NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

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Course paper

On Extremums of Assymetric Random Walk

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Contents

1 Abstract

Once upon a time, during the Advanced Statistics seminar, I was told that no explicit result exists for the maximum of an asymmetric random walk. I accepted this opportunity with great intrigue, and this my result.

2 Random Walk

2.1 Simple Random Walk

Consider

$$R_n = \sum_{k=1}^{n} X_k, \quad R_0 = 0 \tag{1}$$

where

$$X_i = \begin{cases} +1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2} \end{cases} \tag{2}$$

We further define

$$M_n = \max_{k \leq n} R_n \tag{3}$$

By reflection principle one can get

$$P(M_n = k) = \frac{1}{2^n} C_n^{\left[\frac{n-k}{2}\right]} \tag{4}$$

2.2 Non-symmetric Random Walk

Consider

$$R_n = \sum_{k=1}^{n} X_k, \quad R_0 = 0 \tag{5}$$

where

$$X_i = \begin{cases} +1, & p \\ -1, & (1-p) \end{cases} \tag{6}$$

Define

$$M_n = \max_{k \leq n} R_n \tag{7}$$

For future notation $P(M_n = k) = f_{nk}$. Then f_{nk} satisfies

$$\begin{cases} f_{nk} = p f_{n-1,k-1} + (1-p) f_{n-1,k+1} \\ f_{n0} = (1-p) f_{n-1,0} + (1-p) f_{n-1,1} \end{cases} \tag{8}$$

It is easy to check that (proof will be provided at the end using mathematical induction)

$$f_{nk} = p^k (1-p)^{\lceil \frac{n-k}{2} \rceil} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} p^i \binom{m+i}{i} \frac{m-i+1}{m+1}; \quad n>1, \quad k \le n \tag{9}$$

$$m = \left\lfloor \frac{n+k-1}{2} \right\rfloor \tag{10}$$

$$\text{with} \quad f_{00} = 1 \tag{11}$$

satisfies the system above.

2.3 Limit result

Matrix form of recurrence can be provided:

$$\begin{pmatrix} f_{n0} \\ f_{n1} \\ f_{n2} \\ \dots \\ f_{nn} \end{pmatrix} = \begin{pmatrix} 1-p & 1-p & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ \dots & \dots & \dots & \dots & 1-p \\ 0 & 0 & 0 & p & 0 \end{pmatrix} \times \begin{pmatrix} f_{n-1,0} \\ f_{n-1,1} \\ f_{n-1,2} \\ \dots \\ f_{n-1,n} \end{pmatrix} \tag{12}$$

Eigenvalue $\lambda = 1$ and its eigenvector is stationary point:

$$f_k = \left(\frac{p}{1-p}\right)^k \left(1 - \frac{p}{1-p}\right), \quad 0 < p < 0.5 \tag{13}$$

3 Generating Function

Define

$$F(x,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{nk} x^n y^k \tag{14}$$

To solve for generating function:

$$\begin{cases} f_{nk} x^n y^k = pxy f_{n-1,k-1} x^{n-1} y^{k-1} + qx\frac{1}{y} f_{n-1,k+1} x^{n-1} y^{k+1} & | \sum_{n\ge1} \sum_{k\ge1}^{n} \\ y f_{n0} x^n = (1-p)xy f_{n-1,0} x^{n-1} + (1-p)x f_{n-1,1} x^{n-1} y & | \sum_{n\ge1} \end{cases} \tag{15}$$

3.1 Boundary Condition

Define

$$\sum_{n\geq 0} f_{n0}x^n = G(x) \tag{16}$$

Then

$$y\sum_{n\geq 1} f_{n,0}x^n = y(G(x)-1) \tag{17}$$

$$(1-p)xy\sum_{n\geq 1} f_{n-1,0}x^{n-1} = (1-p)xyG(x) \tag{18}$$

Define

$$[y^1F(x,y)] = \sum_{n\geq 0} f_{n1}x^n \tag{19}$$

Then

$$(1-p)xy\sum_{n\geq 1} f_{n-1,1}x^{n-1} = (1-p)xy[y^1F(x,y)] \tag{20}$$

Therefore, boundary condition can be rewritten as

$$y(G(x)-1) = (1-p)xyG(x) + (1-p)xy[y^1F(x,y)] \tag{21}$$

or

$$(G(x)-1) = (1-p)xG(x) + (1-p)x[y^1F(x,y)] \tag{22}$$

We further compute

$$[y^1F(x,y)] = \frac{G(x)-1-(1-p)xG(x)}{(1-p)x} \tag{23}$$

3.2 Main Condition

$$\sum_{n\geq 0}\sum_{k\geq 0}^{n} f_{n,k}x^ny^k = \sum_{n:=0}\sum_{k\geq 0}^{n} f_{n,k}x^ny^k + \sum_{n\geq 1}\sum_{k\geq 0}^{n} f_{n,k}x^ny^k = \tag{24}$$

$$= \sum_{n:=0}\sum_{k\geq 0}^{n} f_{n,k}x^ny^k + \sum_{n\geq 1}\left(f_{n0}x^ny^0 + \sum_{k\geq 1}^{n} f_{n,k}x^ny^k\right) = \tag{25}$$

$$= f_{00} + (G(x)-1) + \sum_{n\geq 1}\sum_{k\geq 1}^{n} f_{n,k}x^ny^k \tag{26}$$

Therefore

$$\sum_{n\geq 1}\sum_{k\geq 1}^{n} f_{n,k}x^ny^k = F(x,y) - G(x) \tag{27}$$

$$pxy\sum_{n\geq 1}\sum_{k\geq 1}^{n} f_{n-1,k-1}x^{n-1}y^{k-1} = pxy\sum_{n\geq 0}\sum_{k\geq 0}^{n} f_{n,k}x^n y^k = pxyF(x,y) \tag{28}$$

$$\sum_{n\geq 0}\sum_{k\geq 0}^{n} f_{n,k}x^n y^k = \sum_{n\geq 0}\left(f_{n,0}x^n + f_{n,1}x^n y + \sum_{k\geq 2}^{n} f_{n,k}x^n y^k\right) \tag{29}$$

$$= G(x) + y[y^1F(x,y)] + \sum_{n\geq 0}\sum_{k\geq 2}^{n} f_{n,k}x^n y^k \tag{30}$$

Therefore,

$$q\frac{x}{y}\sum_{n\geq 1}\sum_{k\geq 1}^{n} f_{n-1,k+1}x^{n-1}y^{k+1} = q\frac{x}{y}\sum_{n\geq 0}\sum_{k\geq 2}^{n} f_{n,k}x^n y^k = \tag{31}$$

$$= q\frac{x}{y}\sum_{n\geq 0}\left(f_{n,0}x^n + f_{n,1}x^n y + \sum_{k\geq 2}^{n} f_{n,k}x^n y^k\right) = q\frac{x}{y}(F(x,y)-G(x)-y[y^1F(x,y)]) \tag{32}$$

Thus, main recurrence can be written as:

$$F(x,y) - G(x) = pxyF(x,y) + q\frac{x}{y}\left(F(x,y) - G(x) - y[y^1F(x,y)]\right) \tag{33}$$

3.3 Kernel Method

We rewrite our system

$$\begin{cases} F(x,y) - G(x) = pxyF(x,y) + q\frac{x}{y}\left(F(x,y) - G(x) - y[y^1F(x,y)]\right) \\ [y^1F(x,y)] = \frac{G(x)-1-(1-p)xG(x)}{(1-p)x} \end{cases} \tag{34}$$

By substituting lower into upper we get

$$F(x,y) = \frac{qx(1-y)G(x) - y}{pxy^2 - y + qx} = \frac{qx(1-y)G(x) - y}{px(y-r_1)(y-r_2)} \tag{35}$$

where

$$r_1 = \frac{1-\sqrt{1-4pqx^2}}{2px} \tag{36}$$

$$r_2 = \frac{1+\sqrt{1-4pqx^2}}{2px} \tag{37}$$

$$r_1r_2 = \frac{q}{p} \tag{38}$$

Now observe that $(y-r_1) \sim 0$ when $x, y \to 0$. Therefore, $\frac{1}{y-r_1}$ has no expansion as power series around $(0,0)$. The idea of kernel method is that if $F(x,y)$ has expansion then "bad" factor $y-r_1$ should disappear i.e. it must be a factor of numerator. From this fact $G(x)$ can be computed.

Therefore,

$$qx(1-r_1)G(x) - r_1 = 0 \tag{39}$$

$$G(x) = \frac{r_1}{qx(1-r_1)} = \frac{1-\sqrt{1-4pqx^2}}{qx(2px-1+\sqrt{1-4pqx^2})} \tag{40}$$

We now go back to the main recurrence and simplify it

$$F(x,y) = \frac{G(x)\left(qx(1-y) - y\frac{qx(1-r_1)}{r1}\right)}{px(y-r_1)(y-r_2)} = \frac{G(x)\left(r_1 qx(1-y) - yqx(1-r_1)\right)}{r_1 px(y-r_1)(y-r_2)} \tag{41}$$

$$= \frac{G(x)qx(r_1-y)}{r_1 px(y-r_1)(y-r_2)} = \frac{-G(x)qx}{r_1 px(y-r_2)} = \frac{-G(x)qx}{r_1 px(y-r_2)} \tag{42}$$

Therefore by plugging $G(x) = \frac{r_1}{qx(1-r_1)}$ and using $r_1 r_2 = \frac{q}{p}$ we get

$$F(x,y) = \frac{1}{xp(r_1-1)(y-r_2)} = \frac{4px}{\left(\sqrt{1-4pqx^2}+2px-1\right)\left(\sqrt{1-4pqx^2}-2ypx+1\right)} \tag{43}$$

$$\boxed{F(x,y) = \frac{4px}{\left(\sqrt{1-4pqx^2}+2px-1\right)\left(\sqrt{1-4pqx^2}-2ypx+1\right)}} \tag{44}$$

4 Explicit Formula

We have mentioned before that coefficient solution for this $F(x, y)$ satisfies:

$$f_{nk} = p^k (1-p)^{\lceil \frac{n-k}{2} \rceil} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} p^i \binom{m+i}{i} \frac{m-i+1}{m+1}; \quad n > 1, \quad k \le n \tag{45}$$

$$m = \left\lfloor \frac{n+k-1}{2} \right\rfloor \tag{46}$$

$$\text{with} \quad f_{00} = 1 \tag{47}$$

Now we prove it.

4.1 Main condition

Check that f_{nk} satisfies:

$$f_{nk} = p f_{n-1,k-1} + (1-p) f_{n-1,k+1} \tag{48}$$

Let's denote

$$p^k (1-p)^{\lceil \frac{n-k}{2} \rceil} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} p^i \binom{m+i}{i} \frac{m-i+1}{m+1}, \text{ where} \tag{49}$$

$$a = \left\lceil \frac{n-k}{2} \right\rceil, \quad b = \left\lfloor \frac{n-k}{2} \right\rfloor, \quad m = \left\lfloor \frac{n+k-1}{2} \right\rfloor \tag{50}$$

For $f_{n-1,k-1}$:

$$a' = \left\lceil \frac{(n-1)-(k-1)}{2} \right\rceil = a, \quad b' = \left\lfloor \frac{(n-1)-(k-1)}{2} \right\rfloor = b \tag{51}$$

$$m' = \left\lfloor \frac{n-1+k-1-1}{2} \right\rfloor = m-1 \tag{52}$$

This leads us to

$$p f_{n-1,k-1} = p^k (1-p)^a \sum_{i=0}^{b} p^i \binom{m+i-1}{i} \frac{m-i}{m} \tag{53}$$

For $f_{n-1,k+1}$:

$$a' = \left\lceil \frac{(n-1)-(k+1)}{2} \right\rceil = a-1, \quad b' = \left\lfloor \frac{(n-1)-(k+1)}{2} \right\rfloor = b-1 \tag{54}$$

$$m' = \left\lfloor \frac{n-1+k+1-1}{2} \right\rfloor = m \tag{55}$$

This leads us to

$$(1-p)f_{n-1,k+1} = p^{k+1}(1-p)^a \sum_{i=0}^{b-1} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} \tag{56}$$

Therefore, we need to verify that

$$p^k (1-p)^a \sum_{i=0}^{b} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} \vee \tag{57}$$

$$p^k (1-p)^a \sum_{i=0}^{b} p^i \binom{m+i-1}{i} \frac{m-i}{m} + p^{k+1}(1-p)^a \sum_{i=0}^{b-1} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} \tag{58}$$

or

$$\sum_{i=0}^{b} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} \vee \sum_{i=0}^{b} p^i \binom{m+i-1}{i} \frac{m-i}{m} + p \sum_{i=0}^{b-1} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} \tag{59}$$

First of all, note that

$$p \sum_{i=0}^{b-1} p^i \binom{m+i}{i} \frac{m-i+1}{m+1} = \sum_{i=1}^{b} p^i \binom{m+i-1}{i-1} \frac{m-i+2}{m+1} \tag{60}$$

We now check coefficients before each p^i

For $\underline{i=0}$

$$\binom{m}{0} \frac{m+1}{m+1} = \binom{m-1}{0} \frac{m}{m} \to 1 = 1 \tag{61}$$

For $\underline{i=k}$ $1 \le k \le b$

$$\binom{m+k}{k} \frac{m-k+1}{m+1} = \binom{m+k-1}{k} \frac{m-k}{m} + \binom{m+k-1}{k-1} \frac{m-k+2}{m+1} \tag{62}$$

$$\frac{(m+k)!}{k!\,m!} \cdot \frac{m-k+1}{m+1} \vee \frac{(m+k-1)!}{(m-1)!\,k!} \cdot \frac{m-k}{m} + \frac{(m+k-1)!}{m!\,(k-1)!} \cdot \frac{m-k+2}{m+1} \tag{63}$$

$$\frac{(m+k)}{k\,m} \cdot \frac{m-k+1}{m+1} \vee \frac{1}{k} \cdot \frac{m-k}{m} + \frac{1}{m} \cdot \frac{m-k+2}{m+1} \tag{64}$$

$$\frac{(m+k)(m-k+1)}{k\,m(m+1)} \vee \frac{(m-k)(m+1)+k(m-k+2)}{k\,m(m+1)} \tag{65}$$

$$\frac{m^2+m-k^2+k}{k\,m(m+1)} = \frac{m^2+m-k^2+k}{k\,m(m+1)} \tag{66}$$

4.2 Boundary condition

Check that $f_{n,0} = (1-p)f_{n-1,0} + (1-p)f_{n-1,1}$ holds.
Consider two cases: when n is even or odd.

Case1: Even

$n = 2l, k = 0$

Then

$$a = \left\lceil \frac{2l}{2} \right\rceil = l, \quad b = \left\lfloor \frac{2l}{2} \right\rfloor = l - 1, \tag{67}$$

$$m = \left\lfloor \frac{2l-1}{2} \right\rfloor = \left\lfloor l - \frac{1}{2} \right\rfloor = l - 1 \tag{68}$$

This leads us to

$$f_{n,0} = (1-p)^l \sum_{i=0}^{l} p^i \binom{l-1+i}{i} \frac{l-i}{l} \tag{69}$$

For $f_{n-1,0}$:

$$a' = \left\lceil \frac{2l-1}{2} \right\rceil = l, \quad b' = \left\lfloor \frac{2l-1}{2} \right\rfloor = l - 1, \tag{70}$$

$$m' = \left\lfloor \frac{2l-1-1}{2} \right\rfloor = \lfloor l-1 \rfloor = l - 1 \tag{71}$$

Therefore

$$(1-p)f_{n-1,0} = (1-p)^{l+1} \sum_{i=0}^{l-1} p^i \binom{l-1+i}{i} \frac{l-i}{l} \tag{72}$$

For $f_{n-1,1}$:

$$a' = \left\lceil \frac{2l-1-1}{2} \right\rceil = l - 1, \quad b' = \left\lfloor \frac{2l-1-1}{2} \right\rfloor = l - 1, \tag{73}$$

$$m' = \left\lfloor \frac{2l-1+1-1}{2} \right\rfloor = \left\lfloor l - \frac{1}{2} \right\rfloor = l - 1 \tag{74}$$

Therefore

$$(1-p)f_{n-1,1} = p\,(1-p)^l \sum_{i=0}^{l-1} p^i \binom{l-1+i}{i} \frac{l-i}{l} \tag{75}$$

Thus we need to verify the following

$$(1-p)^l \sum_{i=0}^{l} p^i \binom{l-1+i}{i} \frac{l-i}{l} \vee \tag{76}$$

$$(1-p)^{l+1}\sum_{i=0}^{l-1}p^i\binom{l-1+i}{i}\frac{l-i}{l}+p\,(1-p)^l\sum_{i=0}^{l-1}p^i\binom{l-1+i}{i}\frac{l-i}{l} \tag{77}$$

Simplifying yields

$$\sum_{i=0}^{l}p^i\binom{l-1+i}{i}\frac{l-i}{l}\vee(1-p)\sum_{i=0}^{l-1}p^i\binom{l-1+i}{i}\frac{l-i}{l}+p\sum_{i=0}^{l-1}p^i\binom{l-1+i}{i}\frac{l-i}{l} \tag{78}$$

Since both sums on the right side are same we get

$$S\vee pS+(1-p)S \tag{79}$$

Therefore, the formula satisfies recurrence for even n.

Case2: Odd

$n=2l+1,k=0$

Then

$$a=\left\lceil\frac{2l+1}{2}\right\rceil=l+1,\quad b=\left\lfloor\frac{2l+1}{2}\right\rfloor=l, \tag{80}$$

$$m=\left\lfloor\frac{2l+1-1}{2}\right\rfloor=\lfloor l\rfloor=l \tag{81}$$

This leads us to

$$f_{n,0}=(1-p)^{l+1}\sum_{i=0}^{l}p^i\binom{l+i}{i}\frac{l-i+1}{l+1} \tag{82}$$

For $f_{n-1,0}$:

$$a'=\left\lceil\frac{2l+1-1}{2}\right\rceil=l,\quad b'=\left\lfloor\frac{2l+1-1}{2}\right\rfloor=l, \tag{83}$$

$$m'=\left\lfloor\frac{2l-1}{2}\right\rfloor=l-1 \tag{84}$$

Therefore

$$(1-p)f_{n-1,0}=(1-p)^{l+1}\sum_{i=0}^{l}p^i\binom{l-1+i}{i}\frac{l-i}{l} \tag{85}$$

For $f_{n-1,1}$:

$$a'=\left\lceil\frac{2l-1}{2}\right\rceil=l,\quad b'=\left\lfloor\frac{2l-1}{2}\right\rfloor=l-1, \tag{86}$$

$$m' = \left\lfloor \frac{2l+1-1}{2} \right\rfloor = \lfloor l \rfloor = l \tag{87}$$

Therefore

$$(1-p)f_{n-1,1} = p\,(1-p)^{l+1} \sum_{i=0}^{l-1} p^i \binom{l+i}{i} \frac{l-i+1}{l+1} \tag{88}$$

It remains to verify that

$$(1-p)^{l+1} \sum_{i=0}^{l} p^i \binom{l+i}{i} \frac{l-i+1}{l+1} \vee \tag{89}$$

$$(1-p)^{l+1} \sum_{i=0}^{l-1} p^i \binom{l-1+i}{i} \frac{l-i}{l} + p\,(1-p)^{l+1} \sum_{i=0}^{l-1} p^i \binom{l+i}{i} \frac{l-i+1}{l+1} \tag{90}$$

By canceling common terms it yields

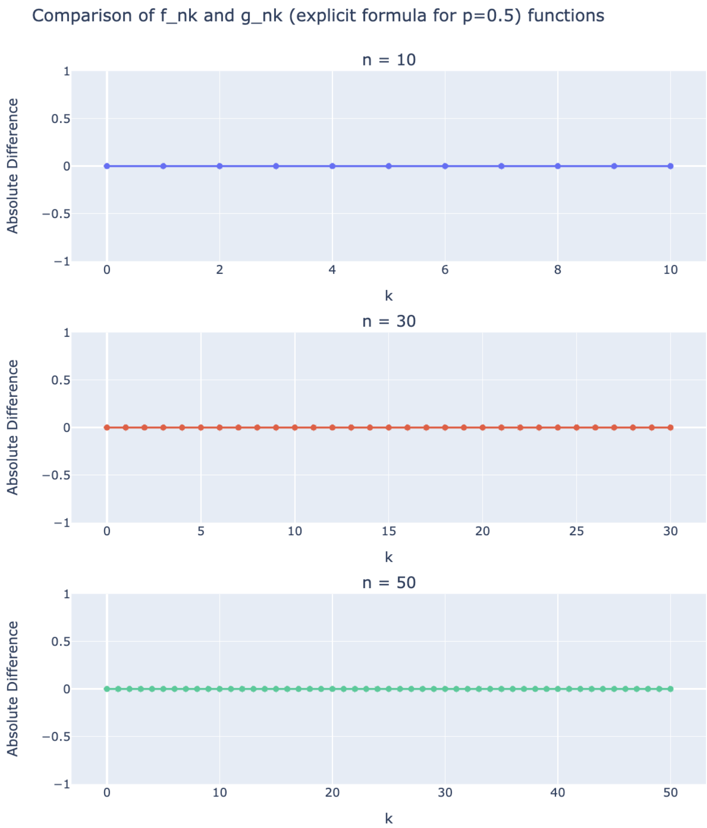
$$\sum_{i=0}^{l} p^i \binom{l+i}{i} \frac{l-i+1}{l+1} \vee \sum_{i=0}^{l-1} p^i \binom{l-1+i}{i} \frac{l-i}{l} + p \sum_{i=0}^{l-1} p^i \binom{l+i}{i} \frac{l-i+1}{l+1} \tag{91}$$

One can observe that this is exactly the sum proved for main condition. Therefore, we put equality sign and finish the proof.

Q.E.D.

5 Graphical Representation

The following results can be obtained for absolute difference between reflection principle result and my formula. Code in appendix



6 Bibliography

References

[1] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1. Wiley, 2nd edition, 1968.

[2] Henry McKean. *Probability: The Classical Limit Theorems*. Cambridge University Press, 2014.

[3] Helmut Prodinger. The kernel method: A collection of examples. *Séminaire Lotharingien de Combinatoire*, 50:Article B50f, 2003.

7 Appendix

```python
import sympy as sp
import plotly.graph_objects as go
from plotly.subplots import make_subplots

def g_nk(n, k, p):
    floor_val = (n - k) // 2
    binom = sp.binomial(n, floor_val)
    return float(binom / (2 ** n))

def f_nk(n, k, p):
    p_expr = sp.sympify(p)
    if n == 0:
        return sp.S(1) if k == 0 else sp.S(0)
    m_val = (n + k - 1) // 2
    ceil_val = (n - k + 1) // 2
    floor_val = (n - k) // 2
    s = sp.S(0)
    for i in range(0, floor_val + 1):
        binom_term = sp.binomial(m_val + i, i)
        term = (p_expr ** i) * binom_term * (m_val - i + 1) / (m_val + 1)
        s += term
    result = (p_expr ** k) * ((1 - p_expr) ** ceil_val) * s
    return result


n_values = [10, 30, 50]
p_val = 0.5


subplot_titles = [f"n = {n}" for n in n_values]

#Create subplots with titles
fig = make_subplots(
    rows=len(n_values),
    cols=1,
    subplot_titles=subplot_titles,
    vertical_spacing=0.1
)

for idx, n_val in enumerate(n_values, start=1):
    diff = []
    for k_i in range(0, n_val + 1):
        f_val = f_nk(n_val, k_i, p_val)
        f_val_float = float(f_val) if isinstance(f_val, (int, float)) else float(f_val.evalf())
        g_val = g_nk(n_val, k_i, p_val)
        diff.append(abs(f_val_float - g_val))

    fig.add_trace(
        go.Scatter(
            x=list(range(0, n_val + 1)),
            y=diff,
            mode='lines+markers',
            name=f'n = {n_val}'
        ),
        row=idx, col=1
    )

    fig.update_xaxes(title_text="k", row=idx, col=1)
    fig.update_yaxes(title_text="Absolute Difference", row=idx, col=1)

fig.update_layout(
    height=900,
    width=800,
    title_text="Comparison of f_nk and g_nk (explicit formula for p=0.5) functions",
    showlegend=False
)

fig.show()
```

Figure 1: Python Implementation