Problem Set 1

Problem 1

Prove the properties of convolution. First, the linear operator and shift-invariant we have is

$$f * h = \int f(t)h(x-t)dt$$

(a) Associativity: $(f * g) * h = f * (g * h)$

$$(f * g) * h = \int_0^t (f * g)(s)h(t-s)ds$$

$$= \int_{s=0}^{t} \left(\int_{u=0}^{s} f(u)g(s-u)du \right) h(t-s)ds$$

$$= \int_{s=0}^{t} \int_{u=0}^{s} f(u)g(s-u)\, h(t-s)\, duds$$

$$= \int_{u=0}^{t} \int_{s=0}^{s} f(u)g(s-u)\, h(t-s)\, dsdu$$

$$= \int_{u=0}^{t} \int_{s=0}^{t-u} f(u)g(s)\, h(t-s-u)\, dsdu$$

$$= \int_{u=0}^{t} f(u) \left(\int_{s=0}^{t-u} g(s)\, h(t-u-s) \right) du$$

$$= \int_{u=0}^{t} f(u)(g * h)(t-u)\, du$$

$$= f * (g * h)$$

Therefore we have $(f * g) * h = f * (g * h)$

(b) Distributivity: $f * (g + h) = f * g + f * h$

$$f * (g + h) = \int f(x-t)\big(g(t) + h(t)\big)dt$$

$$= \int f(x-t)g(t)dt + \int f(x-t)h(t)dt$$

$$= f * g + f * h$$

Therefore we have $f * (g + h) = f * g + f * h$

(c) Differentiation Rule: $(f * g)' = f' * g = f * g'$

$$(f * g)' = \left(\int f(t)g(x-t)\,dt\right)'$$

$$= \int f(t)g'(x-t)\,dt$$

$$= f * g'$$

Invoking the commutativity of convolution we have

$$(f * g)' = f' * g$$

Therefore we have $(f * g)' = f' * g = f * g'$

(d) Convolution Theorem

$$F(f * g)(s) = \int \left[\int f(u)g(t-u)du\right] e^{-j2\pi st} dt$$

Changing the integration order we obtain

$$= \int f(u) \left[\int g(t-u)e^{-j2\pi st} dt\right] du$$

Because

$$\int g(t-u)e^{-j2\pi st} dt = e^{-j2\pi st} F(g)(s)$$

We have

$$F(f * g)(s) = \int f(u)e^{-j2\pi st} F(g)(s) du$$

$$= F(g)(s) \int f(u)e^{-j2\pi st} du$$

$$= F(f)F(g)$$

Therefore we have $F(f * g) = F(f)F(g)$

Problem 2

Frequency smoothing

(a) Compute Fourier transform
(b) Keep different number of low frequencies
(c) Reconstruct the original image

Please refer to the *problem2.m* for implementation details

The images reconstructed are displayed below. The number of low frequencies kept increase from left to right and from top to bottom. (Top left: 10x10. Top right: 20x20. Bottom left: 40x40. Bottom right: 512x512)

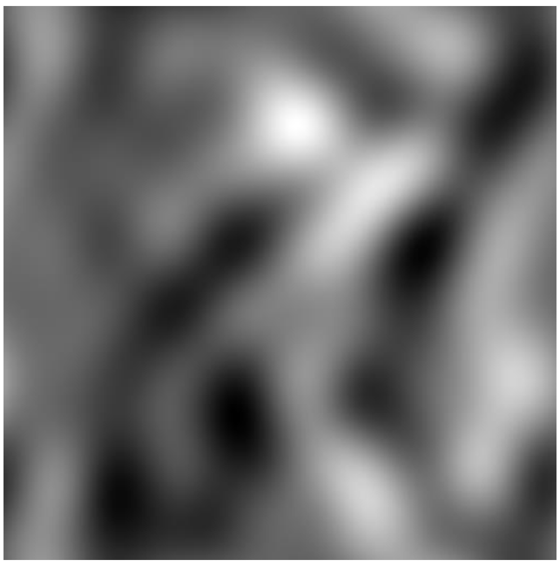



Image reconstructed with different number of low frequencies


Problem 3: Implement gradient decent algorithm for ROF

Problem description

In ROF model the problem to be solved is the minimization on the following equation:

$$\int \sqrt{u_x^2 + u_y^2}\, dx\, dy$$

Such that

$$\int u = \int u_0 \ and \ \int \frac{1}{2}(u - u_0)^2 dxdy = \sigma^2$$

To convert this constrained problem to unconstrained problem, LaGrange multiplier λ is used and the equations becomes:

$$J(u) = \int \sqrt{u_x^2 + u_y^2}\, dxdy + \frac{\lambda}{2}\int (u_0 - u)^2 - \sigma^2\, dx\, dy$$

Minimizing the energy function by take its Euler-langrage

$$u_t = div.\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda(u - u_0)$$

Where u_t is the step size of the gradient descent for the primal.

Implementation Solution

The gradient descent approach to total variation problem is straight forward. Overall the broad idea is that, the algorithm repeatedly computes the new gradient and moves towards the direction of the gradient, while keeping track of tolerance and energy values until convergence.

The criteria for convergence is determined by the maximum number of iterations or the tolerance value for the duality gap. Please refer to the *rof.m* for algorithm details such as computation equations. In order to execute the experiments, please run *problem3.m* to observe the data in the workspace. The variable names are explained in the comments.

Experiment Results

Overall the gradient descent algorithm converges very fast. When the tolerance level set of the duality gap is small (10^{-7}), the algorithm quickly approaches the minimum energy but spends a large number of iterations (~ 1000) to approach the desired duality gap level. As illustrated below, the convergence graph for the ROF model quickly approaches the minimum energy within one hundred iterations, and continue in order for the duality gap to decrease below the tolerance level. This reflects a very common problem of the gradient descent algorithm, which is diminishing gradient. The convergence graph for tolerance level (10^{-5}) is also attached below for comparison. In both cases, the algorithm approaches the minimum energy very quickly. However, a tolerance level with two more significance digit than 10^{-5} takes only about ten times more iterations.

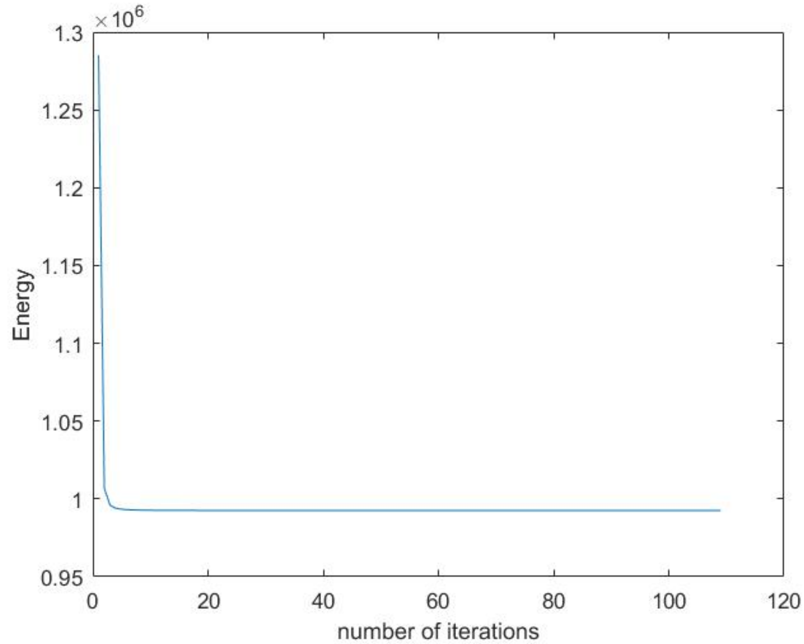
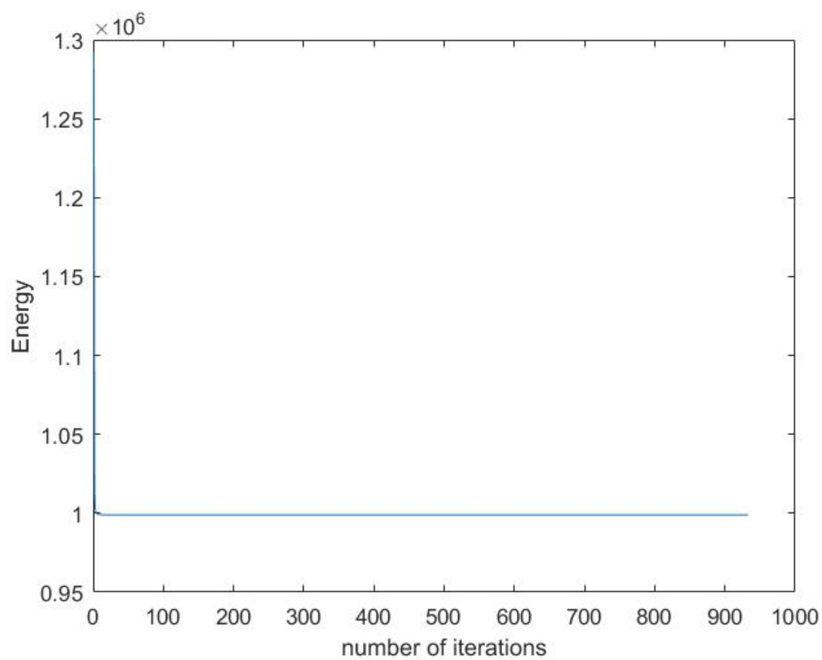


Figure above: convergence graph for $tolerence = 10^{-5}$
Figure blow: convergence graph for $tolerence = 10^{-7}$



The actual results of the experiment is very similar. However, the difference is observable. The denoised images are attached below. Image denoised with smaller tolerance level (on the right) has a high contrast and a slightly clear boundaries between the objects. If observing carefully, the face in the left denoised picture is completely blurred while in the right denoised picture we can observe darkened area which can be interpreted as eyes. In either cases, the denoised images are both better than the original noised image.

Denoised image. (Left with 10^{-5} tolerance; right with 10^{-7} tolerance for duality gap)
Below is the original noised image

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Reference:

University of Utah Computer Science Department:

http://www.cs.utah.edu/~ssingla/AdvIP/Image%20Denoising/Rof.html