1 Background

Thermal convection is a pervasive phenomenon in our universe. Simply put, thermal convection is the force that drives fluid motion due to a temperature gradient. The ingredients needed for thermal convection to get going are the following: a fluid, a gravitational field, and a hot and cold surface confining the fluid. Such fluid flows are found in every aspect of human life and beyond. A limited set of examples include boiling water, geophysical flows causing plate tectonics, the sun and all of its consequences, and atmospheric flows. For fun, let's consider the example of geophysics. The center of the Earth is very hot (call this the bottom surface) while the surface of the Earth (where we live) is cool (call this the top surface). Gravity is pointing "down" from the top surface to the bottom surface. The fluid in-between these two surfaces is not a fluid as such; in fact, it is rock. However, over geologic eons, the rock behaves like a fluid. The fluid in this configuration is unstable. That is, the hot fluid wants to rise while the cold fluid wants to fall. For a large enough temperature gradient, this is precisely what happens. Ultimately, the fluid (rock here) moves giving rise to plate tectonics and volcanoes, which together shape the planet and give rise to life on Earth. Now, this was a drastic simplificiation of the incredibly complex physical processes that actually take place. We neglected the effects of the Earth's magnetic field, for one. Nevertheless, the underlying ingredients for thermal convection to drive such a complex flow are all there.

The study of thermal convection had much humbler beginnings. In fact, a simpler problem called Rayleigh-Bénard convection (RBC) was studied first. Rayleigh was a famous mathematician and scientist who made major contributions to the study of fluid mechanics in the 19^{th} and early 20^{th} century. Bénard was an experimentalist who studied fluid mechanics, among other things. Bénard conducted an experiment wherein he placed fluid in a Petri dish with the top of the fluid exposed to the air. Upon applying heat to the bottom of the shallow dish, Bénard noticed that fluid patterns arose in the initially quiescent fluid. Rayleigh embarked upon a mathematical analysis to explain this remarkable phenomenon. He studied an infinite slab of fluid confined between two parallel plates with gravity acting in the downward direction (see figure 1). The mathematical equations he used to describe the fluid motion were the Boussinesq equations (see Section 2). He was seeking

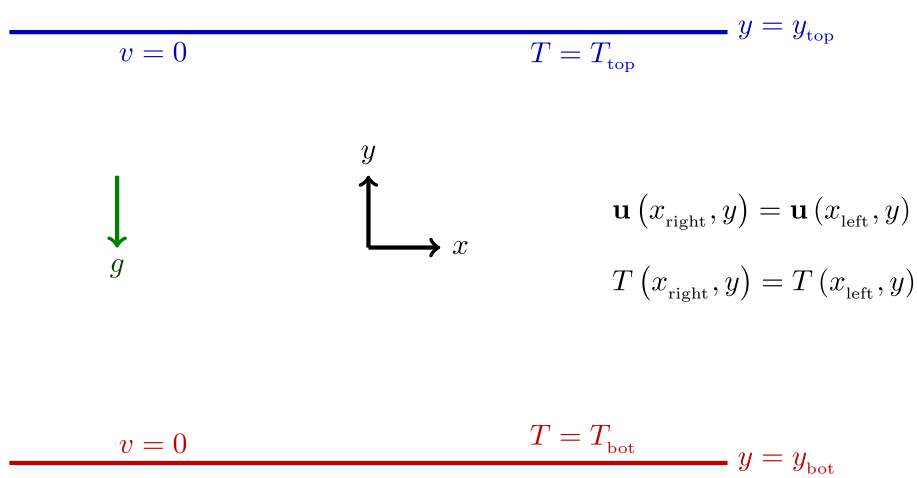

Figure 1: A schematic of two-dimensional Rayleigh-Bénard convection with no-slip boundary conditions.

to determine precisely under what conditions the fluid would be set in motion. Rayleigh succeeded in this endeavor. However, there is a twist to this story. It turns out that the phenomenon that

Bénard observed was due to surface tension effects, rather than the pure temperature difference between the top and bottom of the dish. When Rayleigh performed his analysis, he did not take surface tension into account. Nevertheless, the name Rayleigh-Bénard convection stuck for this phenomenon.

2 Governing Equations

Consider a fluid of density ρ_0 at temperature T_0. The equations governing the behavior of this fluid in a gravitational field are

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \frac{1}{\rho_0}\nabla P + \nu \nabla^2 \mathbf{u} + \frac{\rho}{\rho_0} g \widehat{\mathbf{y}} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where the term $\rho\mathbf{g}$ represents the buoyancy force and $\widehat{y}$ is the unit vector pointing in the $y-$direction. We know that the fluid density can be affected by the temperature of fluid and that this can lead to compressibility effects. However, the incompressible equations are simpler to work with (indeed, that is what we have written above) and as such we will *assume* that density only varies in the buoyancy term. Moreover, we will assume that density exhibits a linear dependence on the temperature in the buoyancy term and is constant in all other terms. This leads to the Boussinesq equations. These equations have been analyzed extensively over the years and their validity has been discussed at length. For our purposes, we expand the density in a Taylor series about T_0 and keep only the linear terms,

$$\rho\left(T_0 + \delta T\right) \approx \rho\left(T_0\right) - \left.\frac{\partial \rho}{\partial T}\right|_{T=T_0} \delta T \tag{3}$$

where $\delta T = T - T_0$ and we call $\alpha_V = -\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_{T=T_0} > 0$ the coefficient of volume expansion (a property of the fluid). Plugging (3) into (1) gives

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \frac{1}{\rho_0}\nabla P + \nu \nabla^2 \mathbf{u} + \alpha_V g T \widehat{\mathbf{y}} \tag{4}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{5}$$

where the pressure has been modified to account for the reference temperature terms. Without going into any details, the temperature can be determined via an advection-diffusion equation of the form,

$$\frac{\partial T}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} T) = \kappa \nabla^2 T \tag{6}$$

where κ is the thermal diffusivity. The final system of equations is,

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \frac{1}{\rho_0}\nabla P + \nu \nabla^2 \mathbf{u} + \alpha_V g T \widehat{\mathbf{y}} \tag{7}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{8}$$

$$\frac{\partial T}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} T) = \kappa \nabla^2 T. \tag{9}$$

The problem configuration is that shown in Figure 1. The boundary conditions at the top and bottom walls are,

$$\mathbf{u}\left(\mathbf{x}, y_{\text{top}}\right) = \mathbf{u}\left(\mathbf{x}, y_{\text{bot}}\right) \tag{10}$$

$$T\left(\mathbf{x}, y_{\text{top}}\right) = T_{\text{top}}, \quad T\left(\mathbf{x}, y_{\text{bot}}\right) = T_{\text{bot}} \tag{11}$$

and the boundary conditions on the sides are periodic.

3 Nondimensional Parameters

3.1 Nondimensional Equations

There are a variety of possible choices to perform the non-dimensionalization. The temperature can be non-dimensionalized by the temperature difference across the slab of fluid, $\Delta T = T_{\text{bot}} - T_{\text{top}}$. Non-dimensionalize space by the height of the channel $H = y_{\text{top}} - y_{\text{bot}}$. As usual, the pressure can be non-dimensionlized by the dynamic pressure $\rho_0 U^2$ where U is some velocity scale to be specified later. Time and velocity are non-dimensionalized by τ and $U = H/\tau$, respectively. The only scale left unspecified now is the time scale. Three choices for τ are:

1. $\tau = \dfrac{H^2}{\kappa}$: This is the classical choice, also referred to as "thermal" scaling
2. $\tau = \dfrac{H^2}{\nu}$: This is referred to as "viscous" scaling
3. $\tau = \left(\dfrac{H}{g\alpha_V \Delta T}\right)^{1/2}$: This is called the inertial scale. It is a more modern way of performing the non-dimensionalization and may make more sense for highly turbulent flows.

For pedagogical purposes, we will use the classical scaling (i.e. the thermal scaling). This leads to the non-dimensional Boussinesq equations,

$$\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = -\nabla P + \mathrm{Pr}\nabla^2 \mathbf{u} + \mathrm{RaPr}T\widehat{\mathbf{y}} \tag{12}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{13}$$

$$\frac{\partial T}{\partial \mathrm{t}} + \nabla \cdot (\mathbf{u}T) = \nabla^2 T \tag{14}$$

where all quantities appearing in the equations are understood to be dimensionless. Two new parameters appear: Pr and Ra. The Prandtl number, $\mathrm{Pr} = \dfrac{\nu}{\kappa}$, is a fluid property and is a ratio of the fluid viscosity to thermal diffusivity. The Rayleigh number Ra is given by,

$$\mathrm{Ra} = \frac{\alpha_V \Delta T g H^3}{\nu\kappa} \tag{15}$$

and can be thought of as a non-dimensional temperature difference. Moreover, it roughly quantifies the ratio of temperature-driven inertial forces to viscous forces. Hence, very high Ra results in turbulent flow fields. You are encouraged to work out this non-dimensionalization for yourself and to see what the other choices of τ give.

3.2 Heat Transport

We will be primarily concerned with the amount of vertical heat transport in the Rayleigh-Bénard system. The Nusselt number is the primary diagnostice quantity used to measure vertical heat transport. It is defined as the ratio of total heat transport to conduction heat transport,

$$Nu = \frac{\mathcal{H}}{\mathcal{H}_0} \tag{16}$$

where the conduction heat transport is

$$\mathcal{H}_0 = \frac{\kappa \Delta T}{H}. \tag{17}$$

The form of (17) can be worked out by solving the steady heat equation in the absence of forcing and neglecting the convection term. The total heat transport from the bottom wall is defined as the average temperature gradient perpendicular to the bottom wall,

$$\mathcal{H} = -\kappa \left. \frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \right|_{y_{\mathrm{bot}}} \tag{18}$$

where the overline $(\overline{\cdot})$ denotes an average in the x direction. Hence, the Nusselt number is

$$Nu = -\frac{H}{\Delta T} \left. \frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \right|_{y_{\mathrm{bot}}}. \tag{19}$$

Note that with this definition, the Nusselt number is a time-dependent quantity.

Another commonly-used form for the Nusselt number can be derived directly from the governing equations. The basic steps are:

- Assume a steady state
- Average the equations in the x direction
- Integrate in y to remove the y derivative. In case you are trying to derive the results independently, the resulting expression is,

$$-\kappa \frac{\mathrm{d}\overline{T}}{\mathrm{d}y} + \overline{vT} = -\kappa \left. \frac{\mathrm{d}\overline{T}}{\mathrm{d}y} \right|_{y=y_{\mathrm{bot}}} = \mathcal{H}_0 Nu. \tag{20}$$

- Integrate both sides over the height of the channel

These steps eventually lead to the following expression for the Nusselt number,

$$Nu = 1 + \frac{H}{\kappa \Delta T} \langle v\ T \rangle \tag{21}$$

where the notation $\langle \cdot \rangle$ represents a volume average (here an $x-y$ average). The Nusselt number is a dimensionless quantity. Note that in the absense of convection (i.e. $v = 0$) $Nu = 1$. Hence, conduction heat transport has unity Nusselt number. In steady state, the two expressions (19) and (21) are equal. Finally, we can express (21) in terms of non-dimensional velocity and temperature. Using the nondimensionalization from section 3.1 yields,

$$Nu = 1 + \langle v\ T \rangle \tag{22}$$

where now v and T are non-dimensional quantities.

4 Problem Statement

4.1 Parameters

Simulate two-dimensional turbulent Rayleigh-Bénard convection using the Drekar code. Use the following parameters:

- $Ra = 10^{10}$
- $Pr = 1$
- $\Gamma = 2$
  - Remember, $\Gamma = \dfrac{L}{H}$ where L is the length of the domain in the x direction and H is the height of the channel.
- No-slip boundary conditions for velocity
- Prescribe temperature on top and bottom boundary
- Periodic boundary conditions in the x direction
- Use a custom, non-uniform mesh in y and a uniform mesh in x. For example, if $y \in [0,\ 1]$ a classic non-uniform mesh is the Chebyshev mesh,

  $$y_j = \frac{1}{2}\left(1 - \cos\left(\frac{j}{N_y - 1}\pi\right)\right), \quad j = 0, \ldots, N_y - 1 \tag{23}$$

  where N_y is the number of points in the y direction and j is the index of point j.
- Use 1024 cores for the simulation for 7 days.

4.2 Deliverables

Once the simulation is completed, you should combine the parallel files into a single output file *or* leave them as is. In the latter case, you will need to take advantage of parallel postprocessing either in Python or Paraview. Write postprocessing scripts and utilities (or libraries) to compute various quantities of interest including (but not limited to):

- The Nusselt number (use both definitions of the Nusselt number). Note that one version will give you a single number while the other will give you a function of time. In this second case, you should make a plot of $Nu\left(t\right)$.
- Visualize the temperature field at a single time instant. You can do this in various ways including:
  - Two-dimensional surface plot of $T\left(x, y\right)$
  - Contour plot of T
  - Filled contour plot
- Visualize the velocity field using the same techniques that you did for the temperature field. Note that the velocity field is a vector field, so you should visualize the x and y components independently.

- Visualize the velocity field by computing and plotting streamlines
- Try to visualize the vorticity field
- Plot the temperature profile averaged in x and t. That is $T_{prof}\left(y\right) = \left\langle T\left(x, y, t\right)\right\rangle_{xt}$
- Calculate the Reynolds number using different velocity scales including the RMS velocity
- Average the horizontal velocity over x and plot it as a function of y. You may do this for a single snapshot or for all times
- Make a movie of the temperature field! Warning: This may take a while.

Other things you may want to try are:

- Compute and visualize the thermal and kinetic energies,

$$K^{T}\left(t\right) = \frac{1}{2}\int_{y_{\text{bot}}}^{y_{\text{top}}}\int_{x_L}^{x_R} T^{2}\left(x, y, t\right)\ \mathrm{d}x\ \mathrm{d}y \tag{24}$$

$$K^{V}\left(t\right) = \frac{1}{2}\int_{y_{\text{bot}}}^{y_{\text{top}}}\int_{x_L}^{x_R} \mathbf{u}\left(x, y, t\right)\cdot\mathbf{u}\left(x, y, t\right)\ \mathrm{d}x\ \mathrm{d}y \tag{25}$$

- Visualize Lagrangian tracer particles in the flow. Warning: You'll have to write some additional ODE solvers.
- Calculate turbulence quantities such as the Reynolds stress tensor.
- Calculate and visualize the boundary layers and boundary layer thickness.