1 Problem 1: Couette Flow

Couette flow is a classic basic flow pattern in which a fluid is confined between two, smooth parallel plates. The bottom plate (at $y = -h$) is stationary while the top plate (at $y = h$) is moving with a constant velocity U. It is assumed that the flow is:

1. Steady
2. Fully developed
3. Two-dimensional (independent of z)
4. Zero pressure-gradient

Use no-slip boundary conditions. Figure 1 shows a diagram of the flow configuration.

Figure 1: The Couette flow configuration.

Calculate the following:

1. The velocity field
2. The vorticity field
3. The shear stress
4. The volume flow rate
5. The average and maximum velocities in the channel
    - Hint: $u_{\text{ave}} = Q/h$ where Q is the volume flow rate.

Plot: Make a plot of the velocity profile at a few different values of U. Put y on the $y-$ axis and $u(y)$ on the $x-$ axis.

Solution

1.1 Mathematical Formulation

The Couette Flow obeys the following equations of motion:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mu \nabla^2 \mathbf{u} \tag{2}$$

Eq.(1) is known as the equation of continuity, and Eq.(2) describes the conservation of momentum. Other assumptions in this problem are translated as follows:

1. Steady
$$\frac{\partial (\cdot)}{\partial t} = 0 \tag{3}$$
2. Fully developed
$$\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{4}$$
3. Two-dimensional
$$\frac{\partial (\cdot)}{\partial z} = 0 \tag{5}$$
4. Zero pressure-gradient
$$\nabla P = 0 \tag{6}$$
5. No-slip boundary
$$\mathbf{u}|_{y=-h} = 0, \quad \mathbf{u}|_{y=h} = U\hat{\mathbf{e}}_x \tag{7}$$

Putting them all together, we have the well-posed problem for $\mathbf{u} = u(y)\hat{\mathbf{e}}_x + v(y)\hat{\mathbf{e}}_y$:

$$\frac{\partial v}{\partial y} = 0 \tag{8}$$

$$v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{9}$$

$$v|_{y=-h} = 0, \quad v|_{y=h} = 0 \tag{10}$$

$$u|_{y=-h} = 0, \quad u|_{y=h} = U \tag{11}$$

Solution

1.2 The Velocity Field

We first note that v has its own independent boundary problem, *i.e.*

$$\frac{\partial v}{\partial y} = 0 \tag{12}$$

$$v|_{y=-h} = 0, \quad v|_{y=h} = 0 \tag{13}$$

whose solution is:

$$v = 0 \tag{14}$$

Plugging it back, we rewrite the problem for u:

$$\frac{\partial^2 u}{\partial y^2} = 0 \tag{15}$$

$$u|_{y=-h} = 0, \quad u|_{y=h} = U \tag{16}$$

So the solution for u is:

$$u = \frac{U}{2}\left(1 + \frac{y}{h}\right) \tag{17}$$

Putting them together we have the full solution for the velocity field:

$$\mathbf{u} = \frac{U}{2}\left(1 + \frac{y}{h}\right)\hat{\mathbf{e}}_x \tag{18}$$

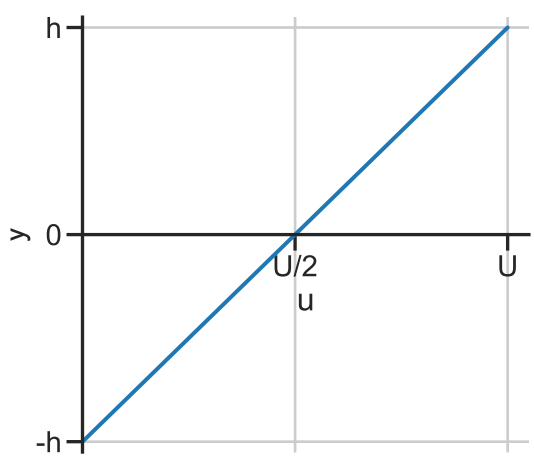


Figure 2: Velocity Profile

Solution

1.3 The Vorticity Field

$$\omega = \nabla \times \mathbf{u} = -\frac{U}{2h}\hat{\mathbf{e}}_z \tag{19}$$

1.4 The Shear Stress

$$\tau = \mu\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) = \frac{\mu U}{2h}\left(\hat{\mathbf{e}}_{xy} + \hat{\mathbf{e}}_{yx}\right) \tag{20}$$

1.5 The Volume Flow Rate

$$Q = \int_{-h}^{h} u \; \mathrm{d}y = Uh \tag{21}$$

1.6 The Average and Maximum Velocity

$$u_{ave} = \frac{Q}{2h} = \frac{U}{2} \tag{22}$$

$$u_{max} = U \tag{23}$$

2 Problem 2: Rectangular Duct Flow

Consider flow through a duct of rectangular cross section as shown in Figure 3. Make the following assumptions:

- 1-component flow (that is $v = w = 0$)
- Fully developed flow in x
- Steady flow

Use no-slip boundary conditions on all surfaces. Finally, note that the flow is driven by a prescribed (known) pressure gradient in the $x-$ direction.

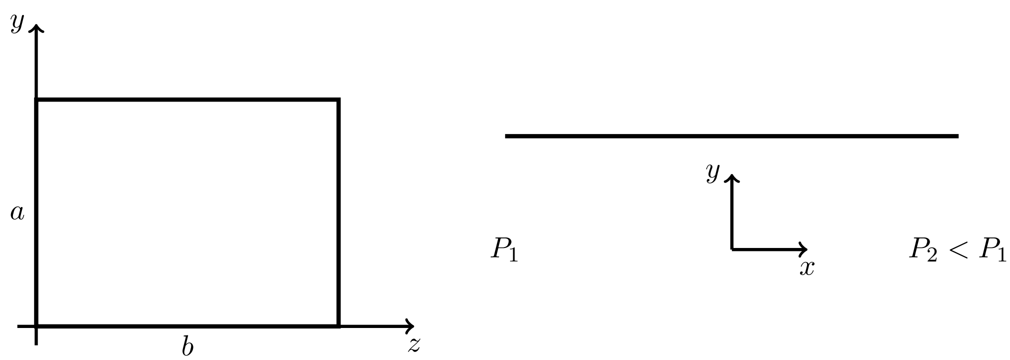


Figure 3: Left: Cross-section of the rectangular duct. Right: Profile of rectangular duct.

2.1 Governing Equations

Using the provided assumpsions, show that the Navier-Stokes equations reduce to,

$$\partial_y^2 u + \partial_z^2 u = \frac{1}{\mu}\partial_x P. \tag{24}$$

What are the boundary conditions?

Solution

Because $v = w = 0$, we can only consider the momentum equation for the u component:

$$\rho(\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u) = -\partial_x P + \mu(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u)$$

- $v\partial_y u + w\partial_z u$ can be removed because $v = w = 0$
- "Fully developed in x" means $\partial_x u = 0$, thus the $u\partial_x u$ and $\partial_x^2 u$ terms above can be removed. Steady flow means
- "Steady flow" means $\partial_t u = 0$

The remaining terms are:

$$0 = -\partial_x P + \mu(\partial_y^2 u + \partial_z^2 u)$$

Re-arranging them leads to:

$$\partial_y^2 u + \partial_z^2 u = \frac{1}{\mu}\partial_x P$$

The no-slip boundary conditions are

$$u(y=0) = u(y=a) = u(z=0) = u(z=b) = 0$$

2.2 Solving for u: Part 1

We have an inhomogenous, linear PDE on our hands. Let's solve this using the method of eigenfunction expansions. We know that with the current boundary conditions, the homogenous solution to our equation has eigenfunctions that are sines. Hence, let's assume a solution of the form,

$$u\left(y,z\right) = \sum_{n=1}^{\infty} \beta_n\left(z\right) \sin\left(\frac{n\pi}{a}y\right). \tag{25}$$

Plug this assumed form into the governing PDE and show that the unknown coefficients ($\beta_n\left(z\right)$) are governed by the following inhomogenous ODE,

$$\beta_n'' - \gamma_n^2 \beta_n = q_n \tag{26}$$

where $(\cdot)'$ denotes d/dz,

$$\gamma_n = \frac{n\pi}{a} \tag{27}$$

and

$$q_n = \frac{2}{a}\int_0^a Q \sin\left(\gamma_n y\right)\ \mathrm{d}y, \qquad Q = \frac{1}{\mu}\partial_x P. \tag{28}$$

2.2.1 Some Specific Details

Here is a breakdown of the required steps.

1. First show that after plugging the assumed form into the governing PDE the resulting expression is

$$\sum_{n=1}^{\infty} \beta_n'' \sin\left(\frac{n\pi}{a} y\right) - \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi}{a} y\right) = Q. \tag{29}$$

2. Next, multiply by $\sin\left(\frac{m\pi}{a} y\right)$ and integrate over y.

$$\int_0^a \sum_{n=1}^{\infty} \beta_n'' \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right)\ \mathrm{d}y - \int_0^a \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right)\ \mathrm{d}y = \int_0^a Q \sin\left(\frac{m\pi}{a} y\right)\ \mathrm{d}y. \tag{30}$$

3. That's a pretty big mess. Now, we use a beautiful result: the orthogonality of sines:

$$\int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right)\ \mathrm{d}y = \begin{cases} 0 & n \neq m \\ \frac{a}{2} & n = m \end{cases} \tag{31}$$

   The main point here is that every single term in the infinite sum is zero *except* for the term where $n = m$.

4. The final result is

$$\beta_n'' - \gamma_n^2 \beta_n = \frac{2}{a} \int_0^a Q \sin(\gamma_n y)\ \mathrm{d}y \tag{32}$$

What are the boundary conditions on β_n?

Solution

Start with

$$u\left(y,z\right)=\sum_{n=1}^{\infty}\beta_n\left(z\right)\sin\left(\frac{n\pi}{a}y\right).$$

Take second-order derivatives:

$$\partial_y^2 u=-\sum_{n=1}^{\infty}(\frac{n\pi}{a})^2\beta_n\left(z\right)\sin\left(\frac{n\pi}{a}y\right)$$
$$\partial_z^2 u=\sum_{n=1}^{\infty}\beta_n''\left(z\right)\sin\left(\frac{n\pi}{a}y\right)$$

Bring into the original PDE:

$$\partial_y^2 u+\partial_z^2 u=Q$$

Leads to:

$$\sum_{n=1}^{\infty}\beta_n''\sin\left(\frac{n\pi}{a}y\right)-\sum_{n=1}^{\infty}\beta_n\left(\frac{n\pi}{a}\right)^2\sin\left(\frac{n\pi}{a}y\right)=Q$$

Multiply by $\sin\left(\frac{m\pi}{a}y\right)$ and integrate over y:

$$\int_0^a\sum_{n=1}^{\infty}\beta_n''\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}y\right)\,\mathrm{d}y-\int_0^a\sum_{n=1}^{\infty}\beta_n\left(\frac{n\pi}{a}\right)^2\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{a}y\right)\,\mathrm{d}y=\int_0^a Q\sin\left(\frac{m\pi}{a}y\right)\,\mathrm{d}y$$

Using the orthogonality of sines, it becomes:

$$\frac{a}{2}\beta_m''-\frac{a}{2}(\frac{m\pi}{a})^2\beta_m=\int_0^a Q\sin\left(\frac{m\pi}{a}y\right)\,\mathrm{d}y$$

Replace m by n (they are interchangeable at the above step) and use $\gamma_n=\frac{n\pi}{a}$, the above reduces to the desired result:

$$\beta_n''-\gamma_n^2\beta_n=\frac{2}{a}\int_0^a Q\sin\left(\gamma_n y\right)\,\mathrm{d}y$$

2.3 Solving for β_n: The Homogeneous Part

To solve for β_n, we first need to solve the homogeneous equation,

$$\beta_n''-\gamma_n^2\beta_n=0. \tag{33}$$

Show that the homogeneous solution is

$$\beta_n^H = c_1 w_1 + c_2 w_2 \tag{34}$$

where $w_1 = \sinh(\gamma_n z)$ and $w_2 = \cosh(\gamma_n z)$ are the two linearly independent solutions.

Solution

The homogeneous ODE can be factored (in terms of operators) as:

$$(\frac{d}{dz} - \gamma_n)(\frac{d}{dz} + \gamma_n)\beta_n = 0$$

The solution needs to satisfy either:

$$(\frac{d}{dz} - \gamma_n)\beta_n = 0 \text{ or } (\frac{d}{dz} + \gamma_n)\beta_n = 0$$

The solution to each first-order ODE is:

$$\beta_n = e^{\gamma_n z} \text{ or } \beta_n = e^{-\gamma_n z}$$

Any of their linear combinations are correct solutions:

$$\beta_n^H = k_1 e^{\gamma_n z} + k_2 e^{-\gamma_n z}$$

Or, equivalently

$$\beta_n^H = c_1 \sinh(\gamma_n z) + c_2 \cosh(\gamma_n z)$$

2.4 Solving the Inhomogeneous Part

You can use the method of variation of parameters to solve the inhomogeneous equation. Let

$$\beta_n(z) = v_1(z)\, w_1(z) + v_2(z)\, w_2(z) \tag{35}$$

where $v_1(z)$ and $v_2(z)$ are the coefficients (parameters) to be varied. Now *assume* that

$$w_1 v_1' + w_2 v_2' = 0. \tag{36}$$

Then, by plugging (35) into (26) and using (36) yields two equations for v_1' and v_2'. These two equations are,

$$w_1 v_1' + w_2 v_2' = 0 \tag{37}$$
$$w_1' v_1' + w_2' v_2' = q_n. \tag{38}$$

Show that

$$v_1 = \frac{q_n}{\gamma_n^2} \sinh(\gamma_n z) + \alpha_1 \tag{39}$$
$$v_2 = -\frac{q_n}{\gamma_n^2} \cosh(\gamma_n z) + \alpha_2 \tag{40}$$

where α_1 and α_2 are constants of integration.

Finally, show that

$$\beta_n(z) = C_n\left[-\sinh(\gamma_n b) + \sinh(\gamma_n z) - (\sinh(\gamma_n z)\cosh(\gamma_n b) - \cosh(\gamma_n z)\sinh(\gamma_n b))\right] \tag{41}$$

$$= C_n\left[\sinh(\gamma_n z) - \sinh(\gamma_n b) - \sinh(\gamma_n(z-b))\right] \tag{42}$$

$$= 2C_n \sinh\left(\frac{\gamma_n}{2}(z-b)\right)\left[\cosh\left(\frac{\gamma_n}{2}(z+b)\right) - \cosh\left(\frac{\gamma_n}{2}(z-b)\right)\right]. \tag{43}$$

where

$$C_n = \frac{q_n}{\gamma_n^2 \sinh(\gamma_n b)}. \tag{44}$$

Solution

Start with

$$w_1 v_1' + w_2 v_2' = 0$$
$$w_1' v_1' + w_2' v_2' = q_n$$

Or equivalently,

$$\sinh(\gamma_n z) v_1' + \cosh(\gamma_n z) v_2' = 0$$
$$\cosh(\gamma_n z) v_1' + \sinh(\gamma_n z) v_2' = \frac{q_n}{\gamma_n}$$

Solve for v_1' and v_2':

$$v_1' = \frac{q_n}{\gamma_n}\cosh(\gamma_n z)$$
$$v_2' = -\frac{q_n}{\gamma_n}\sinh(\gamma_n z)$$

Integrate:

$$v_1 = \frac{q_n}{\gamma_n^2}\sinh(\gamma_n z) + \alpha_1$$
$$v_2 = -\frac{q_n}{\gamma_n^2}\cosh(\gamma_n z) + \alpha_2$$

Bring them back to the full solution:

$$\beta_n(z) = v_1(z)\, w_1(z) + v_2(z)\, w_2(z)$$
$$= (\frac{q_n}{\gamma_n^2}\sinh(\gamma_n z) + \alpha_1)\sinh(\gamma_n z) - (\frac{q_n}{\gamma_n^2}\cosh(\gamma_n z) - \alpha_2)\cosh(\gamma_n z)$$

It should satisfy the boundary condition:

$$\beta_n(0) = \beta_n(b) = 0$$

Solution

(continued)

$$
\begin{aligned}
0 = \beta_n(0) &= (\frac{q_n}{\gamma_n^2}\sinh(\gamma_n 0) + \alpha_1)\sinh(\gamma_n 0) - (\frac{q_n}{\gamma_n^2}\cosh(\gamma_n 0) - \alpha_2)\cosh(\gamma_n 0) \\
&= -(\frac{q_n}{\gamma_n^2} - \alpha_2)
\end{aligned}
$$

So

$$\alpha_2 = \frac{q_n}{\gamma_n^2}$$

$$0 = \beta_n(b) = (\frac{q_n}{\gamma_n^2}\sinh(\gamma_n b) + \alpha_1)\sinh(\gamma_n b) - (\frac{q_n}{\gamma_n^2}\cosh(\gamma_n b) - \alpha_2)\cosh(\gamma_n b)$$

So

$$\alpha_1 = \frac{q_n}{\gamma_n^2\sinh(\gamma_n b)}(1 - \cosh(\gamma_n b))$$

Bring back to $\beta_n(z)$:

$$
\begin{aligned}
\beta_n(z) = {} & [\frac{q_n}{\gamma_n^2}\sinh(\gamma_n z) + \frac{q_n}{\gamma_n^2\sinh(\gamma_n b)}(1 - \cosh(\gamma_n b))]\sinh(\gamma_n z) - (\frac{q_n}{\gamma_n^2}\cosh(\gamma_n z) - \frac{q_n}{\gamma_n^2})\cosh(\gamma_n z) \\
&= \frac{q_n}{\gamma_n^2\sinh(\gamma_n b)}[\sinh^2(\gamma_n z)\sinh(\gamma_n b) + \sinh(\gamma_n z) - \cosh(\gamma_n b)\sinh^2(\gamma_n z) \\
&\text{(line continue)} \ - \sinh(\gamma_n b)\cosh^2(\gamma_n z) + \sinh(\gamma_n b)\cosh(\gamma_n z)] \\
&= C_n[[\sinh^2(\gamma_n z) - \cosh^2(\gamma_n z)]\sinh(\gamma_n b) + \sinh(\gamma_n z) - \cosh(\gamma_n b)\sinh^2(\gamma_n z) + \sinh(\gamma_n b)\cosh(\gamma_n z)] \\
&= C_n[-\sinh(\gamma_n b) + \sinh(\gamma_n z) - \cosh(\gamma_n b)\sinh^2(\gamma_n z) + \sinh(\gamma_n b)\cosh(\gamma_n z)]
\end{aligned}
$$

which is the desired result.

2.5 The Velocity Field

Put everything together to show that,

$$u(y,z) = \sum_{n=1}^{\infty} 2C_n \sinh\left(\frac{\gamma_n}{2}(z-b)\right)\left[\cosh\left(\frac{\gamma_n}{2}(z+b)\right) - \cosh\left(\frac{\gamma_n}{2}(z-b)\right)\right]\sin(\gamma_n y). \tag{45}$$

A good sanity check is to make sure the boundary conditions are satisfied by this solution.

2.5.1 Other Thoughts

There are a variety of things that can be computed and inspected from here. For example:

- Calculate the vorticity and streamlines. Any surprises here?

- How does the solution change if you have different boundary conditions?
  - Neumann boundary conditions on each surface
  - Dirichlet boundary conditions on all surfaces *except* at $y = a$ for which we have a Neumann condition. (This would be like fluid flowing through an open channel.)

Note: You are not required to compute any of these for this assignment!

Solution

Bring the complete form of $\beta_n(z)$ (obtained in the previous question) into the original expansion:

$$u(y,z) = \sum_{n=1}^{\infty} \beta_n(z) \sin\left(\frac{n\pi}{a} y\right).$$

Leads to the desired result that

$$u(y,z) = \sum_{n=1}^{\infty} 2C_n \sinh\left(\frac{\gamma_n}{2}(z-b)\right)\left[\cosh\left(\frac{\gamma_n}{2}(z+b)\right) - \cosh\left(\frac{\gamma_n}{2}(z-b)\right)\right] \sin(\gamma_n y).$$

- The boundary condition $u(y=0) = u(y=a) = 0$ is satisfied as $\sin\left(\frac{n\pi}{a}y\right) = 0$ when $y = 0$ or $y = a$
- $u(z=b) = 0$ is satisfied as $\sinh\left(\frac{\gamma_n}{2}(z-b)\right) = 0$ when $z = b$.
- $u(z=b) = 0$ is satisfied as $[\cosh\left(\frac{\gamma_n}{2}(z+b)\right) - \cosh\left(\frac{\gamma_n}{2}(z-b)\right) = 0$ when $z = 0$.

So all boundary conditions are satisfied.

2.6 Vizualization

Now that we have an explicit formula for the velocity field, we can visualize it. Try to make the following plots:

- Plot $u(y, z = z^*)$ at a few values of z^* (perhaps near the boundary, 1/4 of the channel width, and at half the channel width).
- Plot $u(y = y^*, z)$ at a few values of y^* (perhaps near the boundary, 1/4 of the channel height, and at half the channel height).
- Make a surface plot of $u(y, z)$.

Solution

To label the axis, we assume those parameter values (arbitrary choice, do not affect the shape of the curve):

$$a = 2$$
$$b = 1$$
$$Q = -1$$

The plots are shown in Fig 4, 5, 6.

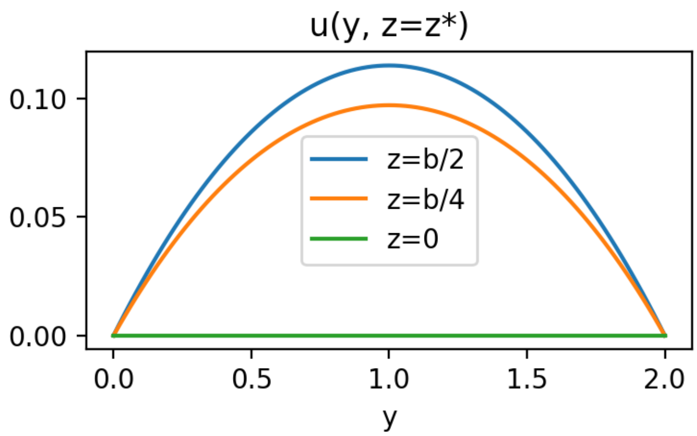


Figure 4: u(y)

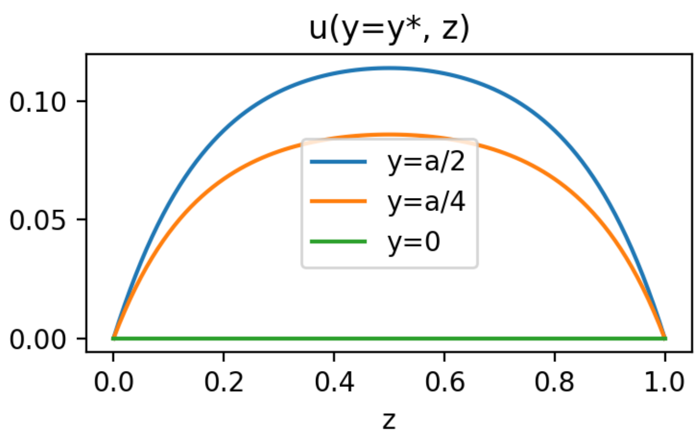


Figure 5: u(z)

Solution

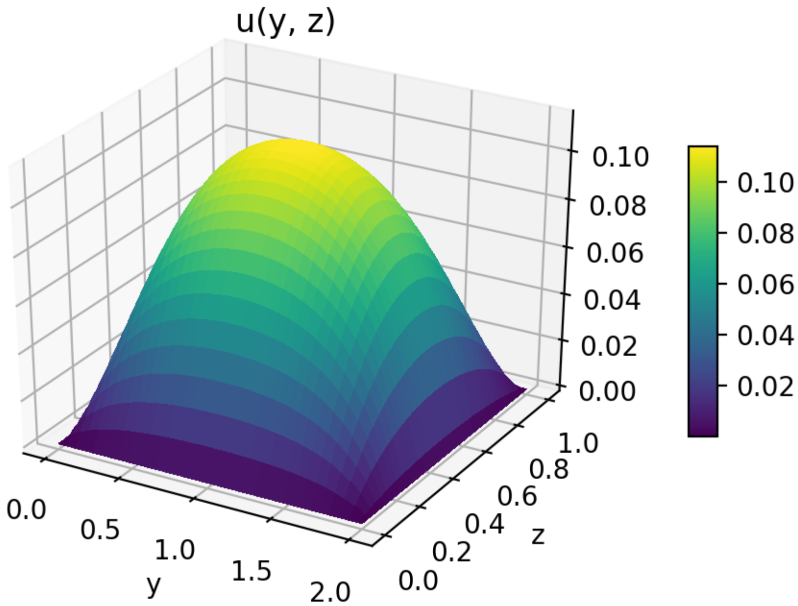


Figure 6: Surface plot of u(y, z)

See HW1P2_plot.ipynb for the code generating the plots.

3 Problem 3: The Finite Element Method

Consider the steady one-dimensional advection-diffusion-reaction equation,

$$a\partial_x u - \lambda u - k\partial_x^2 u = f, \qquad x \in (0,1) \tag{46}$$

with boundary conditions

$$u(0) = u_0, \quad -k\partial_x u(1) = b_1. \tag{47}$$

In (46), a is the constant advection speed, $\lambda > 0$ is the constant reaction coefficient, and $k > 0$ is the constant diffusion coefficient.

3.1 Weak Form

Write the weak form corresponding to the strong form (46). Be sure to specify all function spaces. Don't forget to include any boundary terms.

Solution

Let w be a test function which we will multiply times the stronghold function, and integrate over the domain, where $w(0) = 0$.

$$\int_0^1 w(a\partial_x u - \lambda u - k\partial_x^2 u)dx = \int_0^1 fwdx \tag{48}$$

$$\int_0^1 wa\partial_x udx - \int_0^1 w\lambda udx - \int_0^1 kw\partial_x^2 udx = \int_0^1 fwdx \tag{49}$$

$$\int_0^1 wa\partial_x udx - \int_0^1 w\lambda udx - wk\partial_x u\Big|_0^1 + \int_0^1 k\partial_x w\partial_x udx = \int_0^1 fwdx \tag{50}$$

$$\int_0^1 wa\partial_x udx - \int_0^1 w\lambda udx - (kw(1)\partial_x u(1)) + \int_0^1 k\partial_x w\partial_x udx = \int_0^1 fwdx \tag{51}$$

$$a\int_0^1 w\partial_x udx - \lambda\int_0^1 wudx + k\int_0^1 \partial_x w\partial_x udx = \int_0^1 fwdx + b_1 w(1) \tag{52}$$

Let the following function spaces be defined as follows:

$S = \{u | u \in H^{(1)}, u(0) = u_0\}$ and $V = \{v | v \in H^{(1)}, v(0) = 0\}$

Additionally, let $u^h \in S^h \subset S$, $w^h \in V^h \subset V$, and $u^h \in V^h$. We can now proceed to write the Galerkin statement below.

3.2 Galerkin Statement

From the weak form, write the Galerkin statement. Again, specify all function spaces.

Solution

Find $u^h \in V^h$ such that $\forall\ w^h \in V^h$:

$a\int_0^1 w^h \partial_x u^h dx - \lambda \int_0^1 w^h u^h dx + k\int_0^1 \partial_x w^h \partial_x u^h dx = \int_0^1 f w^h dx + b_1 w^h(1)$

Now, let us assume that u^h has the following $u^h = v^h + g^h$ where $g^h(0) = u_0$, still meeting the conditions to be in the V^h function subspace. We proceed to rewrite the function in the following form:

$$a\int_0^1 w^h(\partial_x v^h + \partial_x g^h)dx - \lambda \int_0^1 w^h(v^h + g^h)dx + k\int_0^1 \partial_x w^h(\partial_x v^h + \partial_x g^h)dx = \int_0^1 f w^h dx + b_1 w^h(1)$$

$$\begin{aligned} &a\int_0^1 w^h \partial_x v^h dx - \lambda \int_0^1 w^h v^h dx + k\int_0^1 \partial_x w^h \partial_x v^h dx \\ &= \int_0^1 f w^h dx + b_1 w^h(1) - a\int_0^1 w^h \partial_x g^h dx + \lambda \int_0^1 w^h g^h dx - k\int_0^1 \partial_x w^h \partial_x g^h dx \end{aligned} \tag{53}$$

3.3 Finite Element Discretization

Introduce the finite element basis and arrive at a linear algebra problem. Clearly define the form of all resulting matrices. You may write everything in terms of the basis functions (i.e. there is no need to directly compute the entries of the matrices).

Solution

Now, let

$$w^h = \sum_{A=1}^{n} W_A N_A(x) \qquad N_A(1) = 0, \forall\, A = [1, ..., n]$$

$$u^h = \sum_{B=1}^{n} U_B N_B(x) + u_0 N_{n+1}(x) \qquad N_B(0) = 0, \forall\, B = [1, ..., n],\, N_{n+1}(0) = 0$$

Therefore, defining:

$$v^h = \sum_{B=1}^{n} U_B N_B(x)$$

$$g^h = u_0 N_{n+1}(x)$$

We proceed now to plug these values into the Galerkin Statement:

$$\begin{aligned}
&a \int_0^1 \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \partial_x \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx - \lambda \int_0^1 \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx \\
&\quad + k \int_0^1 \partial_x \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \partial_x \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx = \int_0^1 f \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) dx \\
&\quad + b_1 \Big(\sum_{A=1}^{n} W_A N_A(1) \Big) - a \int_0^1 \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \partial_x \Big(u_0 N_{n+1}(x) \Big) dx \\
&\quad + \lambda \int_0^1 \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \Big(u_0 N_{n+1}(x) \Big) dx - k \int_0^1 \partial_x \Big(\sum_{A=1}^{n} W_A N_A(x) \Big) \partial_x \Big(u_0 N_{n+1}(x) \Big) dx
\end{aligned} \tag{54}$$

$$\begin{aligned}
\sum_{A=1}^{n} W_A \Bigg[&a \int_0^1 N_A(x) \partial_x \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx - \lambda \int_0^1 N_A(x) \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx \\
&+ k \int_0^1 \partial_x N_A(x) \partial_x \Big(\sum_{B=1}^{n} U_B N_B(x) \Big) dx - \int_0^1 f N_A(x) dx \\
&- b_1 N_A(1) + a \int_0^1 N_A(x) \partial_x \Big(u_0 N_{n+1}(x) \Big) dx \\
&- \lambda \int_0^1 N_A(x) \Big(u_0 N_{n+1}(x) \Big) dx + k \int_0^1 \partial_x N_A(x) \partial_x \Big(u_0 N_{n+1}(x) \Big) dx \Bigg] = 0
\end{aligned} \tag{55}$$

Solution

$$\sum_{A=1}^{n} W_A\Bigg[\sum_{B=1}^{n} U_B\Bigg(a\int_0^1 N_A(x)\partial_x N_B(x)dx - \lambda\int_0^1 N_A(x)N_B(x)dx \\ + k\int_0^1 \partial_x N_A(x)\partial_x N_B(x)dx\Bigg) - \int_0^1 fN_A(x)dx \\ - b_1 N_A(1) + a\int_0^1 N_A(x)\partial_x\Big(u_0 N_{n+1}(x)\Big)dx \\ - \lambda\int_0^1 N_A(x)u_0N_{n+1}(x)dx + k\int_0^1 \partial_x N_A(x)\partial_x\Big(u_0N_{n+1}(x)\Big)dx\Bigg] = 0 \tag{56}$$

To simplify the formula, let us redefine some of the expressions above:

$$\kappa_{AB} = a\int_0^1 N_A(x)\partial_x N_B(x)dx - \lambda\int_0^1 N_A(x)N_B(x)dx + k\int_0^1 \partial_x N_A(x)\partial_x N_B(x)dx \tag{57}$$

$$F_a = \int_0^1 fN_A(x)dx \tag{58}$$

$$G_a = a\int_0^1 N_A(x)\partial_x\Big(u_0N_{n+1}(x)\Big)dx \tag{59}$$

$$H_a = b_1N_A(1) \tag{60}$$

$$I_a = \lambda\int_0^1 N_A(x)u_0N_{n+1}(x)dx \tag{61}$$

$$J_a = k\int_0^1 \partial_x N_A(x)\partial_x\Big(u_0N_{n+1}(x)\Big)dx \tag{62}$$

$$\mathbf{F}_A = F_a + G_a - H_a + I_a - J_a \tag{63}$$

Therefore, rewriting our expression above as:

$$\sum_{A=1}^{n} W_A\Big[\Big(\sum_{B=1}^{n} U_B(\kappa_{AB}) - \mathbf{F}_A\Big] = 0 \tag{64}$$

In order for the expression to hold, the following must be true:

$$\sum_{B=1}^{n} U_B(\kappa_{AB}) = \mathbf{F}_A$$

4 Problem 4: Implementation

We wish to use the finite element method to solve

$$-\partial_x^2 u = fx, \qquad x \in 0, 1 \tag{65}$$

$$-\partial_x u0 = \langle, \quad u1 = \} \tag{66}$$

for ux where fx is a known forcing function and $\}$ and $\langle$ are constant boundary data.

Write a one-dimensional finite element code that uses piecewise linear finite elements to solve for ux. The following specifications are required:

- The code should work for any constant $\}$ and $\langle$ as well as arbitrary fx. These will be inputs to the code.
- Other inputs should include the domain size. You have some flexibility on how to do this. For example, you may pass in a fully-formed mesh if you wish. Alternatively, you can require the user to specify the number of elements from which your code can compute the uniform mesh size.
  - **Note:** You may assume a uniform mesh.
- The code **must** use the local point of view. That is, loop over individual elements and perform the finite element assembly operation to form the global stiffness matrix and force vector.
- Use Gaussian quadrature to perform the integrals. Although not strictly necessary here, it will give you some intuition for how things are actually done.
- You may use an external library to solve the resulting linear system.
- Use must use a compiled language such as `C`, `C++`, or (modern) Fortran.
- Try to submit your job on Odyssey!

Some other things you may want to consider are the following:

- Start simple. A natural progression may be the following:
  - Select $f = 0$, $g = 0$ and $h = 1$ to begin. The exact solution will be $u = 1 - x$.
  - Then make $g = 1$, $h = 1$, $f = 0$. The exact solution will be $u = 2 - x$.
  - Finally, try $g = 1$, $h = 1$, and $f = 1$. The exact solution will be $u = 2 - x + \dfrac{1}{2}1 - x^2$.
  - At this point, you will have some confidence in your code. This is called code verification. You can try more complicated fx if you'd like to.
- Organize your code into separate files. Use a `Makefile` to do the linking.
  - You may want to have files for the following:
    * Quadrature routines
    * Stiffness matrix routines
    * Right-hand-size (RHS) routines
  - Feel free to consider other code designs.

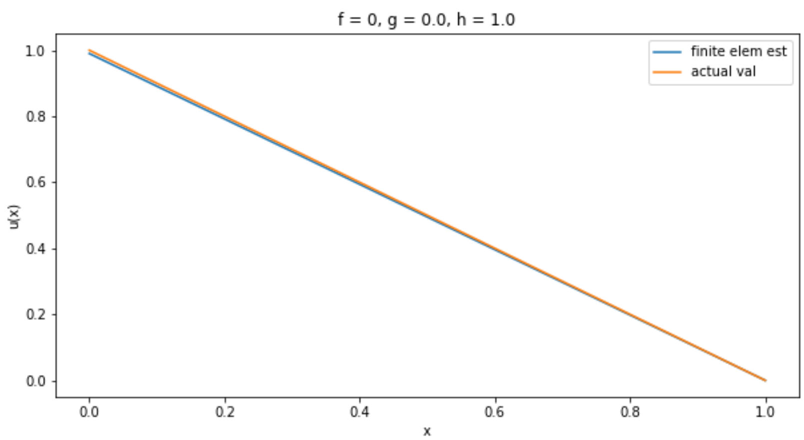
Solution

Using the C programming language, we constructed the following finite element method implementation:

> **main.c** : allows the user to specify his/her values for the function f, and boundary conditions for g and h.
>
> **construct_matrix.c** : computes κ_{AB}
>
> **gaussian_quadrature.c** : uses gaussian quadrature to estimate integration values for various functions specified within the file
>
> **tridiag_solver.c** : uses the *lapacke* library to solve a linear system described by Ax = y for x.

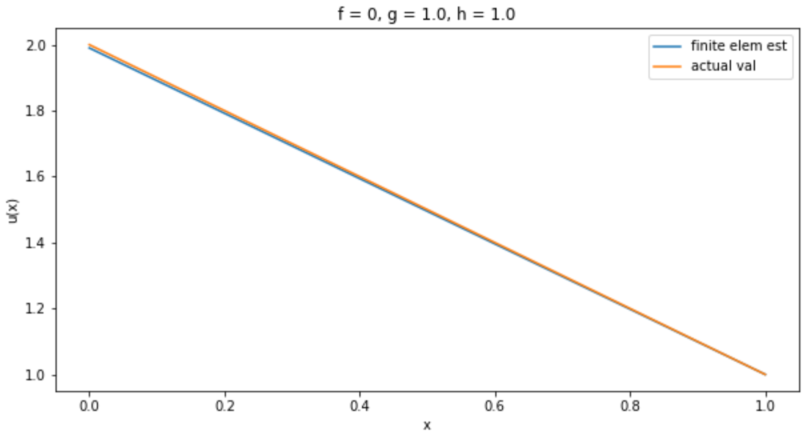
To specify the values for f, g, and h, there is a section in the main function in **main.c** file, under the section for `Variable initialization`.
To verify that our solution made sense we graphed our output in a Jupyter notebook that can be found in the same directory as the rest of our code.

For $f = 0$, $g = 0$ and $h = 1$, the exact solution will be $u = 1 - x$.

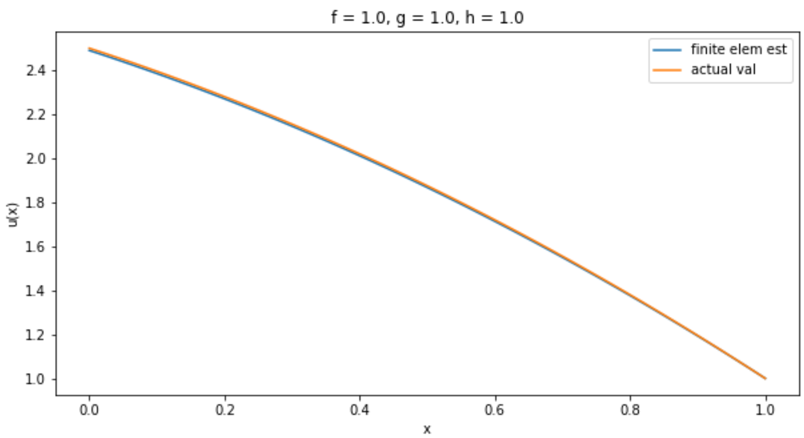


Solution

For $g = 1$, $h = 1$, $f = 0$, the exact solution will be $u = 2 - x$.



For $g = 1$, $h = 1$, and $f = 1$, the exact solution will be $u = 2 - x + \frac{1}{2}(1 - x^2)$.



Therefore, we verified our solution and have confidence in our code.