Graduate Macro - BC: Spring 2009 - Lecture Notes

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April 29, 2009

Contents

Introduction

The reading list for this course in on my BC webpage.

The website is `http://www2.bc.edu/~iacoviel/teach/0809/EC751.html`.

Background on the solution of DSGE models

The term dynamic stochastic general equilibrium model defines a class of models widely used in macro which have the following features:

1. They describe and characterize the comovement the evolution of (random) economic variables over time
2. They are normally based on microfoundations (although this word did not make the final cut) and they are based on the hypothesis that markets clear through adjustment of prices and quantities
3. They rely on random fluctuations in technology, preferences and other exogenous sources as the primary impulse for the movement of the economic variables over time.

An example

An example of a dynamic stochastic general equilibrium model is the famous real business cycle (RBC) model. For simplicity, we assume that capital does not depreciate, that labor is supplied inelastically (and normalized to 1), that utility is logarithmic in consumption and that technology shocks affect society's ability to produce goods. Then the representative agent problem can be written as:

$$\max E_t \left(\sum_{s=t}^{\infty} \beta^{s-t} \log C_s \right)$$

subject to

$$K_t - K_{t-1} = A_t^{1-\alpha} K_{t-1}^{\alpha} - C_t \tag{1}$$

where we assume technology follows an AR(1) process in logs, that is

$$\log A_t = \rho \log A_{t-1} + \log U_t \tag{2}$$

and where ρ is the autocorrelation of the shock and we assume that the innovation to technology $\log U_t$ has mean zero, finite variance.

Optimal consumption across two successive periods can be described by the consumption Euler equation:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(\alpha \left(\frac{A_{t+1}}{K_t} \right)^{1-\alpha} + 1 \right) \right) \tag{3}$$

The final condition for optimality is the transversality condition that states that the present discounted value of capital has to be equal to zero, that is $\lim_{\tau \to \infty} E_t \left(C_{t+\tau}^{-1} K_{t+\tau} \beta^\tau \right) = 0$.

The system made by (1) to (3) is a non-linear system with rational expectations. In compact notation, the system can be represented as follows:

$$E_t \left[f_\theta \left(\mathbf{x}_{t+1}, \mathbf{x}_t, \mathbf{x}_{t-1}, \varepsilon_t \right) \right] = 0 \tag{4}$$

where

- $\mathbf{x}$ is the vector collecting all the exogenous variables
- ε is the vector collecting the exogenous stochastic shocks
- θ is the vector collecting the "deep parameters" of the model

The solution to a model like this is normally a set of functions relating controls to states such that (1) to (3) are satisfied at all times. For most of the problems we are interested in (which are recursive), the "state" is normally summarized by the realization of the shocks and by some or all of the the variables dated at time $t-1$ in our model. For instance, our solution for $\mathbf{x}_t$ is a function

$$\mathbf{x}_t = g \left(\mathbf{x}_{t-1}, \varepsilon_t \right)$$

such that the equations in (4) hold. The problem is that there is an infinite number of values of $\mathbf{x}_{t-1}$ and ε_t and an infinite number of values of $\mathbf{x}_t$ to find. Hence, with a few rare exceptions what we do to solve DSGE models is to approximate g by a linear or quadratic function around a well defined point. More precisely, we usually take the following steps

1. Find the steady state.

   In steady state variables are constant, that is

$$\begin{aligned} \log A &= 0->A=1 \\ C &= K^\alpha \\ 1-\beta &= \alpha\beta \left(\frac{1}{K} \right)^{1-\alpha} -> \left(\frac{\alpha\beta}{1-\beta} \right)^{\frac{1}{1-\alpha}} = K \end{aligned}$$

2. Linearize or log-linearize around the steady state

(a) equation 1

$$C_t = A_t^{1-\alpha} K_{t-1}^{\alpha} - K_t + K_{t-1}$$

$$\begin{aligned}
\log C_t &= \log\left(A_t^{1-\alpha} K_{t-1}^{\alpha} - K_t + K_{t-1}\right) \\
&\quad \text{take total differential around steady state} \\
\frac{1}{C} dC_t &= \frac{1}{C}\left(\left(1-\alpha\right) A^{-\alpha} K^{\alpha} dA_t + \alpha A^{1-\alpha} K^{\alpha-1} dK_{t-1} - dK_t + dK_{t-1}\right) \\
c_t &= \frac{\left(1-\alpha\right) K^{\alpha}}{C} dA_t + \alpha \frac{K^{\alpha-1}}{C} dK_{t-1} - \frac{1}{C} dK_t + \frac{1}{C} dK_{t-1} \\
c_t &= \left(1-\alpha\right) a_t + \alpha k_{t-1} - \frac{K}{C} k_t + \frac{K}{C} k_{t-1} \\
\frac{K}{C} &= \frac{K}{K^{\alpha}} = \frac{\alpha\beta}{1-\beta} \\
0 &= -c_t + \left(1-\alpha\right) a_t - \frac{\alpha\beta}{1-\beta} k_t + \frac{\alpha}{1-\beta} k_{t-1}
\end{aligned}$$

(b) equation 2

$$a_t = \rho a_{t-1} + u_t$$

(c) equation 3

$$\begin{aligned}
E_t\left(\frac{C_{t+1}}{C_t}\right) &= \beta E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
\log E_t\left(C_{t+1}\right) - \log\left(C_t\right) &= \log\beta + \log E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
E_t \log\left(C_{t+1}\right) - \log\left(C_t\right) &= \log\beta + \log E_t\left(\alpha\left(\frac{A_{t+1}}{K_t}\right)^{1-\alpha} + 1\right) \\
&\quad \text{steady state of both sides is 1} \\
E_t c_{t+1} - c_t &= \beta\left(\alpha\left(1-\alpha\right)\left(\frac{A}{K}\right)^{-\alpha}\left(\frac{1}{K} dE_t A_{t+1} - \frac{A}{K^2} dK_t\right)\right) \\
E_t c_{t+1} - c_t &= \alpha\beta\left(1-\alpha\right)\left(\frac{1}{K}\right)^{1-\alpha}\left(E_t a_{t+1} - k_t\right) \\
E_t c_{t+1} - c_t &= \left(1-\alpha\right)\left(1-\beta\right)\left(E_t a_{t+1} - k_t\right) \\
0 &= -E_t c_{t+1} + c_t + \left(1-\alpha\right)\left(1-\beta\right)\left(E_t a_{t+1} - k_t\right)
\end{aligned}$$

Here, a lowercase variable indicates the log-deviation of a variable from its steady state. For a variable X_t in levels, we have that

$$x_t = \log\left(\frac{X_t}{X}\right) \simeq \frac{X_t - X}{X}$$

This is a dynamic system of 3 equations in 3 unknowns. To use a more compact notation, we prefer to write it in the following form

$$\begin{aligned} 0 &= E_t \left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \right] && (5) \\ \mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t && (6) \end{aligned}$$

where:

- $\mathbf{x}_t$ is the column vector collecting all the endogenous variables of the model. This variables can be further divided into state and jump variables (see Walsh, 2nd edition, pages 88 and 89, and Uhlig's paper)
- $\mathbf{z}_t$ collects all the exogenous stochastic processes.

In our above example:

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} c \\ k \end{bmatrix} \\ \mathbf{z} &= [a] \end{aligned}$$

and

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \\ \mathbf{G} &= \begin{bmatrix} -1 & -\frac{\alpha\beta}{1-\beta} \\ 1 & -(1-\alpha)(1-\beta) \end{bmatrix} \\ \mathbf{H} &= \begin{bmatrix} 0 & \frac{\alpha}{1-\beta} \\ 0 & 0 \end{bmatrix} \\ \mathbf{L} &= \begin{bmatrix} 0 \\ (1-\alpha)(1-\beta) \end{bmatrix} \\ \mathbf{M} &= \begin{bmatrix} 1-\alpha \\ 0 \end{bmatrix} \\ \mathbf{N} &= [\rho] \end{aligned}$$

The solution of a DSGE model

To summarize, a DSGE model can be written in the following form

$$\begin{aligned} 0 &= E_t \left[\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \right] \\ \mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{e}_t \end{aligned}$$

What one is looking for is the recursive equilibrium law of motion that describes the endogenous variables as a function of the STATE, that is

$$\begin{aligned} \mathbf{x}_t &= \mathbf{P}\mathbf{x}_{t-1} + \mathbf{Q}\mathbf{z}_t && (7) \\ \mathbf{z}_t &= \mathbf{N}\mathbf{z}_{t-1} + \mathbf{u}_t \end{aligned}$$

i.e., matrices $\mathbf{P}$, $\mathbf{Q}$ such that the equilibrium is described by these rules. Notice that (7) describes the solution, in that, once we know the state described by current shocks $\mathbf{z}_t$ and last period values of our variables $\mathbf{x}_{t-1}$, we also know the values of $\mathbf{x}_t$.

If the Blanchard-Kahn conditions are met, it is possible to find a unique solution of the form described in (7) [1]

Remark 1 *Blanchard-Kahn condition. When the dynamic system is expressed in the form*

$$E_t\left(\left[\begin{array}{c}\mathbf{x}_{t+1}^J\\ \mathbf{x}_t^S\end{array}\right]+\mathbf{A}\left[\begin{array}{c}\mathbf{x}_t^J\\ \mathbf{x}_{t-1}^S\end{array}\right]+\mathbf{M}\mathbf{z}_t\right)$$

where we partition $\mathbf{x}$ *into* $\left[\begin{array}{c}\mathbf{x}^J\\ \mathbf{x}^S\end{array}\right]$, $\mathbf{x}_0^S$ *is given and* $\mathbf{x}_0^S$ *is the vector of predetermined variables (determined in* $t-1$ *or earlier) and* $\mathbf{x}_t^J$ *is the vector of jump variables (determined in* t*), and* $\mathbf{z}_t$ *is the vector of exogenous shocks (which can be either iid on stationary autoregressive processes), the solution of a linear rational expectations system is unique if the number* n *of unstable eigenvalues in* $\mathbf{A}$ *(greater than one in absolute value) is exactly equal to the number* n^J *of jump (forward-looking, control) variables.*

Too many stable roots ($n < n^J$) *: multiple stable solutions, equilibrium path not unique (need alternative solution techniques)*

Too many unstable roots ($n > n^J$)*: no stable solution, all paths are explosive (for instance, this might happen if transversality conditions of the model are violated)*

Let me be a little loose here.

The equations in (5) and (6) are in a sense already a solution to most economic problems. For instance, we know that the famous consumption Euler equation that relates consumption to its expected value and to interest rates is the solution to a dynamic optmitization problem. However, this is not a solution in that it does not tell us what next consumption is going to be as a function of the state vector only.

The equation in (7) is instead a particular solution, because it expresses the endogenous variables as a function of history only.

As DSGE economists, what we mostly do for a living is to write down some sort of model, express it in linearized form (5) and (6) (and we mostly do this by hand, but things are changing quickly here).

Next, what we do is to plug the matrices in (5) and (6) in a computer, to obtain (7).

In our toy example above, set

$$\begin{aligned}\alpha &= 0.33\\ \beta &= 0.99\\ \rho &= 0.98\end{aligned}$$

[1] A note on my webpage discusses in detail how to verify the Blanchard-Kahn conditions for this model. See `http://www2.bc.edu/~iacoviel/dsgemodels_files/rational_expectations_models.pdf`

Then

$$\begin{bmatrix} c_t \\ k_t \end{bmatrix} = \begin{bmatrix} 0 & 0.6589 \\ 0 & 0.9899 \end{bmatrix} \begin{bmatrix} c_{t-1} \\ k_{t-1} \end{bmatrix} + \begin{bmatrix} 0.1755 \\ 0.0151 \end{bmatrix} [a_t]$$

Notice that both consumption and capital (in log deviation from the steady state) are a distributed lag of the single innovation to productivity a_t. That is, there is just one factor driving both variables. Also, think about what the numbers above mean: they have an interpretation in terms of elasticities: why?

In this model, we have neglected output. However, since output is simply:

$$y_t = 0.67 z_t + 0.33 k_{t-1}$$

we can immediately expand our solution to include output.

The solution above can be used to calculate many objects of interest, for instance (1) impulse responses; (2) dynamic simulations; (3) theoretical moments; (4) simulated moments; (5) variance decompositions.

The files `rbcsimple.m` and `rbcsimple_go.m` simulate this model. They are available from `http://www2.bc.edu/~iacoviel/dsgemodels.htm`

Chapter 1

Vector autoregressions

We begin by taking a look at the data of macroeconomics. A way to summarize the dynamics of macroeconomic data is to make use of vector autoregressions. VAR models have become increasingly popular in recent decades. They are estimated to provide empirical evidence on the response of macroeconomic variables to various exogenous impulses in order to discriminate between alternative theoretical models of the economy.

This simple framework provides a systematic way to capture rich dynamics in multiple time series, and the statistical toolkit that came with VARs was easy to use and to interpret. As Sims (1980) and others argued in a series of influential early papers, VARs held out the promise of providing a coherent and credible approach to data description, forecasting, structural inference and policy analysis.

1.1 VARs and the identification problem

A VAR is an $n-$equation, $n-$variable model in which each variable is in turn explained by its own lagged values, plus (current) and past values of the remaining $n-1$ variables. A VAR can be thought of as the reduced form of a dynamic economic system involving a vector of variables z_t.[1] That is, starting from the so-called structural form:

$$\begin{aligned} Az_t &= B_1 z_{t-1} + B_2 z_{t-2} + ... + B_p z_{t-p} + u_t \\ Euu' &= \Sigma_u = \begin{bmatrix} \sigma_{u1}^2 & 0 & & 0 \\ 0 & \sigma_{u2}^2 & & 0 \\ & & ... & \\ 0 & 0 & & \sigma_{un}^2 \end{bmatrix} \end{aligned}$$

a VAR of lag length p ($VAR(p)$) can be written as

$$z_t = A^{-1}B_1 z_{t-1} + A^{-1}B_2 z_{t-2} + ... + A^{-1}B_p z_{t-p} + A^{-1}u_t$$

[1] We assume that each of the variables in z_t is demeaned prior to estimation, so we can neglect constant terms in each of the equations.

where $E(u_t) = 0$, $E(u_t u'_\tau) = \Sigma_u$ for $t = \tau$, and 0 otherwise. Thus, a vector autoregression is a system in which each variable is expressed as a function of own lags as well as lags of each of the other variables.

VAR's come in three varieties: reduced form, recursive and structural. In each case, what we want to solve is the identification problem. That is, our goal is to recover estimates of A, B, and Σ_u.

In the examples below, we assume there is only one lag in the VAR. This comes without loss of generality, since higher order lags can be dealt exactly in the same way.

REDUCED FORM Each variable is expressed as a linear function of its own past values and past values of all other variables. Each equation is estimated by OLS. The error terms are the surprise movements in the variables after taking past values into account. If the variables are correlated with each other, then the error terms will also be correlated. In practice, given the structural form, we estimate $A^{-1}B_1, ...A^{-1}B_p$ and $A^{-1}\Sigma_u A^{-1\prime}$, but we cannot easily revert to the A's and the Σ_u that are our object of interest.

RECURSIVE VAR A recursive VAR tries to identify the structure of the model by constructing the error term in each regression to be uncorrelated with the error in the preceding equations. This is done by estimating the equations of the VAR by carefully including in some of the equations the *contemporaneous* values of other variables as regressors.

Consider for instance this bivariate VAR with 2 equations and 2 variables. Think about, say, a simple bi-variate model with only income and interest rates:

$$\begin{aligned} Y_t &= -a_{yr}R_t + b_{yr}R_{t-1} + b_{yy}Y_{t-1} + u_{yt} \qquad (1) \\ R_t &= -a_{ry}Y_t + b_{rr}R_{t-1} + b_{ry}Y_{t-1} + u_{rt} \qquad (2) \end{aligned}$$

in this model:

- $a_{ry}, a_{yr}, b_{..}$ are the structural parameters;
- u_{rt}, u_{yt} are the uncorrelated structural shocks with standard deviation σ_r and σ_y.

The 2 equations cannot be estimated by OLS, since they violate the assumptions of the classical regression model of uncorrelation between the regressors and the error term. Suppose in fact that we estimate the second equation by OLS: the variable Y will be correlated with the error term, since:

$$\begin{aligned} cov(Y_t, u_{rt}) &= cov(-a_{yr}R_t + b_{yr}R_{t-1} + b_{yy}Y_{t-1} + u_{yt}, u_{rt}) = \\ &= cov(-a_{yr}(b_{rr}R_{t-1} + b_{ry}Y_{t-1} - a_{ry}Y_t + u_{rt}) + b_{yr}R_{t-1} + b_{yy}Y_{t-1} + u_{yt}, u_{rt}) = \\ &= a_{yr}a_{ry}cov(Y_t, u_{rt}) - a_{yr}\sigma_{ur}^2 \\ cov(Y_t, u_{rt}) &= \frac{-a_{yr}}{1 - a_{yr}a_{ry}}\sigma_{ur}^2. \end{aligned}$$

OLS estimates of equation 2 will yield inconsistent estimates of the parameters of the model unless we assume that $a_{yr} = 0$, that is the contemporaneous effect of interest rate shocks on output is zero. Suppose then we impose $a_{yr} = 0$. Equation 2 can be estimated by OLS. An "interest rate shock" $u_{rt} = 1$ will have an effect on R_t equal to 1, and on Y_t of 0. An "income shock" u_{yt} will have an effect on Y_t of 1, and an effect of $-a_{ry}$ on R_t.

This way of proceeding is called *recursive*, since we can estimate one by one the equations by OLS but in a peculiar way: for equation 1, we exclude R_t from the regressors; for equation 2, we include Y_t among the regressors.[2]

To be more precise, consider estimation of the VAR described by Y and R; the structural model cannot be estimated by OLS:

$$\begin{bmatrix} 1 & a_{yr} \\ a_{ry} & 1 \end{bmatrix} \begin{bmatrix} Y_t \\ R_t \end{bmatrix} = \begin{bmatrix} b_{yy} & b_{yr} \\ b_{ry} & b_{rr} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

but the reduced form:

$$\begin{bmatrix} Y_t \\ R_t \end{bmatrix} = \begin{bmatrix} 1 & a_{yr} \\ a_{ry} & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_{yy} & b_{yr} \\ b_{ry} & b_{rr} \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ R_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & a_{yr} \\ a_{ry} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_{yt} \\ u_{rt} \end{bmatrix}$$

can be estimated by OLS; however, the error terms of the two equations are now correlated across equations, which means that we cannot identify the structural parameters from the residuals of the two equations! The residual of the two reduced form equations will be linked to the structural parameters by:

$$\begin{aligned} e_{1t} &= \frac{1}{1 - a_{yr}a_{ry}} (u_{yt} - a_{yr}u_{rt}) \\ e_{2t} &= \frac{1}{1 - a_{yr}a_{ry}} (u_{rt} - a_{ry}u_{yt}) \end{aligned}$$

If $a_{yr} = 0$, then:

$$\begin{aligned} e_{1t} &= u_{yt} \\ e_{2t} &= u_{rt} - a_{ry}u_{yt} \end{aligned}$$

which implies that:

1. first, we estimate the *reduced form* of the first equation (regress Y on lagged Y and R) and we recover the structural shock u_{yt}.

[2] If we had three variables with, say, R, Y and P, we would

1. exclude the contemporaneous values of Y and P them from the equation for R.
2. exclude the contemporaneous value of P from the equation for Y.
3. exclude no variable (that is, include P and Y contemporaneous) from the equation for R.

2. We know that OLS estimates of 2 are consistent once $a_{yr} = 0$. So, in order to recover u_{rt}, we simply estimate $R_t = -a_{ry}Y_t + b_{rr}R_{t-1} + b_{ry}Y_{t-1} + u_{rt}$ and calculate the residual.

This way, from our dynamic system made by (1) and (2), we are able to estimate our structural parameters.

In the VAR jargon, the algorithm for estimating the coefficients when more than one variable is involved does not go about estimating equation 1 in reduced form, equation 2 excluding as contemporaneous regressors the LHS variable of equation 1, and so on. What we normally do goes instead in two steps:

1. Estimate the reduced form for all equations. This can be done easily with any econometric / computational package.

2. Compute the Choleski decomposition of the variance-covariance matrix of the residuals. In general, for each symmetric, positive definite matrix X, the Choleski decomposition is an upper triangular matrix U such that:[3]

$$X = U'U$$

Why? Denote $a_{ry} = \alpha$, $\sigma_{12} = \sigma_x$. Then:

$$z_t = A^{-1}Bz_{t-1} + \underbrace{A^{-1}u_t}_{e_t}$$

$$\Sigma_e = A^{-1}\Sigma_u^{1/2}\Sigma_u^{1/2}A^{-1\prime}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1^2 & \sigma_x \\ \sigma_x & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}\begin{bmatrix} \sigma_y & 0 \\ 0 & \sigma_r \end{bmatrix}\begin{bmatrix} \sigma_y & 0 \\ 0 & \sigma_r \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1^2 & \sigma_x \\ \sigma_x & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_y & 0 \\ -\alpha\sigma_y & \sigma_r \end{bmatrix}\begin{bmatrix} \sigma_y & -\sigma_y\alpha \\ 0 & \sigma_r \end{bmatrix}$$

The last expression defines a system of three equations above that be solved for:

$$\begin{aligned} \sigma_y^2 &= \sigma_1^2 \\ \alpha &= -\frac{\sigma_x}{\sigma_y^2} = -\frac{\sigma_x}{\sigma_1^2} \\ \sigma_r^2 &= \sigma_2^2 - \alpha^2\sigma_1^2 \end{aligned}$$

[3]For instance, the Choleski decomposition of a generic 2×2 matrix is:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} \sqrt{a} & 0 \\ \frac{b}{\sqrt{a}} & \sqrt{c - \frac{1}{a}b^2} \end{bmatrix}\begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}} \\ 0 & \sqrt{c - \frac{1}{a}b^2} \end{bmatrix}$$

Computer programs have an easy way to do all of this. Note that:

$$chol\left(\Sigma_e\right) = \Sigma_u^{1/2} A^{-1\prime}$$

So, first you estimate the VAR in reduced form. Through the Choleski decomposition of Σ_e you recover a matrix that on the main diagonal has the standard deviation of all structural shocks. That is (I am using Matlab language here: given a matrix X, the command `diag(diag(X))` returns a diagonal matrix with the elements of X on the main diagonal):

$$\Sigma_u^{1/2} = diag\left(diag\left(chol\left(\Sigma_e\right)\right)\right)$$

Next, you can calculate A and/or A^{-1}. A^{-1} is given by:[4]

$$\begin{aligned} A^{-1\prime} &= \Sigma_u^{-1/2} chol\left(\Sigma_e\right) \\ A^{-1} &= chol\left(\Sigma_e\right)' \Sigma_u^{-1/2} \end{aligned}$$

STRUCTURAL VAR It uses economic theory to sort out the contemporaneous relationships between the variables. For instance, we might know from economic theory that $a_{yr} = -0.2$. This type of assumption need not to involve only the contemporaneous effect of shocks.

1.2 Impulse response functions

In a VAR, we are often interested in obtaining the impulse response functions. Impulse responses trace out the response of current and future values of each of the variables to a one-unit increase (or to a one-standard deviation increase, when the scale matters) in the current value of one of the VAR errors, assuming that this error returns to zero in subsequent periods and that all other errors are equal to zero. The implied thought experiment of changing one error while holding the others constant makes most sense when the errors are uncorrelated across equations, so impulse responses are typically calculated for recursive and structural VARs.

We begin by noticing that if we know A and Σ_u, we can begin from:

$$z_t = A^{-1} B z_{t-1} + \underbrace{A^{-1} u_t}_{e_t}$$

we can calculate the IRF's to a unit shock of u once we know A^{-1}. Assume the system has been in steady state for a while. Suppose we are interested in tracing the dynamics to a

[4] or, using all of the above

$$A = \Sigma_u^{1/2} \left(chol\left(\Sigma_e\right)'\right)^{-1} = diag\left(chol\left(\Sigma_e\right)\right)\left(chol\left(\Sigma_e\right)'\right)^{-1}$$

shock to the first variable in a two variable VAR: when a shock hits at time 0:

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} Y_0 \\ R_0 \end{bmatrix} = A^{-1}u_0$$

For every $s > 0$,

$$z_s = A^{-1}Bz_{s-1}.$$

To summarize, the impulse response function is a practical way of representing the behavior over time of z in response to shocks to the vector u.

1.3 Forecast error variance decomposition (FEVD)

Variance decomposition separates the variation in an endogenous variable into the component shocks to the VAR. Thus, the variance decomposition provides information about the relative importance of each random innovation in affecting the variables in the VAR. Given our structural model:

$$G(L)\, z_t = u_t$$

the VMA (vector moving average) representation of our VAR is:

$$z_t = \Gamma_0 u_t + \Gamma_1 u_{t-1} + \Gamma_2 u_{t-2} + ...$$

Define the forecasting error at horizon s as the difference between actual realization of the variable in time $t+s$ and its expectation computed at time t. The error in forecasting z_t in the future is, for each horizon s:

$$z_{t+s} - E_t z_{t+s} = \Gamma_0 u_{t+s} + \Gamma_1 u_{t+s-1} + \Gamma_2 u_{t+s-2} + ... + \Gamma_{s-1} u_{t+1};$$

In practice, the equation says that any forecasting error must come from shocks occurred between time $t+1$ and time $t+s$. We want a measure of how large this forecasting error is. The variance of the forecasting error is:

$$Var\left(z_{t+s} - E_t z_{t+s}\right) = \Gamma_0 \Sigma_u \Gamma_0' + \Gamma_1 \Sigma_u \Gamma_1' + \Gamma_2 \Sigma_u \Gamma_2' + ... + \Gamma_{s-1} \Sigma_u \Gamma_{s-1}'$$

on the basis of this formula, we can compute the share of the total variance of the forecast error for each variable attributable to the variance of each structural shock.

1.4 Vector autocovariances (VACF)

A structural VAR is just one of the ways of summarize the second moment properties of a time-series process, and relies heavily on the implied assumption about the contemporaneous effects of the shocks. In more general terms, however, we can find another useful way to summarize all the second moments for a vector of time series z_t[5] following a $VAR(p)$ process. The $VAR(p)$ reduced form representation is:

$$z_t = \Phi_1 z_{t-1} + \Phi_2 z_{t-2} + ... + \Phi_p z_{t-p} + e_t.$$

Define the following variables:

$$\underset{np\times 1}{\xi_t} = \begin{bmatrix} z_t \\ z_{t-1} \\ ... \\ z_{t-p+1} \end{bmatrix}$$

$$\underset{np\times 1}{F} = \begin{bmatrix} \Phi_1 & \Phi_2 & ... & \Phi_{p-1} & \Phi_p \\ I_n & 0 & ... & ... & 0 \\ 0 & I_n & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & .. & I_n & 0 \end{bmatrix}$$

$$\underset{np\times 1}{v} = \begin{bmatrix} e_t \\ 0 \\ ... \\ 0 \end{bmatrix}$$

Then the $VAR(p)$ above can be written as:

$$\xi_t = F\xi_{t-1} + v_t$$

$$Evv' = \underset{np\times np}{Q} = \begin{bmatrix} \Sigma_e & 0 & ... & 0 \\ 0 & 0 & .. & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & 0 \end{bmatrix} \text{ <− contemporaneous VCOV matrix}$$

where note that with just one lag: $\xi_t = z_t$, $F = \Phi_1 = A^{-1}B$, $v = e$.

Under stationarity conditions (we need to check that the eigenvalues of F all lie inside the unit circle, so that we can invert the VAR process), the VAR has a covariance stationary MA representation.

We can then define the jth autocovariance as the following $n \times n$ matrix:

$$\Gamma_j = Ez_t z'_{t-j}$$

Remark 2 *Consider for instance the simple case $j = 0$. Then $\Gamma_0 = Ez_t z'_t$ is the contemporaneous covariance matrix of our time series, that is in the example with Y and R only:*

$$\Gamma_0 = \begin{bmatrix} VAR(Y_t) & cov(Y_t, R_t) \\ cov(Y_t, R_t) & VAR(R_t) \end{bmatrix}$$

[5] In general, each of the series has to be demeaned so as to avoid dealing with the constant term of the regression.

notice that $\gamma_j = \gamma_{-j}$ for a scalar, but not for a time series, for which in general

$$\Gamma_j \neq \Gamma_{-j}$$

Remark 3 *Consider for instance the simple case $j = 1$. Then $\Gamma_1 = E z_t z'_{t-1}$: in the example with R and Y only*

$$\Gamma_1 = E_t \begin{bmatrix} Y_t \\ R_t \end{bmatrix} \begin{bmatrix} Y_{t-1} & R_{t-1} \end{bmatrix} = \begin{bmatrix} cov\left(Y_t, Y_{t-1}\right) & cov\left(Y_t, R_{t-1}\right) \\ cov\left(R_t, Y_{t-1}\right) & cov\left(R_t, R_{t-1}\right) \end{bmatrix}$$

since for instance $cov\left(Y_t, R_{t+1}\right) \neq cov\left(R_t, Y_{t+1}\right)$

In practice, the autocovariances are calculated as follows:

$$\begin{gathered} \xi_t = F\xi_{t-1} + v_t \\ \text{post-multiply by own transpose and take expectations} \\ E\left(\xi_t \xi'_t\right) = FE\left(\xi_{t-1}\xi'_{t-1}\right)F' + E\left(v_t v'_t\right) \\ \Lambda = F\Lambda F' + Q \\ \text{where} \\ \Lambda = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \ldots & \Gamma_{p-1} \\ \Gamma'_1 & \Gamma_0 & \ldots & \Gamma_{p-2} \\ \ldots & \ldots & \ldots & \ldots \\ \Gamma'_{p-1} & \ldots & \ldots & \Gamma_0 \end{bmatrix} \end{gathered}$$

for lags greater than p, we postmultiply ξ_t by ξ'_{t-1}

$$\xi_t \xi'_{t-j} = F\xi_{t-j}\xi'_{t-j} + v_t \xi'_{t-j}$$

and then take expectations to obtain:

$$\begin{gathered} E\left(\xi_t \xi'_{t-j}\right) = FE\left(\xi_{t-j}\xi'_{t-j}\right) + E\left(v_t \xi'_{t-j}\right) \\ \Lambda_j = F\Lambda_{j-1} \end{gathered}$$

Note the intuition behind the autocovariance function for a VAR. For all the variables, we have an easy way to summarize all the dynamic cross-correlations between them, without making any sort of causality statement.

1.5 Evidence

With this background in mind, we try and see what the empirical VAR evidence shows: to this purpose, we run a simple 4-variable, 4-lag VAR on US quarterly data running from 1960 to 2002. Data are on $100 \times \log\left(GDP\ deflator\right)$, $100 \times \log\left(GDP\right)$, $100 \times \log\left(M2\right)$, Fed

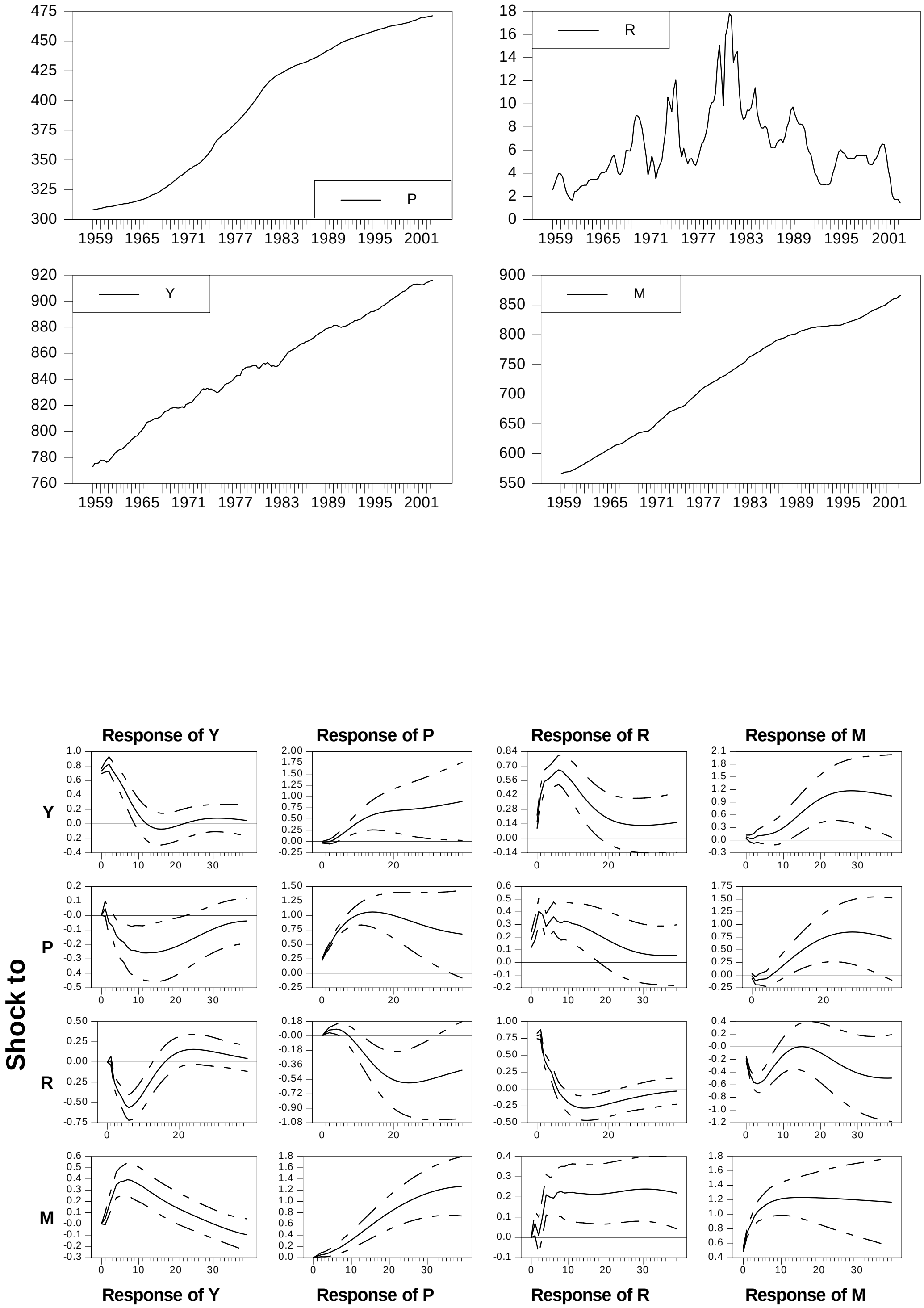

Figure 1.1: VAR evidence on US data

Funds rate. See plot of the data and make sure you know what distances on the vertical axis measure.

Impulse responses trace out the response of current and future values of each of the variables to a unit increase in the current value of one of the VAR structural errors, assuming that this error returns to zero thereafter.

What shall we remember about the VAR evidence?

- Delays of monetary policy actions: peak response of output occurs several quarters after shock
- endogenous responses of interest rate (countercyclical policy): when shocks lead to a rise in output or prices, the nominal interest rate follows in the same direction; this might represent attempts on part of the policymaker to stabilize the economy.
- output-inflation trade-off: there are shocks that generate a positive correlation between output and prices, whereas other shocks move the variables in opposite directions.
- price puzzle: prices temporarily rise after a (contractionary) interest rate increase.
- money shocks are very persistent and suggest long-run elasticity of prices to money supply of 1, whereas at the same time they are neutral with respect to output in the long run.

1.6 Implications for macroeconomic modelling.

Today's macroeconomics is quantitative in many respects. Macroeconomists try and build dynamic, micro-founded models that are as close as possible to the evidence we presented today. The advantage of having micro-founded models are several:

1. they are in general elegant and nice
2. they can be used to evaluate the welfare effects of alternative policy
3. we can do "what if" experiments with these models since their deep parameters, being based on preferences and technology, are in general robust to Lucas' critique.

Chapter 2

Money in Flexible Price Environments

Money plays the following roles in the economy: (1) medium of exchange; (2) store of value; (3) unit of account

The fact that, compared to the old days, unbacked, non-commodity money is used derives from the fact that it can used as a medium of exchange. But why is unbacked paper money used? How can money affect real decisions?

The main approaches that have been followed in the literature to model a need/demand for money are:

(1) to assume that money yields direct utility or production services by incorporating money balances directly into the utility (Sidrauski 1967) or production function.

(2) to impose transaction costs of some form that give rise to a demand for money, either assuming that exchanging assets is costly (Baumol-Tobin), or that exchanging commodities is costly (MATCHING: Kiyotaki and Wright models), or that money is needed for certain types of transactions (CIA: Clower, 1967)

(3) treating money as an asset to transfer resources intertemporally (OLG model: Samuelson, 1958), while at the same money starving agents of alternative forms of saving.

Of these models, we consider the money in utility model.

2.1 The model

We construct a general equilibrium model in which people hold money because real balances are an argument of the utility function. The idea is that real balances allow agents to save time in conducting their transactions: purchase of goods requires in other words the input of transaction services, and these transaction services are produced by money and time (see the discussion in Walsh, Chapter 3, Paragraph 2 for more on this).

The model we consider is a close relative of the real business cycle model.

2.1.1 The structure of the model

In this model we have:

Households-businesses

- produce final good
- supply labor
- own capital
- consume and invest
- purchase one-period real bonds
- hold money

Government

- issues money
- makes lump-sum transfers to the private sector with the real income generated by money creation.

agents/mkts	goods	money	bonds	*bc*
Household	$-c - K + f(K_{-1}, L) + (1-\delta) K_{-1} + T$	$\frac{M_{-1}-M}{P}$	$R_{-1}B_{-1} - B$	$= 0$
Govt	$-T$	$\frac{M-M_{-1}}{P}$		$= 0$
equilibrium	$= 0$	$= 0$	$= 0$	

Advantage of lumping household-business together: no need to keep track of the real wage or of the rental rate of capital.

2.1.2 The setup

The representative household chooses $\{C_{t+i}, K_{t+i}, L_{t+i}, M_{t+i}/P_{t+i}\}_{i=0}^{\infty}$ to maximize:

$$\begin{aligned}\max_{C_t, L_t, K_t, M_t/P_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, \frac{M_t}{P_t}, L_t\right) \\ \text{with } u_C > 0,\ u_m > 0,\ u_L < 0\end{aligned}$$

subject to:

$$C_t + K_t + \frac{M_t}{P_t} + B_t = R_{t-1}B_{t-1} + f(K_{t-1}, L_t) + (1-\delta) K_{t-1} + \frac{M_{t-1}}{P_t} + T_t$$

some considerations

- Why do we have real money $m_t \equiv M_t/P_t$ (as opposed to nominal) in the utility? If we wish to keep the assumption that economic agents are rational, what matters for agents' utility cannot be just the number of dollars that these individual hold, but rather how dollars can be exchanged for goods. Typically if you like good 1 and good 2, you write utility function as $u(x_1, x_2)$, rather than $u(x_1, p_2x_2)$. Here instead, the idea is that the services of money are proportional to how many goods you can buy with them allows to write $u = u(c_t, money\_services_t) = u(c_t, M_t/P_t)$

- We also assume that $du/dm > 0$: this implies that keeping consumption constant, an increase in m_t makes us happier. All in all, having m in the utility function is thus just a shortcut.

- In this formulation, we assume that agents trade a real bond (claim on one unit of the consumption good) that offer a return in real terms R.

- In the competitive version of this model, we can separate households and firms with households renting production factors to firms, and firm maximizing period by period profits solving a static problem. In this case, the RHS of the household constraint includes $r_tK_{t-1} + w_tL_t$ instead of $f(K_{t-1}, L_t)$ (r_t would be the real return on capital).

2.1.3 Optimality conditions

From the household problem, we form the Lagrangean to find:

$$
\begin{aligned}
L &= u\left(C_t, \frac{M_t}{P_t}, L_t\right) + \beta E_t u\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, L_{t+1}\right) + \ldots \\
&\quad -\lambda_t\left(\left[C_t + K_t + \frac{M_t}{P_t} + B_t\right] - \left(R_{t-1}B_{t-1} + f(K_{t-1}, L_t) + (1-\delta)K_{t-1} + \frac{M_{t-1}}{P_t} + T_t\right)\right) \\
&\quad -\beta E_t\lambda_{t+1}\left([\cdots] - \left(R_tB_t + f(K_t, L_{t+1}) + (1-\delta)K_t + \frac{M_t}{P_t}\frac{P_t}{P_{t+1}} + T_{t+1}\right)\right) - \ldots
\end{aligned}
$$

The first order conditions for this problem are:

$$
\begin{aligned}
u_{C,t} &= \lambda_t && \text{(C)} \\
u_{m,t} &= \lambda_t - \beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) && \text{(m)} \\
u_{Lt} &= -\lambda_t f_{Lt} && \text{(L)} \\
\lambda_t &= \beta E_t(R_t\lambda_{t+1}) && \text{(B)} \\
\lambda_t &= \beta E_t((1-\delta+f_{Kt})\lambda_{t+1}) && \text{(K)}
\end{aligned}
$$

And dropping the multiplier:

$$\begin{aligned}
u_{C,t} &= \beta R_t E_t u_{C,t+1} && (1)\\
u_{C,t} &= \beta E_t \left(u_{C,t+1}\left(1-\delta+f_{Kt}\right)\right) && (2)\\
0 &= u_{L,t}+u_{C,t} f_{Lt} && (3)\\
u_{C,t} &= u_{m,t}+\beta E_t\left(u_{C,t+1}\frac{1}{\Pi_{t+1}}\right) && (4)
\end{aligned}$$

Some considerations:

1. Pricing nominal bonds

   Combining the first two conditions shows that bonds offer in equilibrium the same return as capital. Suppose in the model you also have nominal bonds Z_t traded offering a return I_t. In this case, the individual budget constraint becomes:

$$C_t+K_t+\frac{M_t}{P_t}+B_t+\frac{Z_t}{P_t}=I_{t-1}\frac{Z_{t-1}}{P_t}+R_{t-1}B_{t-1}+f\left(K_{t-1},L_t\right)+\left(1-\delta\right)K_{t-1}+\frac{M_{t-1}}{P_t}+T_t$$

   Optimality with respect to Z_t requires:

$$u_{C,t}=\beta I_t E_t\left(\frac{P_t}{P_{t+1}}u_{C,t+1}\right)$$

   combining this equation with (1), $u_{C,t}=\beta R_t E_t u_{C,t+1}$, we get, assuming that the covariance between expected inflation and the marginal utility of consumption is zero,

$$R_t=E_t\left(I_t\frac{1}{\Pi_{t+1}}\right)$$

   which is the Fisher parity, after Fisher. If the covariance is not zero,

$$\begin{aligned}
R_t E_t u_{C,t+1} &= I_t E_t\left(\frac{P_t}{P_{t+1}}u_{C,t+1}\right)\\
R_t E_t u_{C,t+1} &= I_t\left(E_t\left(\frac{P_t}{P_{t+1}}\right)E_t\left(u_{C,t+1}\right)+cov\left(\frac{P_t}{P_{t+1}},u_{C,t+1}\right)\right)\\
R_t &= I_t E_t\left(\frac{P_t}{P_{t+1}}\right)+\frac{I_t}{E_t\left(u_{C,t+1}\right)}cov\left(\frac{P_t}{P_{t+1}},u_{C,t+1}\right)
\end{aligned}$$

2. Pricing a consol

   A consol is a bond that pays one unit of consumption over forever. Suppose we want to price such an asset: in the budget constraint, a consol is an asset H whose ex-dividend price is q and that pays 1 every period from $t+1$ on:

$$C_t+A_t+q_t H_t=R_t A_{t-1}+\left(1+q_t\right)H_{t-1}$$

Optimality with respect to H_t requires

$$u_{C,t}q_t = \beta E_t \left(u_{C,t+1}\left(1+q_{t+1}\right)\right)$$

which implies

$$R_t = E_t\left(\frac{1+q_{t+1}}{q_t}\right)$$

in log-linear terms, this expression can be rewritten as:

$$\widehat{q}_t = \beta E_t \widehat{q}_{t+1} - \widehat{R}_t$$

which shows how the price of a consol is negatively related to current and future short-term interest rates.

3. Pricing money

   The Euler condition for money is a typical expression for the price of an asset: if I give up consumption today and decide to hold money forever from then on, I will enjoy the stream of utility services in square brackets, which will be eroded from the rise in prices between t and the future.

$$\begin{aligned} u_{C,t} &= u_{m,t} + \beta E_t\left(u_{C,t+1}\frac{1}{\Pi_{t+1}}\right) \\ &= u_{m,t} + E_t\left(\frac{\beta}{\Pi_{t+1}}u_{m,t+1} + \frac{\beta^2}{\Pi_{t+1}\Pi_{t+2}}u_{m,t+2} + \frac{\beta^3}{\Pi_{t+1}\Pi_{t+2}\Pi_{t+3}}u_{C,t+3}\right) \end{aligned}$$

4. The money demand equation

   From the optimality condition for m

$$u_{C,t} = u_{m,t} + \beta E_t\left(u_{C,t+1}\frac{1}{\Pi_{t+1}}\right)$$

using:

$$u_{C,t} = \beta E_t\left(\left(I_t\frac{1}{\Pi_{t+1}}\right)u_{C,t+1}\right)$$

we obtain:

$$u_{C,t} = u_{m,t} + \frac{u_{C,t}}{I_t}$$

which implicitly defines a money demand function, so that we may write

$$\begin{aligned} u_{m,t} &\equiv g\left(m_t\right) = u_{C,t} - \frac{u_{C,t}}{I_t} \equiv h\left(C_t, I_t\right) \\ \frac{M_t}{P_t} &= \underset{g^{-1}h}{\varphi}\left(C_t, I_t\right) = \frac{\varphi\left(C_t, I_t\right)}{C_t}C_t \end{aligned}$$

rearranging, this can be written as:

$$
\begin{aligned}
M_t v_t &= P_t C_t \\
v_t &= \frac{C_t}{\varphi\left(C_t, I_t\right)}
\end{aligned}
$$

if changes in money supply do not affect C_t and I_t, there is a proportional relationship between money and prices.

5. The transversality conditions

   Together with this, we also have appropriate transversality conditions that state:

$$\lim_{t\to\infty} \beta^t u_{C,t} x_t = 0$$

   for $x = B, K, m$.

   An interpretation of the transversality condition is that it is not optimal to end up at infinity with keeping net assets if these are valuable. Loosely speaking, we impose the transversality condition by solving "forward" the forward-looking equations of the model.

2.1.4 The equilibrium

In each period the Government rebates to the public any real resources created by printing money:

$$T_t = \frac{M_t}{P_t} - \frac{M_{t-1}}{P_t}$$

To complete the description of the government problem, we also need to specify a money creation process (a rule to print money). A typical specification assumes that the growth rate of money supply follows a stationary process around a constant mean, that is:

$$M_t = \theta_t M_{t-1} \tag{5}$$

Below, we will assume that $\log\theta_t$ is random, has mean $\log\overline{\theta}$, and follows an AR(1) process around this mean. That is:

$$\ln\theta_t = (1-\rho)\ln\overline{\theta} + \rho\ln\theta_{t-1} + \varepsilon_t^M$$

where ε_t^M is zero mean, iid, with variance σ_ε^2. At quarterly frequency, a typical number for $\overline{\theta}$ would be 1.01.

Once we have the optimality conditions, the only thing we need to close the model is the market clearing conditions. In equilibrium net supply of bonds is zero, since all agents are identical. Therefore:

$$C_t + K_t = Y_t + (1-\delta) K_{t-1} \tag{6}$$

where

$$Y_t = A_t f\left(K_{t-1}, L_t\right) \tag{7}$$

and

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_t^A$$

We can then define a recursive equilibrium as follows:

An equilibrium is a sequence of values for $(C_t, K_t, L_t, m_t, \Pi_t, R_t, Y_t)$ *, given* (K_{t-1}, m_{t-1}) *and the sequence of monetary and technology shocks* $\left(\varepsilon_t^M, \varepsilon_t^A\right)$ *, satisfying at all times equations* (1) *to* (7) *as well as the transversality conditions for* K_t, B_t and m_t.

2.1.5 Functional forms and steady state

We want to consider the properties of this economy when it is in a steady state equilibrium in which nominal money grows at rate $\overline{\theta}$ and technology is constant around its mean. The steady state values of consumption, capital, real balances, inflation and the nominal interest rate must satisfy the conditions (1) to (7) above. Starting with the real side of the model, these conditions can be written as:

$$\begin{aligned} C + \delta K &= Y \\ R &= \beta^{-1} \end{aligned}$$

moving to the monetary side, from $M_t = \theta_t M_{t-1}$, real balances M/P will be constant in steady state only if

$$\Pi = \overline{\theta}.$$

To move on, it is easier to assume some specific functional forms. Suppose we parametrize the utility as follows (here $m_t \equiv M_t/P_t$ denotes real money):

$$u_t = \frac{C_t^{1-\phi} m_t^{b(1-\phi)}}{1-\phi} - \frac{\tau L_t^{\eta}}{\eta}$$

ϕ is a parameter that dictates the separability between consumption and money balances. For the production function, we assume

$$Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

where time variation in A allows for random shocks to productivity.

In steady state, the marginal utility of consumption, labor and money balances are:

$$\begin{aligned} u_C &= C^{-\phi} m^{b(1-\phi)} \\ u_L &= -\tau L^{\eta-1} \\ u_m &= b C^{1-\phi} m^{b(1-\phi)-1} \end{aligned}$$

Using the steady state version of (2) :

$$u_C = \beta u_C \left(1 - \delta + f_K\left(K\right)\right) \tag{2}$$

we obtain:

$$\begin{aligned} 1 &= \beta\left(1-\delta+\frac{\alpha Y}{K}\right) \\ \frac{K}{Y} &= \frac{\alpha}{\beta^{-1}-(1-\delta)} \end{aligned}$$

Also, using $C = Y - \delta K$

$$\frac{C}{Y} = 1 - \frac{\delta K}{Y} = 1 - \frac{\alpha\delta}{\beta^{-1}-(1-\delta)} = c_0$$

On the monetary side:

$$I = R\Pi = \frac{\overline{\theta}}{\beta}$$

Finally, money holdings will be:

$$\begin{aligned} \frac{u_m}{u_C} &= \frac{\overline{\theta}-\beta}{\overline{\theta}} = \frac{bC^{1-\phi}m^{b(1-\phi)-1}}{C^{-\phi}m^{b(1-\phi)}} \\ \frac{m}{C} &= b\frac{\overline{\theta}}{\overline{\theta}-\beta} \end{aligned}$$

notice: in the data, $\frac{m}{C}$ has an economic interpretation.

The lower $\overline{\theta}$, the lower I, the higher m/C. Hence real money holdings depend negatively on inflation in this model. This intuition has generated lots of research into how big the welfare costs of inflation are. You will do an exercise on this: the basic idea is that one can measure the welfare costs of inflation by the percentage increase in steady-state consumption necessary to compensate the decrease in utility associated with an increase in the steady state nominal interest rate.

As for labor supply:

$$\begin{aligned} \tau L^{\eta-1} &= u_C f_L \\ \tau L^{\eta-1} &= C^{-\phi}m^{b(1-\phi)}(1-\alpha)\frac{Y}{L} \end{aligned}$$

Knowing that $C = c_0 Y$, we can solve for L as a function of C with some use of algebra.

$$\tau L^{\eta} = C^{-\phi}\left(b\frac{\overline{\theta}}{\overline{\theta}-\beta}C\right)^{b(1-\phi)}(1-\alpha)\frac{C}{c_0}$$

2.1.6 Summarizing

For given values of the structural parameters, we can find the steady state values of all the variables of the model. However, for local dynamics what we are interested in are mostly the ratios of these variables relative to Y. These are the "so-called" big ratios. In most applications, what we do is to treat the ratios (m/C), (K/Y) and so on as observable to imply restrictions on the structural parameters that satisfy the big ratios: this approach is referred to as calibration.

$$\begin{aligned}
\tau L^{\eta} &= \zeta C^{1-\phi+b(1-\phi)} \\
\frac{C}{Y} &= 1-\frac{\alpha\delta}{\beta^{-1}-(1-\delta)} \\
\frac{m}{C} &= b\frac{\overline{\theta}}{\overline{\theta}-\beta} \\
\frac{K}{Y} &= \frac{\alpha}{\beta^{-1}-(1-\delta)} \\
Y &= AK^{\alpha}L^{1-\alpha}
\end{aligned}$$

where

$$\zeta = \left(b\frac{\overline{\theta}}{\overline{\theta}-\beta}\right)^{b(1-\phi)}\frac{(1-\alpha)}{c_0}$$

Note the following:

- In steady state, the ratios $\frac{K}{Y}$, $\frac{C}{Y}$ are independent of all parameters of the utility function and of the inflation rate. However, the level of output may depend on the inflation rate: this because the level of Y depends on L, which in steady state is a function of $\overline{\theta}$.

- Inflation in steady state just equals the growth rate of money supply.

- When $\phi = 1$, the model displays "superneutrality" of money in steady state: that is, when $\phi = 1$, one can solve for the levels of C, L, Y and K independently of the steady state growth rate of money $\overline{\theta}$.

2.2 The log-linear equilibrium

To study local dynamics, we describe the log-linear version of the system made by equations (1) to (7). For each variable X_t, let:

$$\widehat{X}_t = \frac{X_t - X}{X}$$

The log-linear will be given by:

$$\widehat{Y}_t = \alpha \widehat{K}_{t-1} + (1-\alpha)\widehat{L}_t + \widehat{A}_t \tag{L1}$$

$$\widehat{Y}_t = \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}\left(\widehat{K}_t - (1-\delta)\widehat{K}_{t-1}\right) \tag{L2}$$

$$\widehat{R}_t = \frac{\alpha\beta Y}{K}\left(E_t\widehat{Y}_{t+1} - \widehat{K}_t\right) \tag{L3}$$

$$\widehat{R}_t = \phi\left(E_t\widehat{C}_{t+1} - \widehat{C}_t\right) - b(1-\phi)\left(E_t\widehat{m}_{t+1} - \widehat{m}_t\right) \tag{L4}$$

$$\eta\widehat{L}_t = \widehat{Y}_t - \phi\widehat{C}_t + b(1-\phi)\widehat{m}_t \tag{L5}$$

$$\widehat{R}_t + E_t\widehat{\pi}_{t+1} = \frac{\overline{\theta}-\beta}{\beta}\left(\widehat{C}_t - \widehat{m}_t\right) \tag{L6}$$

$$\widehat{m}_t - \widehat{m}_{t-1} = \widehat{\theta}_t - \widehat{\pi}_t \tag{L7}$$

To these 7 equations in 7 endogenous variables, we need to add laws of motion for the exogenous stochastic processes θ_t and A_t.

2.2.1 Derivation of some of the equations

Some rules to remember for linearizing:

1. Sometimes it is convenient to switch to logs. The advantage is that the the derivative of $\log X_t$ gives us $\widehat{X}$

   - The log-sum rule

$$\begin{aligned} X_t &= A_t + B_t \\ \ln X_t &= \ln(A_t + B_t) \\ \frac{1}{X}dX_t &= \frac{1}{A+B}(dA_t + dB_t) \\ \widehat{X}_t &= \frac{A}{X}\widehat{A}_t + \frac{B}{X}\widehat{B}_t \end{aligned}$$

   - The log-product rule

$$\begin{aligned} X_t &= A_tB_t \\ \ln X_t &= \ln(A_tB_t) \\ \widehat{X}_t &= \widehat{A}_t + \widehat{B}_t \end{aligned}$$

2. The total differential rule: Sometimes it is easier to take the total differential of the expression we wish to log-linearize, and then multiply and divide by the variable to get its log-deviation from steady state. See equation L3 below

2.2.1.1 Equation L3

From

$$\begin{aligned} u_{C,t} &= \beta R_t E_t u_{C,t+1} && (1) \\ u_{C,t} &= \beta E_t u_{C,t+1} \left[1 - \delta + f_K \left(K_t\right)\right] && (2) \end{aligned}$$

get

$$R_t = 1 - \delta + \alpha E_t \frac{Y_{t+1}}{K_t}$$

so that, taking total differential and dividing both sides by $R = 1/\beta$:

$$\begin{aligned} dR_t &= \alpha \left(\frac{1}{K} dE_t Y_{t+1} - \frac{Y}{K^2} dK_t \right) \\ \widehat{R}_t &= \alpha\beta \left(\frac{Y}{K} \left(E_t \widehat{Y}_{t+1} - \widehat{K}_t \right) \right) \end{aligned}$$

2.2.1.2 Equation L5

From:

$$\tau L_t^{\eta-1} = C_t^{-\phi} m_t^{b(1-\phi)} (1-\alpha) \frac{Y_t}{L_t}$$

notice that this expression is linear in the log of the variables, hence it lends itself to easy application of the log-product rule.

2.2.1.3 Equation L6

In steady state:

$$\begin{aligned} u_C &= C^{-\phi} m^{b(1-\phi)} \\ u_L &= -\tau L^{\eta-1} \\ u_m &= b C^{1-\phi} m^{b(1-\phi)-1} \end{aligned}$$

Start from equation 4 and combine it with equation 1. Then, assuming that the terms covariance terms are negligible, so that we can approximate $E(xy) = E(x) E(y)$ for any variable x and y:

$$\begin{aligned}
u_{C,t} &= u_{m,t} + \beta E_t \left(u_{C,t+1} \frac{1}{\Pi_{t+1}} \right) = \beta E_t \left(u_{C,t+1} R_t \right) \\
u_{C,t} &= u_{m,t} + E_t \left(\frac{u_{C,t}}{R_t \Pi_{t+1}} \right) \\
C_t^{-\phi} m_t^{b(1-\phi)} &= b C_t^{1-\phi} m_t^{b(1-\phi)-1} + E_t \left(\frac{C_t^{-\phi} m_t^{b(1-\phi)}}{R_t \Pi_{t+1}} \right) \\
1 &= b \frac{C_t}{m_t} + E_t \left(\frac{1}{R_t \Pi_{t+1}} \right) \\
0 &= b \frac{C}{m} \left(\widehat{C}_t - \widehat{m}_t \right) - \frac{1}{R\Pi} \left(\widehat{R}_t + \widehat{\pi}_{t+1} \right) \\
0 &= \frac{\overline{\theta} - \beta}{\overline{\theta}} \left(\widehat{C}_t - \widehat{m}_t \right) - \frac{\beta}{\overline{\theta}} \left(\widehat{R}_t + E_t \widehat{\pi}_{t+1} \right)
\end{aligned}$$

In the penultimate row, we use the differential rule to switch to log deviations.

2.2.1.4 Equation L7

Start from equation 5:

$$M_t = \theta_t M_{t-1}$$

convert nominal money to real money and then use the log-product rule:

$$\begin{aligned}
\frac{M_t}{P_t} &= \theta_t \frac{M_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} \\
\widehat{m}_t - \widehat{m}_{t-1} &= \widehat{\theta}_t - \widehat{\pi}_t
\end{aligned}$$

2.2.1.5 Derivation of the other equations

Other equations require straightforward applications of the rules above.

2.2.2 An interesting case: separability

In the special case in which utility is separable in consumption and money balances, $\phi = 1$. This is an interesting case because now the equations of the model $(L1)$ to $(L7)$ can be separated in two independent blocks. The first block includes equations $(L1)$ to $(L5)$

adequately modified:

$$
\begin{aligned}
\widehat{Y}_t &= \alpha \widehat{K}_{t-1} + (1-\alpha)\,\widehat{L}_t + \widehat{A}_t \\
\widehat{Y}_t &= \frac{C}{Y}\widehat{C}_t + \frac{K}{Y}\left(\widehat{K}_t - (1-\delta)\,\widehat{K}_{t-1}\right) \\
\widehat{R}_t &= \frac{\alpha\beta Y}{K} E_t\left(\widehat{Y}_{t+1} - \widehat{K}_t\right) \\
\widehat{R}_t &= \phi E_t\left(\widehat{C}_{t+1} - \widehat{C}_t\right) \\
\eta \widehat{L}_t &= \widehat{Y}_t - \phi \widehat{C}_t
\end{aligned}
$$

where can solve for C_t, K_t, Y_t, R_t and L_t independently of the rest of the model. Money is thus completely neutral for the real variables, in and out of the steady state. $(L1)$ to $(L5)$ represent the basic Real Business Cycle Model that you might have already seen.

Similarly, the equations $(L6)$ and $(L7)$ to study inflation and money growth independently of the real variables. To this end, assume technology is constant, so that the real interest rate and consumption will be constant too. Then, from $(L6)$:

$$\widehat{R}_t + E_t \widehat{\pi}_{t+1} = \frac{\overline{\theta} - \beta}{\beta}\left(\widehat{C}_t - \widehat{m}_t\right)$$

we obtain

$$\widehat{m}_t \equiv \widehat{M_t} - \widehat{P}_t = \frac{\beta}{\overline{\theta} - \beta}\left(\widehat{P}_t - E_t \widehat{P}_{t+1}\right)$$

this is Cagan money demand. If people expect high inflation in the future, they will reduce their real model holdings now. This equation can alternatively be solved for P_t

$$\widehat{P}_t = \frac{\beta}{\overline{\theta}} E_t \widehat{P}_{t+1} + \frac{\overline{\theta} - \beta}{\overline{\theta}} \widehat{M_t} \qquad (*)$$

to show how the price level today depends on current and future expected money growth (which is described by $(L7)$).

The model can also be used in order to:

- analyse the welfare losses from inflation
- optimal rate of inflation (marginal benefit of money is minus the nominal interest rate, marginal cost is zero, want the nominal rate to be zero).

2.3 Prices and money

Equation $(*)$ can be solved by standard methods to express the price level as a function of the money supply process.

In the model we developed above, we maintained the assumption that there was positive inflation in steady state (given by the steady state growth rate of M, so that $\Pi = \overline{\theta} \geq 1$). In other words, we assumed that money was growing over time at some constant rate $\overline{\theta}$: when we look at deviations from steady state, we perturb the path of money around the steady state in which nominal money is growing over time at rate $\overline{\theta}$, real money is constant, and there are innovations to θ_t at each point in time. Let $\sigma = \frac{\beta}{\overline{\theta}-\beta}$. The monetary side of our model can be written as:

$$\begin{aligned} \widehat{m}_t &= -\sigma E_t \widehat{\pi}_{t+1} && \text{(m1)} \\ \widehat{m}_t - \widehat{m}_{t-1} &= \widehat{\theta}_t - \widehat{\pi}_t && \text{(m2)} \end{aligned}$$

together with the linearized version of $\ln \theta_t = (1-\rho)\ln \overline{\theta} + \rho \ln \theta_{t-1} + \varepsilon_t$, namely

$$\widehat{\theta}_t = \rho \widehat{\theta}_{t-1} + \widehat{\varepsilon}_t$$

where $\widehat{\varepsilon}_t = \varepsilon_t$.

What is the effect on inflation of a one-period, transitory increase in the growth rate of money with persistence ρ? From the expression above, we guess that a solution for inflation is:

$$\widehat{\pi}_t = \varepsilon_1 \widehat{m}_{t-1} + \varepsilon_2 \widehat{\theta}_t \tag{guess}$$

Using this guess, we get, using $E_t \widehat{\theta}_{t+1} = \rho \widehat{\theta}_t$ and plugging $(guess)$ into $(m1)$:

$$\begin{aligned} \widehat{m}_t &= -\sigma E_t \left(\varepsilon_1 \widehat{m}_t + \varepsilon_2 \widehat{\theta}_{t+1} \right) \\ (1+\sigma\varepsilon_1)\, \widehat{m}_t &= -\sigma \varepsilon_2 \rho \widehat{\theta}_t \\ (1+\sigma\varepsilon_1) \left(\widehat{m}_{t-1} + \widehat{\theta}_t - \widehat{\pi}_t \right) &= -\sigma \varepsilon_2 \rho \widehat{\theta}_t \\ (1+\sigma\varepsilon_1) \left(\widehat{m}_{t-1} + \widehat{\theta}_t - \varepsilon_1 \widehat{m}_{t-1} - \varepsilon_2 \widehat{\theta}_t \right) &= -\sigma \varepsilon_2 \rho \widehat{\theta}_t \end{aligned}$$

These two equations must hold at all times for all values of $\widehat{m}_{t-1}$ and $\widehat{\theta}_t$, thus implying that the values of the undetermined coefficients ε_1 and ε_2 will be equal to:

$$\begin{aligned} 1 - \varepsilon_1 &= 0 \\ (1+\sigma\varepsilon_1)(1-\varepsilon_2) &= -\sigma \varepsilon_2 \rho \end{aligned}$$

hence

$$\begin{aligned} \varepsilon_1 &= 1 \\ \varepsilon_2 &= \frac{1+\sigma}{1+\sigma(1-\rho)} \end{aligned}$$

Since σ is positive, the impact response of inflation to a shock in $\widehat{\theta}_t$ will be in general larger than one, so long as $\rho > 0$. This also tells us than in increase in $\widehat{\theta}_t$ leads to a fall in $\widehat{m}_t$.

$$\widehat{\pi}_t = \widehat{m}_{t-1} + \frac{1+\sigma}{1+\sigma(1-\rho)} \widehat{\theta}_t$$

Bottom line: how much inflation rises in response to a monetary expansion depends on how temporary or persistent the increase in money supply is, and on the underlying process for money. But, so long as prices are forward looking, an increase in the rate of money growth should cause an even larger response in inflation.[1]

2.4 The Effect of Monetary and Technology Shocks

2.4.1 The real effects of money shocks when $\phi \neq 1$.

When we relax separability between consumption and money balances things become more cumbersome, and analytical solutions are hard to get. Assume a *persistent money growth shock*. This shock will lead to an *increase in expected inflation,* which will move agents away from money towards other goods. *Real money demand m thus falls.* What happens to output depends on the substitutability between money and consumption.

If m falls, do agents consume more goods (consumption) or more spare time (leisure)? The choice depends on whether money and consumption are complements or substitutes. Note that:

$$-\frac{u_C}{u_L} = \frac{C^{-\phi} m^{b(1-\phi)}}{\tau L^{\eta-1}}$$

1. when $\phi < 1$, a fall in m lowers u_C. Hence, in choosing C versus L, agents consume more leisure (consumption does not make you feel happier), work less, and Y falls

2. when $\phi > 1$, a fall in m raises u_C. Hence L rises, Y rises.

2.4.2 Technology shocks and inflation

By affecting consumption and the real rate, a technology shock will also affect the nominal variables. To see this, assume $\widehat{\theta}_t = 0$, so that $\widehat{M}_t = 0$ is constant for all t. Assume also $\overline{\theta} = 1$ (net money growth is zero in steady state) and $\phi = 1$. Hence $\widehat{m}_t = \widehat{M}_t - \widehat{P}_t = -\widehat{P}_t$. Equation

[1] The solution for π_t can be further manipulated using

$$\widehat{m}_t = -\sigma E_t \widehat{\pi}_{t+1} = -\sigma \left(\widehat{m}_t + \frac{1+\sigma}{1+\sigma(1-\rho)} \rho \widehat{\theta}_t \right)$$

to obtain a solution for $\widehat{\pi}_t$ which is:

$$\widehat{\pi}_t = \frac{\rho}{1+\sigma(1-\rho)} \widehat{\theta}_{t-1} + \frac{1+\sigma}{1+\sigma(1-\rho)} \widehat{\varepsilon}_t.$$

For empirically plausible values of σ, the term multiplying $\widehat{\theta}_{t-1}$ is small, the the autocorrelation of inflation delivered by the model is quite small, unlike in the data, where there seems to be more inflation persistence that can be rationalized by this model.

L6 then becomes:

$$\begin{aligned}
\beta\left(\widehat{R}_t + E_t\widehat{\pi}_{t+1}\right) &= (1-\beta)\left(\widehat{C}_t - \widehat{m}_t\right) \\
\beta\left(\widehat{R}_t + E_t\widehat{P}_{t+1} - \widehat{P}_t\right) &= (1-\beta)\left(\widehat{C}_t + \widehat{P}_t\right) \\
\widehat{P}_t &= \beta E_t\widehat{P}_{t+1} + \beta\widehat{R}_t - (1-\beta)\,\widehat{C}_t \\
\widehat{P}_t &= \beta E_t\widehat{P}_{t+1} - \beta\left(\widehat{C}_t - E_t\widehat{C}_{t+1}\right) - (1-\beta)\,\widehat{C}_t \\
\widehat{P}_t - \beta E_t\widehat{P}_{t+1} &= -\left(\widehat{C}_t - \beta E_t\widehat{C}_{t+1}\right)
\end{aligned}$$

As shown by the last equation, prices will move in the opposite direction as consumption if money supply is held constant after a technology shock. In other words. If a technology shock raises consumption, prices fall by the same amount.

Can you give an intuition? The key is the Cobb-Douglas specification of preferences. With Cobb-Douglas preferences, individuals aim at consuming constant shares of the goods entering the utility function. When consumption rises, the only way agents can "consume" more real balances is through a fall in the price level or through an increase in money supply. If M is constant, then P has to fall.

Chapter 3

Credit and Business Cycles

Here I present a model of the interaction between credit and business cycles. In representative agent models, remember, no lending takes place!

The literature on the topic has emphasized the idea that a credit market imperfections can magnify the effects on the real economy of given technology or monetary shocks. This happens because a positive shock increases income of owners of production technology; this rise in net worth lowers the costs associated with external financing of investment projects, allowing for increased investment. This serves to amplify and propagate the effects of a shock through time...

3.1 The modelling choices

We now develop a model that illustrates the role of debt, net worth and asset price fluctuations on equilibrium output. You can consider this model as an extension of the real business cycle model which allows for financial factors to play a role in business fluctuations.

The model has to be as simple as possible. In fact it can be made quite simple even though it features heterogenous agents by adding to it the following features:

1. A constant interest rate (one less variable)
2. No labor supply decision
3. No variable capital
4. Only one asset that can be used for production, and is available in fixed supply in the aggregate (call it land).

3.2 A basic model

The reference is the paper "Credit Cycles" by Kiyotaki and Moore (1997), JPE, 211-247, available on JSTOR. We use a slightly modified version.

There are two types of agents, farmers (productive) and gatherers (unproductive). They both have linear preferences over consumption, but they discount the future differently. Farmers have a discount factor of γ, gatherers have a discount factor of β, where $\beta > \gamma$. They produce a final good y_t using land h_t.

The production functions are respectively described by:

$$\begin{aligned} y_t &= A_t h_{t-1}^{\nu} \\ y'_t &= A'_t h'^{\mu}_{t-1} \end{aligned}$$

3.2.1 Gatherers

The gatherers will be the "unproductive" agents, that is they will earn a lower return on their asset holdings in equilibrium. They solve the following problems.

$$\max_{h'_t, b'_t} \; E_0 \left(\sum_{t=0}^{\infty} \beta^t c'_t \right)$$
$$\text{s.t } c'_t + q_t h'_t + R b'_{t-1} = y'_t + q_t h'_{t-1} + b'_t$$

The Lagrangean for this problem is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(y'_t + q_t h'_{t-1} + b'_t - q_t h'_t - R b'_{t-1} \right)$$

for each t, where $A'_t h'^{\mu}_{t-1}$. The first order conditions are, choosing b'_t and h'_t respectively:

$$\begin{aligned} \beta R &= 1 \\ q_t &= \beta E_t \left(q_{t+1} + \mu \frac{y'_{t+1}}{h'_t} \right) \end{aligned}$$

Linear preferences over consumption imply that in equilibrium the gatherers must be indifferent about any path of consumption and debt, so that the interest rate will equal their rate of time preference, and $R = 1/\beta$.

3.2.2 Farmers

Farmers maximize

$$\max_{b_t, h_t} E_0 \left(\sum_{t=0}^{\infty} \gamma^t c_t \right)$$

their flows of funds is

$$c_t + q_t h_t + R b_{t-1} = y_t + q_t h_{t-1} + b_t$$

where on the right-hand side we have the sources of funds, on the left-hand side we have its uses. The amount of claims farmers can issue is bound by:

$$Rb_t \leq mE_t\left(q_{t+1}h_t\right)$$

Remark 4 *Why can't the farmers borrow more than the* $mE_t\left(q_{t+1}h_t/R\right)$*? The idea is as follows: the Gatherer lends some goods to the Farmer, who in turn promises him to pay back at some future date. The assumption here is that the farmer's input is critical for production: once the farmer starts producing, no one can replace him/her, and the farmer cannot commit to repay his debt. If the creditor tries to extract too much from the farmer, the farmer can simply walk away from the land. Current production is lost, and the farmer only recovers the value of the land. This in turn limits his/her ability to borrow.*

The Lagrangean for this problem is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \gamma^t \left(\left(y_t + q_t h_{t-1} + b_t - q_t h_t - Rb_{t-1}\right) - \lambda_t \left(Rb_t - mq_{t+1}h_t\right)\right)$$

for each t, where $A_t h_{t-1}^{\nu}$. The first order conditions are, choosing b_t and h_t respectively:

$$\begin{aligned} 1 &= \gamma R + \lambda_t R \\ q_t &= \gamma E_t\left(q_{t+1} + \nu \frac{y_{t+1}}{h_t}\right) + E_t\left(\lambda_t m q_{t+1}\right) \end{aligned}$$

3.2.3 Equilibrium

There are three markets: the market for loans b_t, goods $y_t + y_t'$ and market for h.

From the Euler equation for gatherers, we know that:

$$\beta R = 1$$

coupled with that of farmers

$$\lambda_t = \lambda = \frac{1-\gamma R}{R} = \beta - \gamma > 0$$

which implies that the borrowing constraint will be binding in steady state.

Once we collect all of them, we obtain the following equations:

$$\begin{aligned} c_t' + q_t h_t' + Rb_{t-1}' &= y_t' + q_t h_{t-1}' + b_t' && (1) \\ c_t + q_t h_t + Rb_{t-1} &= y_t + q_t h_{t-1} + b_t && (2) \\ q_t &= \left(m\beta + (1-m)\gamma\right) E_t q_{t+1} + \gamma\nu E_t\left(\frac{y_{t+1}}{h_t}\right) && (3) \\ q_t &= \beta E_t q_{t+1} + \beta\mu E_t\left(\frac{y_{t+1}'}{h_t'}\right) && (4) \\ Rb_t &= mE_t\left(q_{t+1}h_t\right) && (5) \end{aligned}$$

To this group of equations, we add the market clearing conditions. Normalize the total supply of land H to 1.

$$\begin{aligned} h_t + h'_t &= 1 && (6) \\ b_t + b'_t &= 0 && (7) \end{aligned}$$

$$\begin{aligned} y_t &= A_t h_{t-1}^{\nu} && (8) \\ y'_t &= A'_t h_{t-1}^{\prime\mu} && (9) \end{aligned}$$

Remember that if the bond market clears, so will the goods one, so we will not need to write down the market clearing condition for goods.

A quick check: 9 equations, whereas the variables are $c'_t \quad c_t \quad y_t \quad y'_t \quad h_t \quad h'_t \quad b_t \quad b'_t \quad q_t$.

We restrict our attention to perfect-foresight equilibria, where, absent unanticipated shocks, the expectations of future variables realize themselves. An equilibrium can be defined as follows: for given levels of h_{t-1} and b_{t-1}, an equilibrium from date t onwards is characterized by paths for the endogenous variables $\{\mathsf{x}_{t+s}|s \geq 0\}$ satisfying equations (1) to (9) at dates t, $t+1$, $t+2, ...$ A transversality condition rules out exploding paths in asset prices, that is

$$\lim_{s \to \infty} E_t \left(\frac{q_{t+s}}{R^s} \right) = 0$$

If the transversality condition is satisfied, there is a locally unique perfect-foresight equilibrium path starting from initial values of h_{t-1} and b_{t-1} in the neighborhood of the steady state.

3.2.4 Simplify

Most of the equations are trivial. Also notice that c'_t and c_t only appear in 1 and 2, so they will adjust so that 1 and 2 hold.

To simplify matters, Kiyotaki and Moore also assume that $\nu = 1$, so that the production technology of the farmers is linear. This implies that all the dynamics of the system can be analyzed with reference to two equations only. Define $\phi = m\beta + (1-m)\gamma$ (you can think of ϕ as an average of the two discount factors, the higher m, the higher ϕ)

$$\begin{aligned} q_t &= \phi E_t q_{t+1} + \gamma E_t A_{t+1} && (3) \\ q_t &= \beta E_t q_{t+1} + \beta\mu E_t \left(\frac{A'_{t+1}}{(1-h_t)^{1-\mu}} \right) && (4) \end{aligned}$$

3.2.5 Steady state

Let us look at the steady state first. From 3

$$q = \frac{\gamma A}{1-\phi}$$

from 4

$$q\left(1-\beta\right)=\frac{\beta\mu A'}{\left(1-h\right)^{1-\mu}}$$

which gives us h.

$$1-h=\left(\frac{\beta\mu A'\left(1-\phi\right)}{\gamma A\left(1-\beta\right)}\right)^{\frac{1}{1-\mu}}.$$

Note from this expression that one needs to place restrictions of A'/A need to placed to ensure an interior steady state equilibrium with $0<h<1$.

Remark 5 *What is the marginal product of land used in the farming sector (define it as MPK)? Because of their linear production function (we assume $\nu=1$ now), it is simply A*

What is MPK in the gatherers' sector? Start from the optimality conditions of gatherers:

$$\begin{aligned} q &= \beta q+\beta MPK \\ MPK &= \frac{q\left(1-\beta\right)}{\beta}=\frac{\gamma\left(1-\beta\right)}{\beta\left(1-\phi\right)}A \end{aligned}$$

Given $\gamma<\phi<\beta$, $\frac{\gamma(1-\beta)}{\beta(1-\phi)}<1$, so that the steady state MPK in the gatherers' sector is below that of the farmers' sector. This implies that in the competitive equilibrium the allocation of land is inefficient since its marginal product is not equated across the two sectors.

From 5 we can also derive:

$$b=\beta mqh$$

as for the other variables

$$\begin{aligned} c &= y-\left(R-1\right)b=Ah-\left(1-\beta\right)mqh=A\left(1-\frac{m\gamma\left(1-\beta\right)}{1-\phi}\right)h \\ c' &= y'+\left(R-1\right)b \end{aligned}$$

The main idea here is that credit market imperfections might cause a misallocation of resources in equilibrium.

3.2.6 The dynamics effects of an unexpected productivity shock

To consider the effects on the economy of a productivity shock, we linearize the model equations around the steady state and calculate the effect of a 1% increase in productivity.

We consider a proportional increase in productivity for both agents, that is $\widehat{A}_t=\widehat{A}'_t$ (where $\widehat{X}_t\equiv\frac{X_t-X}{X}$ is the percent deviation from steady state of a variable). Let us also assume that once a technology shock occurs, it follows an AR(1) process of the form:

$$\widehat{A}_t=\rho\widehat{A}_{t-1}+\widehat{e}_t$$

From equation 3 we obtain:

$$
\begin{aligned}
q_t &= \phi E_t q_{t+1} + \gamma E_t A_{t+1} \\
dq_t &= \phi E_t dq_{t+1} + \gamma E_t dA_{t+1} \\
\frac{dq_t}{q} &= \phi E_t \frac{dq_{t+1}}{q} + \gamma E_t \frac{dA_{t+1}}{A} \frac{A}{q} \\
\text{use } q &= \frac{\gamma A}{1-\phi} \rightarrow \frac{A}{q} = \frac{1-\phi}{\gamma} \\
\widehat{q}_t &= \phi E_t \widehat{q}_{t+1} + (1-\phi) E_t \widehat{A}_{t+1} \qquad \text{(L1)}
\end{aligned}
$$

which can be solved forward to obtain:

$$
\begin{aligned}
\text{use } E_t \widehat{q}_{t+1} &= (1-\phi) E_t \widehat{A}_{t+2} + \phi E_t \widehat{q}_{t+2} \\
\widehat{q}_t &= (1-\phi) E_t \widehat{A}_{t+1} + \phi \Big((1-\phi) E_t \widehat{A}_{t+2} + \phi E_t \widehat{q}_{t+2} \Big) \\
\text{use } E_t \widehat{q}_{t+2} &= (1-\phi) E_t \widehat{A}_{t+3} + \phi E_t \widehat{q}_{t+3} \\
\widehat{q}_t &= (1-\phi) E_t \widehat{A}_{t+1} + \phi \Big((1-\phi) E_t \widehat{A}_{t+2} + \phi \Big((1-\phi) E_t \widehat{A}_{t+3} + \phi E_t \widehat{q}_{t+3} \Big) \Big) \\
\widehat{q}_t &= (1-\phi) E_t \widehat{A}_{t+1} + \phi (1-\phi) E_t \widehat{A}_{t+2} + \phi^2 (1-\phi) E_t \widehat{A}_{t+3} + ... \\
\widehat{q}_t &= (1-\phi) E_t \widehat{A}_{t+1} + \phi (1-\phi) \rho E_t \widehat{A}_{t+1} + \phi^2 (1-\phi) \rho^2 E_t \widehat{A}_{t+1} + ... \\
\widehat{q}_t &= (1-\phi) \big(1 + \phi\rho + \phi^2 \rho^2 + ... \big) E_t \widehat{A}_{t+1} + ... \\
\widehat{q}_t &= \frac{1-\phi}{1-\phi\rho} E_t \widehat{A}_{t+1} = \frac{1-\phi}{1-\phi\rho} \rho \widehat{A}_t \qquad \text{(dr1)}
\end{aligned}
$$

where ρ is the persistence of the technology shock.

To linearize equation 4:

$$q_t = \beta E_t q_{t+1} + \beta\mu E_t \left(\frac{A'_{t+1}}{(1-h_t)^{1-\mu}}\right)$$

$$q_t = \beta E_t q_{t+1} + \beta\mu E_t \left(A'_{t+1}(1-h_t)^{\mu-1}\right)$$

take total differential

$$dq_t = \beta E_t (dq_{t+1}) + \beta\mu (1-h)^{\mu-1} dE_t A'_{t+1} - \beta\mu A' (\mu - 1)(1-h)^{\mu-2} dh_t$$

$$dq_t = \beta E_t (dq_{t+1}) + \beta\mu (1-h)^{\mu-1} dE_t A'_{t+1} - \beta\mu A' (\mu - 1)(1-h)^{\mu-1}(1-h)^{-1} dh_t$$

use steady state result $q = \beta q + \beta\mu A'(1-h)^{\mu-1} \rightarrow q(1-\beta) = \beta\mu A'(1-h)^{\mu-1}$

$$dq_t = \beta E_t (dq_{t+1}) + q(1-\beta)\frac{dE_t A'_{t+1}}{A'} - q(1-\beta)(\mu-1)(1-h)^{-1} dh_t$$

divide everything by q

$$\frac{dq_t}{q} = \beta E_t\left(\frac{dq_{t+1}}{q}\right) + (1-\beta)\frac{dE_t A'_{t+1}}{A'} - (1-\beta)\frac{(\mu-1)}{1-h} h \frac{dh_t}{h}$$

define $\eta = \dfrac{1-h}{(1-\mu)h}$

$$\frac{dq_t}{q} = \beta E_t\left(\frac{dq_{t+1}}{q}\right) + (1-\beta)\frac{dE_t A'_{t+1}}{A'} + (1-\beta)\frac{1}{\eta}\frac{dh_t}{h}$$

$$\widehat{q}_t = \beta E_t \widehat{q}_{t+1} + (1-\beta)\left(E_t \widehat{A}_{t+1} + \frac{1}{\eta}\widehat{h}_t\right) \tag{L2}$$

where $\eta = \frac{1-h}{(1-\mu)h} > 0$ can be interpreted as an elasticity of asset demand with respect to the price (roughly speaking, it tells you by how much $\widehat{h}$ changes in the long-run for a given change in $\widehat{q}$, this is why we interpret it as an elasticity).

Combining $(dr1)$ and $(L2)$ together yield:

$$\frac{1-\phi}{1-\phi\rho}\rho\widehat{A}_t = \frac{1-\phi}{1-\phi\rho}\beta\rho^2\widehat{A}_t + (1-\beta)\left(\rho\widehat{A}_t + \frac{1}{\eta}\widehat{h}_t\right)$$

which can be solved for the response of $\widehat{h}_t$ to a productivity shock, that is:

$$\widehat{h}_t = \frac{(1-\phi)(1-\beta\rho) - (1-\phi\rho)(1-\beta)}{(1-\phi\rho)(1-\beta)}\eta\rho\widehat{A}_t \tag{dr2}$$

$(dr1)$ and $(dr2)$ are the two linearized decision rules of our problem (that is, they express the endogenous variables as a function of shocks only). $(dr2)$ shows that h rises with A so long as $0 < \rho < 1$ (the numerator can be shown to be positive because $\phi < \beta$).

Hence in this problem a productivity shocks raises asset prices which in turn raises asset demand of the most productive agents, who can then produce more and accumulate more: the effects on aggregate activity are therefore amplified by the asset price effect.

The main idea here is that credit market imperfections might amplify the effects of given shocks to the economy.

It is easy to show how aggregate output rises more than the increase in productivity: that is, in log deviations from the steady state:

$$\widehat{Y}_t = \frac{y}{y+y'}\widehat{y}_t + \frac{y'}{y+y'}\widehat{y}'_t > \widehat{A}_t.$$

3.3 Forward-lookingness and history dependence

The model here displays forward-lookingness; an increase in asset prices implies that the creditor will be able to recover more from selling the asset whenever the debtor defaults, therefore for each asset price increase he is willing to supply more credit. As credit increases, asset prices rise, aggregate investment rises (since the marginal productivity of debtors' is higher than creditors') and so on and so forth, in a cumulative fashion.

However, it does not display history-dependence (one way to see this is that the equilibrium decision rules do not depend on h_{t-1} and b_{t-1} explicitly). Kiyotaki and Moore say when a shock hits, it is too late to renegotiate, therefore repayments are the same, but current borrowing increases. If consumption is bound by some upper limit, all the new borrowing goes into investment, and this strongly magnifies the response of output to a given productivity shock. The homework questions asks you to analyze this in more detail.

Chapter 4

The dynamic new-keynesian model

At least since Keynes, it has been thought that in order to have real effects from monetary actions, it is key to have some degree of nominal rigidity. The question we want to explore is: can a model based on microfoundations based on these features describe some important features of the link between monetary actions and the business cycle?

What do we need in order to get nominal rigidities in the traditional, dynamic general equilibrium model? Well, we need some form of pricing power, for instance coming from monopolistic competition, and therefore some heterogeneity among goods.

The main actors of the DNK model are:

agents/mkts	final good	intermediate	labor	profit	money	nom.bonds	
Household	$-PC$		WL	PF	$M_{-1} - M + PT$	$R_{-1}B_{-1} - B$	$= 0$
final firm	PY	$-\int_0^1 P_j Y_j dj$					$= 0$
interm. firms		$\int_0^1 P_j Y_j dj$	$-WL$	$-PF$			$= 0$
Govt					$M - M_{-1} - PT$		$= 0$
equilibrium	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	

- households: make consumption and labor supply decisions, demand money and bonds
- final good firms: produce homogeneous final goods Y_t from intermediate goods Y_{jt}
- intermediate good firm: use labor to produce intermediate goods Y_{jt}. Over each of this goods they have monopoly power. Demand labor. Can set price of good Y_j
- government: runs monetary policy.

4.1 Households

There is a continuum of infinitely-lived individuals, whose total is normalized to 1. They choose consumption C_t, labor L_t, money M_t and nominal bonds B_t in order to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\rho}}{1-\rho} - \frac{1}{\eta} (L_t)^\eta + \chi \ln \frac{M_t}{P_t} \right)$$

where E_0 denotes the expectation operator conditional on time 0 information, β is the discount factor, subject to:

$$C_t + \frac{B_t}{P_t} = \frac{R_{t-1}}{P_t} B_{t-1} + \frac{W_t}{P_t} L_t + F_t - \frac{M_t - M_{t-1}}{P_t} + T_t$$

where the wage is $w_t \equiv W_t/P_t$, and bonds pay the predetermined nominal interest rate R_{t-1}; F_t denotes lump-sum dividends received from ownership of intermediate goods firms (whose problems are described below); the last three terms indicate net transfers from the central bank that are financed by printing money. Here, P_t is an appropriately aggregated price index for output and consumption (that we derive below) that converts nominal into real quantities.

Let $\Pi_t \equiv P_t/P_{t-1}$ denote the gross rate of inflation. Solving this problem yields first order conditions for consumption/saving, labour supply and money demand:

$$\frac{1}{C_t^\rho} = \beta E_t \left(\frac{R_t}{\Pi_{t+1} C_{t+1}^\rho} \right) \quad \text{(Euler)}$$

$$\frac{w_t}{C_t^\rho} = L_t^{\eta-1} \quad \text{(LS)}$$

$$\frac{1}{C_t^\rho} = E_t \left(\beta \frac{1}{\Pi_{t+1}} \frac{1}{C_{t+1}^\rho} \right) + \frac{\chi}{m_t} \quad \text{(MD)}$$

where m_t are real balances (M_t/P_t).

4.2 Final-goods firm

There is a final-goods sector where a representative firm produces the final good Y_t using intermediate goods Y_{jt}. Total final goods are given by the CES aggregator of the different quantities of intermediate goods produced:

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where $\varepsilon > 1$.

The firm buys inputs Y_{jt} and produces the final good in order to maximize profits. Alternatively, the firm tries to minimize expenditure given the production constraint. We can write the Lagrangean of the firm as:

$$\mathcal{L} = \int_0^1 P_{jt} Y_{jt} dj + P_t \left(Y_t - \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)$$

Optimal choice of Y_{jt} solves $\frac{\partial \mathcal{L}}{\partial Y_{jt}} = 0$, that is

$$P_{jt} = P_t \frac{\partial Y_t}{\partial Y_{jt}}$$

$$\text{use } \frac{\partial Y_t}{\partial Y_{jt}} = \frac{\varepsilon}{\varepsilon - 1} \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{1}{\varepsilon-1}} \frac{\varepsilon - 1}{\varepsilon} Y_{jt}^{-\frac{1}{\varepsilon}} = \left(\frac{Y_t}{Y_{jt}} \right)^{1/\varepsilon}$$

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t.$$

This expression can be solved for the unknown value of the multiplier. Use the definition of Y_t and use the solution for Y_{jt} to write:

$$Y_t = \left(\int_0^1 \left(\left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} Y_t \right)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} = Y_t \left(\int_0^1 \left(\frac{P_{jt}}{P_t} \right)^{1-\varepsilon} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

Because the production function has constant returns to scale, Y_t drops from both sides of the expression, and we can then solve for P_t as:

$$P_t = \left(\int_0^1 P_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}.$$

P_t represents the minimum cost of producing one unit of the final-goods bundle Y_t (and, because of the constant-returns-to-scale assumption, is independent of the quantity produced). For this reason we interpret P_t (the Lagrange multiplier) as the aggregate price index. The intuition: P_t tells us at the optimum what is the value for the firm (in dollar terms) of relaxing the constraint on production.

4.3 Intermediate goods

The intermediate goods sector is made by a continuum of monopolistically competitive firms owned by consumers, indexed by $j \in (0, 1)$.

4.3.1 The constraints

Each firm, as we saw above, faces a downward sloping demand for its product. It uses labor to produce output according to the following technology:

$$Y_{jt} = A_t L_{jt}$$

Each producer chooses her own sale price P_{jt} taking as given the demand curve. He can reset his price only when given the chance of doing so, which occurs with probability $1 - \theta$ in every period.

To summarize, each intermediate firm faces the following constraints:

1. The production constraint: $Y_{jt} = A_t L_{jt}$
2. The demand curve $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} Y_t$
3. The fact that prices can be adjusted only with probability $1 - \theta$. We follow Calvo (1983) and assume that every period only a random fraction of firms is optimally setting prices. Each period, this fraction is independent from the previous period.

We can break this problem down into two sub-problems. As a cost minimizer and as a price setter.

4.3.2 Producer as a cost minimizer

Consider the cost minimization problem first, conditional on the output Y_{jt} produced. This problem involves minimizing $W_t L_{jt}$ subject to producing $Y_{jt} = A_t L_{jt}$ (there is no sub-index on W_t since all sectors where labor is employed must pay same wage in equilibrium). In real terms this problem can be written as

$$\min_{L_{jt}} \frac{W_t}{P_t} L_{jt} + Z_t \left(Y_{jt} - A_t L_{jt}\right) \qquad [Z_t]$$

where Z_t is multiplier associated with the constraint.[1] The first order condition implies $\frac{W_t}{P_t} = Z_t A_t$, or, using $Y_{jt}/L_{jt} = A_t$:

$$\frac{Y_{jt}}{L_{jt}} = \frac{1}{Z_t} \frac{W_t}{P_t} \tag{LD}$$

notice that this first order condition suggests than we can write the cost function in real terms as

$$COST_{jt} \equiv \frac{W_t}{P_t} L_{jt} = Z_t Y_{jt}$$

For this reason, we can think of Z_t as real marginal cost; we can likewise define its inverse $X_t = 1/Z_t$ as the markup. Given cost minimization, the firm takes Z_t as given, when choosing the output price, to which we turn now.

4.3.3 Producer as a price setter

4.3.3.1 Digression: the problem with flexible prices (and the macro equilibrium with flexible prices)

In order to warm yourself up, consider the problem of a monopolistic producer who has the chance to change her prices every period. The cost minimization problem is the same as before. On the revenue side, define $r_{jt} \equiv P_{jt}/P_t$ the relative price that the producer charges. The maximization problem will be:

$$\max_{r_{jt}} r_{jt} Y_{jt} - Z_t Y_{jt}$$

[1] There is no subscript on Z_t either because W_t is the only cost measure and is common to all firms.

where $Y_{jt} = r_{jt}^{-\varepsilon} Y_t$. Optimal choice of r_{jt} will imply $\partial \left(r_{jt} Y_{jt} - Z_t Y_{jt} \right) / \partial r_{jt} = 0$, or:

$$Y_{jt}^* + r_{jt}^* \frac{\partial Y_{jt}^*}{\partial r_{jt}^*} - Z_t \frac{\partial Y_{jt}^*}{\partial r_{jt}^*} = 0$$

$$Y_{jt}^* \left(1 + \frac{r_{jt}^*}{Y_{jt}^*} \frac{\partial Y_{jt}^*}{\partial r_{jt}^*} - \frac{Z_t}{r_{jt}^*} \frac{r_{jt}^*}{Y_{jt}^*} \frac{\partial Y_{jt}^*}{\partial r_{jt}^*} \right) = 0$$

$$r_{jt}^* = \frac{P_{jt}^*}{P_t} = \frac{\varepsilon}{\varepsilon - 1} Z_t$$

that is, the relative price would be a constant markup $X \equiv \frac{\varepsilon}{\varepsilon-1} > 1$ over the real marginal cost.

Remark 6 *This condition is crucial because with monopolistic competition but flexible prices we would derive a neutrality result similar to that of Sidrauski model. In the symmetric equilibrium, $P_{jt} = P_t$, hence $Z_t = \frac{\varepsilon-1}{\varepsilon} = \frac{1}{X} < 1$ for all t. Combining labor supply (equation LS) and labor demand (equation LD) and imposing market clearing $Y_t = C_t$ would give*

$$w_t = \underset{labor\ supply}{Y_t^{\rho} L_t^{\eta-1}} = \underset{labor\ demand}{A_t / X}$$

using $L_t = Y_t / A_t$ in the symmetric equilibrium we will have:

$$Y_t = \left(\frac{A_t^{\eta}}{X} \right)^{\frac{1}{\rho+\eta-1}}$$

so that output in the model is a function only of technology (and money and nominal rigidities do not have any effect on the dynamics of the model around the steady state, like in the Sidrauski model with utility separable in consumption and real balances). In static terms, output would be suboptimally low.

4.3.3.2 The problem with sticky prices

To begin with, at any point in time if some intermediate good producers can change prices and others cannot, the average price level will be a CES aggregate of all prices in the economy, and will be

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta) \left(P_t^* \right)^{1-\varepsilon} \tag{*}$$

where P_{t-1} is previous price level, and P_t^* is avg price level chosen by those who have the chance to change prices (Call P^* the reset price). It is at these guys that we look now.

Consider the intermediate goods producer who has a chance $1 - \theta$ to reset prices at time t. The demand curve is:

$$Y_{jt+k}^* = \left(P_{jt}^* / P_{t+k} \right)^{-\varepsilon} Y_{t+k}$$

for any period $k \geq 0$ for which he will keep that price.

His maximization problem is:

$$\max_{P_{jt}^*} \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left(\Lambda_{t,k} \left(\frac{P_{jt}^*}{P_{t+k}} - Z_{t+k} \right) Y_{jt+k}^* \right) \tag{\#}$$

$$\Lambda_{t,k} = (C_t / C_{t+k})^{\rho}$$

where Z_t is the real marginal cost. θ represents the probability that the price P_j^* chosen at t will still apply in later periods. This expression is the "expected discounted sum of all profits that the price setter will make conditional on his choice of P_{jt}^* and weighted by how likely P_{jt}^* is to stay in place in future periods".

At time t, the price setter chooses P_{jt}^* to maximize profit. Differentiate the profit function with respect to P_{jt}^* to obtain

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \left(\frac{Y_{jt+k}^*}{P_{t+k}} + \frac{P_{jt}^*}{P_{t+k}} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - Z_{t+k} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0$$

$$\text{take } \frac{Y_{jt+k}^*}{P_{t+k}} \text{ out, isolate elasticity of } Y_{jt+k}^* \text{ wrt } P_{jt}^*$$

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 + \frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} - \frac{Z_{t+k} P_{t+k}}{P_{jt}^*} \frac{P_{jt}^*}{Y_{jt+k}^*} \frac{\partial Y_{jt+k}^*}{\partial P_{jt}^*} \right) \right] = 0$$

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(1 - \varepsilon + \frac{Z_{t+k} P_{t+k}}{P_{jt}^*} \varepsilon \right) \right] = 0$$

$$\text{multiply everything by } P_{jt}^*$$

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} \frac{Y_{jt+k}^*}{P_{t+k}} \left(P_{jt}^* - \frac{\varepsilon}{\varepsilon - 1} Z_{t+k} P_{t+k} \right) \right] = 0$$

In equilibrium, all the firms that reset the price choose the same price (and face the same demand), hence

$$P_{jt}^* = P_t^*$$

These two expressions enter the equilibrium (using $X = \frac{\varepsilon}{\varepsilon-1}$ = steady state markup). One is $*$ that we derived above, the other is:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \left(\frac{P_t^*}{P_{t+k}} - X Z_{t+k} \right) \right] = 0 \tag{**}$$

Use $Z_{t+k}^n \equiv Z_{t+k} P_{t+k}$ to rearrange the expression above to obtain:

$$\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* \frac{P_t^*}{P_{t+k}} \right] = X \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* Z_{t+k} \right]$$

$$P_t^* = X \frac{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* Z_{t+k}^n P_{t+k}^{-1} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left[\Lambda_{t,k} Y_{t+k}^* P_{t+k}^{-1} \right]} = X \sum\nolimits_{k=0}^{\infty} \phi_{t,k} Z_{t+k}^n$$

where $\phi_{t,k} = \frac{(\theta\beta)^k E_t\left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k}\right]}{\sum_{k=0}^{\infty}(\theta\beta)^k E_t\left[\Lambda_{t,k} Y^*_{t+k} P^{-1}_{t+k}\right]}$. This expression says that the optimal price is a weighted average of current and expected future nominal marginal costs. Weigths depend on expected demand in the future, and how quickly firm discounts profits.

Therefore you can notice the following:

- Under purely flexible prices, $\theta = 0$: the markup is a constant. $P^*_t = XZ^n_t$ and optimal prices are a multiple X of the current marginal cost.
- When $\theta > 0$, the optimal price depends on future expected values of aggregate variables as well as future nominal marginal costs Z^n_{t+k}. Put differently, one can see that all the fluctuations in the markup are due to firms being unable to adjust prices.

4.4 The equilibrium

4.4.1 Closing the model

We need to combine everything, impose market clearing, and linearize around the steady state.

Total output in economy is:

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\int_0^1 (A_t L_{jt})^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

It is not possible to simplify this expression since input usages across firms differs. However the linear aggregator $Y'_t = \int_0^1 Y_{jt} dj$ is approximately equal to Y_t within a local region of the steady state. Hence for local analysis we can simply use

$$Y_t = A_t L_t$$

Goods market clearing is simply $Y_t = C_t$. Trivially, bond market clearing implies $B_t = 0$.

4.4.2 Monetary policy

We assume that the central bank policy sets the dynamics of money supply in a way to achieve a target level of the interest rate. This way, the money demand equation becomes redundant since it only serves to determine the behavior of endogenous money.

To better gain insight into this, consider the money demand equation of the model. Because utility is separable between consumption and money balances, this is the only instance where money appears. Given that M_t is a function of R_t, this implies that the central bank can always choose the supply of money to implement the interest rate it desires. From

$$\frac{M_t}{P_t} = \chi C_t^{\rho} \frac{R_t}{R_t - 1}$$

we can see that, for given values C_t and P_t, this equation implies a one-to-one mapping between M_t and R_t. We find it convenient to work with so called interest rate rules, closing the model specifying an exogenous process for R_t rather than M_t.

4.4.3 The equilibrium in levels

Let us look at the equation summarising the model:

$$\frac{1}{Y_t^{\rho}} = \beta E_t\left(\frac{R_t P_t}{P_{t+1} Y_{t+1}^{\rho}}\right) \tag{1}$$

$$Y_t^{\eta+\rho-1} = A_t^{\eta} Z_t \tag{2}$$

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta)\, E_t\left(X \sum\nolimits_{k=0}^{\infty} \phi_{t,k} Z_{t+k} P_{t+k}\right)^{1-\varepsilon} \tag{3}$$

$$R_t = R_{t-1}^{\phi_r}\left(\overline{rr}\left(\frac{P_t}{P_{t-1}}\right)^{1+\phi_\pi}\left(\frac{Z_t}{Z}\right)^{\phi_z}\right)^{1-\phi_r} \varepsilon_{r,t} \tag{4}$$

1. Eq.1 is the aggregate demand equation: it combines goods market clearing with the Euler equation for bonds

2. Eq. 2 is the equilibrium in the labor market. Take labour demand (LD) and labor supply (LS) and impose market clearing. Then equate (LD) and (LS) so as to eliminate of the real wage w from that expression. Whenever you have L, remember to replace it with Y/A.

3. Eq. 3 is the equation that describes how the aggregate price level is a weigthed average of (1) previous price level P_{t-1} and (2) reset prices P_t^*, which are in turn a function of future expected marginal costs.

4. The last expression is the monetary policy rule. $\overline{rr}$ is the steady state real interest rate. We assume that the central bank chooses money supply so as to set the nominal interest rate to be a function of previous interest rate, current inflation and current real marginal costs (as a fraction of their steady state value). ϕ_r is a smoothing term. This is a Taylor rule, from John Taylor of Stanford University, who was the first to notice in a 1993 seminal paper that central banks set the interest rate as a function of inflation and output gap (output gap=deviation of output from its natural rate). The last term $\varepsilon_{r,t}$ represents a monetary policy shock.

   Notice that under this forumulation, the target (steady state) inflation rate is 1. This happens because in steady state from equation 1 the real interest rate is $\overline{rr} = 1/\beta$. Hence $R = \overline{rr}\Pi$ in steady state (from 1) where Π is the gross inflation rate. Hence 4 in steady state holds only if $\Pi = 1$, which implies a constant price level.

4.5 The log-linear equilibrium

4.5.1 Linearizing the Phillips curve

Use $Y_{t+k}^{*}=\left(P_t^{*}/P_{t+k}\right)^{-\varepsilon}Y_{t+k}$ and cancel out P_t^{*} in numerator and denominator to obtain:

$$P_t^{*}=X\frac{\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left[\Lambda_{t,k}P_{t+k}^{\varepsilon-1}Y_{t+k}Z_{t+k}^{n}\right]}{\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left[\Lambda_{t,k}P_{t+k}^{\varepsilon-1}Y_{t+k}\right]}$$

To gain insight into this expression, it is convenient to loglinearize it. Intuitively, we can see that numerator and denominator only differ up to a multiple given by Z_{t+k}^{n}, which in turn multiplies $\left(\theta\beta\right)^{k}\left(1-\theta\beta\right)$. Hence we can expect that in log-linearising Y, $P^{\varepsilon-1}$ and Λ will cancel out and disappear. Use the steady state result that $Z^{n}X=P$. Rearranging and dividing by P_t:

$$\frac{P_t^{*}}{P_t}\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left[\Lambda_{t,k}Y_{t+k}P_{t+k}^{\varepsilon-1}\right]=\frac{1}{P_t}X\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left[\Lambda_{t,k}Y_{t+k}P_{t+k}^{\varepsilon-1}Z_{t+k}^{n}\right]$$

LHS first

$$\left(\widehat{P}_t^{*}-\widehat{P}_t\right)\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}\right]+\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}\right]E_t\left(\widehat{\Lambda}_{t,k}+\widehat{Y}_{t+k}+\left(\varepsilon-1\right)\widehat{P}_{t+k}\right)$$

RHS next

$$-\widehat{P}_t\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}\right]+\frac{X}{P}\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}Z^{n}\right]E_t\left(\widehat{\Lambda}_{t,k}+\widehat{Z^{n}}_{t+k}+\widehat{Y}_{t+k}+\left(\varepsilon-1\right)\widehat{P}_{t+k}\right)=$$
$$-\widehat{P}_t\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}\right]+\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}\left[\Lambda YP^{\varepsilon-1}\right]E_t\left(\widehat{\Lambda}_{t,k}+\widehat{Z^{n}}_{t+k}+\widehat{Y}_{t+k}+\left(\varepsilon-1\right)\widehat{P}_{t+k}\right)$$

where we have used $\frac{XZ^{n}}{P}=1$. Hence, using $\widehat{Z^{n}}_{t+k}=\widehat{P}_{t+k}+\widehat{Z}_{t+k}$

$$\widehat{P}_t^{*}\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}=\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left(\widehat{P}_{t+k}+\widehat{Z}_{t+k}\right)$$
$$\widehat{P}_t^{*}=\left(1-\theta\beta\right)\sum_{k=0}^{\infty}\left(\theta\beta\right)^{k}E_t\left(\widehat{P}_{t+k}+\widehat{Z}_{t+k}\right)\qquad(@)$$

Equation (@) simply states in log-linear terms that the optimal price has to be equal to a weighted average of current and future marginal costs, weighted by the probability that this price will hold in later periods too. So you assign weight 1 to today, weight $\theta\beta$ to tomorrow, $\theta^{2}\beta^{2}$ to the day after tomorrow, and so on. Notice the complete forwardlookingness of this expression, and the fact that these weights need to be normalized (the sum of all of them is $\frac{1}{1-\theta\beta}$, whose inverse premultiplies the summation - weights sum up to one -)

Using $\widehat{P}_t - \theta \widehat{P}_{t-1} = (1-\theta)\widehat{P}_t^*$, the expression above can be rearranged as follows:

$$\begin{aligned}
\widehat{P}_t^* &= (1-\theta\beta) E_t \left(\left(\widehat{P}_t + \widehat{Z}_t \right) + \theta\beta \left(\widehat{P}_{t+1} + \widehat{Z}_{t+1} \right) + \theta^2 \beta^2 (...) + ... \right) \\
\widehat{P}_t^* &= (1-\theta\beta) \left(\widehat{P}_t + \widehat{Z}_t \right) + \theta\beta E_t \widehat{P}_{t+1}^* \\
(1-\theta)\widehat{P}_t^* &= (1-\theta)(1-\theta\beta) \left(\widehat{P}_t + \widehat{Z}_t \right) + (1-\theta)\theta\beta \widehat{P}_{t+1}^* \\
\widehat{P}_t - \theta \widehat{P}_{t-1} &= (1-\theta)(1-\theta\beta) \left(\widehat{P}_t + \widehat{Z}_t \right) + \theta\beta \left(E_t \widehat{P}_{t+1} - \theta \widehat{P}_t \right) \\
\widehat{P}_t - \widehat{P}_{t-1} &= -(1-\theta)\widehat{P}_{t-1} + (1-\theta)(1-\theta\beta) \left(\widehat{P}_t + \widehat{Z}_t \right) + \theta\beta \left(E_t \widehat{P}_{t+1} - \theta \widehat{P}_t \right) \\
\widehat{P}_t - \widehat{P}_{t-1} &= (1-\theta) \left(\widehat{P}_t - \widehat{P}_{t-1} \right) - \theta\beta (1-\theta) \widehat{P}_t \\
&\quad + (1-\theta)(1-\theta\beta)\widehat{Z}_t + \theta\beta \left(E_t \widehat{P}_{t+1} - \widehat{P}_t \right) + \theta\beta (1-\theta) \widehat{P}_t \\
\widehat{\pi}_t &= (1-\theta)\widehat{\pi}_t + \theta\beta E_t \widehat{\pi}_{t+1} + (1-\theta)(1-\theta\beta)\widehat{Z}_t
\end{aligned}$$

$$\widehat{\pi}_t = \beta E_t \widehat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta} \widehat{Z}_t$$

This equation is nothing else but an "expectations augmented Phillips curve", which states that inflation rises when the real marginal costs rise. It takes a while to derive, but again it is nothing else but an aggregate supply curve for the whole economy. Notice that

$$\begin{aligned}
\partial \left(\frac{\partial \widehat{\pi}_t}{\partial \widehat{Z}_t} \right) / \partial \beta &= -(1-\theta) < 0 \\
\partial \left(\frac{\partial \widehat{\pi}_t}{\partial \widehat{Z}_t} \right) / \partial \theta &< 0
\end{aligned}$$

- the higher β, the higher the weight to future $\widehat{Z}_t$'s, and the lower today's elasticity to current marginal cost

- the higher θ, the higher the chance that I will be stuck with my price for a long period, and the higher the elasticity of $\widehat{P}_t^*$ to $\widehat{Z}_t$. However, few prices will be changed in the aggregate, therefore aggregate inflation will not be sensitive to the marginal cost.

4.5.2 The remaining equations

Equations (1), (2) and (4) are already linear in logs.

We assume that a_t and e_t follows $AR(1)$ processes.

4.5.3 The complete log-linear model

From now on, we denote with lowercase variables deviations of variables from their respective steady states. I now work in terms of the markup rather than the real marginal cost. When we log-linearize the 4 expressions above, what we obtain the following system:

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \frac{1}{\rho}\left(r_t - E_t \pi_{t+1}\right) \\
y_t &= \frac{1}{\eta+\rho-1} z_t + \frac{\eta}{\eta+\rho-1} a_t \\
\pi_t &= \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta} z_t + u_t \\
r_t &= \phi_r r_{t-1} + (1-\phi_r)\left((1+\phi_\pi)\pi_t + \phi_z z_t\right) + e_t
\end{aligned}
$$

Remark 7 *The linearized equations are in the Matlab file, where we use $x_t = -z_t$ (the real marginal cost Z_t is the inverse of the markup X_t in levels, and $x_t = -z_t$ in logs - see equation LD). dnwk.m and dnwk_go.m simulate this model. It has 7 equations rather than four, but you can forget about three of them. One equation is capital demand if you extend this model to have capital as well (you can set the weight on K to be arbitrarily small in the production function, so that equation does not count); the other says that $Y = C$; another defines λ as the marginal utility of consumption.*

It is sometimes convenient to call y_t^n a new variable that defines the equilibrium level of output (the natural output) that would prevail under completely flexible prices ($\theta = 0$). This way z_t can in fact be eliminated. In fact, If firms were able to adjust prices optimally each period, $z_t = 0$ (since $\frac{(1-\theta)(1-\beta\theta)}{\theta} \Rightarrow \infty$) and we would be able to define the flexible price equilibrium values for real interest rate rr (nominal minus expected inflation) and output, which we call their "natural" rates:

$$
\begin{aligned}
y_t^n &= \frac{\eta}{\eta+\rho-1} a_t \\
rr_t^n &= -\frac{\rho\eta}{\eta+\rho-1}\left(a_t - E_t a_{t+1}\right)
\end{aligned}
$$

We can then derive an expression for x_t as a function of the gap between flexible price and sticky price equilibrium, that is:

$$x_t = (\eta+\rho-1)\left(y_t^n - y_t\right)$$

Hence x is positive whenever y is below y^n, output is below its natural level. It is for this reason that we sometimes refer to x_t as the "output gap", since x is proportional to the shortfall of output from its natural level. y_t^n is an exogenous variable, since it depends only on technology. With this convention, the dynamic new-keynesian model can be rewritten as:

$$
\begin{aligned}
y_t &= E_t y_{t+1} - \sigma\left(r_t - E_t \pi_{t+1}\right) && \text{(a)} \\
\pi_t &= \lambda\left(y_t - y_t^n\right) + \beta E_t \pi_{t+1} + u_t && \text{(b)} \\
r_t &= \phi_r r_{t-1} + (1-\phi_r)\left((1+\phi_\pi)\pi_t + \phi_x\left(y_t - y_t^n\right)\right) + e_t && \text{(c)}
\end{aligned}
$$

where $\lambda = (\eta+\rho-1)\frac{(1-\theta)(1-\beta\theta)}{\theta}$, $\sigma = 1/\rho$. Some authors have also postulated cost push shocks u_t, that push inflation up. Some authors refer to the system made by (1), (2), (3) as the "benchmark" dynamic-new keynesian model.

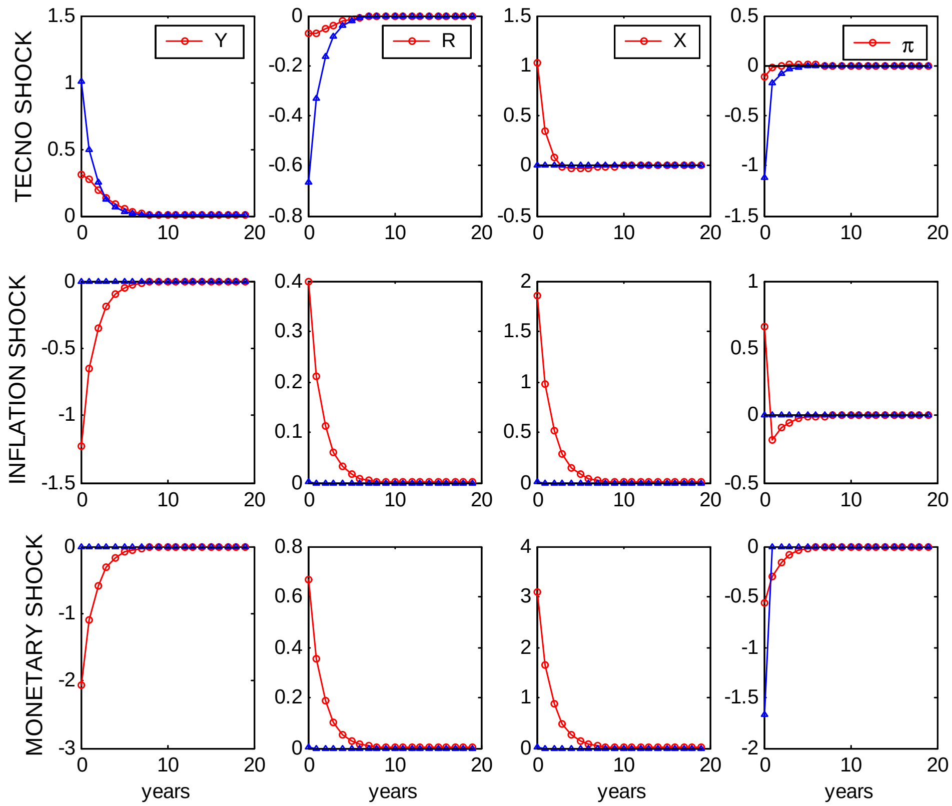

Figure 4.1: Simulations from dnwk.m, flexible (triangles) versus sticky price (circles) model

4.6 The dynamic effects of technology and monetary shocks

The Matlab programs in my webpage will allow you to analyze the dynamics of this model by means of impulse response functions. The plot compares the responses that obtain under flexible prices versus sticky prices. It was generated with *dnwk.m*[2]

Technology shocks.

1. Following a rise in technology, marginal costs fall. Since not all prices are free to fall immediately, markups will rise. A fraction $1-\theta$ of "flex price" firms will lower their prices and hire more factors of production. A fraction θ of "fixed price" firms will be unable to lower their prices and to increase their sales and therefore will hire less factors of production. Production rises less than with flexible prices

[2] $\rho_a = 0.5$; $\rho_e = 0.0$; $\rho_u = 0.0$; $\beta = .99$; $\eta = 1.5$; $\theta = .75$ and $.0001$; $X = 1.1$; $\rho = 1$; $\phi_r = 0.8$; $\phi_\pi = 2$; $\phi_x = 0.0$;

2. The theory of endogenous markup variations provides the crucial link that allows the concerns of RBC models and conventional monetary models to be synthetised. In addition, the markup directly measures the extent to which a condition for efficient resource allocation fails to hold.

Monetary shocks. A monetary contraction leads to drop in output, rise in the nominal interest rate, fall in inflation, and a rise in the output gap. These predictions, which are qualitatively in line with the VAR evidence, are hard to obtain in the flexible price model.

Inflation ("cost-push") shocks Inflation shocks are important in this setup because they generate a trade-off between output gap versus inflation stabilization.

Chapter 5

Discretionary policy and time inconsistency

Up to now we have assumed that monetary policy authorities can commit to a rule, and that this commitment is perceived credible by the public. In practice, credibility is a key problem in thinking about monetary issues. If monetary expansions can raise output, will central banks always be tempted to raise output above its natural level?

Here we present a very basic introduction to the credibility problems that arise in the making of monetary policy.

5.1 The model and the central bank problem

We are going to use a slightly modified version of the Dynamic new-keynesian model. In absence of technology shocks, so that the natural rate of output is constant ($y_t^n = 0$), this model can be summarized in log-linear terms by the following two equations:

$$y_t = E_t y_{t+1} - \sigma \left(r_t - E_t \pi_{t+1} \right) \tag{1}$$

$$\pi_t = \xi y_t + \beta E_t \pi_{t+1} + u_t' \tag{2}$$

plus a monetary policy rule. We assume that u_t' is a zero-mean cost-push shock that creates a trade-off between stabilization of output and stabilization of inflation.

By choosing an appropriate nominal interest rate r_t, monetary policy can set aggregate demand to the desired level. Therefore, we can think of (2) as the only constraint on the central bank's menu of choices available. Let us simplify (2) further by assuming $\beta = 1$.[1]

[1] Under the assumption that $\beta = 1$, the Phillips curve becomes "vertical" in the long-run. From (2), long-run output is

$$y_\infty = \pi_\infty \frac{1-\beta}{\xi}$$

In principle, positive steady state inflation could generate long-run output permanently above its zero steady-state (recall, these are variables in deviations from their steady state). Assuming $\beta = 1$ rules this possibility out.

Equation (2) then can be written as:

$$y_t = a\left(\pi_t - \pi_t^e\right) - u_t \tag{2'}$$

where we denote π_t^e expected next period inflation at time t.[2] By setting the appropriate level of money supply or by choosing the interest rate, the central bank can indirectly choose π_t as an instrument of policy. In this chapter, we make and keep this assumption throughout. However, whenever the central bank chooses its own policy, the public forms its expectations in a consistent way about what policy is going to be in the future.

Though the problem looks easy in principle, Kydland and Prescott showed how central banks who are concerned with both curbing inflation and stimulating the economy face a time inconsistency problem that can result in excessively high inflation without any effect on the natural rate of output, even if the policymakers share the same incentives as private agents. The literature that Kydland and Prescott started has been very influential in determining the goals of central bankers, the extent of their independence from the government, and the design in general of monetary policy.

Throughout this chapter, we assume that the central bank chooses a given policy so as to minimize the expected value of a loss function that depends on output and inflation fluctuations:

$$\min_{\pi_t} L_t = \frac{\lambda}{2}\left(y_t - k\right)^2 + \frac{1}{2}\pi_t^2$$

where

$$k > 0$$

reflects the monopolistic competition distortions: since output is suboptimally low because of monopolistic competition, the central bank target level of y_t might not be its steady state level (in steady state $y_t = 0$), but a higher level.

5.2 Solution under discretion

Under discretion, we assume that the central bank chooses its instrument of policy and the private sector adjusts its expectations taking into account that the central bank is free to reoptimize every period. Because the central bank is unable to affect the expectations of the private sector, the problem of the central bank is:

$$\min_{\pi_t} L^D = \frac{\lambda}{2}\left(a\left(\pi_t - \pi_t^e\right) - u_t - k\right)^2 + \frac{1}{2}\pi_t^2 .$$

The solution to this problem yields:

$$a\lambda\left(a\left(\pi_t - \pi_t^e\right) - u_t - k\right) + \pi_t = 0$$

[2] Note that $a = 1/\xi$ and u is a scaled version of u', that is $u = u'/\xi$.

or:

$$\pi_t = \frac{a^2\lambda\pi_t^e + a\lambda u_t + a\lambda k}{1+a^2\lambda}$$

We solve the above equation using the method of undetermined coefficients. The solution to this equation must have the form:

$$\pi_t = \gamma_1 u_t + \gamma_2 k$$

for some undetermined γ. If the above is the solution, then

$$\pi_t^e = \gamma_1 \rho_u u_t + \gamma_2 k = \gamma_2 k$$

where we assume that the cost-push shock is autocorrelated, that is $\rho_u = 0$, so that $E_t(u_{t+1}) = \rho_u E(u_t) = 0$. And therefore:

$$\begin{aligned}
\pi_t &= \frac{a^2\lambda(\gamma_2 k) + a\lambda u_t + a\lambda k}{1+a^2\lambda} \\
&\text{match coefficients} \\
\gamma_1 &= \frac{a\lambda}{1+a^2\lambda} \\
\gamma_2 &= \frac{a\lambda}{1+a^2\lambda}(1+\gamma_2 a) = a\lambda \\
&\text{hence solutions are} \\
\pi_t^D &= \frac{a\lambda}{1+a^2\lambda}u_t + a\lambda k \\
y_t^D &= a(\pi_t - \pi_t^e) - u_t = a\left(\frac{a\lambda}{1+a^2\lambda}u_t + a\lambda k - a\lambda k\right) - u_t = -\frac{1}{1+a^2\lambda}u_t
\end{aligned}$$

The solutions for π and y have the property that in equilibrium the central bank has no reason to change its policy in an unexpected way, even if it has the discretion to do so. For this reason, the discretionary equilibrium is also called time-consistent.

From the equations above, we note that we have two crucial results:

- $E_t\pi_t^D > 0$: inflation bias: even in absence of supply shocks, inflation is permanently above zero;
- $E_t y_t^D = 0$: no systematic offsetting to underproduction

Under this outcome, the *expected* loss for the central bank is:

$$\begin{aligned}
E(2L^D) &= E\left(\lambda\left(y_t^D - k\right)^2 + \left(\pi_t^D\right)^2\right) = \\
&= E\left(\lambda\left(-\frac{1}{1+a^2\lambda}u_t - k\right)^2 + \left(\frac{a\lambda}{1+a^2\lambda}u_t + a\lambda k\right)^2\right) = \\
&= \lambda\left(\frac{1}{(1+a^2\lambda)^2}\sigma_u^2 + k^2\right) + \left(\frac{a^2\lambda^2}{(1+a^2\lambda)^2}\sigma_u^2 + a^2\lambda^2 k^2\right) = \\
&= \frac{\lambda}{1+a^2\lambda}\sigma_u^2 + \lambda\left(1+a^2\lambda\right)k^2
\end{aligned}$$

Notice that we have no specified the policy rule of the central bank that supports this equilibrium. However, we can solve equation (1) above for the value of r_t that supports this equilibrium.

5.3 Solution under commitment

Now suppose the central bank makes a binding commitment to what inflation will be before inflation is realized. Assume, in particular, that the central bank commits, prior to the formation of private expectations, to a policy rule for the target variable of the form:

$$\pi_t = b_0 + b_1 u_t$$

where b_0 and b_1 are policy coefficients chosen by the central bank. Notice how this "policy rule" includes the discretionary policy as a special case.

In this case, expected inflation is a constant $\pi_t^e = b_0$ which depends on the specific policy adopted. Substituting the rule in the loss function gives:

$$L_t^C = \frac{\lambda}{2}\left(ab_1 u_t - u_t - k\right)^2 + \frac{1}{2}\left(b_0 + b_1 u_t\right)^2$$

Under commitment, the central bank commits itself to particular values of b_0 and b_1 prior to the public forming its expectations of inflation and prior to observing particular realizations of the shock u_t. Thus, b_0 and b_1 are chosen to **minimize the unconditional expectation** of the loss function, that is:

$$E\left(L_t^C\right) = \frac{\lambda}{2}\left(\left(ab_1 - 1\right)^2 \sigma_u^2 + k^2\right) + \frac{1}{2}\left(b_0^2 + b_1^2 \sigma_u^2\right)$$

Solving the minimization problem (choosing b_0 and b_1) yields:

$$\begin{aligned} b_0 &= 0 \\ \lambda a \left(ab_1 - 1\right)\sigma_u^2 + b_1 \sigma_u^2 &= 0 \end{aligned}$$

which yields:

$$\begin{aligned} b_1 &= \frac{\lambda a}{1 + \lambda a^2} \\ \pi_t^C &= \frac{\lambda a}{1 + \lambda a^2} u_t \\ y_t^C &= a\left(\pi_t - \pi_t^e\right) - u_t = a\left(\frac{\lambda a}{1 + \lambda a^2} u_t\right) - u_t = -\frac{1}{1 + \lambda a^2} u_t \end{aligned}$$

The unconditional expectation of the loss under commitment is therefore:

$$E\left(2L^C\right) = E\left(\lambda\left(-\frac{1}{1 + \lambda a^2} u_t - k\right)^2 + \left(\frac{\lambda a}{1 + \lambda a^2} u_t\right)^2\right) = \frac{\lambda}{1 + \lambda a^2}\sigma_u^2 + \lambda k^2$$

5.4 Comparison of discretion versus commitment

Comparison of the loss under discretion with the loss under commitment shows that

$$E\left(L^D - L^C\right) = \frac{1}{2}a^2\lambda^2k^2$$

we can interpret the amount $\frac{1}{2}a^2\lambda^2k^2$ as the "cost" of discretion, since this is the loss attributable to a nonzero average rate of inflation. The so-called "inflation bias" occurs because the central bank is unable to precommit to a zero inflation rate. Under discretion, if the public believes that $\pi^e = 0$, the central bank has an incentive to inflate, so the central bank zero inflation policy will not be believed in the first place.

Remark 8 *Suppose instead the central bank is forced to set $\pi = b_0$ only. Twice the loss function will be simply*

$$L^{C0} = \lambda\left(a\left(b_0 - b_0\right) - u - k\right)^2 + b_0^2$$

Here, $E\left(L^{C0}\right)$ is minimized when $b_0 = 0$, so that the loss is

$$E\left(L^{C0}\right) = \lambda\left(\sigma_u^2 + k^2\right)$$

comparing this to the loss under discretion:

$$E\left(L^D\right) = \frac{\lambda}{1+a^2\lambda}\sigma_u^2 + \lambda\left(1+a^2\lambda\right)k^2$$

one can see that commitment will dominate when $E\left(L^D - L^{C0}\right) > 0$, which occurs when $k^2 > \frac{1}{1+a^2\lambda}\sigma_u^2$, so that a is large (intuitively, a flat supply curve means that there is a big incentive to inflate, so there is a big incentive to commit to a zero-inflation policy), k is large, σ_u^2 is small (commitment is preferrable when supply shocks are not big).

5.5 A graphical interpretation

Assuming $a = 1$, $\beta = 1$, $\lambda = 1$, when $u_t = 0$, we can draw the supply curve

$$\pi_t = \pi^e + y_t$$

and consider the indifference curves of the central bank:

$$\overline{L} = \left(y_t - k\right)^2 + \pi_t^2$$

which are circles in the (y, π) space with origin $(1, 0)$.

Here I normalize $k = 1$. Using our parameterization above, REE indicates the discretionary equilibrium. COM indicates the commitment equilibrium.

The central bank is unable to precommit to a zero inflation rate. To see why, consider diagram above and see that in COM there is a clear incentive to inflate.

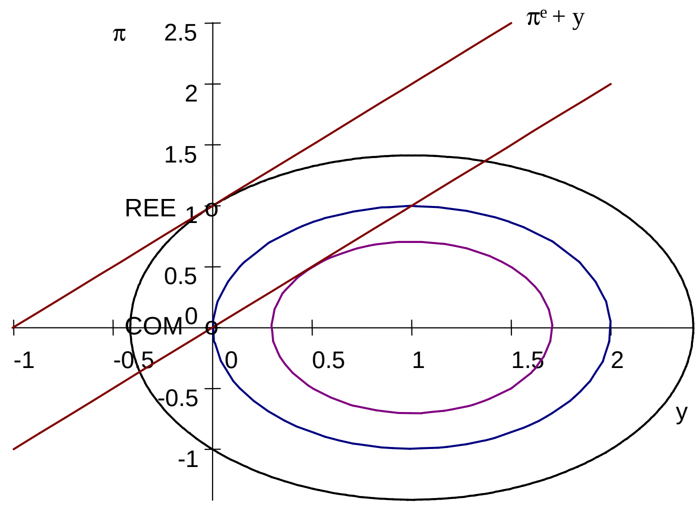

Figure 5.1: Equilibrium inflation under discretion

5.6 How to overcome the inflation bias?

If credibility is a problem, are there institutional reforms a country can adopt to lower inflation expectations without sacrificing the flexibility of monetary policy?

5.6.1 Preferences

Rogoff (QJE 1985) suggests that society can hire a conservative central banker whose loss function is:

$$\begin{aligned} L^{CB} &= \lambda\phi\left(y_t - k\right)^2 + \pi_t^2 \\ \phi &< 1 \end{aligned}$$

in this case the equilibrium under discretion:

$$\begin{aligned} \pi_t^D &= \frac{a\lambda\phi}{1+a^2\lambda\phi}u_t + a\lambda\phi k \\ y_t^D &= -\frac{1}{1+a^2\lambda\phi}u_t \end{aligned}$$

and the society's loss (where the weight is λ only) under this outcome is:

$$L_t^S = \lambda\left(\frac{1}{1+a^2\lambda\phi}u_t + k\right)^2 + \left(\frac{a\lambda\phi}{1+a^2\lambda\phi}u_t + a\lambda\phi k\right)^2$$

The expected value of this function is

$$E\left(L^S\right) = \lambda\left(\frac{1}{1+a^2\lambda\phi}\right)^2\sigma^2 + \lambda k^2 + \left(\frac{a\lambda\phi}{1+a^2\lambda\phi}\right)^2\sigma^2 + \left(a\lambda\phi k\right)^2$$

Minimizing this function with respect to ϕ gives us the "optimal degree of conservativism", which can be shown to be between 0 and 1 (proving this requires heavy algebra, see Walsh, 2nd edition, page 395). That is:

$$\phi^* = \arg\min_{\phi} E\left(L^S\right) \in [0,1]$$

One can see a conservative central banker has two effects:

1. he responds less to supply shocks, thereby letting output to fluctuate too much in response to supply shocks (in other words, a given supply shock u has a larger effect on output y under a conservative central banker). The first term of the loss function increases the lower ϕ.

2. he reduces the inflation bias, which is beneficial (second term).

This means that the inflation problem is solved through delegation: *if aggregate supply shocks are small*, the gains in terms of low inflation clearly dominate the distortion in stabilization policy.

5.6.2 Contracts

Suppose that the central bank loss function is:

$$L^{CB} = \lambda\left(y_t - k\right)^2 + \pi_t^2 + 2\omega\pi_t$$

that is, there is a central banker who has now the same preferences as society's but receives a bonus which is reduced (therefore his loss increases) as inflation rises. The FOC is:

$$a\lambda\left(a\left(\pi_t - \pi_t^e\right) - u_t - k\right) + \pi_t + \omega = 0$$

and the solution for inflation is

$$\pi_t^D = \frac{a\lambda}{1 + a^2\lambda}u_t + a\lambda k - \omega$$

so that if $\omega = a\lambda k$, the central banker will be induced to adopt the same inflation rate that would derive in the presence of a commitment technology.

A similar result can be obtained by appointing a central banker with a loss function centered around an optimal value of π which is lower than the social optimum (say $\overline{\pi}$). This is the currently the case in many countries and notably the United Kingdom. You can show that there is an optimal value of $\overline{\pi}$ that replicates the commitment solution.

5.6.3 Reputation

The basic model we studied so far is a one-shot game. The central bank's actions had no effect on the future so every problem was simplifying to an infinite sequence of one-period problems.

Barro and Gordon (1983) evaluate the role of reputation by considering a repeated game in which the choice of inflation at time t can affect future inflation expectations. Barro and Gordon examine whether there are inflation rates lower than the solution of the one-shot game that can be sustained in a trigger strategy[3] equilibrium.

Suppose we modify the central bank preferences by assuming that

$$\begin{aligned} U &= E_t \sum\nolimits_{i=0}^{\infty} \beta^i u_{t+i} \\ u_t &= \lambda a \left(\pi_t - \pi_t^e\right) - \frac{1}{2}\pi_t^2 \end{aligned}$$

we assume (1) no supply shock, (2) no quadratic term in output, (3) $k = 0$. In this setup, $\pi = \pi^e = a\lambda$ is the equilibrium of the "one-shot game" under discretion; to see why, consider that for given inflation expectations:

$$\pi_t^{OS} = \arg\max \ \lambda a \left(\pi_t - \pi_t^e\right) - \frac{1}{2}\pi_t^2 = a\lambda$$

In the context of this problem, we modify the way private sector expectations are formed. In particular, we assume that expectations are given by the following formula:

$$\pi_t^e = \left\{ \begin{matrix} \overline{\pi} < a\lambda \text{ if } \pi_{t-1} = \pi_{t-1}^e \\ a\lambda \text{ otherwise} \end{matrix} \right\}$$

this "trigger strategy" works as follows: if in period $t-1$ the central bank delivers an inflation rate equal to what the public had expected, the public expects a low inflation rate $\overline{\pi}$. If, instead, the central bank did fool the public, the public "punishes" the central bank by expecting the inflation rate that would arise under pure discretion. The punishment lasts one period only: that is, after one period, the public will expect again a lower rate $\overline{\pi}$.

The question is: *are there sustainable equilibria for the inflation rate that are less than the outcome under pure discretion?* To answer this question, we assume that in period $t = 0$ expected inflation is equal to some value $\overline{\pi}$, and we ask: what are the costs and the benefits of cheating (creating unexpected inflation) for a central bank? There are 4 objects that we need to calculate: $u_t^d, \beta u_{t+1}^d, u_t^n, \beta u_{t+1}^n$: they represent current and expected maximized value of the utility function for central bank from deviating d or from not deviating n.

Suppose the central bank sets inflation at some value $\pi \neq \overline{\pi}$. The utility will be:

$$\lambda a \left(\pi - \overline{\pi}\right) - \frac{1}{2}\pi^2$$

[3] A trigger strategy is a class of strategies employed in the repeated prisoner's dilemma. In a trigger strategy, one player initially cooperates but, if the other player defects, the trigger strategy punishes the other player by defecting against her for some number of plays.

which is maximized when

$$\pi = a\lambda$$

hence if central bank deviates, its gain today is:

$$u_t^d = \lambda a \left(\lambda a - \overline{\pi}\right) - \frac{1}{2}\lambda^2 a^2 \tag{ud0}$$

Next period, $\pi_t^e = a\lambda$, the utility for the central bank is

$$\lambda a \left(\pi - a\lambda\right) - \frac{1}{2}\pi^2$$

which is maximized setting $\pi = a\lambda$, so that

$$\beta u_{t+1}^d = -\frac{\beta}{2}\left(a\lambda\right)^2 \tag{ud1}$$

Consider now what happens if the central bank does not cheat. The utility both today and tomorrow is simply:

$$u_t^n = u_{t+1}^n = -\frac{1}{2}\overline{\pi}^2 \tag{un}$$

What is the gain/temptation today from deviating? Subtract (un) from $(ud0)$ to obtain:

$$G\left(\overline{\pi}\right) = u_t^d - u_t^n = \lambda a \left(\lambda a - \overline{\pi}\right) - \frac{1}{2}\lambda^2 a^2 + \frac{1}{2}\overline{\pi}^2 = \frac{1}{2}\left(\lambda a - \overline{\pi}\right)^2 \tag{gain}$$

What is instead the expected cost/punishment? Subtract $\beta\,(un)$ from $(ud1)$ to obtain:

$$C\left(\overline{\pi}\right) = \beta\left(u_{t+1}^n - u_{t+1}^d\right) = \frac{\beta}{2}\left(\left(a\lambda\right)^2 - \overline{\pi}^2\right) \tag{cost}$$

We can plot the G and the C function as a function of initial expected inflation $\overline{\pi}$. The G function is concave (U shaped, its second derivative is $1 > 0$) in $\overline{\pi}$, equals $\left(\lambda a\right)^2/2$ when $\overline{\pi} = 0$, and reaches a zero minimum when $\overline{\pi} = a\lambda$. The idea is that the gain from inflating for the central bank is higher the lower expected inflation is.

The function C, on the other hand, is convex ($\cap$ shaped, its second derivative is $-\beta < 0$) in $\overline{\pi}$ for all values, equals $\beta\left(\lambda a\right)^2/2$ when $\overline{\pi} = 0$, and equals zero when $\overline{\pi} = a\lambda$. Notice that the cost is also higher the lower the expected inflation is, but is smaller the smaller β.

G and C intersect twice, and the central bank will find an incentive deviate from any value of π such that G is greater than C. Instead, inflation rates in the range $(\pi_{\min}, a\lambda)$ are all sustainable. $\pi_{\min}$, in particular, is the minimum sustainable inflation rate, and equals:

$$\pi_{\min} = \frac{1-\beta}{1+\beta}a\lambda < a\lambda$$

The higher β, the higher the weight that the central bank places on the future, and the lower the inflation rate that can be sustained as an equilibrium outcome.

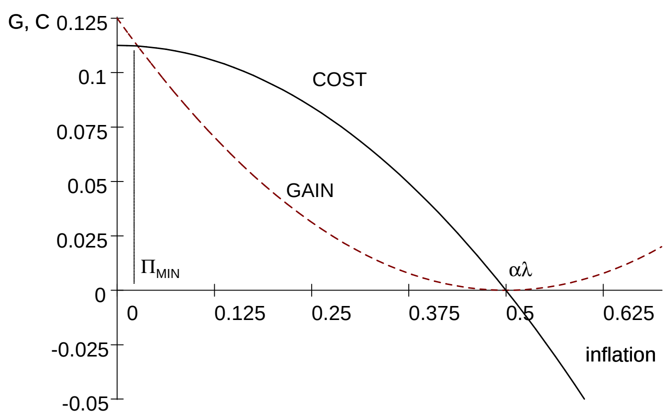

Figure 5.2: Gain/temptation from inflating (U-shaped) and cost/enforcement (∩−shaped) from doing so.