# Stress, Mach & Multi-Grid LBM

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# Projects Updates

Group1:

- ???

Group2:

- ???

Recommendations: align with repository

# Inviscid flow (recap)

Euler Equation $\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \mathbf{F}$

rewritten as:

Cauchy Momentum Equation $\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot \Pi = \mathbf{F}$

with

Momentum Flux Density Tensor $\Pi_{\alpha\beta} = \rho u_\alpha u_\beta - \sigma_{\alpha\beta}$

and

Inviscid Stress Tensor $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$

# Viscous flow (recap)

Viscous Stress Tensor = Shear Stress + Normal Shear Stress:

$$\sigma'_{\alpha\beta} = \eta \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right) + \left( \eta_B - \frac{2}{3}\eta \right) \delta_{\alpha\beta} \frac{\partial u_\gamma}{\partial x_\gamma}$$

with $\eta$ Shear Viscotisy and $\eta_B$ Bulk Viscosity
For incompressible flow:

$$\sigma'_{\alpha\beta} = \eta \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)$$

# Viscous flow (recap)

Stress Tensor = Deviatoric Stress + Pressure

$$\sigma_{\alpha\beta} = \sigma'_{\alpha\beta} - p\delta_{\alpha\beta}$$

Strain Rate Tensor:

$$\mathbf{S} = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

Constitutive Law (Stress-Strain relationship): $\sigma = 2\mu\mathbf{S}$
Strain Rate $\dot{\gamma} = 2\sqrt{\mathbf{S} : \mathbf{S}}$
Directly from LBM (with forces):
$\mathbf{S} = -\frac{3\omega}{2}\sum_p \mathbf{c}_p\mathbf{c}_p(f_p - f_p^{eq})$
$\sigma' = -\left(1 - \frac{\omega}{2}\right)\sum_p \mathbf{c}_p\mathbf{c}_p(f_p - f_p^{eq}) - \frac{1}{2}\left(1 - \frac{\omega}{2}\right)\left(\mathbf{Fu} + \mathbf{Fu}^T\right)$

# Stress on Walls

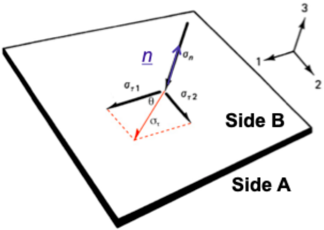
Stress acting on a plan at given point. GIven the force exerted on a plane $d\mathbf{F}$, it can be decomposed as:

Normal Stress

$$\sigma_n = \frac{d\mathbf{F}}{da} = \mathbf{n} \cdot \sigma \mathbf{n}$$

Shear Stress

$$|\sigma_\tau| = \sqrt{(\sigma \mathbf{n})^2 - (\mathbf{n} \cdot \sigma \mathbf{n})^2}$$



# Wall Shear Stress and Physiology

In hemodynamics $||\sigma||$ and $|\sigma_\tau|$ are important indicators for

- atherosclerotic plaque rupture
- plaque build-up
- aneurism

Computing this quantities locally and accurately is a crucial point in LBM-based hemodynamics.

# Low Mach LBM

Reynolds number $Re = \frac{t_{diff}}{t_{conv}} = \frac{ul}{\nu}$ Mach number $Ma = \frac{t_{sound}}{t_{conv}} = \frac{u}{c_S}$

LBM is a quasi-compressible method. $c_S$is fixed to $c_S = 1/\sqrt{3}$, so lowering $u$ can simulate low $Ma$. How to reproduce very low $Ma$?

In this case, we do not match the real and simulated$Ma = \frac{u}{c_S}$ , while we always want to match $Re$.

This approximation works well because flow structure depends very weakly on Mach for $Ma \ll 1$.

The big advantage is for the timestep $\Delta t = \frac{c_S \Delta x Ma}{u_{ph}} = \Delta x \frac{c_S Ma}{c_{ph} Ma_{ph}}$ (where $c_{ph}$is the physical speed of sound)

# Post-Processing for pressure

Working at a different *Ma* however has a main difference on pressure.
Utilizing Bernoulli's equation for an incompressible flow, one can remap pressure to be substantially insensitive to *Ma*:

$$\frac{1}{2}u^2 + \frac{p}{\rho} = \frac{1}{2}u_{ref}^2 + \frac{p_{ref}}{\rho_{ref}}$$

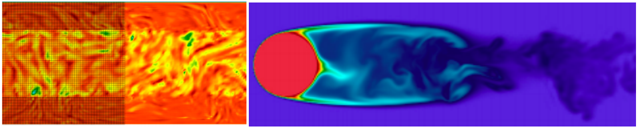
and by taking the reference values at the inlet, the post-processed pressure is

$$p = \rho \left[ \frac{p_{in}}{\rho_{in}} + \frac{1}{2}\left(u_{in}^2 - u^2\right) \right]$$

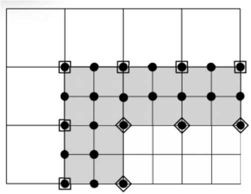
You can reconsider your homework to estimate the effect of this correction.

# Multi-Grid LBM

Simulating generic systems require choosing carefully the levels of detail and consequently the local resolutions.



Multi-Grid LBM is based on cartesian meshes with $2^n$ spacing with:



- meshes have some overlap for proper interpolations (especially to handle $f - f^{eq}$)
- time-advances in $2^n$
- viscosity scales as $2^n$ since $\sim \Delta x^2 / \Delta t$

Enforcing CFL condition everywhere is challenging, it is a good idea to change viscosity in low-resolution regions.