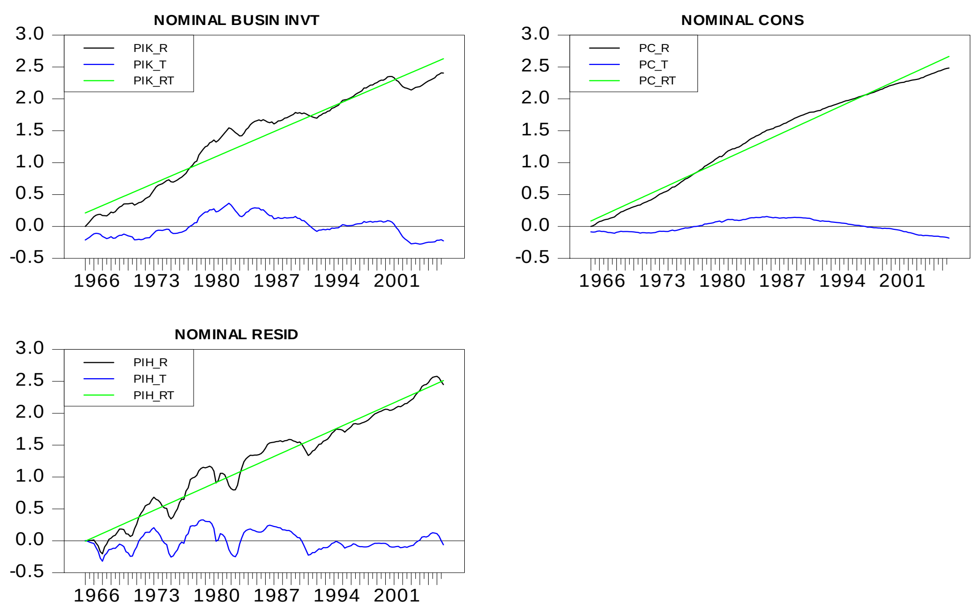
$$c_t = f\left(s_t k_{t-1}, n_{ct}\right)$$
$$k_t = g\left(\left(1-s_t\right) k_{t-1}, n_{kt}\right) + \left(1-\delta\right) k_{t-1}$$

capital is homogeneous, can be costly relabeled as belonging to one sector or another. This is equivalent to having a single constraint as: $c_t + k_t = f\left(k_{t-1}, n_t\right) + \left(1-\delta\right) k_{t-1}$

Line		2006	
1	**Gross domestic product**	**13,246.6**	% of GDP
2	**Personal consumption expenditures**	**9,268.9**	70%
3	Durable goods	1,070.3	8%
4	Nondurable goods	2,714.9	20%
5	Services	5,483.7	41%
6	**Gross private domestic investment**	**2,212.5**	17%
7	Fixed investment	2,162.9	16%
8	Nonresidential	1,396.2	11%
9	Structures	411.2	3%
10	Equipment and software	985.0	7%
11	Residential	766.7	6%
12	Change in private inventories	49.6	0%
13	**Net exports of goods and services**	**-762.5**	-6%
14	Exports	1,466.2	11%
15	Goods	1,035.4	8%
16	Services	430.8	3%
17	Imports	2,228.7	17%
18	Goods	1,879.5	14%
19	Services	349.2	3%
20	**Government consumption expenditures and gross investment**	**2,527.7**	19%
21	Federal	926.6	7%
22	National defense	621.0	5%
23	Nondefense	305.6	2%
24	State and local	1,601.1	12%

Table 5.1: NIPA data, in billion of current dollars


CHAINW BUSIN INVT
IK1_R
IK1_T
IK1_RT
CHAINW CONS
C1_R
C1_T
C1_RT
CHAINW RESID
IH1_R
IH1_T
IH1_RT


NOMINAL BUSIN INVT
PIK_R
PIK_T
PIK_RT
NOMINAL CONS
PC_R
PC_T
PC_RT
NOMINAL RESID
PIH_R
PIH_T
PIH_RT

Write Lagrangean as:

$$
\begin{aligned}
L = & \; u(c_t, n_{ct}, n_{kt}) + ... \\
& -\lambda_t (c_t - f(s_t k_{t-1}, n_{ct})) - \beta\lambda_{t+1}(c_{t+1} - f(s_{t+1}k_t, n_{ct+1})) - .. \\
& -\mu_t (k_t - g((1-s_t)k_{t-1}, n_{kt}) - (1-\delta)k_{t-1}) \\
& -\beta\mu_{t+1}(k_{t+1} - g((1-s_{t+1})k_t, n_{kt}) - (1-\delta)k_t)
\end{aligned}
$$

optimality conditions are

$$
\begin{aligned}
u_{ct} &= \lambda_t \\
-u_{nct} &= \lambda_t f_{nct} \\
-u_{nkt} &= \mu_t g_{nkt} \\
\lambda_t f_{st} + \mu_t g_{st} &= 0 \\
\mu_t &= \beta\lambda_{t+1} f_{kt} + \beta\mu_{t+1}(g_{kt} + 1 - \delta).
\end{aligned}
$$

So far, everything qualifies this model as a two-sector model. Let us ask, however, under which conditions this model is equivalent to the canonical one-sector growth model. Note that, above, the ratio μ_t/λ_t denotes the price of capital in units of consumption. For this model to have a one-sector interpretation, we require that $\lambda = \mu$ for all t. We can combine the second, third and fourth equation above to show that optimality requires:

$$\frac{\lambda}{\mu} = \frac{u_{nc}/f_{nc}}{u_{nk}/g_{nk}} = -\frac{g_s}{f_s}$$

and assuming $u_{nc} = u_{nk}$ and using $-g_s = g_{kk}k_{t-1}$ and $f_s = f_{kc}k_{t-1}$, we have that $\lambda/\mu = 1$ requires that:

$$\frac{f_{kc}}{f_{nc}} = \frac{g_{kk}}{g_{nk}}$$

In the equality above, we have that the ratio of MPK and MPL across sectors must be equalized. Because of the CRTS assumption, we know that MPK and MPL are only function of the k/n ratio, so that we can write:

$$\frac{f_{kc}(k_c/n_c)}{f_{nc}(k_c/n_c)} = \frac{g_{kk}(k_k/n_k)}{g_{nk}(k_k/n_k)}$$

If f and g are the same function, equality between LHS and RHS requires that $\frac{k_c}{n_c} = \frac{k_k}{n_k} = \frac{k}{n}$. Hence, if f and g are the same function and if labor is fully mobile across sectors, the two-sector model is equivalent to a one-sector one.

When this happens, we can combine the two constraints into a single one of the form:

$$
\begin{aligned}
c_t + k_t &= f(s_t k_{t-1}, n_{ct}) + g((1-s_t)k_{t-1}, n_{kt}) + (1-\delta)k_{t-1} = \\
&= n_{ct} f\left(\frac{s_t k_{t-1}}{n_{ct}}\right) + n_{kt} g\left(\frac{(1-s_t)k_{t-1}}{n_{kt}}\right) + (1-\delta)k_{t-1} = \\
&= (n_{ct} + n_{kt}) f\left(\frac{k_{t-1}}{n_{ct} + n_{kt}}\right) + (1-\delta)k_{t-1}
\end{aligned}
$$

so that we are back into the one-sector world.

Exercise 5 *A simple homework for you to check you have the intuition for the workings of this model: write down the optimality and equilibrium conditions for this model under the assumption that there are competitive markets, and that households rent capital and labor to two types of firms, a consumption-good firm and an investment-good firm. Normalize the price of consumption to* 1 *and denote with q the price of investment.* (1) *Show that in equilibrium the relative price of investment is* 1 *so long as f and g are equal.* (2) *What is the relationship between q and the λ/μ ratio?*

5.2.2 Same as above, with adjustment costs

The social planner maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, n_{ct}, n_{kt}\right)$$

subject to

$$c_t = f\left(s_t k_{t-1}, n_{ct}\right)$$
$$k_t = g\left(\left(1-s_t\right) k_{t-1}, n_{kt}\right)\left(1 - G\left(\frac{k_t}{k_{t-1}}\right)\right) + \left(1-\delta\right) k_{t-1}$$

where G is such that

$$\begin{aligned} G &= G\left(\frac{k_t}{k_{t-1}}\right), G\left(1\right) = 0 \\ G'\left(1\right) &= 0 \\ G''\left(1\right) &> 0. \end{aligned}$$

The idea is that this function penalizes any change in the capital stock between one period and the next.

An alternative version replaces the function G with the function Γ, where:

$$\begin{aligned} \Gamma &= \Gamma\left(\frac{k_t - \left(1-\delta\right) k_{t-1}}{k_{t-1} - \left(1-\delta\right) k_{t-2}}\right), \Gamma\left(1\right) = 0 \\ \Gamma'\left(1\right) &= 0, \Gamma''\left(1\right) > 0 \end{aligned}$$

this function penalizes changes in investment growth.

5.2.3 A simple two-sector model with intermediate inputs

The social planner maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, n_{ct}, n_{kt}\right)$$

subject to

$$c_t + m_t = f\left(s_t k_{t-1}, n_{ct}\right)$$
$$k_t = g\left(\left(1-s_t\right) k_{t-1}, m_t, n_{kt}\right) + \left(1-\delta\right) k_{t-1}$$

Notice that m is an intermediate good, and we now break the symmetry between the two sectors. What about the accounting of this model? Clearly, defined in units of c,

$$f + p_g g = c + m + p_g \left(k' - (1 - \delta)\, k\right)$$

However, $f + p_g g$ is not total GDP. Why? Because we want to consider only final goods in the definition of GDP.

The corrected definition of GDP is:

$$GDP = f + p_g g - m$$

5.2.4 Baxter (RESTAT, 1996)

The model by Baxter assumes that the sector producing durables also produces capital for the production of the consumption good. In this sense, it is the opposite of the Greenwood and Hercowitz (see below) model:.

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, h_t, l_t\right)$$

subject to

$$\begin{aligned} y_t &= f\left(k_{ct-1}, l_{ct}\right) = c_t \\ i_t &= g\left(k_{kt-1}, l_{ht}\right) = h_t - (1 - \delta_h)\, h_{t-1} + k_{ct} - (1 - \delta_{kc})\, k_{ct-1} + k_{kt} - (1 - \delta_{kk})\, k_{kt-1} \end{aligned}$$

Bexter also discusses assumption relative to the substitutabilty between c and h.

5.2.5 Greenwood and Hercowitz (JPE, 1991)

GH is a two sector model with a key asymmetry between the two (market and home) sectors: only the market sector (the one producing the consumption good) can produce capital.

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c, h_t, l_t\right)$$

subject to

$$c_t + k_{ct} + k_{ht} - (1 - \delta_k)\left(k_{ct-1} + k_{ht-1}\right) = y_t = f\left(k_{ct-1}, z_t l_t\right)$$
$$h_t = g\left(k_{ht-1}, z_t\left(1 - l_t\right)\right)$$

where z_t is a labor productivity shock.

Measurement:

$$\begin{aligned} y_t &: \text{ GNP less gross housing product} \\ c_t &: \text{ PCE on nondurables, services less housing} \\ \Delta k_{ht} &: \text{ PCE on durables + residential investment} \\ \Delta k_{ct} &: \text{ Nonresidential investment} \\ h_t &: \text{ non-measured home/household production} \end{aligned}$$

There are several variants of this model: for instance Gomme, Kydland and Rupert (JPE, 2001) introduce a time-to-build technology for the production of market capital. All these models belong to the so-called home production literature.

5.2.6 Greenwood, Hercowitz and Krusell (EER 2000)

The social planner maximizes:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c, n_t)$$

subject to:

$$c_t + \frac{1}{q}\left(k_{et} - (1-\delta_{ke}) k_{et-1}\right) + k_{ht} - (1-\delta_{kh}) k_{ht-1} = y_t$$

$$y_t = f(k_{ht-1}, k_{et-1}, l_t)$$

Measurement

$$\begin{aligned} y_t &: \text{ GNP} \\ k_{ht} - (1-\delta) k_{ht-1} &: \text{ Structures investment} \\ k_{et} - (1-\delta) k_{et-1} &: \text{ Equipment investment} \end{aligned}$$

So in a sense this is a one sector model with equipment specific shocks. In particular, GHK use data on the equipment price deflator to back out a series for q which can then be used to calibrate the model.

5.2.7 Christiano and Fisher (2003)

Christiano and Fisher (NBER WP 10031, Stock Market and Investment Good Prices) use a multi sector model to study the behavior of stock prices. The period utility of the representative agent is:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_{ct}, n_{it})$$

$$\begin{aligned} c_t &= A_{ct} k_{ct-1}^{\alpha_c} n_{ct}^{1-\alpha_c} \\ i_{ct} + i_{it} &= A_{ct} A_{kt} k_{it-1}^{\alpha_i} n_{it}^{1-\alpha_i} \\ k_{ct} &= (1-\delta) k_{ct-1} + i_{ct}(1-G_{ct}) \\ k_{it} &= (1-\delta) k_{it-1} + i_{it}(1-G_{it}) \end{aligned}$$

where now G is the adjustment cost is on investment, modelled as:

$$G = \frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2$$

with the following properties $G = G\left(\frac{i_t}{i_{t-1}}\right) \Longrightarrow G(1) = 0,\ G'(1) = 0,\ G''(1) > 0.$

The planner's problem is

$$\begin{aligned}
&\max E_t(u(c_t, n_{ct}, n_{it}) + \beta u(c_{t+1}, n_{ct+1}, n_{it+1}) \\
&-\lambda_t\left(c_t - A_{ct}k_{ct-1}^{\alpha_c}n_{ct}^{1-\alpha_c}\right) - \beta\lambda_{t+1}\left(c_{t+1} - A_{ct+1}k_{ct}^{\alpha_c}n_{ct+1}^{1-\alpha_c}\right) \\
&-v_t\left(i_{ct} + i_{it} - A_{ct}A_{kt}k_{it-1}^{\alpha_i}n_{it}^{1-\alpha_i}\right) - \beta v_{t+1}\left(i_{ct+1} + i_{it+1} - A_{ct+1}A_{kt+1}k_{it}^{\alpha_i}n_{it+1}^{1-\alpha_i}\right) \\
&-\mu_{ct}\left(k_{ct} - (1-\delta)k_{ct-1} - i_{ct}(1 - G_{ct})\right) - \beta\mu_{ct+1}\left(k_{ct+1} - (1-\delta)k_{ct} - i_{ct+1}(1 - G_{ct+1})\right) \\
&-\mu_{it}\left(k_{it} - (1-\delta)k_{it-1} - i_{it}(1 - G_{it})\right) - \beta\mu_{it+1}\left(k_{it+1} - (1-\delta)k_{it} - i_{it+1}(1 - G_{it+1})\right) - ...)
\end{aligned}$$

The Euler equations for $c_t, k_{ct}, k_{it}, i_{ct}, i_{it}, n_{ct}, n_{it}$ are:

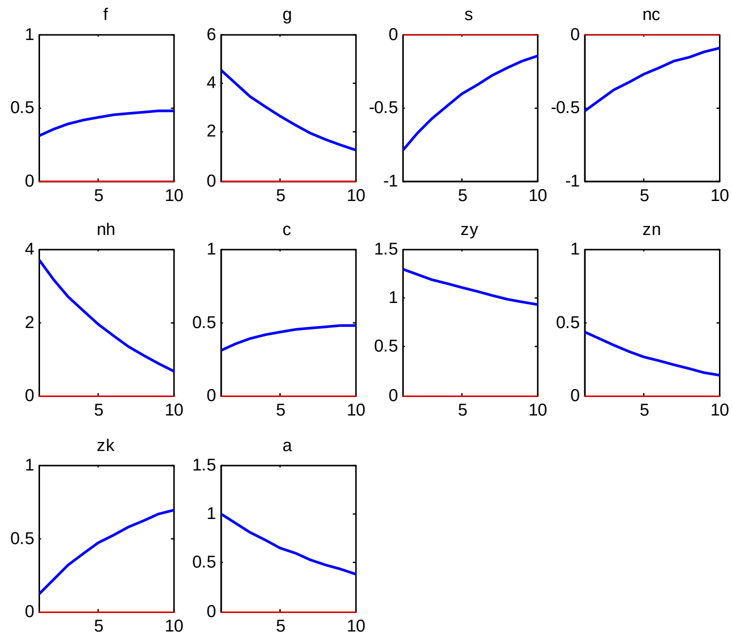
$$\begin{aligned}
u_{ct} &= \lambda_t \\
\mu_{ct} &= \beta\lambda_{t+1}\alpha_c\frac{c_{t+1}}{k_{ct}} + \mu_{ct+1}(1-\delta) \\
\mu_{it} &= \beta\lambda_{t+1}\alpha_i\frac{i_{ct+1} + i_{it+1}}{k_{it}} + \mu_{it+1}(1-\delta) \\
v_t &= \mu_{ct}\left(1 - G_{ct} - i_{ct}\frac{dG_{ct}}{di_{ct}}\right) - \beta\mu_{ct+1}i_{ct+1}\frac{dG_{ct+1}}{di_{ct}} \\
v_t &= \mu_{it}\left(1 - G_{it} - i_{it}\frac{dG_{it}}{di_{it}}\right) - \beta\mu_{it+1}i_{it+1}\frac{dG_{it+1}}{di_{it}} \\
\frac{du_t}{dn_{ct}} &= \lambda_t(1-\alpha_c)\frac{c_t}{n_{ct}} \\
\frac{du_t}{dn_{it}} &= \nu_t(1-\alpha_i)\frac{i_{ct} + i_{it}}{n_{it}}
\end{aligned}$$

They use the solution of the planner problem to study the behavior of stock prices in their model. In their model, the stock price is the value of the economy's capital stock, defined as:

$$S_t = \frac{\mu_{ct}k_{ct} + \mu_{it}k_{it}}{\lambda_t}$$

5.2.8 The Comovement Puzzle

The variable names correspond to those of the first model of this chapter. Following a 1% productivity shock, you can see that: (1) s falls, that is capital is shifted away from the consumption sector to the investment sector (even if it rises in the aggregate, see zk); (2) that while aggregate labor (zn) increases, it falls in the consumption sector (nc), it rises in the investment sector (nh). This is a puzzle because in the data we do not observe lots of reallocation in response to business cycle shocks. See paper by Christiano and Fitzgerald (2000) in the reading list for more details.



Simulation from twosecrbc1.mod

5.3 The Iacoviello and Neri (2009) paper

5.3.1 Overview

Build and estimate a quantitative, small-scale model of the housing market and the macroeconomy with financing frictions on the household side.

Two sectors: consumption-business investment sector (sticky prices and wages), housing sector (flexible prices, sticky wages).

Features that we want to capture:

Role of housing as collateral for loans, and potential wealth effects on consumption

Long-run trends and cyclical movements in housing prices and housing investment

Contribution of the recent housing boom to consumption and investment growth

Contribution of monetary policy to house price and housing investment dynamics

Four main elements in our paper:

(1) multi-sector structure with housing;

(2) nominal rigidities and monetary policy;

(3) financing frictions;

(4) lots of shocks and structural estimation.

Main modeling choices

Two Sectors

$Y-$sector produces consumption and business investment (using capital and labor)

$IH-$sector produces new homes (using capital, labor and land)

Two Types of Households

Patient Households work, consume, buy homes, rent capital and land to firms and lend to impatient households

Impatient/Credit Constrained Households work, consume, buy homes and borrow against the value of their home

(We set up preferences in a way that, for small shocks, the constraint is always binding)

Other modeling choices

Different trend technological progress across sectors (C, IK, IH)

Sticky prices in the non-housing sector (Calvo-style price rigidity and indexation), flexible house prices

Sticky wages in both sectors

Central bank runs monetary policy

Real rigidities: habits in C, imperfect labor mobility across sectors, K adjustment costs, variabile K utilization

Debt contracts are in nominal terms

Different types of shocks

5.3.2 Firms

Maximize profits:

$$\frac{Y_t}{X_t} + q_t IH_t - \left(\sum w_{it} n_{it} + R_{ct} k_{ct-1} + R_{ht} k_{ht-1} + p_{bt} k_{bt} + R_{lt} l_{t-1}\right)$$

where

$$Y_t = \left(\mathsf{A}_{ct}\left(n_{ct}^{\alpha} n_{ct}'^{1-\alpha}\right)\right)^{1-\mu_c}\left(z_{ct} k_{ct-1}\right)^{\mu_c}$$
$$IH_t = \left(\mathsf{A}_{ht}\left(n_{ht}^{\alpha} n_{ht}'^{1-\alpha}\right)\right)^{1-\mu_h-\mu_b-\mu_l}\left(z_{ht} k_{ht-1}\right)^{\mu_h} l_{t-1}^{\mu_l} k_{bt}^{\mu_b}$$

Two types of households/workers of measure 1 (more on this below)

α : wage share accruing to unconstrained households

$1-\alpha$: wage share accruing to constrained households.

Y_t : wholesale good, price of $1/X_t$ relative to the final good

Final good produced by "retailers", each producing a differentiated good

The retailer pricing decision (subject to Calvo constraint and indexation constraint) implies standard NK Phillips curve:

$$\log \pi_t - \iota_\pi \log \pi_{t-1} = \beta G_C \left(E_t \log \pi_{t+1} - \iota_\pi \log \pi_t\right) - \varepsilon_\pi \log\left(\frac{X_t}{X}\right) + \log \mathsf{u}_t$$

5.3.3 Unconstrained Households

Maximize utility

$$E_0 \sum_{t=0}^{\infty} \left(\beta G_C\right)^t \mathsf{z}_t \begin{pmatrix} \log\left(c_t - \varepsilon c_{t-1}\right) + \mathsf{j}_t \log h_t \\ -\frac{\tau_t}{1+\eta}\left(n_{ct}^{1+\xi} + n_{ht}^{1+\xi}\right)^{\frac{1+\eta}{1+\xi}} \end{pmatrix}$$

subject to budget constraint:

$$\begin{aligned} & c_t + \frac{k_{ct}}{\mathsf{A}_{kt}} + k_{ht} + k_{bt} + q_t\left(h_t - \left(1-\delta_h\right) h_{t-1}\right) + \phi_t + p_{lt}\left(l_t - l_{t-1}\right) \\ = & \left(R_{ct} z_{ct} + \frac{1-\delta_k}{A_{kt}} - a(\cdot)\right) k_{ct-1} + \left(R_{ht} z_{ht} + 1 - \delta_k - a(\cdot)\right) k_{ht-1} \\ & + p_{bt} k_{bt} + \frac{w_{ct}}{X_{wct}} n_{ct} + \frac{w_{ht}}{X_{wht}} n_{ht} + Div_t + b_t - \frac{R_{t-1} b_{t-1}}{\pi_t} + R_{lt} l_{t-1} \end{aligned}$$

ϕ_t : quadratic adjustment costs for the two types of capital

Div_t : profits from monopolistic competition

k_{bt} : intermediate good (e.g. bricks)

l_t : stock of land rented to firms (fixed)

5.3.4 Constrained Households

Maximize utility, discount future more heavily ($\beta' < \beta$)

$$E_0 \sum_{t=0}^{\infty} \left(\beta' G_C\right)^t \mathsf{z}_t \left(\log\left(c_t' - \varepsilon' c_{t-1}'\right) + \mathsf{j}_t \log h_t' - \frac{\tau_t}{1+\eta'}\left(n_{ct}'^{1+\xi'} + n_{ht}'^{1+\xi'}\right)^{\frac{1+\eta'}{1+\xi'}}\right)$$

subject to budget constraint

$$c'_t + q_t\left(h'_t - (1-\delta_h)\,h'_{t-1}\right) = \frac{w'_{ct}}{X'_{wct}} n'_{ct} + \frac{w'_{ht}}{X'_{wht}} n'_{ht} + b'_t - \frac{R_{t-1}}{\pi_t} b'_{t-1}$$

and to borrowing constraint

$$b'_t \leq mE_t\left(q_{t+1}h'_t\pi_{t+1}/R_t\right)$$

5.3.5 Monetary Policy

$$\begin{aligned}
R_t &= (R_{t-1})^{r_R}\left(\pi_t^{r_\pi}\left(\frac{GDP_t}{g_C GDP_{t-1}}\right)^{r_Y}\overline{rr}\right)^{1-r_R}\frac{\mathsf{e}_{Rt}}{\mathsf{s}_t} \\
\mathsf{e}_{Rt} &: \text{ iid monetary policy shock} \\
\mathsf{s}_t &: \text{ highly persistent inflation objective shock}
\end{aligned}$$

5.3.6 Shocks

Stationary AR(1)

z_t : preference (discount factor) shock

j_t : housing demand shock

τ_t : labor supply shock

e_{Rt} : monetary shock (iid)

s_t : inflation objective shock

u_t : markup/inflation shock (iid)

Trend-stationary shocks

$$\begin{aligned}
\ln \mathsf{A}_{ct} &= t\ln(1+\gamma_{AC}) + \ln A_{ct}, \quad \ln A_{ct} = \rho_{AC}\ln A_{ct-1} + \varepsilon_{ct} \\
\ln \mathsf{A}_{ht} &= t\ln(1+\gamma_{AH}) + \ln A_{ht}, \quad \ln A_{ht} = \rho_{AH}\ln A_{ht-1} + \varepsilon_{ht} \\
\ln \mathsf{A}_{kt} &= t\ln(1+\gamma_{AK}) + \ln A_{kt}, \quad \ln A_{kt} = \rho_{AK}\ln A_{kt-1} + \varepsilon_{kt}
\end{aligned}$$

5.3.7 Market clearing

$$C_t + IK_{ct}/\mathsf{A}_{kt} + IK_{ht} + k_{bt} = Y_t - \phi_t$$

$$h_t + h'_t - (1-\delta_h)\left(h_{t-1} + h'_{t-1}\right) = IH_t.$$

By Walras' law, $b_t + b'_t = 0$.

5.3.8 Model workings

1. At a basic level, model works like an RBC model with sticky prices/wage in the $Y-$sector, like an RBC with flex prices/sticky wages in the $IH-$sector

2. Sector specific shocks or preference shocks can shift resources from one sector to the other

3. Role of housing, debt and borrowing constraints

Housing as collateral generates wealth effects on consumption from fluctuations in house prices

Debt in nominal terms creates potential for debt deflation effects

ROLE OF TRENDS

1. Log preferences and Cobb-Douglas yield balanced growth

2. C and qIH grow at the same rate over time.

3. IK can grow faster than C, thanks to A_k progress

4. IH can grow slower than C, if land is a limiting factor and A_h is slow

5. Long-run growth rates

$$\begin{aligned}
G_C &= G_{IK_h} = G_{q\times IH} = 1+\gamma_{AC}+\frac{\mu_c}{1-\mu_c}\gamma_{AK} \\
G_{IK_c} &= 1+\gamma_{AC}+\frac{1}{1-\mu_c}\gamma_{AK} \\
G_{IH} &= 1+(\mu_h+\mu_b)\gamma_{AC}+\frac{\mu_c(\mu_h+\mu_b)}{1-\mu_c}\gamma_{AK}+(1-\mu_h-\mu_l-\mu_b)\gamma_{AH} \\
G_q &= 1+(1-\mu_h-\mu_b)\gamma_{AC}+\frac{\mu_c(1-\mu_h-\mu_b)}{1-\mu_c}\gamma_{AK}-(1-\mu_h-\mu_l-\mu_b)\gamma_{AH}.
\end{aligned}$$

5.3.9 The Complete Model

We summarize here the equations describing the equilibrium of the model. Let u_c denote the marginal utility of consumption, u_{nc} (u_{nh}) the marginal disutility of working in the goods (housing) sector, and u_h the marginal utility of housing (with analogous definitions holding for impatient households). We drop the t subscript to denote the steady-state value of a particular variable. The budget constraint for patient households is:

$$\begin{aligned} c_t + \frac{k_{c,t}}{\mathsf{A}_{k,t}} + k_{h,t} + k_{b,t} + q_t h_t + p_{l,t} l_t - b_t = \frac{w_{c,t}}{X_{wc,t}} n_{c,t} + \frac{w_{h,t}}{X_{wh,t}} n_{h,t} - \phi_t \\ + \left(R_{c,t} z_{c,t} + \frac{1-\delta_{kc}}{\mathsf{A}_{k,t}} \right) k_{c,t-1} + \left(R_{h,t} z_{h,t} + 1 - \delta_{kh} \right) k_{h,t-1} + p_{b,t} k_{b,t} - \frac{R_{t-1} b_{t-1}}{\pi_t} \\ + \left(p_{l,t} + R_{l,t} \right) l_{t-1} + q_t \left(1 - \delta_h \right) h_{t-1} + Div_t - \frac{a\left(z_{c,t}\right)}{\mathsf{A}_{k,t}} k_{c,t-1} - a\left(z_{h,t}\right) k_{h,t-1}. \end{aligned} \tag{A.1}$$

The first-order conditions for patient households are:

$$u_{c,t} q_t = u_{h,t} + \beta G_C E_t \left(u_{c,t+1} q_{t+1} \left(1 - \delta_h \right) \right) \tag{A.2}$$

$$u_{c,t} = \beta G_C E_t \left(u_{c,t+1} R_t / \pi_{t+1} \right) \tag{A.3}$$

$$u_{c,t} \left(\frac{1}{\mathsf{A}_{k,t}} + \frac{\partial \phi_{c,t}}{\partial k_{c,t}} \right) = \beta G_C E_t u_{c,t+1} \left(R_{c,t+1} z_{c,t+1} - \frac{a\left(z_{c,t+1}\right) + 1 - \delta_{kc}}{\mathsf{A}_{k,t+1}} - \frac{\partial \phi_{c,t+1}}{\partial k_{c,t}} \right) \tag{A.4}$$

$$u_{c,t} \left(1 + \frac{\partial \phi_{h,t}}{\partial k_{h,t}} \right) = \beta G_C E_t u_{c,t+1} \left(R_{h,t+1} z_{h,t+1} - a\left(z_{h,t+1}\right) + 1 - \delta_{kh} - \frac{\partial \phi_{h,t+1}}{\partial k_{h,t}} \right) \tag{A.5}$$

$$u_{c,t} w_{c,t} = u_{nc,t} X_{wc,t} \tag{A.6}$$

$$u_{c,t} w_{h,t} = u_{nh,t} X_{wh,t} \tag{A.7}$$

$$u_{ct} \left(p_{bt} - 1 \right) = 0 \tag{A.8}$$

$$R_{ct} \mathsf{A}_{kt} = a' \left(z_{ct} \right) \tag{A.9}$$

$$R_{ht} = a' \left(z_{ht} \right) \tag{A.10}$$

$$u_{c,t} p_{l,t} = \beta G_C E_t u_{c,t+1} \left(p_{l,t+1} + R_{l,t+1} \right). \tag{A.11}$$

The budget and borrowing constraint for impatient households are:

$$c'_t + q_t h'_t = \frac{w'_{c,t}}{X'_{wc,t}} n'_{c,t} + \frac{w'_{h,t}}{X'_{wh,t}} n'_{h,t} + b'_t - \frac{R_{t-1}}{\pi_t} b'_{t-1} + q_t \left(1 - \delta_h \right) h'_{t-1} + Div'_t \tag{A.12}$$

$$b'_t = m E_t \left(q_{t+1} h'_t \pi_{t+1} / R_t \right) \tag{A.13}$$

and the first-order conditions are:

$$u_{c',t} q_t = u_{h',t} + \beta' G_C E_t \left(u_{c',t+1} \left(q_{t+1} \left(1 - \delta_h \right) \right) \right) + E_t \left(\lambda_t \frac{m q_{t+1} \pi_{t+1}}{R_t} \right) \tag{A.14}$$

$$u_{c',t} = \beta' G_C E_t \left(u_{c',t+1} \frac{R_t}{\pi_{t+1}} \right) + \lambda_t \tag{A.15}$$

$$u_{c',t} w'_{c,t} = u_{nc',t} X'_{wc,t} \tag{A.16}$$

$$u_{c',t} w'_{h,t} = u_{nh',t} X'_{wh,t} \tag{A.17}$$

where λ_t denotes the multiplier on the borrowing constraint, which is greater than zero in a neighborhood of the equilibrium.

The production technologies are:

$$Y_t = \left(\mathsf{A}_{c,t}\left(n_{c,t}^{\alpha} n_{c,t}^{\prime 1-\alpha}\right)\right)^{1-\mu_c}\left(z_{c,t}k_{c,t-1}\right)^{\mu_c} \tag{A.18}$$

$$IH_t = \left(\mathsf{A}_{h,t}\left(n_{h,t}^{\alpha} n_{h,t}^{\prime 1-\alpha}\right)\right)^{1-\mu_h-\mu_l-\mu_b} k_{b,t}^{\mu_b}\left(z_{h,t}k_{h,t-1}\right)^{\mu_h} l_{t-1}^{\mu_l}. \tag{A.19}$$

The first-order conditions for the wholesale goods firms are:

$$\left(1-\mu_c\right)\alpha Y_t = X_t w_{c,t} n_{c,t} \tag{A.20}$$

$$\left(1-\mu_c\right)\left(1-\alpha\right)Y_t = X_t w'_{c,t} n'_{c,t} \tag{A.21}$$

$$\left(1-\mu_h-\mu_l-\mu_b\right)\alpha q_t IH_t = w_{h,t} n_{h,t} \tag{A.22}$$

$$\left(1-\mu_h-\mu_l-\mu_b\right)\left(1-\alpha\right)q_t IH_t = w'_{h,t} n'_{h,t} \tag{A.23}$$

$$\mu_c Y_t = X_t R_{c,t} z_{c,t} k_{c,t-1} \tag{A.24}$$

$$\mu_h q_t IH_t = R_{h,t} z_{h,t} k_{h,t-1} \tag{A.25}$$

$$\mu_l q_t IH_t = R_{l,t} l_{t-1} \tag{A.26}$$

$$\mu_b q_t IH_t = p_{b,t} k_{b,t}. \tag{A.27}$$

The Phillips curve is:

$$\ln \pi_t - \iota_\pi \ln \pi_{t-1} = \beta G_C\left(E_t \ln \pi_{t+1} - \iota_\pi \ln \pi_t\right) - \varepsilon_\pi \ln\left(X_t/X\right) + u_{p,t}. \tag{A.28}$$

Denote with $\omega_{i,t}$ nominal wage inflation, that is, $\omega_{i,t} = \frac{w_{i,t}\pi_t}{w_{i,t-1}}$ for each sector/household pair. The four wage equations are:

$$\ln \omega_{c,t} - \iota_{wc} \ln \pi_{t-1} = \beta G_C\left(E_t \ln \omega_{c,t+1} - \iota_{wc} \ln \pi_t\right) - \varepsilon_{wc} \ln\left(X_{wc,t}/X_{wc}\right) \tag{A.29}$$

$$\ln \omega'_{c,t} - \iota_{wc} \ln \pi_{t-1} = \beta' G_C\left(E_t \ln \omega'_{c,t+1} - \iota_{wc} \ln \pi_t\right) - \varepsilon'_{wc} \ln\left(X_{wc,t}/X_{wc}\right) \tag{A.30}$$

$$\ln \omega_{h,t} - \iota_{wh} \ln \pi_{t-1} = \beta G_C\left(E_t \ln \omega_{h,t+1} - \iota_{wh} \ln \pi_t\right) - \varepsilon_{wh} \ln\left(X_{wh,t}/X_{wh}\right) \tag{A.31}$$

$$\ln \omega'_{h,t} - \iota_{wh} \ln \pi_{t-1} = \beta' G_C\left(E_t \ln \omega'_{h,t+1} - \iota_{wh} \ln \pi_t\right) - \varepsilon'_{wh} \ln\left(X_{wh,t}/X_{wh}\right) \tag{A.32}$$

where $\varepsilon_{wc} = \left(1-\theta_{wc}\right)\left(1-\beta G_C \theta_{wc}\right)/\theta_{wc}$, $\varepsilon'_{wc} = \left(1-\theta_{wc}\right)\left(1-\beta' G_C \theta_{wc}\right)/\theta_{wc}$,
$\varepsilon_{wh} = \left(1-\theta_{wc}\right)\left(1-\beta G_C \theta_{wc}\right)/\theta_{wc}$ and $\varepsilon'_{wh} = \left(1-\theta_{wh}\right)\left(1-\beta' G_C \theta_{wh}\right)/\theta_{wh}$.

The Taylor rule is:

$$R_t = \left(R_{t-1}\right)^{r_R} \pi_t^{r_\pi(1-r_R)}\left(\frac{GDP_t}{G_C GDP_{t-1}}\right)^{r_Y(1-r_R)} \overline{rr}^{1-r_R}\frac{u_{R,t}}{\mathsf{s}_t} \tag{A.33}$$

where GDP_t is the sum of the value added of the two sectors, that is $GDP_t = Y_t - k_{b,t} + \overline{q} IH_t$. Two market-clearing conditions are

$$C_t + IK_{c,t}/\mathsf{A}_{k,t} + IK_{h,t} + k_{b,t} = Y_t - \phi_t \tag{A.34}$$

$$h_t + h'_t - \left(1-\delta_h\right)\left(h_{t-1} + h'_{t-1}\right) = IH_t. \tag{A.35}$$

By Walras' law, $b_t + b'_t = 0$. Finally, total land is normalized to unity:

$$l_t = 1. \tag{A.36}$$

In equilibrium, dividends paid to households equal respectively:

$$Div_t = \frac{X_t - 1}{X_t} Y_t + \frac{X_{wc,t} - 1}{X_{wc,t}} w_{c,t} n_{c,t} + \frac{X_{wh,t} - 1}{X_{wh,t}} w_{h,t} n_{h,t}$$
$$Div'_t = \frac{X'_{wc,t} - 1}{X'_{wc,t}} w'_{c,t} n'_{c,t} + \frac{X'_{wh,t} - 1}{X'_{wh,t}} w'_{h,t} n'_{h,t}.$$

In addition, the functional forms for the capital adjustment cost and the utilization rate are:

$$\phi_t = \frac{\phi_{kc}}{2G_{IK_c}} \left(\frac{k_{c,t}}{k_{c,t-1}} - G_{IK_c} \right)^2 \frac{k_{c,t-1}}{(1+\gamma_{AK})^t} + \frac{\phi_{kh}}{2G_{IK_h}} \left(\frac{k_{h,t}}{k_{h,t-1}} - G_{IK_h} \right)^2 k_{h,t-1}$$
$$a(z_{c,t}) = R_c \left(\varpi z_{c,t}^2/2 + (1-\varpi) z_{c,t} + (\varpi/2 - 1) \right)$$
$$a(z_{h,t}) = R_h \left(\varpi z_{h,t}^2/2 + (1-\varpi) z_{h,t} + (\varpi/2 - 1) \right)$$

where R_c and R_h are the steady-state values of the rental rates of the two types of capital. In the estimation of the model, we specify our prior for the curvature of the capacity utilization function in terms of $\zeta = \varpi/(1+\varpi)$. With this change of variables, ζ is bounded between 0 and 1, since ϖ is positive.

Equations A.1 to A.36 together with the values for IK_c, IK_h, GDP_t, ϕ_t, $a(z)$, Div_t and Div'_t and the laws of motion for the exogenous shocks (reported in the main text) define a system of 36 equations in the following variables: c, h, k_c, k_h, k_b, n_c, n_h, b, l, z_c, z_h, c', h', n'_c, n'_h, b', IH, Y, q, R, π, λ, X, w_c, w_h, w'_c, w'_h, X_{wc}, X_{wh}, X'_{wc}, X'_{wh}, R_c, R_h, R_l, p_b, and p_l.

After detrending the variables by their balanced growth trends, we linearize the resulting system around the non-stochastic steady-state and compute the decision rules using standard methods.

5.3.10 Estimation

See results in the paper.

Chapter 6

Money in Flexible Price Environments

Money plays the following roles in the economy: (1) medium of exchange; (2) store of value; (3) unit of account.

The fact that, compared to the old days, unbacked, non-commodity money is used derives from the fact that it can used as a medium of exchange. But why is unbacked paper money used? How can money affect real decisions?

The main approaches that have been followed in the literature to model a need/demand for money are:

(1) to assume that money yields direct utility or production services by incorporating money balances directly into the utility (Sidrauski 1967) or production function.

(2) to impose transaction costs of some form that give rise to a demand for money, either assuming that exchanging assets is costly (Baumol-Tobin), or that exchanging commodities is costly (MATCHING: Kiyotaki and Wright models), or that money is needed for certain types of transactions (CIA: Clower, 1967)

(3) treating money as an asset to transfer resources intertemporally (OLG model: Samuelson, 1958), while at the same money starving agents of alternative forms of saving.

We look at two popular models that endogenoulsy generate a demand for money. The Sidrauski model and the Cash-in-advance model.

6.1 The Sidrauski money-in-utility model

6.1.1 The setup

We put now money in the utility function. The advantage of doing so is that money is in this way not dominated by bonds (or capital) that pay a positive return in equilibrium.

$$\max_{C_t, K_t, M_t/P_t, L_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, \frac{M_t}{P_t}, L_t\right)$$

$$\begin{aligned} s.t.\ C_t + K_t + \frac{M_t}{P_t} + B_t &= R_{t-1}B_{t-1} + f\left(K_{t-1}, L_t\right) + (1-\delta)\,K_{t-1} + \frac{M_{t-1}}{P_t} + T_t \\ Y_t &= f\left(K_{t-1}, L_t\right) \end{aligned}$$

notice that once you choose K, M/P, and real bonds B, you are automatically choosing C as well. Notice that we could also drop *real bonds* from this formulation since they in equilibrium simply offer the same return as capital and define the gross return on capital as R.

We can think of $\frac{M}{P}$ as the service flow (assumed to be proportional to the stock) which is provided by money holdings. The budget constraint simply states that given the current income, its assets, and any transfers received by the government T_t, the households allocates its resources between (1) consumption; (2) gross investment in physical capital; gross accumulation of (3) real money and (4) bonds.

Notice also that the timing in discrete time models matters: Carlstrom and Fuerst (2001) have argued that M_{t-1} rather than M_t should enter the utility function, since - they say - it is the money balances that individuals hold before purchasing consumption goods that matter for utility.

The first order conditions for this problem are:

$$\begin{aligned} u_{C,t} &= \beta E_t\left(u_{C,t+1}R_t\right) && \text{(a)} \\ u_{C,t} &= \beta E_t\left(u_{C,t+1}\left(1-\delta+f_K\left(K_t\right)\right)\right) && \text{(b)} \\ u_{L,t} &= u_{C,t}f_L\left(L_t\right) && \text{(c)} \\ u_{C,t} &= u_{m,t} + \beta E_t\left(u_{C,t+1}\frac{1}{\pi_{t+1}}\right) && \text{(d)} \end{aligned}$$

(a) to (c) are familiar; (d) is the typical expression for the price of an asset: if I give up consumption today and decide to hold money forever from then on, I will enjoy the stream of utility services in square brackets, which will be eroded from the rise in prices between t and the future. To better notice this, notice that you can use repeated substitution on the right-hand side of (d) to write

$$u_{m,t} + \beta E_t\left(u_{C,t+1}\frac{1}{\pi_{t+1}}\right) = u_{m,t} + E_t\left(\frac{\beta}{\pi_{t+1}}u_{m,t+1} + \frac{\beta^2}{\pi_{t+1}\pi_{t+2}}u_{m,t+2} + \frac{\beta^3}{\pi_{t+1}\pi_{t+2}\pi_{t+3}}u_{C,t+3}\right)$$

Together with this, we also have appropriate transversality conditions that state that the present value of the stock of real resources of the economy approaches zero as the horizon goes to infinity (otherwise, the economy would be saving either too little or too much or would be accumulating too much debt.

$$\begin{aligned} \lim_{t\to\infty} \beta^t u_{C,t}m_t &= 0 \\ \lim_{t\to\infty} \beta^t u_{C,t}K_t &= 0 \\ \lim_{t\to\infty} \beta^t u_{C,t}B_t &= 0 \end{aligned}$$

Another important issue: suppose you also have nominal bonds Z_t traded offering I_t. Optimality requires:

$$\begin{aligned} u_{C,t} &= \beta E_t \left(u_{C,t+1} R_t\right) \\ u_{C,t}\frac{1}{P_t} &= \beta E_t \left(u_{C,t+1}\frac{1}{P_{t+1}} I_t\right) \end{aligned}$$

which implies:

$$R_t = E_t \left(I_t \frac{1}{\pi_{t+1}}\right)$$

which is the Fisher relationship.

6.1.2 Parametrization

Suppose we parameterize the model as follows (here $m_t \equiv M_t/P_t$ denotes real money):

$$\begin{aligned} u &= \frac{\left(aC_t^{1-b} + (1-a)\, m_t^{1-b}\right)^{\frac{1-\phi}{1-b}}}{1-\phi} - \frac{\tau L_t^{\eta}}{\eta} \\ Y_t &= A_t K_{t-1}^{\alpha} L_t^{1-\alpha} \end{aligned}$$

When $\phi = b = 1$, this specification becomes $u = a \log C + (1-a) \log m - \tau L_t^{\eta}/\eta$

Then, defining with $x_t = aC_t^{1-b} + (1-a)\, m_t^{1-b}$

$$\begin{aligned} \frac{\partial u}{\partial C} &\equiv u_C = a x^{\frac{b-\phi}{1-b}} C^{-b} \\ \frac{\partial u}{\partial m} &\equiv u_m = (1-a)\, x^{\frac{b-\phi}{1-b}} m^{-b} \end{aligned}$$

so that the MRS between consumption and money becomes:

$$MRS_{cm} = \frac{a}{1-a}\left(\frac{C_t}{m_t}\right)^{-b}$$

so that the EOS between C and m

$$\frac{d \log \left(C_t/m_t\right)}{d \log MRS} = -\frac{1}{b}$$

so that when $b = 0$ consumption and money are perfect substitutes. As b rises, they become more and more complements.

Another important thing to observe - which matters when we consider near steady state dynamics - . Taking derivative of u_C with respect to m gives:

$$u_{cm} = a x_t^{\frac{b-\phi}{1-b}-1} C_t^{-b} (b-\phi)(1-a)\, m_t^{-b}$$

which is positive if $b > \phi$. That is if $b > \phi$ consumption and money are complements.

6.1.3 Analyzing the model I: The steady state

6.1.3.1 Monetary side

Changes in money supply are made through transfers to the public. That is

$$T_t = \frac{M_t - M_{t-1}}{P_t} = \frac{M_t}{P_t} - \frac{M_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t}$$

Steady state transfers are simply:

$$T = \frac{M}{P} - \frac{M}{P}\frac{1}{\Pi} = m\left(\frac{\Pi - 1}{\Pi}\right) = \frac{M}{P} - \frac{M}{\theta P} = m\left(\frac{\theta - 1}{\theta}\right)$$

so that inflation in steady state is a monetary phenomenon.

$$\Pi = \theta$$

For money demand, using the nominal interest rate:

$$u_{C,t} = u_{m,t} + \frac{1}{I_t} u_{C,t}$$

simple algebra shows that

$$\begin{aligned}
u_C\left(\frac{I-1}{I}\right) &= u_m \\
\frac{a}{1-a}\left(\frac{C}{m}\right)^{-b} &= \frac{I}{I-1} \\
I &= \frac{\theta}{\beta} \\
\frac{a}{1-a}\left(\frac{C}{m}\right)^{-b} &= \frac{\theta}{\theta - \beta}
\end{aligned}$$

6.1.3.2 Real side

From (a) we can see that the capital output ratio and the consumption output ratio are not affected by anything that has to do with money. When the marginal utility of consumption is constant, the steady state capital stock solves, assuming $Y = K^{\alpha} L^{1-\alpha}$:

$$\frac{\alpha Y}{RK} = 1 - \frac{1-\delta}{R}$$

Also:

$$\begin{aligned}
\beta R &= 1 \\
\frac{K}{Y} &= \frac{\alpha}{R - (1-\delta)} \\
\frac{C}{Y} &= 1 - \frac{\delta K}{Y} = 1 - \frac{\alpha\delta}{R - (1-\delta)}
\end{aligned}$$

One can see that K/Y is independent of all parameters of the utility function; the inflation rate. As in Tobin model, inflation in steady state just equals the growth rate of money supply. Model displays **superneutrality of money in steady state**: **the real equilibrium (capital-output, consumption-output ratios) is independent of the rate of growth of money**. (neutrality: level of money does not affect real allocations; superneutrality: rate of growth of money does not affect real allocations)

It remains to be seen whether L is independent of money supply in this setup. Using (c):

$$\tau L^{\eta} = a\left(aC^{1-b} + (1-a)\, m^{1-b}\right)^{\frac{b-\phi}{1-b}} C^{-b} (1-\alpha)\, Y$$

unless $b = \phi$, L depends on m, which depends on θ. If $b > \phi$, faster money growth reduces m. Consumption falls (since C and m are complements) and therefore L falls.

As a matter of fact, we do not need to calculate L when we are interested in analyzing the model in deviations from the steady state. All that matters for labor supply dynamics is η, which dictates the elasticity of labor supply with respect to the real wage.

6.1.4 Analyzing the model II: The dynamics around the steady state

6.1.4.1 The consumption Euler equation

The only awkward stuff is how to linearize the u_C and u_m terms.

We know that

$$\begin{aligned} u_{ct} &= a x_t^{\frac{b-\phi}{1-b}} C_t^{-b} \\ u_{mt} &= (1-a)\, x_t^{\frac{b-\phi}{1-b}} m_t^{-b} \end{aligned}$$

$$\begin{aligned} \widehat{u}_{ct} &= \frac{b-\phi}{1-b}\widehat{x}_t - b\widehat{C}_t \\ \widehat{u}_{mt} &= \frac{b-\phi}{1-b}\widehat{x}_t - b\widehat{m}_t \end{aligned}$$

Simply from

$$\begin{aligned} x_t &= aC_t^{1-b} + (1-a)\, m_t^{1-b} \\ \widehat{x}_t &= \frac{aC^{1-b}}{aC^{1-b} + (1-a)\, m^{1-b}} (1-b)\, \widehat{C}_t + \frac{(1-a)\, m^{1-b}}{aC^{1-b} + (1-a)\, m^{1-b}} (1-b)\, \widehat{m}_t \\ \widehat{x}_t &= \gamma\, (1-b)\, \widehat{C}_t + (1-\gamma)\, (1-b)\, \widehat{m}_t \end{aligned}$$

so that the Euler equation for consumption is:

$$\omega_1 \left(\widehat{C}_t - \widehat{C}_{t+1}\right) - \omega_2 \left(\widehat{m}_t - \widehat{m}_{t+1}\right) = -\widehat{R}_t$$

where

$$\begin{aligned} \gamma &= \frac{aC^{1-b}}{aC^{1-b} + (1-a)\, m^{1-b}} = \left(1 + \frac{1-a}{a}\left(\frac{m}{C}\right)^{1-b}\right)^{-1} \\ \omega_1 &= \gamma\phi + (1-\gamma)\, b \\ \omega_2 &= (b-\phi)\, (1-\gamma) \end{aligned}$$

6.1.4.2 The money demand

From

$$\begin{aligned} \frac{a}{1-a}\left(\frac{C_t}{m_t}\right)^{-b} &= \frac{I_t}{I_t - 1} \\ -b\left(\log C_t - \log m_t\right) &= \log I_t - \log\left(I_t - 1\right) \\ \widehat{m}_t &= \widehat{C}_t - \frac{1}{b\,(I-1)}\widehat{I}_t \end{aligned}$$

6.1.4.3 The labor supply schedule

$$\begin{aligned} u_{L,t} &= u_{C,t} f_L(L_t) \\ (\eta - 1)\widehat{L}_t &= -\omega_1 \widehat{C}_t + \omega_2 \widehat{m}_t + \widehat{Y}_t - \widehat{L}_t \\ \widehat{Y}_t - \omega_1 \widehat{C}_t + \omega_2 \widehat{m}_t &= \eta \widehat{L}_t \end{aligned}$$

6.1.4.4 The evolution of money supply

We assume money balances evolve according to

$$M_t = \theta_t M_{t-1}$$

in this formulation one can immediately notice that, even if θ_t follows a stationary process, the log of nominal money does not. To achieve stationarity, we express the variables which are constant in steady state we divide everything by P_t. Next we multiply and divide the I by P_{t-1}, so that linearizing brings

$$\widehat{m}_t = \widehat{\theta}_t - \widehat{\pi}_t + \widehat{m}_{t-1}$$

6.1.4.5 The complete model

The log-linear equilibrium will be (here I change notation: the variables denote percentage deviations from the steady state):

$$\begin{aligned} Y_t &= \alpha k_{t-1} + (1-\alpha) L_t + A_t && (1) \\ Y_t &= \frac{C}{Y} C_t + \frac{K}{Y}\left(K_t - (1-\delta) K_{t-1}\right) && (2) \\ R_t &= \frac{\alpha Y}{Rk}\left(E_t Y_{t+1} - K_t\right) && (3) \\ -R_t &= \omega_1 \left(C_t - E_t C_{t+1}\right) - \omega_2 \left(m_t - E_t m_{t+1}\right) && (4) \\ \eta L_t &= Y_t - \omega_1 C_t + \omega_2 m_t && (5) \\ m_t &= C_t - \frac{1}{b(I-1)}\left(R_t + E_t \pi_{t+1}\right) && (6) \\ m_t - m_{t-1} &= -\pi_t + \theta_t && (7) \end{aligned}$$

In the special case in which utility is separable in consumption and money balances, $b = \phi$, $\omega_1 = \phi$ and $\omega_2 = 0$. This is an interesting case because now the equations of the model (1) to (7) can be separated in two independent blocks. The first block includes equations (1) to (5) adequately modified:

$$\begin{aligned} Y_t &= \alpha K_{t-1} + (1-\alpha) L_t + A_t && (1) \\ Y_t &= \frac{C}{Y} C_t + \frac{K}{Y}\left(K_t - (1-\delta) K_{t-1}\right) && (2) \\ R_t &= \left(1 - \frac{1-\delta}{R}\right) E_t \left(Y_{t+1} - K_t\right) && (3) \\ -R_t &= \omega_1 \left(C_t - E_t C_{t+1}\right) && (4) \\ \eta L_t &= Y_t - \omega_1 C_t && (5) \end{aligned}$$

we can see that we can solve for the dynamics of C_t, K_t, Y_t, R_t and L_t independently of the rest of the model. Money is thus completely neutral for the real variables, in and out of the steady state.

Similarly, the equations (6) and (7) to study inflation and money growth independently of the real variables. To this end, fix the real interest rate to some constant, so that consumption will be constant too. Then, combining (6) and (7) modified

$$\begin{aligned} m_t &= -\frac{1}{b\,(I-1)}\,(E_t\pi_{t+1}) && (6)\\ m_t - m_{t-1} &= -\pi_t + \theta_t && (7) \end{aligned}$$

we obtain:

$$\pi_t = \frac{1}{b\,(I-1)}\,(E_t\pi_{t+1} - E_{t-1}\pi_t) + \theta_t$$

(6) is a modified version of Cagan's money demand: if people expect high inflation in the future, they will reduce their real model holdings now.

6.1.5 Calibration

We want to calibrate the model so as to get reasonable values for the big real ratios, for the elasticity of money demand to the interest rate, and for the money consumption ratio.

From $\frac{a}{1-a}\left(\frac{C}{m}\right)^{-b} = \frac{\theta}{\theta-\beta}$ and $m_t = C_t - \frac{1}{b(I-1)}\,(R_t + \pi_{t+1})$ we can see that the elasticity of money demand to the nominal interest rate is $\frac{1}{b(I-1)}$. Assuming $I = \theta/\beta = 1.0125/.99 = 1.\,023$ and given that this elasticity is in the neighborhood of 3, we need a value of b such that $\frac{1}{b(I-1)} = 3$. Hence $b = 14.5$.

Estimates of $\frac{C}{m}$ depend on which measure of m we use.

In the US: http://www.federalreserve.gov/releases/H6/hist/h6hist1.txt:

$$\begin{aligned} M1 &= 1.285 \text{ trillion \$}\\ M2 &= 6.132\\ M3 &= 8.956 \end{aligned}$$

quarterly consumption is as of 2003Q2 is $7690/4 = 1.92$ trillion \$. Hence if we take a combo of $M1$ and $M2$ we can use $C/m = 1$ which yields

$$a = 0.978$$

This gives us $\gamma = 0.978$. As the intertemporal elasticity of substitution in consumption is $\omega_1 = \gamma\phi + (1-\gamma)\,b = .978\phi + 14.5\,(1-.978) = 0.978\,\phi + 0.319$, a value of $\phi = 2$ gives us $\omega_1 = 2.3$, which is plausible.

6.1.6 Dynamics

- Model generates non-trivial dynamics only when $u_{cm} <> 0$ (marginal utility of consumption falls when you have more money). Remember that

$$\begin{aligned} \text{sign } u_{cm} &= \text{sign } a x_t^{\frac{b-\phi}{1-b}-1} C_t^{-b} (b-\phi)(1-a) m_t^{-b} = \text{sign } (b-\phi) \\ -u_{L,t} &= u_{C,t} f_L (L_t) \end{aligned}$$

  1. $b > \phi$. $u_{cm} > 0$. When M rises persistently (say, autocorrelation 0.8), $E_t\pi_{t+1}$ rises, hence m falls , hence u_C falls. Hence individuals consume more leisure ($u_{Lt} = L^{\eta-1}$ falls, hence L falls given that $\eta > 1$), and work less. Output falls.
  2. $b < \phi$. $u_{cm} < 0$. When M rises persistently, $E_t\pi_{t+1}$ rises, hence m falls, hence u_C rises, consume less leisure, more work.

- So far, we have talked about persistent rise in M that raises $E_t\pi_{t+1}$. Start from a zero inflation steady state. Assume a purely temporary money supply shock. Given that future money growth rates are unaffected, $E_t\pi_{t+1}$ is also unaffected. None of the variables in (1) to (6) is affected. Hence $m_t = 0$. From:

$$\begin{aligned} m_t - m_{t-1} &= -\pi_t + \theta_t \\ 0\% - 0\% &= -1\% + 1\% \end{aligned}$$

  we see that inflation rises by 1% in the period of the shock, and then reverts to the baseline.

6.1.7 Welfare losses from inflation

Lucas proposes to estimate the welfare costs of inflation as the % increase in consumption that is required to make the household indifferent between a nominal interest rate of i_1 and a nominal interest rate of 0. Assume labor is supplied inelastically $(\eta = \infty)$, so we focus on the consumption-real balances margin only. Look at steady states only:

$$\begin{aligned} u(C,m,L) &= \frac{\left(aC_t^{1-b} + (1-a) m_t^{1-b}\right)^{\frac{1-\phi}{1-b}}}{1-\phi} - \frac{\tau L_t^{\eta}}{\eta} \\ m &= C\left(\frac{1-a}{a}\frac{\theta}{\theta-\beta}\right)^{\frac{1}{b}} \text{ in steady state} \\ u(C, m(C,\theta)) &= u(C,\theta) = \frac{\left(aC^{1-b} + (1-a) C^{\frac{1-b}{b}} \left(\frac{1-a}{a}\frac{\theta}{\theta-\beta}\right)^{\frac{1-b}{b}}\right)^{\frac{1-\phi}{1-b}}}{1-\phi} \end{aligned}$$

The welfare loss from any positive inflation rate $\overline{\theta} > 1$is the value of C that solves

$$\overline{u} = u(1,1) = u\left(\overline{C}, \overline{\theta}\right)$$

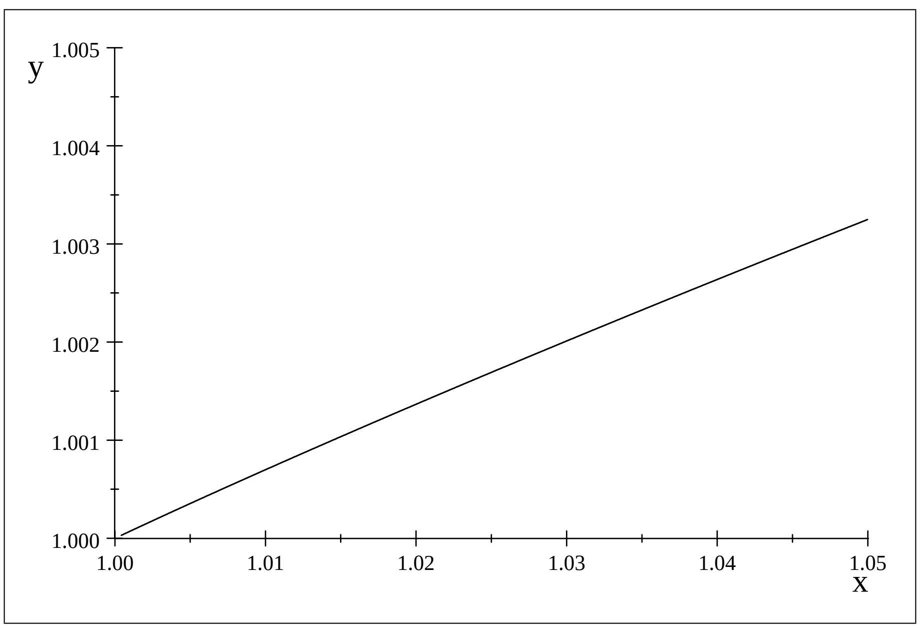
The value of $\overline{C}$ that solves the expression above is the compensation needed to be as well off with an interest rate $\overline{\theta}$ as with 1. Note that $\overline{C}$ is the % change in C require to compensate for the higher nominal interest rate.

Some numerical examples. Suppose we start from a steady state in which $\theta = 1$. Assume same parameters as above. The utility achieved when $C = 1$ and $\theta = 1$ is equal to

$$\begin{aligned} (1-\phi)\, u\,(C=1,\theta=1)^{\frac{1-b}{1-\phi}} &= .978C^{1-14.5} + .022C^{\frac{1-14.5}{14.5}} \left(\frac{.022}{.978} \frac{\theta}{\theta - .99} \right)^{\frac{1-14.5}{14.5}} \\ \overline{u} &= .978 + .022 \left(\frac{.022}{.978} \frac{1}{1 - .99} \right)^{\frac{-13.5}{14.5}} = 0.98834 \end{aligned}$$

Hence we look for all other (C, θ) pairs such that utility is equal to $\overline{u}$. The solution to this equations are in the graph, where y is $\overline{C}$ and θ is the steady state inflation rate.

$$.98834 = .978y^{1-14.5} + .022y^{\frac{1-14.5}{14.5}} \left(\frac{.022}{.978} \frac{x}{x - .99} \right)^{\frac{1-14.5}{14.5}}$$



Consumption (y) required to keep utility constant for every level of the nominal interest rate (x)

A quarterly inflation rate of 2.5% (10% per annum) would require an increase in steady state consumption of 0.2% per quarter (0.8% at an annual rate). Lucas obtains slightly higher numbers (1%), but the substance is the same.

6.1.8 The optimal rate of inflation

Marginal benefit of money is minus the nominal interest rate, marginal cost is zero, want the nominal rate to be zero. It is also know as the optimum quantity of money. Consider the following simple problem:

$$\begin{aligned} & \max_{m} u\left(C, m\right) \\ \text{s.t. } C &= Y - \delta K \\ &\rightarrow u_m = 0 \rightarrow I_t = 1 \text{ and } R_t = \frac{1}{\pi_{t+1}} \end{aligned}$$

The idea is that the optimal rate of inflation is a rate of deflation $\frac{P_t}{P_{t+1}}$ approximately equal to the real return on capital. This result is known as the Friedman rule.

6.2 Cash in advance model

We want to construct a model in which money is used as a medium of exchange that facilitates transactions yielding utility indirectly. In other words, money is needed to make transactions.

$$\max_{K_t, M_t/P_t, L_t, Z_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, L_t\right)$$

$$\begin{aligned}
\text{s.t. } C_t + K_t + \frac{M_t}{P_t} + \frac{Z_t}{P_t} &= \frac{P_{t-1}}{P_t} I_{t-1} \frac{Z_{t-1}}{P_{t-1}} + f\left(K_{t-1}, L_t\right) + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t \\
C_t &\leq \frac{M_{t-1}}{P_t} + T_t \qquad \text{(CIA_GA)}
\end{aligned}$$

- *TIMING DIGRESSION*

  In the formulation above, purchases of consumption goods require cash in hand M_{t-1} : the agent enters the period with money holdings M_{t-1} and receives a lump-sum transfer T_t in real terms, and transfers can be used only to make purchases of consumption goods. This timing convention is due to Svensson, and in such a case we say that the goods market opens first. The household cannot adjust its portfolio holdings after a money supply shock.

  Another possibility (due to Lucas and Stokey) is that households can adjust their portfolios after the money supply shock. In this case:

  $$C_t + \frac{Z_t}{P_t} \leq \frac{M_{t-1}}{P_t} + T_t \qquad \text{(CIA\_AG)}$$

  here after the transfer you can still choose your asset position before deciding your consumption level. Hence the asset market first, then the goods market. (see Woodford's book page 649)

- *MODELING DIGRESSION*

  Another way to generate a money demand is to assume no miu and no cash-in-advance, but to have something like

  $$C_t \left(1 + s\left(v_t\right)\right) + .. + \frac{M_t}{P_t} + .. = .. + f + .. + \frac{M_{t-1}}{P_t} + T_t$$

  where

  $$v_t = \frac{P_t C_t}{M_t}$$

  and s is

  (a) non-negative and twice continuously differentiable

  (b) there is a level $\underline{v} > 0$ of velocity such that $s\left(\underline{v}\right) = 0$, $s'\left(\underline{v}\right) = 0$: this ensures optimality of the Friedman rule without the need for an infinity money demand.

Notice that Z_{t-1} are the holdings of a nominal bond that yields a gross return of I_{t-1} from period $t-1$ to t (unlike the real bond B_t).

Let μ_t denote the multiplier on the cash-in-advance constraint. The Lagrangian for this problem is (replacing C from the flow of funds equation)

$$\begin{aligned} l \;=\; & u\left(\frac{I_{t-1}P_{t-1}}{P_t}\frac{Z_{t-1}}{P_{t-1}} + f\left(K_{t-1}, L_t\right) + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t - K_t - \frac{M_t}{P_t} - \frac{Z_t}{P_t}, L_t\right) + \\ & \beta E_t u\left(\frac{I_t P_t}{P_{t+1}}\frac{Z_t}{P_t} + f\left(K_t, L_{+1}\right) + (1-\delta) K_t + \frac{M_t}{P_t}\frac{P_t}{P_{t+1}} + T_{+1} - K_{+1} - \frac{M_{+1}}{P_{+1}} - \frac{Z_{+1}}{P_{+1}}, L_{+1}\right) \\ & -\mu_t \times \left(\frac{I_{t-1}P_{t-1}}{P_t}\frac{Z_{t-1}}{P_{t-1}} + f\left(K_{t-1}, L_t\right) + (1-\delta) K_{t-1} - K_t - \frac{M_t}{P_t} - \frac{Z_t}{P_t}\right) \\ & -\beta E_t \mu_{t+1} \times \left(\frac{I_t P_t}{P_{t+1}}\frac{Z_t}{P_t} + f\left(K_t, L_{+1}\right) + (1-\delta) K_t \; - K_{+1} - \frac{M_{+1}}{P_{+1}} - \frac{Z_{+1}}{P_{+1}}\right) \end{aligned}$$

The first order conditions are, choosing $Z_t/P_t, K_t, M_t/P_t$ and L_t

$$u_{C,t} - \mu_t = \beta E_t\left(\frac{P_t}{P_{t+1}} I_t\left(u_{C,t+1} - \mu_{t+1}\right)\right) \tag{1}$$

$$u_{C,t} - \mu_t = \beta E_t\left(\left(u_{C,t+1} - \mu_{t+1}\right)\left(1 - \delta + f_K\left(K_t\right)\right)\right) \tag{2}$$

$$u_{C,t} - \mu_t = \beta E_t\left(u_{C,t+1}\frac{P_t}{P_{t+1}}\right) \tag{3}$$

$$u_{C,t} - \mu_t = \frac{u_{L,t}}{f_L\left(L_t\right)} \tag{4}$$

One can use Euler equations for C and for M/P in several ways. Solving for the value of money $(1/P_t)$ from equation (3)

$$u_{C,t} = \mu_t + \beta E_t\left(u_{C,t+1}\frac{P_t}{P_{t+1}}\right) = \mu_t + \beta E_t\left(\frac{P_t}{P_{t+1}}\left(\mu_{t+1} + \beta u_{C,t+2}\frac{P_{t+1}}{P_{t+2}}\right)\right) = \mu_t + E_t\left(P_t\left(\beta\frac{\mu_{t+1}}{P_{t+1}} + \beta^2\frac{\mu_{t+2}}{P_{t+2}} + ...\right)\right)$$

$$\frac{1}{P_t} = \frac{\sum_{i=1}^{\infty}\beta^i E_t\left(\frac{\mu_{t+i}}{P_{t+i}}\right)}{u_{C,t} - \mu_t} \tag{5}$$

whereas the nominal interest rate will equal, combining (1) and (3):

$$\beta E_t\left(u_{C,t+1}\frac{P_t}{P_{t+1}}\right) = \beta E_t\left(\frac{P_t}{P_{t+1}} I_t\left(u_{C,t+1} - \mu_{t+1}\right)\right)$$

$$I_t = E_t\left(\frac{u_{C,t+1}}{u_{C,t+1} - \mu_{t+1}}\right) = E_t\left(1 + \frac{\mu_{t+1}}{u_{C,t+1} - \mu_{t+1}}\right) \tag{6}$$

- Equation (5) says that the value of money (relative to the marginal utility of consumption) is equal to the present value of the marginal utility of money in all future periods. Money is valuable so long as it provides utility services, i.e. $\mu > 0$.

- Similarly, equation (6) shows that the nominal interest rate is positive in (6) so long as $\mu_{t+1} > 0$.

- Looking at the consumption / leisure trade-off in (4), we can see that, whenever $\mu > 0$, $u_{C,t} f_L\left(L_t\right) > u_{L,t}$. Consumption is therefore lower (and labor supply lower) given the distortion towards leisure that the cash-in-advance induces. Thus in a cash-in-advance model a positive net nominal interest rate $I_t - 1$ acts as a tax on consumption, raising the price of consumption above its production cost. This is the sense in which in the cash-in-advance model both leisure and investment can be thought of as credit goods, since they are not subject to the cash-in-advance constraint.

6.2.1 Cooley and Hansen's (AER 1989) stochastic cash-in-advance model

To analyse the model, we parametrize it in the following way:

$$\begin{aligned} u &= \frac{C_t^{1-\phi}}{1-\phi} - \frac{\tau L_t^{\eta}}{\eta} \\ Y_t &= AK_{t-1}^{\alpha} L_t^{1-\alpha} \end{aligned}$$

We assume that money follows:

$$\begin{aligned} M_t &= \theta_t M_{t-1} \\ \ln \theta_t &= (1-\rho)\ln\theta + (1-\rho)\,\theta_{t-1} + \varepsilon_t \end{aligned}$$

where $\theta > \beta$.

6.2.2 The steady state

From 2 derive steady state K/Y:

$$\begin{aligned} u_{C,t} - \mu_t &= \beta E_t \left(\left(u_{C,t+1} - \mu_{t+1} \right) \left(1 - \delta + f_K\left(K_t\right)\right)\right) \\ \frac{K}{Y} &= \frac{\alpha}{R - (1-\delta)} \end{aligned}$$

Then from the economywide constraint $Y = C + \delta K$ derive C/Y. Then we want the levels. Need L. From equation (4):

$$C^{-\phi} = \mu + \frac{\tau L^{\eta}}{(1-\alpha)\,Y}$$

From 3:

$$C^{-\phi}\left(1 - \frac{\beta}{\theta}\right) = \mu$$

As always, $\pi = \theta$; Use this result and combine the previous two equations:

$$\begin{aligned} C^{-\phi}\frac{\beta}{\theta} &= \frac{\tau L^{\eta}}{(1-\alpha)\,Y} \\ \frac{C}{Y} &= 1 - \delta\frac{K}{Y} \to \frac{C^{-\phi}}{Y^{-\phi}}\frac{\beta}{\theta} = \frac{\tau L^{\eta}}{(1-\alpha)\,Y^{1-\phi}} \to \gamma^{-\phi} = \frac{\theta}{\beta}\frac{\tau L^{\eta}}{(1-\alpha)\,Y^{1-\phi}} \\ Y &= AK^{\alpha}L^{1-\alpha} \to Y = A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R-(1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} L = \chi L \\ L &= L\left(\theta\right),\ L_{\theta} < 0 \end{aligned}$$

which shows that labor supply depends negatively on the inflation rate. Again, welfare costs of inflation can be calculated by noting that utility depends on C and L, both of which are a function of inflation.

6.2.3 The linearized model

The log-linear equilibrium will be - here variables with a time subscript denote percentage deviations of that variable from the steady state. (denoting with $R = 1/\beta$ the steady state real interest rate) (you have to play a little with the algebra of log-linearizations to get some of the equations, see Walsh textbook). As always, we proceed by linearizing around a steady state in which money supply growth is constant.

Note: Walsh uses a slightly different version, he defines $\lambda_t = u_{Ct} - \mu_t$ as the marginal utility of one wealth): it follows that in log-linear terms:

$$\widehat{\lambda}_t = -\phi\pi R\widehat{C}_t - (\pi R - 1)\,\widehat{\mu}_t$$

Using my conventions:

$$\begin{aligned}
\widehat{Y}_t &= \alpha\widehat{K}_{t-1} + (1-\alpha)\,\widehat{L}_t && (7)\\
\widehat{Y}_t &= \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}\left(\widehat{K}_t - (1-\delta)\,\widehat{K}_{t-1}\right) && (8)\\
\widehat{R}_t &= \left(1 - \frac{1-\delta}{R}\right)E_t\left(\widehat{Y}_{t+1} - \widehat{K}_t\right) && (9)\\
\widehat{R}_t &= \phi R\pi E_t\left(\widehat{C}_{t+1} - \widehat{C}_t\right) + (R\pi - 1)\,E_t\left(\widehat{\mu}_{t+1} - \widehat{\mu}_t\right) && (10)\\
\eta\widehat{L}_t &= \widehat{Y}_t - \phi R\pi\widehat{C}_t - (R\pi - 1)\,\widehat{\mu}_t && (11)\\
(R\pi - 1)\,\widehat{\mu}_t &= -\phi R\pi\widehat{C}_t + \phi E_t\widehat{C}_{t+1} + E_t\widehat{\pi}_{t+1} && (12)\\
\widehat{C}_t - \widehat{C}_{t-1} &= -\widehat{\pi}_t + \widehat{\theta}_t && (13)
\end{aligned}$$

Equation (7) is the production function.

(8) is the definition of output.

(9) defines the return on a real bond (the real interest rate): we don't have explicitly a real bond in the formulation of the problem, but if we add one we would get that the real interest rate equals in levels: $R_t = E_t\left(\frac{P_t}{P_{t+1}}I_t\right)$, in log-deviations $\widehat{R}_t = \widehat{I}_t - E_t\widehat{\pi}_{t+1}$. We then obtain equation 9 by combining 1 and 2.

(10) is the Euler equation for consumption, derived from equation 2.

(11) is labour supply.

(12) is the money "demand" equation. It says that the multiplier on the cash-in-advance constraint becomes lower if you consume more today (as you consume more, the marginal utility of consumption falls....) and becomes tighter if expected inflation rises.

(13) is the money supply rule. Remember that in the steady state in which the cash-in-advance is binding: $C_t = \frac{M_t}{P_t}$

6.2.4 Dynamics

The monetary transmission mechanism in the cash-in-advance model works as follows: a higher money growth, by raising inflation, makes the cash in advance constraint tighter and shift demand away from the consumption good towards the credit goods (leisure and investment). Inflation hence reduces steady state labor supply.

Unlike in the MIU model, even when $\phi = 1$ and $\pi = 1$ there is interdependence between real and monetary factors. Combining (11) and (12) and forwarding (13) one period ahead:

$$\begin{aligned} \eta \widehat{L}_t - \widehat{Y}_t &= -E_t \widehat{C}_{t+1} - E_t \widehat{\pi}_{t+1} \\ E_t \widehat{C}_{t+1} - \widehat{C}_t &= E_t \widehat{\theta}_{t+1} - E_t \widehat{\pi}_{t+1} \end{aligned}$$

combine the above expressions to obtain:

$$\widehat{Y}_t - \eta \widehat{L}_t - \widehat{C}_t = E_t \widehat{\theta}_{t+1}$$

hence changes in $E_t \theta_{t+1}$ will affect C, L and Y (compare this with the log-log case of Sidrauski's model). In particular, for the case with no capital, $\widehat{Y} = \widehat{C}$ and increases in expected inflation reduce labor supply by a factor which is equal to $1/\eta$.

Comparing the cash-in-advance with the MIU reveals that a money shock in the cash-in-advance model has a much larger impact on output.

To conclude, a recent quote from Lucas (Remarks on the influence of Edward Prescott, Economic Theory, 2007)

"Cooley and Hansen's (1989) AER paper introduced money into the model with a cash-in-advance constraint, and simulated the resulting model (in which optimal and equilibrium allocations do not, of course, coincide) to see how much realistic money supply variability affected the cyclical behavior of the system. The answer was, hardly at all. In this model, there are no goods or labor market rigidities in the system andmoney affects real variables only through inflation tax effects, so substantively this conclusion has been a widely-held conjecture for years. But Cooley and Hansen quantified this fact for the first time, in a framework that permits replication and theoretical experimentation. Now, one even hears people referring to "monetary real business cycle models." The terminology is pretty awful, I know, but the point is well taken: Real general equilibrium theory is a very useful point of departure, even—or maybe especially—for thinking about issues it does not directly address."

Homework

Read the Cooley and Hansen AER 1989 paper, and, of course, study the cash in advance model.

1. Why do you think they use the indivisible labor assumption? What are the main testable implications of their model? How does inflation affect welfare?

2. Assume that technology A_t follows a trend-stationary AR(1) process in logs, and that money growth is stationary. Write the model in stationary form. Write down in a Dynare file the set of non-linear equations describing the CIA model.

3. Calibrate the model as you think appropriate, find the steady state and then consider the dynamic properties of the economy. In particular, given your benchmark calibration, compare the statistics of your economy to their data counterparts (discuss how you computed the statistics of the data counterparts). If they are nowehere near (say, they differ by a factor of 10 or more), this means that your calibration was not appropriate.

4. Consider the response of hours and inflation to monetary and technology shocks. How do the impulse responses of the model change?

   (a) if labor supply elasticity is varied; (b) if the capital share in production changes?; (c) if the persistence of the monetary shock changes?

5. Compare the statistics of your economy (such as output, consumption, prices and hours) standard deviations in two models: one low variance of the money growth process and one with high variance of the money growth process. How do the properties of the variables change when money is supplied erratically? Why?

Chapter 7

Money in Models with Nominal Rigidities

Recent years have seen an explosion of models in which there are nominal rigidities; an advantage of these models is that they have nested the familiar-to-everyone RBC model as a special case.

At least since Keynes, it has been thought that in order to have real effects from monetary actions, it is key to have some degree of nominal rigidity.

What do we need in order to get nominal rigidities in the traditional, dynamic general equilibrium model? Well, we need some form of pricing power, for instance coming from monopolistic competition, and therefore some heterogeneity among goods.

The main actors of the DNK model are:

agents/market	final	interm.	K	L	profit	money	bonds	
HH	$-C$		$ZK_{-1}-$ $q\left(K-(1-\delta)K_{-1}\right)$	$\frac{WL}{P}$	F	$\frac{M_{-1}-M+T}{P}$	$\frac{RB_{-1}-B}{P}$	$=0$
K F	$-I$		$q\phi\left(\cdot\right)K$					$=0$
final F	Y	$\frac{-\int_0^1 P_jY_jdj}{P}$						$=0$
interm F		$\frac{\int_0^1 P_jY_jdj}{P}$	$-ZK_{-1}$	$\frac{-WL}{P}$	$\frac{1-X}{X}Y$			$=0$
G						$\frac{M-M_{-1}-T}{P}$		$=0$
equilibrium	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	

- households: make consumption and labor supply decisions, demand money and bonds. Buy installed capital at market price q and rent it out to intermediate firms which will use it as an intermediate input for production.

- capital producers: after production of Y, they make new capital goods after purchasing raw output. They sell the capital goods output to households.

- final good firms: produce final goods Y_t from intermediate goods Y_{jt}
- intermediate good firm: use labor and capital to produce intermediate goods Y_{jt}. Over each of this goods they have monopoly power. Can set price of good Y_j
- government: runs monetary policy.

7.1 Households

$$\max_{B_t,L_t,K_t,M_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\rho}}{1-\rho} - \frac{1}{\eta} (L_t)^{\eta} + \chi \ln \frac{M_t}{P_t} \right)$$

where E_0 denotes the expectation operator conditional on time 0 information, β is the discount factor, C_t is time t consumption (an aggregate of different consumption goods which is produced by the final goods firms), L_t is labour and M_t/P_t are real money balances.

Every period households derive utility from C_t; rent their labour input to firms at a real wage of $w_t \equiv W_t/P_t$, buy bonds $b_t \equiv B_t/P_t$, pay back $R_{t-1}B_{t-1}/P_t$, where R_{t-1} is the predetermined nominal interest rate paid on loans made between $t-1$ and t; rent capital to capital producing firms at rental rate Z_t

Let $\pi_t \equiv P_t/P_{t-1}$ denote the gross rate of inflation from period $t-1$ to period t. In real terms, the household budget constraint is:

$$C_t + b_t + q_t K_t + \frac{M_t}{P_t} = \frac{R_{t-1}}{\pi_t} b_{t-1} + w_t L_t + F_t + \frac{M_{t-1}}{P_t} + T_t + Z_t K_{t-1} + q_t (1-\delta) K_{t-1}$$

where F_t denotes lump-sum dividends received from ownership of intermediate goods firms (whose problems are described below). Solving this problem yields first order conditions for consumption/saving, labour supply, capital and money demand.

$$\begin{aligned}
\frac{1}{C_t^{\rho}} &= \beta E_t \left(\frac{R_t}{\pi_{t+1} C_{t+1}^{\rho}} \right) && \text{(Euler)} \\
\frac{w_t}{C_t^{\rho}} &= L_t^{\eta-1} && \text{(LS)} \\
R_t &= E_t \left(\pi_{t+1} \left(\frac{Z_{t+1} + q_{t+1}(1-\delta)}{q_t} \right) \right) && \text{(KD)} \\
\frac{1}{C_t^{\rho}} &= E_t \left(\beta \frac{1}{\pi_{t+1}} \frac{1}{C_{t+1}^{\rho}} \right) + \chi (m_t)^{-1} && \text{(MD)}
\end{aligned}$$

where m_t is real balances (M_t/P_t), $w_t = W_t/P_t$, $b_t = B_t/P_t$ (remember, B is a nominal bond; if you divide it by P you get the nominal bond expressed in real terms, but you don't get a real bond).

7.2 Final good firm

There is a final good firm which produces the final good Y_t using intermediate goods Y_{jt}.

Total final goods are given by the CES aggregator of the different quantities of goods produced:

$$Y_t \leq \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $\varepsilon > 1$.

The problem of each firm is to minimize expenditure given the production constraint. We can write the Lagrangean as:

$$L = \int_0^1 P_{jt} Y_{jt} dj + P_t \left(Y_t - \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)$$

Optimal choice of Y_{jt} solves for each j:

$$P_{jt} = P_t \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{1}{\varepsilon-1}} Y_{jt}^{-\frac{1}{\varepsilon}}$$

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t$$

This expression can be solved for the multiplier. Use the definition of Y_t with equality and use the solution for Y_{jt} to write:

$$Y_t = \left(\int_0^1 \left(\left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t \right)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} = Y_t \left(\int_0^1 \left(\frac{P_{jt}}{P_t} \right)^{1-\varepsilon} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Because the production function has constant returns to scale, Y_t drops from both sides of the expression, and we can then solve for P_t as:

$$P_t = \left(\int_0^1 P_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}} .$$

P_t represents the minimum cost of achieving one unit of the final-goods bundle Y_t. For this reason we interpret P_t (the Lagrange multiplier) as the aggregate price index.

7.3 Intermediate goods

7.3.1 The constraints

The intermediate goods sector is made by a continuum of monopolistically competitive firms owned by consumers, indexed by $j \in (0, 1)$. Each firm faces a downward sloping demand for its product. It produces output according to:[1]

$$Y_{jt} = A_t L_{jt}^{\alpha} K_{jt}^{1-\alpha}$$

Each producer chooses own sale price P_{jt} taking as given the demand curve. He can reset his price only when given the chance of doing so, which occurs with probability $1 - \theta$ in every period.

Intermediate goods firms face three constraints:

1. the production constraint;

[1] Capital is not predetermined at the firm level, hence we denote it with K_{jt}. At the aggregate level, it is predetermined, hence $\int K_{jt} dj = K_{t-1}$

2. the demand curve $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} Y_t$

3. prices can be adjusted only with probability $1-\theta$

We can break this problem down into two sub-problems. As a cost minimizer and as a price setter.

7.3.2 Cost minimization

Consider the cost minimization problem first, conditional on the output Y_{jt} produced. This problem involves minimizing $\frac{W_t}{P_t} L_{jt} + Z_t K_{jt}$ subject to producing Y_{jt}. Notice that both factors are purchased in competitive markets. In real terms this problem can be written as

$$\min_{K_{jt},L_{jt}} \frac{W_t}{P_t} L_{jt} + Z_t K_{jt} + \mu_t \left(Y_{jt} - A_t L_{jt}^{\alpha} K_{jt}^{1-\alpha}\right) \qquad [\mu_t]$$

where μ_t is multiplier associated with the constraint (we can think of μ_t as real marginal cost; we likewise define its inverse $X_t = 1/\mu_t$ as the markup). The first order conditions imply:

$$\alpha \frac{Y_{jt}}{L_{jt}} = \frac{1}{\mu_t} \frac{W_t}{P_t} \equiv X_t \frac{W_t}{P_t} \tag{LD}$$

$$(1-\alpha) \frac{Y_{jt}}{K_{jt}} = \frac{1}{\mu_t} Z_t \equiv X_t Z_t \tag{KD}$$

notice that this first order conditions imply than we can write the real cost function as:

$$COST_{jt} = \underbrace{\frac{W_t}{P_t} L_{jt}}_{\alpha \mu_t Y_{jt}} + \underbrace{Z_t K_{jt}}_{(1-\alpha)\mu_t Y_{jt}} = \mu_t Y_{jt}$$

hence this is the sense in which μ_t measure the real marginal cost for each producer.

7.3.3 Producer as a price setter

If some producers can change prices and some others cannot, the average price level will be a CES aggregate of all prices in the economy, and will be given by:

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta) \left(P_t^*\right)^{1-\varepsilon} \tag{*}$$

where P_{t-1} is previous price level, and P^* is average price level chosen by those who have the chance to change prices. It is at these guys that we look now.

Consider the producer who has a chance $1-\theta$ to reset prices at time t. Call P^* the reset price. The demand curve is:

$$Y_{jt+k}^* = \left(P_{jt}^*/P_{t+k}\right)^{-\varepsilon} Y_{t+k}$$

for any k for which he will keep that price.

His maximization problem is:

$$\max_{P_{jt}^*} \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \left(\frac{P_{jt}^*}{P_{t+k}} - \mu_{t+k} \right) Y_{jt+k}^* \right]$$

$$\Lambda_{t,k} = (C_t/C_{t+k})^{\rho}$$

where μ_t is the real marginal cost. θ represents the probability that the price P^* chosen at t will still apply in later periods. The argument of the maximization problem is the "expected discounted sum of all profits that the price setter will make conditional on his choice of P_{jt}^* and weighted by how likely P_{jt}^* is to stay in place in future periods".

At time t, the price setter chooses P_{jt}^* to maximize profit. Differentiate the profit function with respect to P_{jt}^* to obtain

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \left(\frac{Y_{jt+k}^*}{P_{t+k}} + \frac{P_{jt}^*}{P_{t+k}} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - \mu_{t+k} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0$$

take $\frac{Y_{jt+k}^*}{P_{t+k}}$ out, isolate elasticity of Y_{jt+k}^* wrt P_{jt}^*

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 + \frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - \frac{\mu_{t+k} P_{t+k}}{P_{jt}^*} \frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0$$

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 - \varepsilon + \frac{\mu_{t+k} P_{t+k}}{P_{jt}^*} \varepsilon \right) \right] = 0$$

multiply everything by P_{jt}^*

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(P_{jt}^* - \frac{\varepsilon}{\varepsilon - 1} \mu_{t+k} P_{t+k} \right) \right] = 0$$

In the symmetric equilibrium, all the firms that reset the price choose the same price (and face the same demand), hence

$$P_{jt}^* = P_t^*$$

These two expressions enter the equilibrium (using $X = \frac{\varepsilon}{\varepsilon-1}$ = steady state markup). One is $*$ that we derived above, the other is:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \left(\frac{P_t^*}{P_{t+k}} - X \mu_{t+k} \right) \right] = 0 \qquad (**)$$

Use $\mu_{t+k}^n \equiv \mu_{t+k} P_{t+k}$ to rearrange the expression above to obtain:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \frac{P_t^*}{P_{t+k}} \right] = X \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \mu_{t+k} \right]$$

$$P_t^* = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \mu_{t+k}^n P_{t+k}^{-1} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* P_{t+k}^{-1} \right]} = X \sum_{k=0}^{\infty} \phi_{t,k} \mu_{t+k}^n$$

where $\phi_{t,k} \equiv \frac{(\theta\beta)^k E_t\left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k}\right]}{\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k}\right]}$. This expression says that the optimal price is a weighted average of current and expected future nominal marginal costs. Weigths depend on expected demand in the future, and how quickly firm discounts profits.

Therefore you can notice the following:

- Under purely flexible prices, $\theta = 0$: the markup is a constant. $P^*_t = X\mu^n_t$ and optimal prices are a multiple X of the marginal cost.
- When $\theta > 0$, the optimal price depends on future expected values of aggregate variables as well as future nominal marginal costs μ^n_{t+k}. Put differently, one can see that all the fluctuations in the markup are due to firms being unable to adjust prices.

7.4 Deriving the Phillips curve

Use $Y^*_{t+k} = \left(P^*_t/P_{t+k}\right)^{-\varepsilon} Y_{t+k}$ and cancel out P^*_t in numerator and denominator to obtain:

$$P^*_t = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t\left[\Lambda_{t,k} P^{\varepsilon-1}_{t+k} Y_{t+k} \mu^n_{t+k}\right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t\left[\Lambda_{t,k} P^{\varepsilon-1}_{t+k} Y_{t+k}\right]}$$

To gain insight into this expression, it is convenient to loglinearize it. Intuitively, we can see that numerator and denominator only differ up to a multiple given by μ^n_{t+k}, which in turn multiplies $(\theta\beta)^k (1-\theta\beta)$. Hence we can expect that in log-linearising Y, $P^{\varepsilon-1}$ and Λ will cancel out and disappear. Rearranging and dividing by P_t:

$$\frac{P^*_t}{P_t} \sum_{k=0}^{\infty} (\theta\beta)^k E_t\left[\Lambda_{t,k} Y_{t+k} P^{\varepsilon-1}_{t+k}\right] = \frac{1}{P_t} X \sum_{k=0}^{\infty} (\theta\beta)^k E_t\left[\Lambda_{t,k} Y_{t+k} P^{\varepsilon-1}_{t+k} \mu^n_{t+k}\right]$$

LHS first

$$\left(\widehat{P}^*_t - \widehat{P}_t\right) \sum_{k=0}^{\infty} (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1}\right] + \sum_{k=0}^{\infty} (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1}\right] E_t\left(\widehat{\Lambda}_{t,k} + \widehat{Y}_{t+k} + (\varepsilon-1)\widehat{P}_{t+k}\right)$$

RHS next

$$-\widehat{P}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1}\right] + \frac{X}{P} \sum_{k=0}^{\infty} (\theta\beta)^k \left[\frac{\Lambda Y P^{\varepsilon-1} P}{X}\right] E_t\left(\widehat{\Lambda}_{t,k} + \widehat{\mu^n}_{t+k} + \widehat{Y}_{t+k} + (\varepsilon-1)\widehat{P}_{t+k}\right) =$$

$$-\widehat{P}_t \sum_{k=0}^{\infty} (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1}\right] + \sum_{k=0}^{\infty} (\theta\beta)^k \left[\Lambda Y P^{\varepsilon-1}\right] E_t\left(\widehat{\Lambda}_{t,k} + \widehat{\mu^n}_{t+k} + \widehat{Y}_{t+k} + (\varepsilon-1)\widehat{P}_{t+k}\right)$$

where we use $\mu^n = P/X$ in steady state. Hence, using $\widehat{\mu^n}_{t+k} = \widehat{P}_{t+k} + \widehat{\mu}_{t+k}$

$$\widehat{P}^*_t \sum_{k=0}^{\infty} (\theta\beta)^k = \sum_{k=0}^{\infty} (\theta\beta)^k E_t\left(\widehat{P}_{t+k} + \widehat{\mu}_{t+k}\right)$$

$$\widehat{P}^*_t = (1-\theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t\left(\widehat{P}_{t+k} + \widehat{\mu}_{t+k}\right) \qquad (@)$$

Equation (@) simply states in log-linear terms that the optimal price has to be equal to a weighted average of current and future marginal costs, weighted by the probability that this price will hold in later periods too.

So you assign weight 1 to today, weight $\theta\beta$ to tomorrow, $\theta^2\beta^2$ to the day after tomorrow, and so on. Notice the complete forwardlookingness of this expression, and the fact that these weights need to be normalized (the sum of all of them is $\frac{1}{1-\theta\beta}$, whose inverse premultiplies the summation - weights sum up to one -)

But we know that:

$$\begin{aligned}
\widehat{P}_t - \theta\widehat{P}_{t-1} &= (1-\theta)\,\widehat{P}_t^* \\
\widehat{P}_t - \theta\widehat{P}_{t-1} &= (1-\theta)(1-\theta\beta)\,E_t\left(\left(\widehat{P}_t + \widehat{\mu}_t\right) + \theta\beta\left(\widehat{P}_{t+1} + \widehat{\mu}_{t+1}\right) + \theta^2\beta^2(\ldots) + \ldots\right) \\
\widehat{P}_t - \theta\widehat{P}_{t-1} &= (1-\theta)(1-\theta\beta)\left(\widehat{P}_t + \widehat{\mu}_t\right) + \theta\beta\underbrace{\left(E_t\widehat{P}_{t+1} - \theta\widehat{P}_t\right)}_{\text{next period value of LHS}} \\
\widehat{P}_t - \widehat{P}_{t-1} &= -(1-\theta)\,\widehat{P}_{t-1} + (1-\theta)(1-\theta\beta)\,\widehat{P}_t + \theta\beta\left(E_t\widehat{P}_{t+1} - \theta\widehat{P}_t\right) + (1-\theta)(1-\theta\beta)\,\widehat{\mu}_t \\
\widehat{\pi}_t &= (1-\theta)\,\widehat{\pi}_t + \theta\beta E_t\widehat{\pi}_{t+1} + (1-\theta)(1-\theta\beta)\,\widehat{\mu}_t
\end{aligned}$$

$$\widehat{\pi}_t = \beta E_t\widehat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\widehat{\mu}_t$$

This equation is nothing else but an "expectations augmented Phillips curve", which states that inflation rises when the real marginal costs rise. It takes a while to derive, but again it is nothing else but an aggregate supply curve for the whole economy. Notice that

$$\begin{aligned}
\partial\left(\frac{\partial\widehat{\pi}_t}{\partial\widehat{\mu}_t}\right)/\partial\beta &= -(1-\theta) < 0 \\
\partial\left(\frac{\partial\widehat{\pi}_t}{\partial\widehat{\mu}_t}\right)/\partial\theta &< 0
\end{aligned}$$

- the higher β, the higher the weight to future $\widehat{\mu}_t$'s, and the lower today's elasticity to current marginal cost
- the higher θ, the higher the chance that I will be stuck with my price for a long period, and the higher the elasticity of $\widehat{P}_t^*$ to $\widehat{\mu}_t$. However, few prices will be changed in the aggregate, therefore aggregate inflation will not be sensitive to the marginal cost.

7.5 The capital producers

Competitive capital producers purchase raw output as materials input I_{jt}, *rent capital within period* from intermediate firms[2] and produce new capital goods sold at price q_t. The production function for new capital is given by:

$$\begin{aligned}
Y_{jt}^k &= \Phi\left(\frac{I_{jt}}{K_{jt}}\right)K_{jt} \\
\Phi' &> 0,\ \ \Phi'' < 0,\ \ \Phi(0) = 0,\ \ \Phi\left(\frac{I}{K}\right) = \frac{I}{K}
\end{aligned}$$

[2] Capital is rented twice: (1) between periods, households rent capital at rate Z_t to intermediate good firms which produce intermediate goods.

(2) within period, capital is rented from intermediate firms to capital producers after it has produced intermediate goods at rate Z_t^k

These properties imply:

1) constant returns to scale in I and K

2) diminishing returns to I, holding K constant

The representative firm solves:

$$\max_{I_{jt},K_{jt}} q_t\Phi\left(\cdot\right)K_{jt} - I_{jt} - Z_t^k K_{jt}$$

it purchases raw output at 1, rents installed capital at Z^k and produces new capital valued at q using the technology $\Phi\left(\cdot\right)K$. The first order conditions for I_{jt} and K_{jt} are:

$$\begin{aligned} q_t &= \frac{1}{\Phi'\left(\frac{I_{jt}}{K_{jt}}\right)} \\ q_t\left(\Phi - \Phi'\frac{I_{jt}}{K_{jt}}\right) &= Z_t^k \end{aligned}$$

The first condition implies that as I/K rises, Φ' falls and q rises.

The condition for K guarantees that in steady state, since $\Phi\left(I/K\right) = I/K$, $\Phi'\left(I/K\right) = 1$, which implies that $Z^k = 0$ and can be ignored.

7.6 Aggregation, resource constraints and policy

Total output in economy is:

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}} = \left[\int_0^1 \left(A_t K_{jt}^{1-\alpha} L_{jt}^{\alpha}\right)^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

It is not possible to simplify this expression since input usages across firms differs. However the linear aggregator:

$$Y_t' = \int_0^1 Y_{jt} dj$$

is approximately equal to Y_t within a local region of the steady state. Hence for local analysis we can simply use:

$$Y_t = A_t K_{t-1}^{1-\alpha} L_t^{\alpha}$$

with

$$\begin{aligned} L_t &= \int_0^1 L_{jt} dj \\ K_{t-1} &= \int_0^1 K_{jt} dj \end{aligned}$$

Assuming no government consumption:

$$Y_t = C_t + I_t$$

Capital goods mkt clearing

$$K_t = \Phi\left(\frac{I_t}{K_{t-1}}\right) K_t + (1-\delta) K_{t-1}$$

Bond market clearing implies

$$B_t = 0$$

The central bank policy sets the dynamics of money supply in some fashion and closes the model.

Remark 6 *As we know well already, money demand is*

$$\frac{M_t}{P_t} = \chi C_t^{\rho} \frac{R_t}{R_t - 1}$$

assume $\rho = 1$. Notice that in log-linear terms this reads as (R is the nominal rate):

$$\widehat{R}_t = \psi\left(\widehat{C}_t - \widehat{m}_t\right) \tag{md}$$

where $\psi > 0$. This is an interest rate rule. The key point I want to make is the following: we can forget about money demand in the model so long as money enters separably the utility function, and close the model by specifying any process for the policy instrument (M or R) we like.

7.7 Equilibrium

7.7.1 The equilibrium in levels

Equation summarizing the model:

$$
\begin{aligned}
Y_t &= C_t + I_t && (1)\\
\frac{1}{C_t^{\rho}} &= \beta\left(\frac{R_t P_t}{C_{t+1}^{\rho} P_{t+1}}\right) && (2)\\
q_t &= \frac{1}{\Phi'\left(\frac{I_t}{K_{t-1}}\right)} && (3)\\
E_t\left(\frac{R_t}{\pi_{t+1}}\right) &= \frac{1}{q_t} E_t\left(\frac{(1-\alpha) Y_{t+1}\mu_{t+1}}{K_t} + q_{t+1}(1-\delta)\right) && (4)\\
Y_t &= A_t L_t^{\alpha} K_{t-1}^{1-\alpha} && (5)\\
\alpha\frac{Y_t}{L_t} &= \frac{1}{\mu_t} L_t^{\eta-1} C_t^{\rho} && (6)\\
P_t^{1-\varepsilon} &= \theta P_{t-1}^{1-\varepsilon} + (1-\theta)\left(X\sum_{k=0}^{\infty}\phi_{t,k}\mu_{t+k}P_{t+k}\right)^{1-\varepsilon} && (7)\\
K_t &= \Phi\left(\frac{I_t}{K_{t-1}}\right) K_t + (1-\delta) K_{t-1} && (8)\\
R_t &= R_{t-1}^{\phi_r} rr^{1-\phi_r}\left(\left(\frac{P_t}{P_{t-1}}\right)^{1+\phi_\pi}\mu_t^{\phi_z}\right)^{1-\phi_r}\varepsilon_{r,t} && (9)
\end{aligned}
$$

1. market clearing
2. aggregate demand equation
3. supply of capital
4. equilibrium rental rate of K
5. production function
6. equilibrium in the labor market. Take labour demand (LD) and labor supply (LS) and impose market clearing. Then equate (LD) and (LS) so as to eliminate of the real wage w from that expression.
7. aggregate price level is a weighted average of (1) previous price level P_{t-1} and (2) reset prices P_t^*, which depend on future expected marginal costs.
8. capital dynamics
9. monetary policy rule. We assume that the central bank chooses money supply so as to set the nominal interest rate to be a function of previous interest rate, current inflation and current marginal costs. This is a Taylor rule, after John Taylor, who observed in a 1993 seminal paper that central banks set the interest rate as a function of inflation and output gap (output gap=deviation of output from its natural rate). The last term $\varepsilon_{r,t}$ represents a monetary policy shock. The term rr is the target level

of the interest rate: assuming zero inflation in steady state is tantamount to assume that the steady state nominal interest rate equals the real interest rate rr.

7.7.2 The log-linearized model

The steady state values of the system are very similar to those of any standard RBC/MIU model, with the only difference that monopolistic competition drives a wedge between the marginal product of each factor and its marginal cost. We linearize around a steady state with zero inflation and make use of the following

$$\begin{aligned}
q &= 1 \\
R &= \frac{1}{\beta} = (1-\alpha)\frac{Y}{K}\mu + 1 - \delta \\
\frac{C}{Y} &= 1 - \delta\frac{K}{Y}
\end{aligned}$$

When we log-linearize the expressions above, what we obtain the following system:

$$\begin{aligned}
\widehat{Y}_t &= \frac{C}{Y}\widehat{C}_t + \frac{I}{Y}\widehat{I}_t \\
-\rho\left(\widehat{C}_t - E_t\widehat{C}_{t+1}\right) &= \widehat{R}_t - \widehat{\pi}_{t+1} \\
\widehat{q}_t &= -\widehat{R}_t + E_t\widehat{\pi}_{t+1} + \beta(1-\delta)E_t\widehat{q}_{t+1} + (1-\beta(1-\delta))E_t\left(\widehat{\mu}_{t+1} + \widehat{Y}_{t+1} - \widehat{K}_t\right) \\
\widehat{q}_t &= \varphi E_t\left(\widehat{I}_t - \widehat{K}_{t-1}\right) \\
\widehat{Y}_t &= \widehat{a}_t + \alpha\widehat{L}_t + (1-\alpha)\widehat{K}_{t-1} \\
\widehat{Y}_t + \widehat{\mu}_t - \rho\widehat{C}_t &= \eta\widehat{L}_t \\
\widehat{\pi}_t &= \beta E_t\widehat{\pi}_{t+1} + \zeta\widehat{\mu}_t \\
\widehat{K}_t &= \delta\widehat{I}_t + (1-\delta)\widehat{K}_{t-1} \\
\widehat{R}_t &= \phi_R\widehat{R}_{t-1} + (1-\phi_R)\left((1+\phi_\pi)\widehat{\pi}_t + \phi_\mu\widehat{\mu}_t\right) + \widehat{e}_t
\end{aligned}$$

where

$$\begin{aligned}
\varphi &= -\frac{\Phi''\frac{I}{K}}{\Phi'} \\
\zeta &= \frac{(1-\theta)(1-\beta\theta)}{\theta}
\end{aligned}$$

denotes the steady state elasticity of investment to the asset price.

Remark 7 *Models of this kind are now ubiquitous in many textbooks and papers. See for instance:*

Canova, Fabio and Luca Sala (2006), Back to Square One: Identification Issues in DSGE Models, May 2005, revised September 2006, page 26, the 11 equations model

Rubio-Ramirez, Juan and Pau Rabanal (2005), Comparing New Keynesian Models of the Business Cycle : A Bayesian approach (pdf file). Journal of Monetary Economics, 52, pp 1151-1166.

7.7.3 The natural rate of output

It is sometimes convenient to call Y_t^* a new variable that defines the equilibrium level of output (the natural output) that would prevail under completely flexible prices ($\theta = 0$).

This way μ_t can in fact be eliminated. Put differently, if firms were instead able to adjust prices optimally each period, $\mu_t = 0$ (since $\zeta \to \infty$) and we would be able to define the flexible price equilibrium (NATURAL) values for real interest rate and output.

In absence of capital ($\alpha = 1$), we can easily derive an expression for μ as a function of the gap between flexible price and sticky price equilibrium, that is:

$$\widehat{\mu}_t = (\eta + \rho - 1)\left(\widehat{Y}_t - \widehat{Y}_t^*\right) = \lambda\left(\widehat{Y}_t - \widehat{Y}_t^*\right)$$

hence the real marginal cost is positive whenever Y is above Y^*. One can see that Y^* is an exogenous variable, since it depends only on technology, which is exogenous. In fact, setting $\mu = 0$ and solving for Y^* yields:

$$\widehat{Y}_t^* = \frac{\eta}{\eta + \rho - 1}\widehat{a}_t$$

With this convention, the dynamic-new keynesian model becomes:

$$\begin{aligned}
\widehat{Y}_t - \widehat{Y}_t^* &= E_t\left(\widehat{Y}_{t+1} - \widehat{Y}_{t+1}^*\right) - \rho^{-1}\left(\widehat{R}_t - E_t\widehat{\pi}_{t+1}\right) + \widehat{g}_t && \text{(a)}\\
\widehat{\pi}_t &= \lambda\zeta\left(\widehat{Y}_t - \widehat{Y}_t^*\right) + \beta E_t\widehat{\pi}_{t+1} + \widehat{u}_t && \text{(b)}\\
\widehat{R}_t &= \Phi_R\widehat{R}_{t-1} + (1 - \Phi_R)\left((1 + \Phi_\pi)\widehat{\pi}_t + \Phi_x\left(\widehat{Y}_t - \widehat{Y}_t^*\right)\right) + \widehat{e}_t && \text{(c)}
\end{aligned}$$

where g_t represents a composite of shocks, coming from say technology or preferences or government spending. Some authors have also postulated cost push shocks u_t, that drive inflation up independently from changes in μ_t.

The system made by (1), (2), (3) is the "benchmark" dynamic-new keynesian model. For this model to fit the data, some authors have also assumed that consumption is backward and forward looking (with weights θ_b and $1 - \theta_b$), [and that inflation has some backward-looking component too, and so on]. If you assume, say, that output is backward-looking too, equation 1 above becomes:

$$Y_t - Y_t^* = \theta_b\left(Y_{t-1} - Y_{t-1}^*\right) + (1 - \theta_b)E_t\left(Y_{t+1} - Y_{t+1}^*\right) - \rho^{-1}\left(R_t - E_t\pi_{t+1}\right) + g_t \qquad \text{(a')}$$

the idea is simple. The higher θ_b, the higher the dependence of a variable on its past, and the more successful its fit to the data. The model made by (a'), (b) and (c) is in `mc.m` and `mc_go.m`

7.7.4 The main insights of a DNK model

7.7.4.1 Demand shocks, markups and marginal costs

The markup will fall following an aggregate demand shock. When AD rises, not all firms are free to adjust prices upwards. Some will increase quantity, facing increasing marginal costs at same sale prices. Therefore the "average markup" (X) will fall. If the AD increase is sustained, the firms who adjust their prices might increase their prices by more than they would under perfect price flexibility, therefore the "marginal markup" (the markup for the firms that are given the chance to adjust prices) will rise. The intuition is simple: if there is an increase in AD but I am lucky enough to change my price that day, I will increase my price by more given that my price is likely to stay constant for a while.

7.7.4.2 Technology and employment

Following a rise in technology, marginal costs fall (intuitively, you can produce same output with less factors of production). Since not all prices are free to fall immediately, aggregate demand does not increase. In turn, firms need less labor input, since the factors are more productive. So a technology shock can lead - under some conditions - to a fall in employment. Ongoing debate on this issue, mostly because it is never clear how to identify a technology shock in the data (we don't observe A in the real world).

7.7.4.3 Should we care about marginal costs?

The theory of endogenous markup variations provides the crucial link that allows the concerns of RBC models and conventional monetary models to be synthesized. In addition, the markup directly measures the extent to which a condition for efficient resource allocation fails to hold.

7.7.4.4 Margins

Consider the equilibrium condition in the labor market

$$\alpha \frac{Y_t}{L_t} = \frac{1}{\mu_t} L_t^{\eta-1} C_t^{\rho}$$

in log-linear terms:

$$\eta \widehat{L}_t = \widehat{Y}_t - \rho \widehat{C}_t + \widehat{\mu}_t$$

Consider the limiting case in which capital's share in production is very small, so that $\widehat{Y}_t = \widehat{C}_t$ for all t. Notice that, unless $\rho = 1$, hours worked will trend together with GDP. If GDP grows over time, hours should fall over time if $\rho < 1$, and viceversa for $\rho < 1$. For this reason, and in order to be consistent with the balanced growth hypothesis[3], sometimes it is assumed that $\rho = 1$ in a model in which there is exogenous technological progress or random walk behavior in technology.

Notice that, whenever $\rho = 1$:

$$\eta \widehat{L}_t = \widehat{\mu}_t$$

hours will then move in the same direction as real marginal costs.

More in general, one could assume also shocks to the disutility of labor, which would then enter the expression, or shocks to government spending, which would violate the $\widehat{Y}_t = \widehat{C}_t$ equation.

[3] Which states that output, capital, consumption, investment and wages all grow at the same rate over time, whereas hours and interest rates are constant.

7.8 The dynamic effects of a monetary shock

The Matlab programs in my webpage will allow you to analyze the dynamics of this model by means of impulse response functions. What we work with is a modified version of the above model, obtained by dropping I and q. Simple algebra shows that

$$\begin{aligned} q_t &= -R_t + \pi_{t+1} + \beta(1-\delta) q_{t+1} + (1-\beta(1-\delta)) \left(\mu_{t+1} + Y_{t+1} - K_t\right) \\ q_t &= \eta (I_t - K_{t-1}) \\ K_t &= \delta i_t + (1-\delta) K_{t-1} \end{aligned}$$

can be combined into

$$R_t - \pi_{t+1} = -\frac{\varphi}{\delta}(K_t - K_{t-1} + \beta(1-\delta)(K_{t+1} - K_t)) + (1-\beta(1-\delta))\left(\mu_{t+1} + Y_{t+1} - K_t\right)$$

- *mc.m* (which calls *mc_go.m*) allows you to analyze the simple 3 equations model which arises without capital.
- *dnwk.m* (which calls *dnwk_go.m*) allows you to analyze the extended model, if you want to look at the behavior of other variables (labor supply, mark-up...) too.

7.9 Discussion

- The setup here goes under the name of New Neoclassical Synthesis (NNS). According to the NNS, inflation changes because the output gap varies, i.e. there are deviations of output from its level under flexible prices. This concept of the output gap is very different from the one used in many empirical applications. Many traditional studies measure the output gap as deviation of log GDP from some smooth trend, often constructed as a smooth function of time. However, as argued in Sbordone (2002), using detrended GDP as a proxy for the output gap does not have any theoretical justification. The theory exposed above in fact emphasizes that many shocks (e.g. random fluctuations in A_t) may affect the natural rate of output and the latter may turn out to be quite volatile.
- Sbordone (2002, JME): the AS curve relates inflation to real marginal costs. The aggregate price level depends on the discounted sum of expected future values of real marginal costs. Sbordone takes a very simple measure of marginal costs, which assumes that they are proportional to unit labor costs. Using the evolution of ULC, she computes the equilibrium path for the price level implied by the AS curve. She finds that this simple model accounts quite well for the US evolution of the GDP deflator.

Gali and Gertler (1999, JME, Inflation Dynamics: A Structural Econometric Analysis) note that

$$\begin{aligned} \alpha \frac{Y_{jt}}{L_{jt}} &= \frac{1}{\mu_t} \frac{W_t}{P_t} \\ \mu &= \alpha \frac{WL}{PY} \end{aligned}$$

hence the wage share must account well for the behavior of the real marginal cost, which in turn should be sufficient statistic to describe inflation.

Read also the paper by Gali, Gertler and Lopez-Salido (Restat, 2007) in the reading list. It explains how it is possible to derive a measure of the welfare costs of business cycles by looking at the empirical behavior of the markup gap.

7.10 Matching the model to the data

One dimension where the DNK performs poorly is its ability to match the observed slow responses of macro variables to given shocks. Everything happens too soon too quickly, something which does not show up in the data.

Several attempts have been made to get a better fit of the model to the data, using richer versions of adjustment costs, for instance, as well as a variety of other tricks.

7.10.1 Habit formation

See the paper by Boivin and Giannoni (2003) as well as Fuhrer.

With habit formation, we rewrite the momentary utility function (aside for money balances and labor supply) as:

$$u = \frac{\left(C_t - \varepsilon C_{t-1}\right)^{1-\rho}}{1-\rho}$$

this implies that the marginal utility of consumption is

$$\lambda_t = (1-\rho)\left(C_t - \varepsilon C_{t-1}\right)^{-\rho} - \varepsilon\beta\,(1-\rho)\left(C_{t+1} - \varepsilon C_t\right)^{-\rho}$$

and in log-deviations from the steady state, using $\lambda = (1-\rho)\,(1-\beta\varepsilon)\,(C-\varepsilon C)^{-\rho}$

$$\widehat{\lambda}_t = \frac{-\rho}{1-\beta\varepsilon}\left(\left(1+\beta\varepsilon^2\right)\widehat{C}_t - \varepsilon\widehat{C}_{t-1} - \beta\varepsilon\widehat{C}_{t+1}\right)$$

Unlike the DNK w/o habit formation, here $\lambda_t = -\rho C_t$ only if $\varepsilon = 0$.

7.10.2 Investment adjustment costs

See for instance the Christiano, Eichembaum, Evans (JPE, 2005) or the Smets and Wouters AER 2007 paper, or our discussion in the context of two-sector models.

7.10.3 Decisions taken in advance

One simple way to explain the observed delay in the effects of monetary shocks on aggregate expenditure is to assume that expenditure decisions are made in advance.

7.10.3.1 Consumption decisions

As an example, Woodford (Chapter 5) assumes that the consumption for time t is chosen at time $t-k$, so that:

$$C_t = E_{t-k} C_t$$

In Boivin and Giannoni we have a melting pot of the above plus habit formation, so that:

$$E_{t-k}\widehat{\lambda}_t = \frac{-\rho}{1-\beta\varepsilon} E_{t-k}\left(\left(1+\beta\varepsilon^2\right)\widehat{C}_t - \varepsilon\widehat{C}_{t-1} - \beta\varepsilon\widehat{C}_{t+1}\right)$$

How do you plug this into, say, Dynare? One thing to remember is to treat the expectation as of time $t-1$ of variable x as an additional state variable to introduce into your problem. In the specific example above, for $k=2$ we have:

$$E_t\widehat{\lambda}_{t+2} = \frac{-\rho}{1-\beta\varepsilon} E_t\left(\left(1+\beta\varepsilon^2\right)\widehat{C}_{t+2} - \varepsilon\widehat{C}_{t+1} - \beta\varepsilon\widehat{C}_{t+3}\right) \tag{11}$$

Since C_t and C_{t+1} are given as of t, we add the variables C_{t+1} and C_{t+2} as additional variables to our Uhlig's toolkit, and then make C_t and C_{t+1} predetermined by using the definitions:

$$\begin{aligned} C_t &\equiv LC_{t+1} \\ C_{t+1} &\equiv LC_{t+2} \end{aligned}$$

in the matrices of deterministic equations.

For the expectational equations, equation 11 is hard to use. However, since we have created C_{t+1} and C_{t+2}, we also have $E_t C_{t+3}$ defined. For $E_t\lambda_{t+2}$, it suffices to add λ_{t+1} to our system, and then make sure you write the equality between the old and the new λ_t.

7.10.3.2 Pricing decisions

Woodford's book and Boivin and Giannoni (2007) also discuss price decisions taken in advance. Woodford assumes that if prices are not reoptimized, they are indexed to lagged inflation according to

$$\log p_{jt} = \log p_{jt-1} + \gamma\pi_{t-1}$$

notice that Calvo baseline sets $\gamma = 0$. Here we have partial indexation to inflation. This equation linearized becomes

$$\widehat{\pi}_t - \gamma\widehat{\pi}_{t-1} = \beta E_t\left(\widehat{\pi}_{t+1} - \gamma\widehat{\pi}_t\right) + \kappa\widehat{\mu}_t + \widehat{u}_t$$

again, if pricing decisions are taken in advance, one gets:

$$\widehat{\pi}_t - \gamma\widehat{\pi}_{t-1} = \beta E_{t-1}\left(\widehat{\pi}_{t+1} - \gamma\widehat{\pi}_t\right) + \kappa E_{t-1}\widehat{\mu}_t + \widehat{u}_t$$

7.11 Policy rules and real indeterminacy

Clarida, Gali and Gertler (QJE, 2000) analyse the main macroeconomic implications in terms of macroeconomic stability of the workhorse monetary business model with sticky prices.

After log-linearization around the zero inflation steady state, the equilibrium conditions can be written as (ignoring constants and defining $\kappa = \lambda\zeta$)

$$\begin{aligned} R_t &= (1+\phi_\pi)\,\pi_t + \phi_x x_t + e_t \\ \pi_t &= \kappa x_t + \beta E_t \pi_{t+1} + u_t \\ x_t &= E_t x_{t+1} - \sigma(r_t - E_t\pi_{t+1}) + g_t \end{aligned}$$

The exogenous terms are g_t, e_t, u_t.

We assume that monetary policy follows a simple Taylor rule.

Once you eliminate R, which is the only predetermined variable, the system becomes (see Woodford, Appendix C, page 677):

$$\begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} E\pi_{t+1} \\ Ex_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma\,(1+\phi_\pi) & 1+\phi_x\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + Mz_t$$

where z_t is the vector collecting all the shocks. This system can be rewritten as:

$$\begin{bmatrix} E\pi_{t+1} \\ Ex_{t+1} \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\kappa\beta^{-1} \\ \sigma\left(1+\phi_\pi-\beta^{-1}\right) & 1+\sigma\phi_x+\kappa\sigma\beta^{-1} \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \sigma & 1 \end{bmatrix}^{-1} Mz_t$$

A linear rational expectations model of this form (in which all the variables are non predetermined) has a determinate equilibrium if and only if the eigenvalues of the matrix A above are both bigger than one in modulus (outside the unit circle).

Remark 8 *In general cases, you need as many eigenvalues outside the unit circle as the number of non-predetermined variables. If the number of eigenvalues outside the circle is greater, then the equilibrium is explosive. Otherwise, the equilibrium is indeterminate.*

With a fair amount of matrix algebra you can show that the eigenvalues are both greater than one in modulus if:

$$\phi_\pi + \frac{1-\beta}{\kappa}\phi_x > 0$$

This condition can be easily interpreted. Observe that from the Phillips curve a 1% rise in inflation implies that the output gap rises by $\frac{1-\beta}{\kappa}$%. Hence for determinacy the nominal interest rate must rise by at least $\phi_\pi - 1$ to offset the rise in inflation, and by $\frac{1-\beta}{\kappa}$ to offset the rise in the output gap. Otherwise each increase in inflation would be self-fulfilling.

Remark 9 *In a model of this kind an <u>interest rate peg</u> results in indeterminacy of the equilibrium. In other words, there are an infinite number of possible responses of the variables to real disturbances, including some* *in which fluctuations in output and inflation are disproportionately large relative to the size of the disturbances. Hence this implies that the central bank must respond to the real variables to guarantee determinacy of equilibrium.*

Clarida-Gali-Gertler: the policy rule may be a source of instability if the coefficient on inflation is below unity. The intuition is easy: in this case, a rise in anticipated inflation leads to a decline in the real rate. This decline in the real rate leads to an increase in aggregate demand which in turn leads to a rise in inflation. The initial rise in inflation becomes therefore self-fulfilling.

Results of this kind seem to have arisen in the pre-Volcker period.

Homework

1. Read and summarize in two pages the Canova and Sala 2006 paper.

   See `http://www.econ.upf.edu/docs/papers/downloads/927.pdf`

2. Write down into a Dynare file the multi-equation Canova and Sala model described on page 26 of their September 2006 wp version.

3. Calibrate (following their lead), solve and simulate the model. Discuss the responses of the model to (a) monetary and (b) technology shocks. In particular, discuss the response of consumption and investment and hours worked in light of the results of this chapter.

4. Compare the responses that you get with the responses that obtain under flexible prices and wages.

5. Use quarterly US data in deviation from a linear (or quadratic) trend for the period 1960Q1-2006Q4. For the data, use (1) log real consumption C; (2) log business investment I; (3) the nominal interest rate R; (4) inflation π. Use Dynare to perform maximum likelihood or Bayesian estimation of the model parameters.

Chapter 8

Optimal Monetary Policy

Models of monetary economics typically involve as much as four distortions.

1. Violation of Friedman rule

   The first distortion derives from the agents' desire to hold money, given the transaction services that money provides. Since the private cost of holding money is R whereas the social cost of producing it is 0 efficiency would require that the two are set equal by having $R = 0$. Since the real rate is $RR = R - \pi$, this requires inflation to equal minus the real interest rate, that is a steady decline in the price level.

2. Imperfect Competition

   The second distortion derives from imperfect competition in the goods market. When $\alpha = 1$ (no capital), from

   $$Y = \frac{A^{\frac{\eta}{\rho+\eta-1}}}{X^{\frac{1}{\rho+\eta-1}}} < Y^* = A^{\frac{\eta}{\rho+\eta-1}}$$

   we can see that output will be inefficiently low whenever $X > 1$. To correct this distortion, inflation should be permanently above zero (from the Phillips curve), and this creates in itself a trade-off with the objective in (1)

3. Dynamic markup distortion

   Firms' inability to adjust prices at any point in time creates a *dynamic markup distortion*. Markups will fluctuate over time around their constant frictionless level.

4. Static markup distortion

   Even in absence of average inflation ($\pi = 0$), the lack of synchronization in price adjustments will imply the coexistence of different prices for goods that enter symmetrically agents' utility functions and which have a one-to-one marginal rate of transformation.

Modern models normally deal with (3) and (4). Both distortions can be corrected at once by a zero inflation policy. It is in this context, and starting from the utility function of the representative individual producer, that Rotemberg and Woodford (1997, NBER Macro Annual) show that the period utility loss resulting from deviations from the $X = 1$ allocation can by approximated by a quadratic equation of the form:

$$L_t = \frac{u_C C}{2}\left(\sigma_\pi^2 + \frac{\kappa}{\varepsilon}\frac{\rho+\eta-1}{1+\varepsilon(\eta-1)}\sigma_x^2\right)$$

where $\kappa = \lambda\zeta = (\eta+\rho-1)\frac{(1-\theta)(1-\beta\theta)}{\theta}$ is the slope of the Phillips curve.[1] The derivation of the steps is below.

8.1 Deriving the welfare function

Woodford derives a quadratic loss function that represents a second-order Taylor series approximation to the level of expected utility of the representative household in the rational expectations equilibrium associated with a given policy.

I provide a sketch of the derivation. Appendix 11.6 in Walsh provides a good derivation, as well as Woodford's book. Remember that the period utility function for the representative individual is, ignoring the real balances term and remembering that in equilibrium $Y = C$,

$$\frac{Y_t^{1-\rho}}{1-\rho} - \frac{1}{\eta}(L_t)^\eta$$

in a steady state without distortions, letting capital share going to zero:

$$\begin{aligned} Y &= AL && \text{(production function)} \\ wY^{-\rho} &= L^{\eta-1} && \text{(labor supply)} \\ A &= w && \text{(labor demand)} \\ Y^{1-\rho} &= L^\eta && \text{(equilibrium)} \end{aligned}$$

Next take a second-order Taylor approximation of the utility function around the steady state:

8.1.1 Output term

$$u(Y) = \frac{Y^{1-\rho}}{1-\rho} + Y^{-\rho}(Y_t - Y) - \rho\frac{Y^{-\rho-1}}{2}(Y_t - Y)^2$$

define now

$$\begin{aligned} \widetilde{Y} &= \frac{Y_t - Y}{Y} \\ \widehat{Y}_t &= \log(Y_t/Y) \end{aligned}$$

The two are related by

$$\widetilde{Y}_t = \frac{Y_t}{Y} - 1 = \exp\left(\log\frac{Y_t}{Y}\right) - 1 = \exp\left(\widehat{Y}_t\right) - 1$$

[1] The values in the first version of the paper for the relative weights were slightly different. The reason is explained in footnote 23 of the Woodford book, page 400.

Take a Taylor expansion of the above expression to derive

$$\begin{aligned}
\widetilde{Y}_t &= \exp\left(\widehat{Y}\right) + \exp\left(\widehat{Y}\right)\left(\widehat{Y}_t - \widehat{Y}\right) + \frac{1}{2}\exp\left(\widehat{Y}\right)\left(\widehat{Y}_t - \widehat{Y}\right)^2 - 1 = \\
&= \exp(0) + \exp(0)\left(\widehat{Y}_t - 0\right) + \frac{1}{2}\exp(0)\left(\widehat{Y}_t - 0\right)^2 - 1 = \\
\widetilde{Y}_t &= \widehat{Y}_t + \frac{1}{2}\widehat{Y}_t^2
\end{aligned}$$

Using this result, we can rewrite our Taylor expansion for $u(Y)$ as (dropping out constant terms and terms of order greater than 2):

$$u(Y) = Y^{1-\rho}\widetilde{Y} - \frac{1}{2}\rho Y^{1-\rho}\widetilde{Y}^2 = Y^{1-\rho}\left(\widehat{Y}_t + \frac{1}{2}\widehat{Y}_t^2 - \frac{1}{2}\rho\widehat{Y}_t^2 - ...\right) = Y^{1-\rho}\left(\widehat{Y}_t + \frac{1}{2}(1-\rho)\widehat{Y}_t^2\right) \qquad \text{(a)}$$

8.1.2 Labor supply term

Remember that L is the integral of labor supplied by all households in the economy $\int y(z)\,dz$, and each household on the segment produces good $Y(z) = y$ for notational simplicity. A Taylor expansion of $v(L) = (L_t)^\eta/\eta$ yields:

$$v(L) = L^\eta\left(\widehat{L}_t + \frac{\eta}{2}\widehat{L}_t^2\right)$$

integrate wrt z across all households and using $Y^{1-\rho} = L^\eta$

$$\int v(L)\,dz = Y^{1-\rho}\left(E\widehat{y}_t + \frac{\eta}{2}E\widehat{y}_t^2\right) = Y^{1-\rho}\left(E\widehat{y}_t + \frac{\eta}{2}\left(\left(E\widehat{y}_t\right)^2 + VAR\left(\widehat{y}_t\right)\right)\right)$$

Use:

$$\begin{aligned}
Y_t &= \left[\int_0^1 y^{\frac{\varepsilon-1}{\varepsilon}}dz\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
\widehat{Y}_t &= E\widehat{y}_t + \frac{\varepsilon-1}{\varepsilon}\frac{1}{2}VAR\left(\widehat{y}_t\right)
\end{aligned}$$

to write:

$$\int v(L)\,dz = Y^{1-\rho}\left(\widehat{Y}_t - \frac{\varepsilon-1}{2\varepsilon}VAR\left(\widehat{y}_t\right) + \frac{\eta}{2}\underbrace{\left(\widehat{Y}_t^{\,2} + ... + VAR\left(\widehat{y}_t\right)\right)}_{\text{drop } VAR^2 \text{ which is 4th order}}\right) \qquad \text{(b)}$$

8.1.3 Putting things together

We now subtract (b) from (a) and check the relative weight on $\widehat{Y}_t^2$ (dispersion of average output) versus $VAR(y)$ (dispersion of individual output produced by each intermediate goods producer/worker).

$$\begin{aligned}
\text{weight on } Y^2 &\propto \frac{1-\rho}{2} - \frac{\eta}{2} \\
\text{weight on } VAR(y) &\propto \frac{\varepsilon-1}{2\varepsilon} - \frac{\eta}{2}
\end{aligned}$$

hence the ratio:

$$ratio = \frac{\eta - \frac{\varepsilon-1}{\varepsilon}}{\rho + \eta - 1}$$

represents the relative concern on dispersion of output across producers in society's welfare.

The weight on Y^2 reflects concerns for deviations of output from its flexible price level, hence is tantamount to X^2.

The weight on the variance of y, individual output produced, is instead related to the variance of the prices that producers face through the individual demand curve for each product.

$$\begin{aligned} y^* &= (p^*/P)^{-\varepsilon} Y \\ VAR(y) &= \varepsilon^2 VAR(p^*) \end{aligned}$$

The cross-sectional variance of prices is turn linked to the inflation rate through

$$P_t = \theta P_{t-1} + (1-\theta) p^*$$

which can be used to show that, defining weights on inflation and output gap

$$w(p^*) = \frac{\theta}{1-\theta} \frac{1}{1-\theta\beta} w(\pi) = \frac{1}{\kappa} w(\pi)$$

and therefore the relative weight on inflation variance with respect to output variance must satisfy:

$$w(\pi_t) = \kappa w(p^*) = \frac{\kappa}{\varepsilon^2} w(y) = \frac{\kappa}{\varepsilon^2} \frac{\rho+\eta-1}{\eta - \frac{\varepsilon-1}{\varepsilon}} w(X_t) = \frac{\kappa}{\varepsilon} \frac{\rho+\eta-1}{1+\varepsilon(\eta-1)}$$

cfr Walsh page 555 and Woodford page 400.

To sum up one can relate

- $var(y)$ to the inflation variance times a term which depends on ε and θ. So inflation variability captures the dispersion of output levels across producers of different goods. Inflation concerns become more important the greater price rigidities and the less substitutable goods are.
- Y^2 captures the variability of output around its natural rate

What is a plausible value for $\frac{\kappa}{\varepsilon} \frac{\rho+\eta-1}{1+\varepsilon(\eta-1)}$, the relative weight on output (gap) stabilization? Assume

$$\begin{aligned} \theta &= .75 \\ \beta &= .99 \\ \kappa &= \frac{(1-\theta)(1-\theta\beta)}{\theta} \\ \eta &= 1.1 \\ \varepsilon &= 10 \\ \rho &= 1 \\ \xi &= \frac{\lambda}{\varepsilon} \frac{\rho+\eta-1}{1+\varepsilon(\eta-1)} \end{aligned}$$

Then $\xi = 4.7208 \times 10^{-3}$. If we measure output gap in % terms and inflation as an annualized inflation rate, then the appropriate weight on x^2 relative to $(4\pi)^2$ becomes $16\xi = .075$, which is much lower than the values of 1 typically assumed in the literature, on grounds such as "equal weight to the two objectives". In other words, the distortions associated with inflation are far greater than those associated with variation in the output gap.

The losses from inflation can be completely elimated by a zero inflation policy. That is, the price level distortion are minimized by creating an environment where

1. those who choose a new price set the old price
2. if so, then the average price level never changes
3. eventually all good prices are the same
4. hence price stability is a sufficient condition for the absence of price dispersion

8.2 Optimal policy

Back to our earlier model. Aggregate demand and supply are given by:

$$\begin{aligned} x_t - E_t x_{t+1} + \phi\left(R_t - E_t \pi_{t+1}\right) - g_t &= 0 \\ \pi_t - \lambda x_t - \beta E_t \pi_{t+1} - u_t &= 0 \end{aligned}$$

Consider the problem of a central bank:

$$\max W = -\frac{1}{2} E_t \left[\sum_{t=0}^{\infty} \beta^t L_t \right]$$

where:

$$L_t = \pi_t^2 + \alpha x_t^2$$

subject to AD and AS above.

Let me spend a few words about this problem: macroeconomics is full of problems in which the return function is quadratic and the constraint is linear. However not all constraints are the same: in the standard problem, the constraints do not involve expectations of future variables. The resulting problem is the so-called *stochastic optimal linear regulator problem* (see e.g. Ljungqvist-Sargent textbook, chapter 5). More complicated are problems where the constraints involve expectations of future variables, rendering the dynamic programming principle invalid: in this class of problems, in fact, target variables depend not only on policy but also on future expected policy.

Remark 10 *The optimal equilibrium is the one that achieves the lowest possible value of the loss measure W above.*

8.3 Optimal policy under discretion

Under discretion, the central bank expects itself to reoptimize at each successive date, and is unable to commit itself to future paths for inflation and the output gap. That makes the problem rather easy to solve.

Easy way to solve the problem (since we choose x π and R) is first to solve under AS constraint only and then to work out optimal R implied by the aggregate demand curve.

$$\max_{\pi,x} -\frac{1}{2}\left(\pi_t^2+\alpha x_t^2\right)$$

$$\text{s.t. } \pi_t=\lambda x_t+\beta E_t\pi_{t+1}+u_t$$

where $f_t \equiv \beta E_t\pi_{t+1}+u_t$ is taken as given.

The problem then becomes:

$$\max_{\pi} -\frac{1}{2}\left(\pi_t^2+\alpha\left(\frac{\pi_t-f_t}{\lambda}\right)^2\right)$$

yielding (since $\frac{1}{\lambda}(\pi_t-f_t)=x_t$)

$$\pi_t+\frac{\alpha}{\lambda^2}(\pi_t-f_t)=0 \Rightarrow x_t^b=-\frac{\lambda}{\alpha}\pi_t^b$$

where the superscript b indicates that this is the solution under discretion.

To solve the problem now combine this equilibrium condition with the AS curve

$$\pi_t=\lambda x_t+\beta E_t\pi_{t+1}+u_t$$

and impose that expectations are rational. You will get:

$$\pi_t=\frac{\alpha\beta}{\alpha+\lambda^2}E_t\pi_{t+1}+\frac{\alpha}{\alpha+\lambda^2}u_t \equiv c\pi_{t+1}+du_t$$

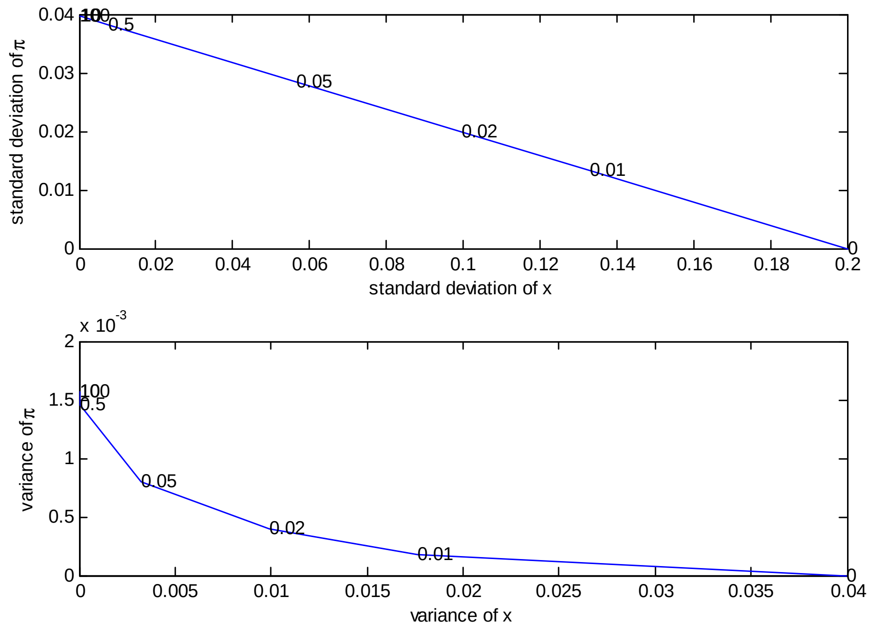
This equation can be solved forward to obtain (under $u_{t+1}=\rho u_t$):

$$\begin{aligned}
\pi_t^b &= c\left(c\pi_{t+2}+du_{t+1}\right)+du_t= \\
&= c\left(c\pi_{t+2}+d\rho u_t\right)+du_t= \\
&= d\left(u_t+c\rho u_t+c^2\rho^2u_t+...\right)=\frac{d}{1-c\rho}u_t= \\
\pi_t^b &= \frac{\frac{\alpha}{\alpha+\lambda^2}}{1-\frac{\alpha\beta\rho}{\alpha+\lambda^2}}u_t=\frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}u_t \\
x_t^b &= -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}u_t
\end{aligned}$$

as $\pi_t=wu_t$, $E\pi_{t+1}=w\rho u_t=\rho\pi_t$.

Go back to IS, solved for R_t:

$$\begin{aligned}
\phi R_t &= E_tx_{t+1}+\phi E_t\pi_{t+1}-x_t+g_t \\
\phi R_t &= -E_t\frac{\lambda}{\alpha}\pi_{t+1}+\phi E_t\pi_{t+1}+\frac{\lambda}{\alpha}\pi_t+g_t \\
\phi R_t &= \left(-\frac{\lambda}{\alpha}+\phi+\frac{\lambda}{\alpha\rho}\right)E_t\pi_{t+1}+g_t \\
R_t &= \left(1+\frac{\lambda(1-\rho)}{\alpha\rho\phi}\right)E_t\pi_{t+1}+\frac{1}{\phi}g_t
\end{aligned}$$



Remark 11 *Optimal policy responds more than one for one to changes in expected inflation.*

8.3.1 Monetary policy trade-offs under discretion

We have found that:

$$
\begin{aligned}
-\frac{\alpha}{\lambda}x_t &= \pi_t = \frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}u_t \\
\sigma_\pi^2 &= \left(\frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}\right)^2\sigma_u^2 \\
\sigma_x^2 &= \left(\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}\right)^2\sigma_u^2 \\
\sigma_x^2 &= \frac{\lambda^2}{\alpha^2}\sigma_\pi^2
\end{aligned}
$$

Remark 12 *α defines the policymaker preferences. For given value of λ, the last equation describes an inverse relationship (Taylor curve) between the two policy objectives. This can be plotted in the $\left(\sigma_x^2, \sigma_\pi^2\right)$ space varying α from 0 to a high number. The plot below illustrates one such example, setting $\lambda = 0.10$, $\rho = 0.50$, $\sigma_u = 0.02$*

8.4 Optimal Policy under Commitment

Literature often divided into two strands. The 1980s literature assumes that the output gap goal is to push output permanently above its natural rate. In the Nash equilibrium, this generates inflation with little output gains. The modern literature considers other issues, i.e. optimal rules.

8.4.1 The Classic Inflationary Bias Problem

Assume demand and supply are given by:

$$\begin{aligned} x_t - E_t x_{t+1} + \phi (R_t - E_t \pi_{t+1}) - g_t &= 0 \\ \pi_t - \lambda x_t - \beta E_t \pi_{t+1} - u_t &= 0 \end{aligned}$$

Problem is:

$$\max W = -\frac{1}{2} E_t \left[\sum_{t=0}^{\infty} \beta^t L_t \right]$$

where:

$$L_t = \pi_t^2 + \alpha (x_t - k)^2$$

$k > 0$ reflects the presence of distortions so that socially efficient output exceeds natural level. In this case a discretionary central bank faces the following problem:

$$\max_{\pi} -\frac{1}{2} \left(\pi_t^2 + \alpha \left(\frac{\pi_t - f_t}{\lambda} - k \right)^2 \right)$$

yielding:

$$\begin{aligned} \pi_t + \frac{\alpha}{\lambda} \left(\frac{\pi_t - f_t}{\lambda} - k \right) &= 0 \\ \pi_t + \frac{\alpha}{\lambda} (x_t - k) &= 0 \\ &\Rightarrow \quad x_t = -\frac{\lambda}{\alpha} \pi_t + k \end{aligned}$$

To solve the problem now combine this equilibrium condition with the AS curve

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t = \frac{\lambda \alpha}{\alpha + \lambda^2} k + \beta \pi_{t+1} + u_t$$

and impose that expectations are rational. You will get:

$$\pi_t^k = \pi_t^b + \frac{\alpha \lambda}{\alpha (1 - \beta) + \lambda^2} k$$

hence inflation equals inflation under the baseline case plus a term related to k. Solving for x_t and remembering that $x_t^b = -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2} u_t$:

$$\begin{aligned} x_t^k &= -\frac{\lambda}{\alpha} \left(\pi_t^b + \frac{\alpha \lambda}{\alpha (1 - \beta) + \lambda^2} k \right) + k \\ &= -\frac{\lambda}{\alpha} \pi_t^b - \frac{\lambda^2}{\alpha (1 - \beta) + \lambda^2} k + k = \\ &= x_t^b + \frac{\alpha (1 - \beta)}{\alpha (1 - \beta) + \lambda^2} k \end{aligned}$$

Remark 13 *Under discretionary policy, Inflation is higher and output is slightly above natural level. However the gain disappears as $\beta \to 1$.*

8.4.2 Gains from commitment: the optimum within simple rules

Here we return to our earlier model but we take into account the possibility that central bank actions might affect private agent expectations. However the general solution to this simple problem is not as simple as it might look like. What we do is therefore to look for the optimum within simple rules.

Consider a rule for the target x of the following form:

$$x_t^c = -\omega u_t$$

this corresponds to the rule under discretion x_t^b whenever the central bank chooses $\omega = \frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}$. Under such a rule inflation is:

$$\pi_t^c = u_t + kx_t^c + \beta E_t \pi_{t+1} = u_t\left(1-\lambda\omega\right) + \beta E_t \pi_{t+1} = \frac{1-\lambda\omega}{1-\beta\rho} u_t$$

or differently:

$$\pi_t^c = \frac{1}{1-\beta\rho}\left(\lambda x_t^c + u_t\right)$$

comparing this with the solution to the discretionary case:

$$\pi_t^b = \lambda x_t^b + \overline{\beta E_t \pi_{t+1}} + u_t$$

when the central bank is unable to manipulate expectations, the scale of trade-off is different. Reducing x_t by 1% reduces π_t^b by λ%, rather than $\frac{\lambda}{1-\beta\rho}$% as in the commitment case.

What is the optimal value of the feedback parameter? Since both π^c and x^c are multiples of u_t, one can write the L as a function of u only.

$$\begin{aligned}
\max W &= -\frac{1}{2}E_t\left(L_t + \beta L_{t+1} + \beta^2 L_{t+2} + ...\right) \\
\max W &= -\frac{1}{2}E_t\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2 u_t^2 + \alpha\omega^2 u_t^2 + \beta L_{t+1} + \beta^2 L_{t+2} + ...\right) = \\
\max_\omega W &= -\frac{1}{2}\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2 \left(1+\rho^2+\rho^4+...\right) u_t^2 + \alpha\left(\omega^2\left(1+\rho^2+\rho^4+...\right)\right) u_t^2\right) = \\
\max_\omega W &= -\frac{1}{2\left(1-\rho^2\right)}\left(\left(\frac{1-\lambda\omega}{1-\beta\rho}\right)^2 u_t^2 + \alpha\omega^2 u_t^2\right) \\
&\Leftrightarrow \frac{\lambda\left(1-\lambda\omega\right)}{\left(1-\beta\rho\right)^2} = \alpha\omega \\
\omega^c &= \frac{\lambda}{\lambda^2 + \alpha\left(1-\beta\rho\right)^2}
\end{aligned}$$

Given the optimal ω^c, the link between x and π is given by:

$$\begin{aligned}
x_t^c &= -\omega^c u_t \\
\pi_t^c &= \frac{1-\lambda\omega^c}{1-\beta\rho} u_t = -\frac{1-\lambda\omega^c}{1-\beta\rho}\frac{1}{\omega^c} x_t^c = -\frac{\alpha\left(1-\beta\rho\right)}{\lambda} x_t^c \\
x_t^c &= -\frac{\lambda}{\alpha\left(1-\beta\rho\right)}\pi_t^c = -\frac{\lambda}{\alpha_c}\pi_t^c
\end{aligned}$$

where $\alpha_c = \alpha(1-\beta\rho) < \alpha$ implies that commitment allows the authority to face a better trade-off. Put differently, under commitment for given rise in inflation the output gap has to fall less $(1-\beta\rho) < 1$. The monetary policy authority can be more aggressive against inflation.

To sum up, these are the outcomes that we obtain under the different assumptions about central bank behavior. Once demand shocks are offset, central bank can achieve a better outcome by responding more aggressively to inflation. That is why under commitment the central bank faces a better trade-off.

	π_t	x_t	trade-off	int.rate R_t^*
disc, no π bias	$\pi_t^b = \frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2} u_t$	$x_t^b = -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2} u_t$	$\sigma_x^2 = \frac{\lambda^2}{\alpha^2}\sigma_\pi^2$	$\left(1+\frac{\lambda(1-\rho)}{\alpha\rho\phi}\right)\pi_{t+1}+\frac{g_t}{\phi}$
disc, π bias	$\pi^k = \frac{\alpha(u_t+\lambda k)}{\alpha(1-\beta\rho)+\lambda^2}$	$x_t^k = \frac{\alpha(1-\beta)k}{\alpha(1-\beta)+\lambda^2} - \frac{\lambda}{\alpha_c+\lambda^2} u_t$		
commitment	$\pi^c = \frac{\alpha(1-\beta\rho)}{\alpha(1-\beta\rho)^2+\lambda^2} u_t$	$x_t^c = -\frac{\lambda}{\alpha(1-\beta\rho)^2+\lambda^2} u_t$	$\sigma_x^2 = \frac{\lambda^2}{\alpha_c^2}\sigma_\pi^2$	$\left(1+\frac{\lambda(1-\rho)}{\alpha(1-\beta\rho)\rho\phi}\right)\pi_{t+1}+\frac{g_t}{\phi}$

8.4.3 The general solution under commitment

The globally optimal rule under commitment is likely not to fall within the restricted family of rules considered in the previous subsection. Remember that the restriction that we had so far was that the central bank was allowed to choose sequences for inflation and output gap that were a function only of *current* period realization of disturbances, so the solutions above were constrained optima within their family.

The constraint we have now is still:

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t$$

Problem is:

$$\max_{x_{t+1},\pi_{t+1}} -\frac{1}{2}E_t \begin{bmatrix} \alpha x_t^2 + \pi_t^2 + 2\xi_t(\pi_t - \lambda x_t - \beta\pi_{t+1} - u_t) + \\ \alpha\beta x_{t+1}^2 + \beta\pi_{t+1}^2 + 2\beta\xi_{t+1}(\pi_{t+1} - \lambda x_{t+1} - \beta\pi_{t+2} - u_{t+1}) + ... \end{bmatrix}$$

The First order condition's are, choosing x_{t+1} and π_{t+1} and scrolling them backward:

$$\begin{aligned} \alpha x_t &= \xi_t \lambda \\ \xi_{t-1} &= \pi_t + \xi_t \end{aligned}$$

You will see the term ξ_{t-1} here. Under commitment (respectively, discretion), this term is non-zero (zero), since past actions are (are not) binding for the social planner. One can drop ξ_t to obtain:

$$\begin{aligned} \pi_t &= p_t - p_{t-1} = -\frac{\alpha}{\lambda}(x_t - x_{t-1}) \\ p_t &= -\frac{\alpha}{\lambda} x_t \end{aligned}$$

notice that this solution implies a form of price level targeting (or, if you like, adjusting the change in the output gap in response to inflation). In fact another way to rewrite it is:

$$\log P_t = \log P^* - \frac{\alpha}{\lambda} x_t$$

where P^* is the target price level.

Definition: Price-level targeting (PT) is a policy systematically responds to deviations of the price level from the price level target path to preclude long-run price-level drift.

This result can be compared with the optimal inflation-targeting policy that we obtained under discretion

$$\pi_t^b = -\frac{\alpha}{\lambda} x_t^b$$

8.4.3.1 Implementation of the general solution under commitment

A problem with such a rule (if policy responds to expected inflation and demand shocks) is that it might not guarantee real determinacy. This can be seen easily replacing the condition $\pi_{t+1} = (-\alpha/\lambda)(x_{t+1} - x_t)$ into the AD curve.

$$\begin{aligned} R_t^* &= \frac{1}{\phi}\left(E_t x_{t+1} - x_t\right) + E_t \pi_{t+1} + \frac{1}{\phi} g_t = \\ &= \left(1 - \frac{\lambda}{\phi\alpha}\right) E_t \pi_{t+1} + \frac{1}{\phi} g_t \end{aligned}$$

A rule of this type might therefore permit self-fulfilling fluctuations in output and inflation that are clearly sub-optimal. Commitment to a rule of this kind might guarantee a REE only under some very stringent conditions.

8.4.4 Robustly optimal rules under commitment

Giannoni and Woodford (2002) solve a problem which is similar in nature to that above, however they explicitly allow for interest rate variability to enter the loss function of the central bank. That is:

$$\begin{aligned} \min W &= E_0\left[\sum_{t=0}^{\infty} \beta^t L_t\right] \\ L_t &= \pi_t^2 + \mu_x {x_t}^2 + \mu_r R_t^2 \end{aligned}$$

The last term may arise if one takes into account the utility services of money, which were set arbitrarily close to zero in the derivation of the welfare function at the beginning of this section (see Woodford, Chapter 6.4.1)

Set up the Lagrangian:

$$\begin{aligned} L \;=\; & \pi_t^2 + \mu_x {x_t}^2 + \mu_r R_t^2 + \\ & \beta\pi_{t+1}^2 + \beta\mu_x {x_{t+1}}^2 + \beta\mu_r R_{t+1}^2 + \\ & -\xi_{1t}\left(\pi_t - \lambda x_t - \beta\pi_{t+1} - u_t\right) - \xi_{2t}\left(x_t - x_{t+1} + \phi R_t - \phi\pi_{t+1} - g_t\right) \\ & -\beta\left(\xi_{1t+1}\left(\pi_{t+1} - \lambda x_{t+1} - \beta\pi_{t+2} - u_{t+1}\right) - \beta\xi_{2t+1}\left(x_{t+1} - x_{t+2} + \phi R_{t+1} - \phi\pi_{t+2} - g_{t+1}\right)\right) \end{aligned}$$

The central bank minimizes over the whole time period, choosing π_{t+1} and x_{t+1} and R_{t+1}. Taking the first-order conditions and scrolling them one period backward yields:

$$\pi_t - \phi\beta^{-1}\xi_{1t-1} + \xi_{2t} - \xi_{2t-1} = 0$$
$$\mu_x x_t + \xi_{1t} - \beta^{-1}\xi_{1t-1} - k\xi_{2t} = 0$$
$$\mu_r R_t + \phi\xi_{1t} = 0$$

together with:

$$\begin{aligned} \pi_t - kx_t - \beta\pi_{t+1} &= 0 \\ x_t - E_t x_{t+1} + \phi R_t - \phi\pi_{t+1} - g_t &= 0 \end{aligned}$$

This dynamic system of 5 equations in 5 unknowns can be solved for R as a function of existing endogenous variables only. Giannoni and Woodford call the deriving rule a "robustly optimal instrument rule". The idea is that you can play with the first three equations and solve for R_t as a function of the existing endogenous variables only.

$$R_t = \left(1 + \frac{k\phi}{\beta}\right) R_{t-1} + \beta^{-1}\Delta R_{t-1} + \frac{k\phi}{\mu_r}\pi_t + \frac{\phi\mu_x}{\mu_r}\Delta x_t$$

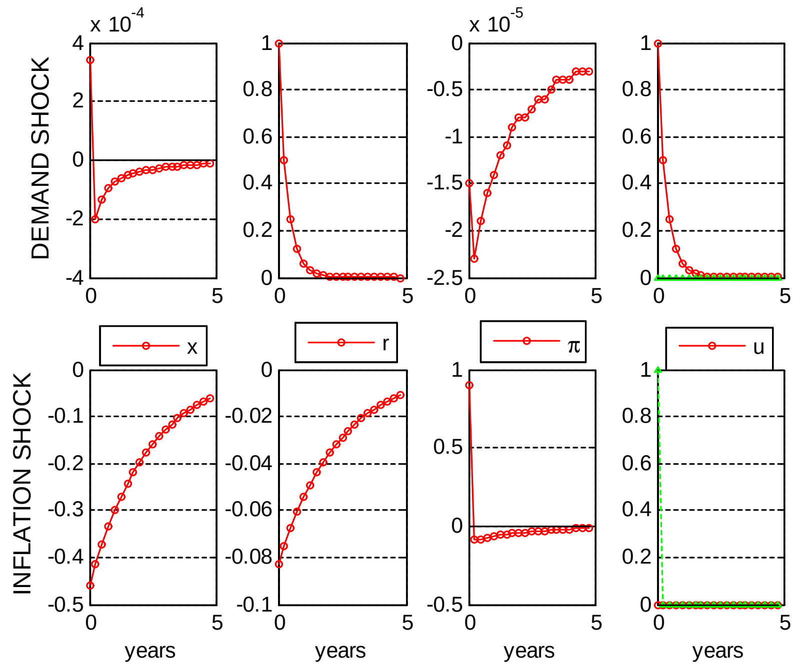
A bunch of comments:

- The optimal interest rate rule is a function only of the variables in the loss function
- The rule requires the interest rate to be positively related to fluctuations in current inflation, in changes of the output gap, and in lagged interest rates
- The rule is super-inertial, in the sense that it requires that the interest rate to vary by more than one for one to past fluctuations of the interest rate.
- The coefficients in the optimal rule do not depend on either the serial correlation of the shocks or the variances of these innovations. It is, therefore, robust with respect to mis-specification of these aspects of the model.

You will notice that, as μ_r approaches zero, we are back into the general solution under commitment result. In fact

$$\begin{aligned} k\pi_t &= \mu_x (x_{t-1} - x_t) \\ P_t &= -\frac{\mu_x}{k} x_t \end{aligned}$$

The outcome under timeless precommitment is shown in the Figure below (for λ_r arbitrarily low, file giannoni.m). Despite the fact that the cost shock has no persistence, the output gap displays positive serial correlation. By keeping output below potential for several periods into the future after the negative shock, the central bank is able to lower expectations of future inflation. A fall in $E_t\pi_{t+1}$ at the time of the shock improves the trade-off between inflation and output gap stabilization faced by the central bank.



This is unlike the case we obtain under discretion, where we find that

$$\pi_t^b = \frac{\alpha}{\alpha(1-\beta\rho)+\lambda^2}u_t$$

$$x_t^b = -\frac{\lambda}{\alpha(1-\beta\rho)+\lambda^2}u_t$$

hence in that case the variables inherit the persistence properties of the cost-push shock, and there is no inertia in the variables following a shock.

8.4.5 Robust Control

Hansen and Sargent have recently developed a different notion of robust policy, and both approaches have been applied in the context of optimal monetary policy. It is possible to show that both approaches lead to exactly the same implicit instrument rule for the policy maker in a standard, forward-looking, new Keynesian model. See Walsh (2004), JMCB.

In Hansen and Sargent's approach, the policy maker's problem can be represented as a game between the policy maker who attempts to minimize the loss function above and nature (or an evil agent) who tries to maximize the same loss function. A robust policy is designed to perform well in this worst-case scenario. Hansen and Sargent show how this game can be represented in what they label the multiplier version of the robust Stackleberg problem.

8.5 What about the Taylor rules?

Suppose we compare:

$$R_t = \left(1+\frac{k\phi}{\beta}\right)R_{t-1} + \beta^{-1}\Delta R_{t-1} + \frac{k\phi}{\mu_r}\pi_t + \frac{\phi\mu_x}{\mu_r}\Delta x_t \quad (1)$$

$$R_t = \phi_\pi P_t + \phi_x x_t \quad (2)$$

$$R_t = \phi_\pi \pi_t + \phi_x x_t \quad (3)$$

Which policy will yield higher welfare? To begin with, we know that 1 dominates 2 and 3 since it is, by construction, the optimal policy. What about 2 (Wicksellian rule, after Wicksell, 1907) versus 3 (Taylor)?

One would believe that 3 is better than 2, since under 2 the policymaker responds to an inflationary shock by bringing about deflation in future periods, hence lowering welfare. However, as shown for instance in Giannoni (2000), "Commitment to an optimal Wicksellian policy allows the policymaker to achieve a response of endogenous variables that is closer to the optimal plan than is the case with the optimal Taylor rule. One particularity of the equilibrium resulting from a Wicksellian policy is that the price level is stationary. This feature turns out to affect the response of endogenous variables in particular when shocks are very persistent [...] [T]he mere expectation of future deflation [..] under the optimal plan and the optimal non-inertial plan already depresses inflation when the shock hits the economy, and is expected to keep inflation below steady-state for several periods. In contrast, under optimal Wicksellian policy, both inflation and the price level rise strongly on impact, but they are expected to return progressively to their initial steady-state"

8.6 A digression on the supply shock

The source of the output-inflation variance trade-off for central bank is the "supply" shock. Were there only demand type shocks, there would be no trade-off. This is summarized in Erceg, Henderson and Levin (2000), Proposition 2.

Proposition 2: With staggered price contracts and completely flexible wages, monetary policy can completely stabilize price inflation and the output gap, thereby attaining the Pareto-optimal social welfare level.

However, a price inflation / output gap variance trade-off arises endogenously in the model above with staggered wage and price setting. When both prices and wages are staggered, it is impossible for monetary policy to attain the Pareto optimum except in the special cases where either wages or prices are completely flexible. Nominal wage inflation and price inflation would remain constant only if the aggregate real wage rate were continuously at its Pareto-optimal level. Such an outcome is impossible because the Pareto-optimal real wage moves in response to various shocks, whereas the actual real wage could never change in the absence of nominal wage or price adjustment. Given that the Pareto optimum is infeasible, the monetary policymaker faces trade-offs in stabilizing wage inflation, price inflation, and the output gap.

(B) With staggered wage contracts and completely flexible prices, monetary policy can completely stabilize wage inflation and the output gap, thereby attaining the Pareto-optimal social welfare level.

Chapter 9

Interaction between Fiscal and Monetary Policy

How do monetary and fiscal policies affect price level determination? We have left this question in the background up to now. It is time to tackle it. We consider a simple economy with MIU, exogenous output and possibility for the government to finance its expenditures through either seigniorage, taxation or issuance of nominal debt.

9.1 A basic model

9.1.1 Households

The household sector is conventional. Households choose $\{C_t, M_t, B_t\}$:

$$\max_{c_t, M_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{m_t^{1-\eta}}{1-\eta} \right)$$
$$\text{s.t. } c_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} + T_t = I_{t-1} \frac{B_{t-1}}{P_t} + Y + \frac{M_{t-1}}{P_t}$$

where now T_t denotes taxes (minus transfers) that the government raises from households.

The government is issuing a nominal bond B_t costing 1\$ and paying in $t+1$ a nominal interest rate of I_t\$.

In equilibrium since only households consume it will be the case that consumption will equal exogenous output Y minus government expenditure G. If Y and G are fixed, the marginal utilities of consumption today and tomorrow will be equal, therefore the household optimality conditions will be:

$$
\begin{aligned}
1 &= \beta I_t \frac{P_t}{P_{t+1}} \qquad (1)\\
C^{-\sigma} &= \left(\frac{M_t}{P_t}\right)^{-\eta} + \beta C^{-\sigma}\frac{P_t}{P_{t+1}}
\end{aligned}
$$

Combining the two equations above yields the usual money demand equation:

$$
\frac{M_t}{P_t} = Y^{\frac{\sigma}{\eta}}\left(\frac{I_t-1}{I_t}\right)^{-\frac{1}{\eta}} \qquad (2)
$$

9.1.2 Government

We begin with the government flow of funds. This flow of funds can be written as:

$$
T_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} = G + I_{t-1}\frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} \qquad (3)
$$

9.1.3 Equilibrium

Goods market clearing implies that in equilibrium, by Walras' law, we only need to consider either the household or the government flows of funds. We choose to pick the latter.

Remark 14 *Equations 1 to 3 involve three equations in the 5 unknowns P I M B T. This implies that we cannot specify independent paths for money supply M, government debt B and primary surplus T if we want to study a unique equilibrium. Precisely, an equilibrium exists only for a restricted set of $\{M, B, T\}$ processes.*

If the government chooses a time path for T_t and M_t (or B_t, anyways 2 out of the 3), then equations (1) to (3) give equilibrium values for P, B and I, provided that $\beta^t M_t/P_t$ and $\beta^t B_t/P_t$ approach zero as time reaches infinity. Notice that goods market clearing already implies $B^g + B^h = 0$, $M^s = M^d$ and $Y = c + G - T$.

9.2 Price level determination without government debt ($B = 0$)

The case without government debt corresponds to what one would call a monetarist regime. Consider first the case in which there is no government debt at all, $B = 0$. In this case equations (1) to (3) simplify further since we can close the model simply by specifying a path for either T_t or M_t. The important lesson we learn is that in any case inflation is always and everywhere a monetary phenomenon.

1. MONEY SUPPLY RULE

Typically, we assume an exogenous stochastic process for money supply. In this case, the linearized versions of (1) to (3) will be:

$$
\frac{M_t - M_{t-1}}{P_t} + T_t = 0
$$

the log-linear equilibrium will be[1]

$$\begin{aligned} m_t - m_{t-1} &= \theta_t - \pi_t && \text{(a)} \\ I_t &= \pi_{t+1} && \text{(b)} \\ m_t &= -\frac{\eta^{-1}}{I-1} I_t && \text{(c)} \end{aligned}$$

do we have a unique determinate REE? The answer is yes. Consider perfect foresight, $\theta = 0$. Solve this system for π_t to obtain

$$\pi_t \left(1 + \frac{I-1}{\eta}\right) = \pi_{t+1}$$

If $\left|1 + \frac{I-1}{\eta}\right| > 1$, the price level is determinate.

2. INTEREST RATE RULE A-LA TAYLOR

In this case money supply becomes endogenous. Assume central bank adjust money supply so as to ensure that

$$I_t = \alpha \pi_t \tag{a}$$

combine this with the Fisher equation

$$I_t = \pi_{t+1} \tag{b}$$

(c) is irrelevant since it dictates only endogenous money supply. Solve for π_t to obtain

$$\alpha \pi_t = \pi_{t+1}$$

hence we have a unique REE only if $|\alpha| > 1$.

3. INTEREST RATE PEG

This case is trivial. This is like 2 but with $\alpha = 0$. In this case we have only an equation in which the price level appears as an expectational error

$$P_t = P_{t+1}$$

hence any expectation of the price level will be self-fulfilling. The price level only appears in the form on an expected rate of change, therefore it is indeterminate. Intuitively, if all agents expect prices to be permanently higher, current prices will be higher as well. The central bank will increase money supply so as to leave the interest rate unchanged, so the real variables will be determinate, but the nominal variable P_t will not. Intuitively, the price level is indeterminate if central bank does not care about nominal variables.

4. PRICE LEVEL TARGETING

Rewrite the two equations

$$\begin{aligned} I_t &= \alpha P_t \\ I_t &= P_t + P_{t+1} \end{aligned}$$

[1] From $\frac{M_t}{P_t} = Y^{\frac{\sigma}{\eta}} \left(\frac{I_t - 1}{I_t}\right)^{-\frac{1}{\eta}}$
$\ln m_t = -\frac{1}{\eta} \ln \left(1 - \frac{1}{I_t}\right)$
$\widehat{m_t} = -\frac{1}{\eta} \frac{\beta}{1-\beta} \widehat{I}_t$, using $\beta = I^{-1}$

determinacy obtains iff $\alpha > 0$

5. ADDING NOMINAL RIGIDITIES

Nominal rigidities solve the problem of nominal indeterminacy since under nominal rigidities yesterday's price level provides the economy with a nominal anchor. For instance, in our model with nominal rigidities:

$$P_t = \theta P_{t-1} + (1-\theta) P_t^*$$

however there is in this models the possibility of real indeterminacy, as argued by Clarida, Gali and Gertler.

9.3 Price level determination with government debt

We return now to the basic setup which includes equations 1 to 3 (and 5 endogenous variables, I, P, T, B, M) unless we make some assumptions), and ask how monetary and fiscal policies affect price level determination. Remember that in order to close the model we need to specify stochastic processes for two variables out of M, B and T.

Before we delve into how we specify these processes, it is important to realise that prices will in general no longer be determined only by monetary conditions. Consider for a moment a perfect foresight equilibrium in which nominal money is constant at M, and rewrite the money demand equation in more general terms as

$$Y^{-\sigma} = g\left(\frac{M_t}{P_t}\right) + \beta Y^{-\sigma}\frac{P_t}{P_{t+1}}$$

where $g = \left(\frac{M}{P_t}\right)^{-\eta}$ in the power utility case. This implies a first order differential equation for $P_{t+1} = f\left(P_t\right)$, where f is convex, and $f\left(0\right) = 0$. In particular, letting P denote the steady state price level, $f'\left(P\right) > 1$, so that the f line cuts the 45degree line from below at P. So, while P is a steady state, any path for P starting at $\vec{P} > P$ and following the law of motion in f is consistent with money demand, but involves ever-growing prices (notice that paths starting at $\overleftarrow{P} < P$ would satisfy money demand, but would violate the transversality condition for m_t. In this case, one would think that the equilibrium price level is indeterminate. However, from the government flow of funds:

$$P_t = \frac{I_{t-1}B_{t-1} - B_t + M_{t-1} - M_t}{T_t - G}$$

even when the money demand equation becomes unsuitable to calculate the price level, the level of government liabilities can allow pinning down an equilibrium price level that depends on the government asset position and on the present value of expected future surpluses. This occurs when we interpret the government equation as a simple *flow of funds* equation rather than an intertemporal budget *constraint.*

9.3.1 Flows of funds versus intertemporal budget constraints

Written as

$$T_t + \frac{M_t}{P_t} + \frac{B_t}{P_t} = G_t + I_{t-1}\frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} \tag{3}$$

the government's budget identity places no restrictions on the government expenditure or tax choices. However, if governments, like individuals, are constrained in their ability to borrow, then this constraint limits the government's choices. In real terms, the same equation can be written as

$$T_t + s_t + b_t = G_t + Rb_{t-1}$$

where s_t denotes seigniorage revenues ($s_t = \Delta M_t/P_t$) and we simplify by assuming that the real gross interest rate is constant and positive. This equation can be solved forward (using $\frac{T_{t+1}+s_{t+1}+b_{t+1}-g_{t+1}}{R} = b_t$ and so on...) to obtain:

$$
\begin{aligned}
T_t + s_t + \left(\frac{T_{t+1} + s_{t+1} + b_{t+1} - g_{t+1}}{r}\right) &= G_t + Rb_{t-1} \\
\sum_{i=0}^{\infty}\left(\frac{T_{t+i}}{R^i}\right) + \sum_{i=0}^{\infty}\left(\frac{s_{t+i}}{R^i}\right) + \lim_{i\to\infty}\left(\frac{b_{t+i}}{R^i}\right) &= \sum_{i=0}^{\infty}\left(\frac{G_{t+i}}{R^i}\right) + Rb_{t-1} \qquad \text{(IBC)}
\end{aligned}
$$

When

$$\lim_{i\to\infty}\left(\frac{b_{t+i}}{R^{t+i}}\right) \to 0 \qquad \text{(NPG)}$$

the government constraint is said to satisfy the intertemporal budget balance (NPG, no Ponzi game condition). In other words, when the NPG condition holds, any change in expenditure must be matched either by an increase in seigniorage or by an increase in taxes. When $b_{t-1} > 0$, the government must plan to raise more taxes in the future compared to expenditure, and viceversa when $b_{t-1} < 0$.

The question is: does the intertemporal budget constraint need the satisfy the NPG condition above? That is, for any given price level and interest rates, does the government need to pick taxes and expenditures so that the NPG holds? Or does the NPG holds only at the equilibrium prices?

1. For some, the NPG is a constraint on government actions. This implies that any government with a current oustanding debt must plan to run primary surpluses in the future This primary surpluses can come either from taxes or from seigniorage. This implies that the government is bound by the NPG condition to follow a "Ricardian policy"

   Example: suppose taxes do not change in response to economic conditions. Hence if seigniorage falls today (because money supply falls), it has to increase tomorrow (money supply is bound to increase). Sargent and Wallace's example of a monetary contraction which to inflation in the future was based on this ideas.

2. For some, the NPG is an equilibrium condition only, and does not need to be satisfied for all price levels, but only at the equilibrium prices. This implies that the government is free to choose (under some conditions) policy paths which are not Ricardian. However, we know from the household optimality conditions that NPG must hold in equilibrium (for household choices to be optimal). Once this result is plugged back inside the (IBC), this gives us an equation that can solve P as a function of fiscal variables only.

9.4 Active and passive policies

Leeper (1992): best model in order to understand at least the basis of all controversies that have surrounded the fiscal theory of the price level. See also survey paper by Leeper himself ("Macro Policy and inflation: an Overview").

See also Woodford's book, Chapter 4.4 (Fiscal requirements for price stability)

Remark 15 *The government flow of funds requires that shocks to the real value of government debt either are financed by future taxes or by inflation (that is, money creation). Having said that, we can dichotomize policies into* (1) *those where future taxes back entirely debt (passive fiscal policies and active monetary policies);* (2) *those where money creation entirely backs debt, like in Sargent and Wallace example of the unpleasant monetarist arithmetic (passive monetary policies and active fiscal policies)*

To the basic setup made by equations (1) to (3), he adds the following rules for monetary and fiscal policy:

$$\widehat{I}_t = \alpha\widehat{\pi}_t + \widehat{e}_t \tag{LL1}$$

$$\widehat{T}_t = \gamma\widehat{b}_{t-1} + \tau_t \tag{LL2}$$

Let us look at the log-linear equilibrium. Log-linearizing (2) we get the usual Fischer equation

$$\widehat{\pi}_{t+1} = \widehat{I}_t \tag{LL3}$$

From money demand (3)

$$\widehat{m}_t = -\frac{\eta^{-1}}{I-1}\widehat{I}_t \tag{LL4}$$

From mkt clearing (4)

$$(I-1)\,\widehat{T}_t + \frac{M}{B}\left(\widehat{m}_t - \widehat{m}_{t-1} + \widehat{\pi}_t\right) + \widehat{b}_t = I\left(\widehat{I}_{t-1} + \widehat{b}_{t-1} - \widehat{\pi}_t\right) \tag{LL5}$$

Use $(LL3)$ and $(LL2)$ the fact that money demand is endogenous and replace T_t.

$$\pi_{t+1} = \alpha\pi_t + e_t \tag{a}$$

Multiply by P/B log-linearized (4)

$$\frac{PT}{B}\left(\gamma\widehat{b}_{t-1} + \tau_t\right) + \frac{M}{B}\left(\widehat{m}_t - \widehat{m}_{t-1} + \widehat{\pi}_t\right) + \widehat{b}_t = I\left(\widehat{I}_{t-1} + \widehat{b}_{t-1} - \widehat{\pi}_t\right)$$

drop $\widehat{m}_t$ using $(LL4)$, define $M/B = \mu$, use steady state for PT/B and drop I_t using Taylor rule to obtain:

$$I\,(I-1)\left(\gamma\widehat{b}_{t-1} + \tau_t\right) - \frac{\eta^{-1}\mu}{I-1}\left(\alpha\left(\widehat{\pi}_t - \widehat{\pi}_{t-1}\right) + \widehat{e}_t - \widehat{e}_{t-1}\right) + \mu\widehat{\pi}_t + \widehat{b}_t = I\left(\alpha\widehat{\pi}_{t-1} + \widehat{e}_{t-1} + \widehat{b}_{t-1} - \widehat{\pi}_t\right) \tag{b}$$

We want to characterize the determinacy properties of the equilibrium: to this purpose, we only consider the perfect foresight equilibrium: the system made up by (a) and (b) can be written as:

$$\left(-\frac{\eta^{-1}\mu\alpha}{I-1} + I + \mu\right)\widehat{\pi}_t + \widehat{b}_t = \left(\alpha\left(I - \frac{\eta^{-1}\mu}{I-1}\right)\right)\widehat{\pi}_{t-1} + \left(I - \gamma\,(I-1)\right)\widehat{b}_{t-1} \tag{b'}$$

$$\pi_t = \alpha\pi_{t-1} \tag{a'}$$

which in matrix form looks like

$$\begin{bmatrix} 1 & -\frac{\eta^{-1}\mu\alpha}{I-1} + I + \mu \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} I - \gamma\,(I-1) & \alpha\left(I - \frac{\eta^{-1}\mu}{I-1}\right) \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} b_{t-1} \\ \pi_{t-1} \end{bmatrix}$$

The system is recursive, and eigenvalues are the two elements on the main diagonal. For a saddle path equilibrium to exists, the eigenvalues must lie on either side of the unit circle, which can happen only if both α and γ exceed or fall short of one in absolute value.

$$\begin{aligned}\mu_1 &= \alpha \\ \mu_2 &= I - \gamma(I-1) = \beta^{-1} - \gamma\left(\beta^{-1}-1\right)\end{aligned}$$

Equilibrium can be characterised in terms of different policies.

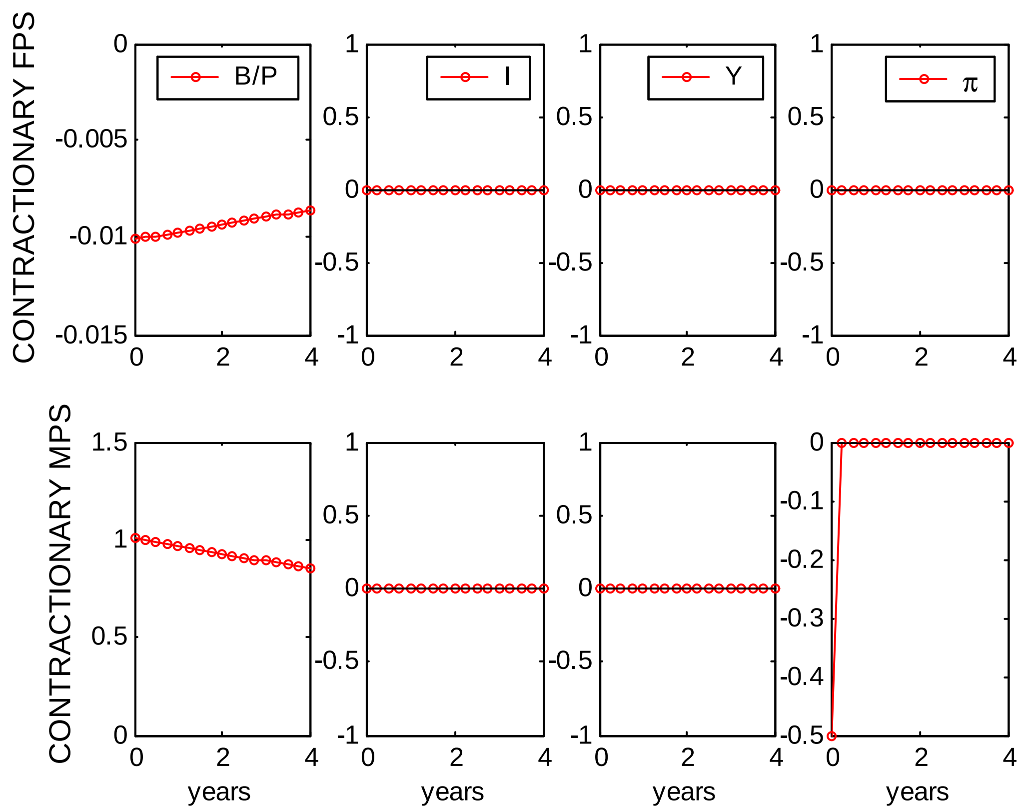
- *Active monetary* $\alpha > 1$ *and Passive Fiscal* $\gamma > 1$. There is a unique REE in which inflation is only a monetary phenomenon. Inflation and nominal interest rate fluctuations depend entirely on the parameters of the policy rule, the discount factor, and the monetary policy shock. In this case (b) is stable difference equation that can be solved backward for b, whereas (a) can be solved forward to obtain, if $e_t = \rho_e e_{t-1} + \varepsilon_t$

$$\pi_t = \frac{1}{\alpha}\pi_{t+1} + \frac{1}{\alpha}e_t = \frac{1}{\alpha}\frac{1}{1-\rho_e\alpha^{-1}}e_t$$

Figure 1 considers this case: (1) in the top row: a transitory increase in taxes causes a persistent fall in real government debt, which keeps the present value of taxes constant.[2] (2) Bottom row: a transitory

[2] With $\gamma > 1$ real debt slowly returns to baseline following a fiscal shock, given the that taxes are "high".

decrease in money supply (positive e_t) causes inflation to fall, and real debt to rise.



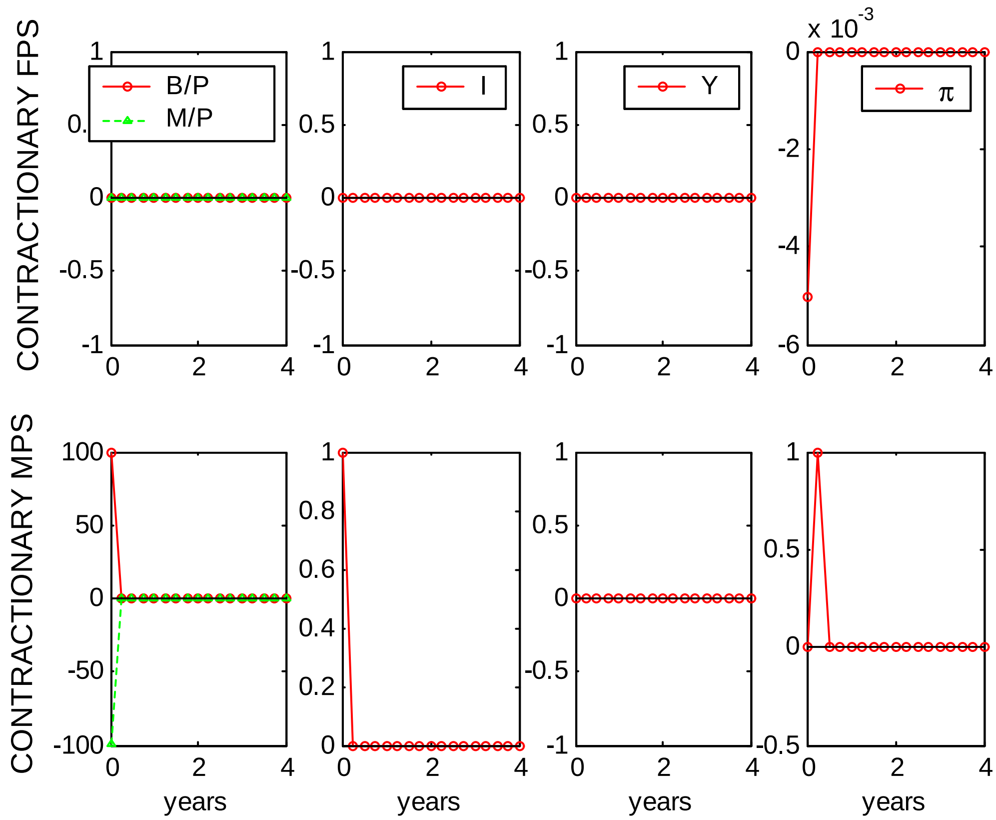
1: Active monetary and passive fiscal

- *Passive Monetary* $\alpha < 1$ *and Active Fiscal* $\gamma < 1$. Here inflation is a monetary and Fiscal phenomenon (FTPL).

In this class we also have exogenous taxes $\gamma = 0$ + interest rate (or money supply) peg, $\alpha = 0$. When $\alpha = 0$ and $\gamma = 0$, the solutions for π_t and b_t are:

$$\begin{aligned}
\pi_t &= -\frac{1-\beta}{1+\beta\mu}\tau_t - \frac{\mu}{\mu+\beta^{-1}}\left(\beta\left(1+\frac{\eta^{-1}}{\beta^{-1}-1}\right) - \frac{\eta^{-1}}{\beta^{-1}-1}\right)e_t + e_{t-1} \\
b_t &= \beta m\left(1+\frac{\eta^{-1}}{\beta^{-1}-1}\right)e_t
\end{aligned}$$

notice that government debt influences the response of inflation to both monetary and fiscal changes. (1) A tax increase (positive τ) lowers inflation, with an elasticity which is greater the smaller $\mu = m/b$. This happens because the monetary authority can now print less money, thus demonetizing the tax cut (top Figure 2). Tax changes do not affect real debt. (2) Interest rate increases (bottom Figure 2) represent a pure asset exchange: a rise in I induces substitution from M into B. Initially, $M + B$ is constant, hence prices do not change. In the next period, the higher real debt must be financed through inflation (since fiscal policy is active), therefore inflation rises with one period lag.



2: Passive monetary and active fiscal

- *Active monetary* $\alpha > 1$ *and Active Fiscal* $\gamma \leq 1$. Public debt will explode over time.
- *Passive Monetary* $\alpha < 1$ *and Passive Fiscal* $\gamma > 1$. Sunspots. Indeterminacy of Equilibria. Each autority acts passively, and there is more than just one interest rate process that is consistent with equilibrium conditions. Notice that this mirrors the indeterminacy results that we obtained in section "Price level determination with government debt".

9.4.1 Evaluation

The standard monetarist recipe for price stability is to make sure that the central bank has a commitment to price stability.

The FTPL argues that this is not a sufficient condition, but price stability also requires an appropriate fiscal policy, which is not *implied* by a strong central bank. The FTPL would therefore imply that central banks must also convince fiscal authorities to behave in an appropriate way. In sum, the difference between the fiscal and monetarist approach boils down to the views on the government budget constraint.

- MONETARISTS: argue that policy must be set in a way that RHS equals LHS, whatever the value of P is (RICARDIAN assumption).
- FTPL: argue that this is just a flow of funds. Whenever LHS or RHS change, the price level will adjust to restore the equality (NON RICARDIAN assumption)

Moving to real world, it seems that the non-ricardian assumption cannot describe always government behaviour in all circumstances. Governments often adjust fiscal variables when their real debts get large (e.g. Maastricht treaty, or US late 1980s: federal debt grew producing political support for raising taxes)

However, as proponents of the FTPL argue, this theory might provide a useful characterisation of actual policies in some contexts (e.g. high inflation in Brazil in late 1970s. In a sense, the FTPL might pose a rationale for the type of budget rules that constitutions or budget rules pose to governments.

Another alleged good point about the FTPL is that is allows to pin price level down even in a cashless economy (when M approaches a near zero level).

9.5 Accounting for price stickiness

It is easy to amend this model to account for price stickiness. What we need to do is the following. Add a Phillips curve (as you can imagine, we add output and we add one equation)

$$\widehat{\pi}_t = \widehat{\pi}_{t+1} + \kappa \widehat{Y}_t \tag{LL6}$$

"replace" the Fisher equation with the AD curve (consumption is no longer fixed now):

$$\widehat{I}_t - \widehat{\pi}_{t+1} = -\sigma^{-1}\left(\widehat{Y}_t - \widehat{Y}_{t+1}\right) \tag{LL3}$$

amend the money demand equation:

$$\widehat{m}_t = \frac{\eta^{-1}}{\sigma}\widehat{Y}_t - \frac{\eta^{-1}}{I-1}\widehat{I}_t \tag{LL4}$$

consider if you want a more general Taylor rule:

$$\widehat{I}_t = \alpha\widehat{\pi}_t + \alpha_y \widehat{Y}_t + \widehat{e}_t \tag{LL1}$$

Fiscal rule and budget constraint for the government remain the same. This setup is due to Woodford (1995), as you can see here taxes are lump-sum. The model nests Leeper if $\kappa = \infty$ and $a_y = 0$

Woodford finds that an "unexpected increase in the fiscal deficit, not offset by any future reduction in future primary deficits, stimulates aggregate demand, temporarily increasing both inflation and output". This happens because an increase in the present value of government deficit increases the present value of total consumption that the representative household can afford, if prices and interest rates do not change, and thus induces and increase in aggregate demand for goods.

Interesting result: a more aggressive monetary policy - i.e., higher ϕ_π and ϕ_y - implies that inflation rises by MORE following a fiscal shock. Try this by yourself with leeper.m

Woodford's book discusses under which conditions the Taylor principle continues to hold.