Control of Black Hole Parameters for Binary Evolutions

Caltech SURF's First Interim Report

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I. Introduction

In the early twentieth century, Einstein revolutionized the study of gravity by connecting spacetime geometry with physical dynamics. As John Wheeler says, "Spacetime tells matter how to move; matter tells spacetime how to curve" [1]. Being a highly complex theory, many problems of interest only have analytic solutions in special cases with symmetry. In this context, Numerical Relativity emerged as an essential field to solve these problems numerically, allowing us to explore general cases that can be found in the universe. Specifically, simulations of Binary Black Holes (BBH) became very important as gravitational wave detectors were developed, needing to use numerical results to identify and characterize signals in their data [2].

Famously, the Einstein equations relate the curvature of spacetime to the stress-energy of matter, forming a system of ten nonlinear partial differential equations (PDEs). With the 3+1 formalism, we can rearrange these equations so that spacetime is described by spacelike three-dimensional slices of constant time [3]. In doing so, we find that four out of the ten equations do not involve time derivatives, implying that they are constraints that must be satisfied at all times. The remaining six equations describe an evolution of the constraint-satisfying fields. Using this formalism, the Spectral Einstein Code (`SpEC`) [4] runs BBH simulations by first finding initial data and then running an evolution on them. Over time, as `SpEC` faced more challenging BBH with high mass ratios and spins, several improvements had to be made to the initial data techniques, which are summarized in [5].

Despite its success in BBH simulations, `SpEC` shows its limitations in more challenging problems, such as binary neutron star mergers and BBH with extreme configurations. In this context, `SpECTRE` [6] was created as a codebase that follows a better parallelism model and aims to be more scalable [7]. Previous work has already shown that `SpECTRE` can be faster and more accurate than `SpEC` when performing similar tasks due to its use of parallelism [8]. This will be especially needed for the upcoming gravitational wave detectors with higher sensitivity, such as the Cosmic Explorer, the Einstein Telescope and LISA.

As part of an effort to allow researchers to fully simulate BBH in `SpECTRE`, an initial data procedure similar to the one in `SpEC` needs to be completed. This is greatly benefitted by a scalable elliptic solver that was recently developed [8], which can now be used to solve the initial data equations. Be that as it may, before the start of the SURF program, `SpECTRE` did not have a way to enforce specific masses and spins for the black holes or to avoid drifts in the orbital trajectory. As described in the next section, this is the problem that this research project aims to address.

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II. Numerical Method

To find initial data, `SpEC` uses the extended conformal thin-sandwich (XCTS) decomposition [5], which transforms the constraint PDEs into a system of five elliptic PDEs. Before solving the XCTS equations, we need to specify the conformal spatial metric $\bar{\gamma}_{ij}$, the extrinsic curvature trace K, and their respective time derivatives $\partial_t \bar{\gamma}_{ij}$ and $\partial_t K$ [9]. There are many methods for specifying these quantities, but the one that has shown to be the most promising for BBH simulations is the superposed Kerr-Schild (SKS) approach [10]. It enforces quasiequilibrium conditions by setting $\partial_t \bar{\gamma}_{ij} = 0$ and $\partial_t K = 0$. Additionally, as the name suggests, it specifies $\bar{\gamma}_{ij}$ and K by superposing two analytic solutions of Kerr-Schild black holes with masses $M_A^{\rm Kerr}$ and $M_B^{\rm Kerr}$ and with dimensionless spins $\vec{\chi}_A^{\rm Kerr}$ and $\vec{\chi}_B^{\rm Kerr}$.

When constructing the SKS analytical expressions for $\bar{\gamma}_{ij}$ and K, we also need to choose the centers of the black holes $\vec{c}_{A,B}$. When setting up a simulation, it is more convenient to specify their relative distance $\vec{D} = \vec{c}_A - \vec{c}_B$. Because of this, for any choice of $\vec{c}_A$, we have $\vec{c}_B = \vec{c}_A + \vec{D}$. In other words, we only have to choose $\vec{c}_A$.

Having $\bar{\gamma}_{ij}$, K, $\partial_t \bar{\gamma}_{ij}$ and $\partial_t K$ specified, we can use the elliptic solver on the XCTS equations, three of which will be solved for the shift β^i. Similar to any elliptic PDE problem, we have to set boundary conditions before solving these equations. In the boundary conditions of $\vec{\beta}$, we can add a constant velocity $\vec{v}_0$, giving us more control over the initial kinematics of the binary system.

Once the XCTS equations are solved, we have all the information that we need about the zero-time slice of spacetime. With this, we can use an apparent horizon finder to get measurements of the black holes in the constructed initial data. Let M_A, M_B, $\vec{\chi}_A$ and $\vec{\chi}_B$ be the masses and spins of the actual black holes that we have been able to create. In general, these values will differ from the "target" quantities M_A^*, M_B^*, $\vec{\chi}_A^*$ and $\vec{\chi}_B^*$. For example, suppose that we wish to simulate an equal-mass non-spinning case with $M_A^* = M_B^* = 0.5$ and $\vec{\chi}_A^* = \vec{\chi}_B^* = 0$. In order to construct the SKS analytical expressions, it is natural to specify $M_{A,B}^{\rm Kerr} = M_{A,B}^*$ and $\vec{\chi}_{A,B}^{\rm Kerr} = \vec{\chi}_{A,B}^*$. With this, we can solve the XCTS equations and find horizons in the resulting initial data. Once we do it in `SpECTRE`, we find that the resulting black hole parameters are actually $M_{A,B} \approx 0.6$ and $\vec{\chi}_{A,B} \approx -0.002$.

Furthermore, if we measure other parameters of the binary, we might find that they are also not desirable. The center of mass $\vec{C}$ can be found by evaluating the integral

$$\vec{C} = \frac{3}{8\pi M_{\rm ADM}} \oint_{S_\infty} \psi^4 \, d\vec{S}, \tag{1}$$

[5], where ψ is the conformal factor and S_∞ is a surface at infinite radius over which we evaluate the surface integral. $M_{\rm ADM}$ is the ADM mass, defined as

$$M_{\rm ADM} = \frac{1}{16\pi} \int_{S_\infty} \sqrt{\gamma} \gamma^{jn} \gamma^{im} (\partial_j \gamma_{mn} - \partial_m \gamma_{jn}) \, dS_j \tag{2}$$

[9], where γ_{ij} is the spatial metric, γ^{ij} is the inverse spatial metric, and γ is the determinant of γ_{ij}. We can also use the ADM formalism to measure the total linear momentum $\vec{P}_{\rm ADM}$ as

$$P^i_{\rm ADM} = \frac{1}{8\pi} \oint_{S_\infty} (K^{ij} - K\gamma^{ij}) \, dS_j \tag{3}$$

[5], where K^{ij} is the inverse extrinsic curvature. Ideally, $\vec{C}$ and $\vec{P}_{\rm ADM}$ should be equal to zero in order to minimize any drifts of the orbit, especially for long simulations. However, similar to the example above, we can only know these quantities after the initial data has been constructed.

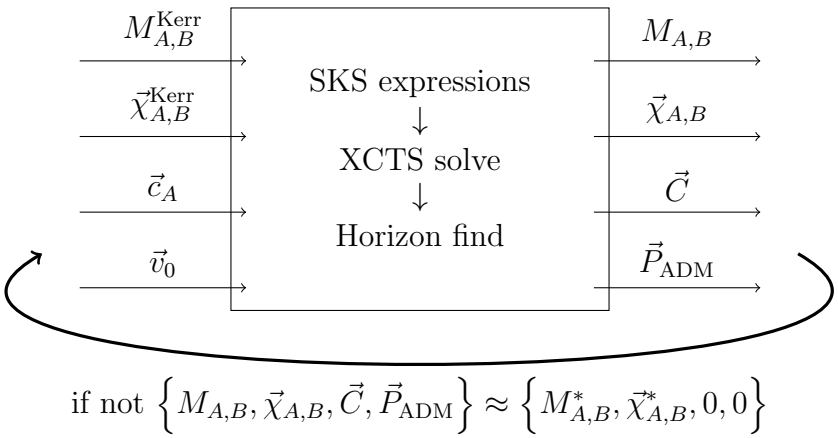

Figure 1: Schematic representation of the control loop.

Note that $M_{A,B}$, $\vec{\chi}_{A,B}$, $\vec{C}$ and $\vec{P}_{\rm ADM}$ are the parameters that we wish to control, but we cannot directly set them to the desired values. The way that `SpEC` handles this issue is by iterating over different choices of the free data ($M^{\rm Kerr}_{A,B}$, $\vec{\chi}^{\rm Kerr}_{A,B}$, $\vec{c}_A$ and $\vec{v}_0$), trying to find the ones that result in the ideal physical parameters. The goal of this project is to reproduce this iterative scheme in `SpECTRE`. Figure 1 shows a schematic representation of this "control loop".

Let the choices of $M^{\rm Kerr}_{A,B}$ and $\vec{\chi}^{\rm Kerr}_{A,B}$ be represented as

$$\mathbf{u} = \left\{M^{\rm Kerr}_A, M^{\rm Kerr}_B, \chi^{{\rm Kerr},x}_A, \chi^{{\rm Kerr},y}_A, \chi^{{\rm Kerr},z}_A, \chi^{{\rm Kerr},x}_B, \chi^{{\rm Kerr},y}_B, \chi^{{\rm Kerr},z}_B\right\}. \tag{4}$$

Also, let the difference between the current and target physical parameters be represented as

$$\begin{aligned}\mathbf{F}(\mathbf{u}) = \Big\{&M_A - M^*_A, M_B - M^*_B, \chi^x_A - \chi^{*,x}_A, \chi^y_A - \chi^{*,y}_A,\\ &\chi^z_A - \chi^{*,z}_A, \chi^x_B - \chi^{*,x}_B, \chi^y_B - \chi^{*,y}_B, \chi^z_B - \chi^{*,z}_B\Big\}.\end{aligned} \tag{5}$$

As discussed in the example above, it is natural to use the target parameters as an initial guess for the Kerr masses and spins:

$$\mathbf{u}_0 = \left\{M^*_A, M^*_B, \chi^{*,x}_A, \chi^{*,y}_A, \chi^{*,z}_A, \chi^{*,x}_B, \chi^{*,y}_B, \chi^{*,z}_B\right\}. \tag{6}$$

If we also assume that $M_{A,B}|_0 \approx M^{\rm Kerr}_{A,B}|_0$ and $\vec{\chi}_{A,B}|_0 \approx \vec{\chi}^{\rm Kerr}_{A,B}|_0$, then a good initial approximation for the Jacobian of $\mathbf{F}(\mathbf{u})$ is $\mathbb{J}_0 \approx \mathbb{I}$ (a 8x8 identity matrix). With this, we can update our free data at every control iteration $k \geq 1$ using a Newton-Raphson scheme [11]:

$$\mathbf{u}_k = \mathbf{u}_{k-1} - \mathbb{J}^{-1}_{k-1} \cdot \mathbf{F}_{k-1}. \tag{7}$$

Then, we can update the Jacobian with Broyden's method [11]:

$$\mathbb{J}_k = \mathbb{J}_{k-1} + \frac{\mathbf{F}_k \otimes \mathbf{\Delta u}_k}{||\mathbf{\Delta u}_k||^2}, \tag{8}$$

where we use $\Delta x_k = x_k - x_{k-1}$ for any variable x from now on.

We can also come up with iterative schemes for the choices of $\vec{c}_A$ and $\vec{v}_0$. To drive the center of mass $\vec{C}$ to zero we can update $\vec{c}_A$ as

$$\vec{c}_{A,k} = \vec{c}_{A,k-1} - \vec{C}_{k-1} - \frac{M_{A,k-1}\Delta M_{B,k-1} - M_{B,k-1}\Delta M_{A,k-1}}{(M_{A,k-1} + M_{B,k-1})^2}\vec{D} \tag{9}$$

[5]. Considering a small perturbation of $\vec{v}_0$, we can drive the linear momentum $\vec{P}_{\rm ADM}$ to zero with

$$\vec{v}_{0,k+1} = \vec{v}_{0,k} - \frac{\vec{P}_{{\rm ADM},k}}{M_k} + \Delta M_k(\vec{v}_{0,k} + \vec{\Omega}_0 \times \vec{c}_{A,k}) - \vec{\Omega}_0 \times \delta\vec{c}_{A,k} - \frac{\Delta M_{B,k}}{M_k}\vec{\Omega}_0 \times \vec{D} \tag{10}$$

[5], where $M_k = M_{A,k} + M_{B,k}$, $\vec{\Omega}_0$ is the user-specified initial orbit angle, and $\delta\vec{c}_{A,k}$ is a small perturbation of $\vec{c}_{A,k}$.

III. Progress Update

IV. Future Work

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