Math Booklet [1]

Iago Mendes [2]

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[1] A booklet with notes of Math.

[2] Oberlin College; double major in Physics (Astrophysics) and Computer Science.

Contents

Algebra

1.1 Linear Algebra

1.1.1 Matrices

- Notation
$$A = [a_{ij}]$$
- Matrix Addition
$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
- Scalar multiplication
$$c[a_{ij}] = [ca_{ij}]$$
- Transpose
$$(aT)_{ij} = a_{ji}$$
- Matrix Multiplication
$$c_{ij} = (\text{ith row of A})(\text{jth column of B}) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Geometry

2.1 Analytic Geometry

2.1.1 Quadric Surfaces

- Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- Elliptic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperbolic paraboloid

$$\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

- Elliptic cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Hyperboloid of two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

2.1.2 Coordinate systems

- Cartesian coordinates ($\mathbb{R}^2$ and $\mathbb{R}^3$)

$$(x, y) \qquad (x, y, z)$$

- Polar coordinates ($\mathbb{R}^2$)

$$(r, \theta)$$

  - Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

  - Polar/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates ($\mathbb{R}^3$)

$$(r, \theta, z)$$

- Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

- Cylindrical/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates ($\mathbb{R}^3$)

$$(\rho, \phi, \theta)$$

- Typical restrictions

$$\rho \geq 0$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

- Spherical/cylindrical conversions

$$\begin{cases} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan\phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

- Spherical/rectangular conversions

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \frac{y}{x} \end{cases}$$

Calculus

3.1 Single Variable Calculus

3.1.1 Limits

- Squeeze Theorem

$$\begin{cases} g(x) \leq f(x) \leq h(x) \\ \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \end{cases}$$
$$\therefore \quad \lim_{x\to a} f(x) = L$$

- Fundamental Trigonometric Limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

- Fundamental Exponential Limit

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

3.1.2 Differentiation

- Definition

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

  - Constant function

$$\frac{d}{dx}c = 0$$

- Derivative of Transcendent Functions
  - Sine function

$$\frac{d}{dx}\sin x = \cos x$$

  - Cosine function

$$\frac{d}{dx}\cos x = -\sin x$$

  - Logarithm function

$$\frac{d}{dx}\log x = \frac{1}{x}$$

  - Exponential function

$$\frac{d}{dx}e^x = e^x$$

- Properties

- Sum and difference

$$(u+v)' = u' + v'$$

- Product

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

- Produc with a constant

$$(c \cdot u)' = c \cdot u'$$

- Quotient

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

  * Polynomial function

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = nx^{n-1}$$

  * Tangent function

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \sec^2 x$$

  * Secant function

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \sec x \cdot \tan x$$

- Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

  - Exponential function (not natural)

$$\frac{d}{dx}a^x = a^x \cdot \log a$$

  - Logarithm of a function

$$\frac{d}{dx}\log g(x) = \frac{g'(x)}{g(x)}$$

- Derivative of The Inverse Function

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

  - Arcsine function

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

  - Arctangent function

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

- Mean Value Theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

  in which $a < c < b$

3.1.3 Applications of Differentiation

- L'Hospital Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f'(a)}{g'(a)}$$

  Cases: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

- Infinity Hierarchy

$$\log x < \sqrt[n]{x} < \sqrt{x} < x < x^2 < x^n < e^x < x! < x^x \to \infty$$

- Curve Sketching

$$f' > 0 \Rightarrow f \text{ is increasing}$$
$$f' < 0 \Rightarrow f \text{ is decreasing}$$
$$f'' > 0 \Rightarrow f' \text{ is increasing} \Rightarrow f \text{ is concave up}$$
$$f'' < 0 \Rightarrow f' \text{ is decreasing} \Rightarrow f \text{ is concave down}$$

  - General Strategy
    1. Plot discontinuities (especially infinite), endpoints (or $x \to \pm\infty$), and easy points (optional);
    2. Solve $f'(x) = 0$, and plot critical points and values;
    3. Decide whether $f' > 0$ or $f' < 0$ on each interval between critical points;
    4. Analyse when the curve is concave up ($f'' > 0$) or down ($f'' < 0$), and what is/are the inflection point(s) ($f''(x_0) = 0$); and
    5. Combine everything.

- Linear Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x \qquad (x \approx 0)$$
  2. Cosine
$$\cos x \approx 1 \qquad (x \approx 0)$$
  3. Exponential
$$e^x \approx 1 + x \qquad (x \approx 0)$$
  4. Logarithm
$$\log(1 + x) \approx x \qquad (x \approx 0)$$
  5. Sum to the power of n
$$(1 + x)^n \approx 1 + n \cdot x \qquad (x \approx 0)$$

- Quadratic Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} \cdot (x - a)^2 \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x \qquad (x \approx 0)$$
  2. Cosine
$$\cos x \approx 1 - \frac{x^2}{2} \qquad (x \approx 0)$$
  3. Exponential
$$e^x \approx 1 + x + \frac{x^2}{2} \qquad (x \approx 0)$$
  4. Logarithm
$$\log(1 + x) \approx x - \frac{x^2}{2} \qquad (x \approx 0)$$
  5. Sum to the power of n
$$(1 + x)^n \approx 1 + n \cdot x + \frac{n(n-1)}{2} x^2 \qquad (x \approx 0)$$

- Taylor's Series

$$f(x) \approx f(a) + \sum_{n=1}^{\infty} \frac{(x - a)^n}{n!} f^{(n)}(a) \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \qquad (x \approx 0)$$

4. Logarithm

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \qquad (x \approx 0)$$

$$\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ... \qquad (x \approx 1)$$

5. Arctangent

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \qquad (x \approx 0)$$

- Power Series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = \sum_{n=0}^{\infty} a_n x^n$$

OBS.: converges when $|x| < R$, where R is the radius of convergence.

1. Geometric Series ($R = 1$)

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

3.1.4 Integration

- Definition

$$\int_a^b f(x)\,dx = \text{Area under the curve}$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

- Properties

1.
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2.
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

3.
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \quad \leftrightarrow \quad f(x) \le g(x)$$

4.
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

5.
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

6.
$$\int_a^a f(x)dx = 0$$

- Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$F(x) = \int_a^x f(t)\,dt$$
$$F'(x) = f(x)$$

- Antiderivatives (Indefinite Integral)

1. Powers (Polynomials)

$$\int x^n \; dx = \frac{x^{n+1}}{n+1} + C \quad \leftrightarrow \quad n \neq -1$$

2. Trigonometric Functions

$$\int \sin x \; dx = -\cos x + C$$
$$\int \cos x \; dx = \sin x + C$$
$$\int \sec^2 x \; dx = \tan x + C$$

3. Important Fractions

$$\int \frac{dx}{x} = \log|x| + C$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$
$$\int \frac{dx}{1-x^2} = \tan^{-1} x + C$$

4. Others

$$\int \log x \; dx = x(\log x - 1) + C$$

- Properties of Some Transcendental Functions

1. $L(x) = \int_1^x \frac{dt}{t}$

$$L(ab) = L(a) + L(b)$$

2. $F(x) = \int_0^x e^{-t^2} dt$

$$\lim_{x \to \infty} F(x) = \frac{\sqrt{\pi}}{2}$$

3. $Li(x) = \int_2^x \frac{dt}{ln\ t}$

$$Li(x) \approx \text{number of primes} < x$$

- Improper Integrals $\left(f(x) = \frac{1}{x^p}\right)$

$$\int_1^\infty \frac{dx}{x^p} \to \infty \qquad \text{(diverges if } p \leq 1)$$
$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \qquad \text{(converges if } p > 1)$$
$$\int_0^1 \frac{dx}{x^p} \to +\infty \qquad \text{(diverges if } p \geq 1)$$
$$\int_0^1 \frac{dx}{x^p} = \frac{1}{p-1} \qquad \text{(converges if } p < 1)$$

3.1.5 Techniques of Integration

- Integration by Substitution

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt$$

- Integration by Parts

$$\int_a^b uv'\, dx = (uv)|_a^b - \int_a^b u'v\, dx$$

- Trigonometric Integration

1. Powers - Easy case (at least one odd exponent) $\rightsquigarrow$ use the fundamental formula and substitution

$$\sin^2 x + \cos^2 x = 1$$

2. Powers - Hard case (only even exponents) $\rightsquigarrow$ use half-angle formulas or other trigonometric identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

   – products $\rightarrow$ sums

$$2 \cdot \sin m \cdot \cos n = \sin(m+n) + \sin(m-n)$$
$$2 \cdot \cos m \cdot \cos n = \cos(m+n) + \cos(m-n)$$
$$2 \cdot \sin m \cdot \cos n = \cos(m-n) - \cos(m+n)$$

3. Tangent

$$\int \tan x\ dx = -ln\,(\cos x) + C$$

4. Secant

$$\int \sec x\ dx = ln\,(\sec x + \tan x) + C$$

- Summary of Trigonometric Substitutions

If integrand contains	**make substitution**	**to get**
$\sqrt{a^2 - x^2}$	$x = a\cos\theta$ or $x = a\sin\theta$	$a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a\tan\theta$

- Partial Fractions (common cases)

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$
$$\frac{p(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$
$$\frac{p(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$
$$\frac{p(x)}{x^2+a} = \frac{Ax+B}{x^2+a}$$
$$\frac{p(x)}{(x^2+a)^2} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{(x^2+a)^2}$$

3.1.6 Applications of Integration

- Volumes

  1. General

  $$V = \int A(x)dx$$

  2. Solid of Revolution (Method of Disks)
     - Method of Disks

     $$V = \int \pi y^2 \, dx \qquad \text{around x-axis}$$

     - Method of Shells

     $$V = \int (\text{circunference})(\text{height}) \, dx \qquad \text{around y-axis}$$

     (a) Sphere

     $$V = \pi \left(ax^2 - \frac{x^3}{3} \right)$$

- Average Value

$$\frac{f(1) + f(2) + ... + f(n)}{n} = \frac{\Sigma_{i=1}^{n}}{n} \approx \frac{1}{n} \int_0^n f(x)dx$$

$$Ave(a, b) = \frac{1}{b-a} \int_a^b f(x)dx$$

$$Ave(a, b) = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

- Arc-length

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + [f'(x)]^2} \, dx$$

$$S = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

- Superficial Area

$$A = \int_{s_0}^{s_n} 2\pi y \, ds = 2\pi \int_{x_0}^{x_n} f(x)\sqrt{1 + [f'(x)]^2} \, dx \qquad \text{(around x-axis)}$$

$$A = \int_{s_0}^{s_n} 2\pi x \, ds = 2\pi \int_{x_0}^{x_n} f(x)\sqrt{1 + [f'(x)]^2} \, dx \qquad \text{(around y-axis)}$$

  - Sphere

  $$A = 2\pi a(x_1 - x_2)$$

- Numerical Integration

  1. Riemann Sums

  $$\int_a^b f(x)dx \approx \Delta x(y_1 + y_2 + ... + y_n) \qquad \text{(right-hand sum)}$$

  $$\int_a^b f(x)dx \approx \Delta x(y_0 + y_1 + ... + y_{n-2} + y_{n-1}) \qquad \text{(left-hand sum)}$$

  If $f(x)$ decreases, lower estimation = right-hand sum, and higher estimation = left-hand sum.
  If $f(x)$ increases, higher estimation = right-hand sum, and lower estimation = left-hand sum.

  2. Trapezoidal Rule

  $$\int_a^b f(x)dx \approx \Delta x \left(\frac{y_0}{2} + y_1 + ... + y_{n-1} + \frac{y_n}{2} \right)$$

3. Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + ... + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

OBS.: $|\text{simpson's value} - \text{exact value}| \approx (\Delta x)^2$

3.2 Multivariable Calculus

$\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

3.2.1 Partial Derivatives

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_n)}{h}$$

- Gradient

$$\nabla f = (f_{x_1},\ ...,\ f_{x_n})$$
$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}),\ ...,\ f_{x_n}(\mathbf{a}))$$

- Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

- Tangent plane

$$z = h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

  - Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

  - Hyperplane

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$
$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

- Differentiability

  1. $D\mathbf{f}(\mathbf{a})$ exists
  2. $$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{||\mathbf{x} - \mathbf{a}||} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_1}} f(x_1, \dots, x_n)$$

  - Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$$

- Chain rule

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0) D\mathbf{x}(\mathbf{t}_0)$$
$$f'(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)$$

- Directional derivative

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = ||\nabla f(\mathbf{a})|| \cos\theta$$

3.2.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems
  1. If f is a scalar-valued function of class C^2, then

  $$\nabla \times (\nabla f) = \mathbf{0}$$

  2. If $\mathbf{F}$ is a vector-valued function of class C^2 on $X \subseteq \mathbb{R}^3$, then

  $$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

3.2.3 Maxima and Minima

- Taylor Polynomials
  - First-order

  $$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i)$$
  $$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

  - Second-order

  $$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$
  $$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion
  - Hessian matrix

  $$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1 x_1}(\mathbf{a}) & \cdots & f_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(\mathbf{a}) & \cdots & f_{x_n x_n}(\mathbf{a}) \end{bmatrix}$$

- Principal minor

$$d_k = \text{determinant of the upperleftmost } k \times k \text{ submatrix of } Hf(\mathbf{a})$$

1. If all $d_k > 0$, then the critical point $\mathbf{a}$ gives a local minimum.
2. If $d_1 < 0,\ d_2 > 0,\ d_3 < 0,\ \ldots$, then the critical point $\mathbf{a}$ gives a local maximum.
3. If neither case 1 nor case 2 occurs, then $\mathbf{a}$ is a saddle point.

If $d_n = 0$, the critical point $\mathbf{a}$ is degenerate and the test fails.

- Extrema Value Theorem

If D is a compact region in $\mathbb{R}^n$ and $f : D \to R$ is continuous, then f must have a (global) maximum and minimum values on D.

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

  - Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

3.2.4 Multiple Integration

- Double Integrals

$$\iint_R f\, dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem ($\mathbb{R}^2$)

$$\iint_R f\, dA = \int_c^d \int_a^b f(x,y)\, dxdy = \int_a^b \int_c^d f(x,y)\, dydx$$

- Elementary Regions ($\mathbb{R}^2$)
  - Type 1
    - Boundaries

$$\begin{aligned} x &= a & x &= b \\ y &= \gamma(x) & y &= \delta(x) \end{aligned}$$

    - Theorem

$$\iint_D f\, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\, dydx$$

  - Type 2
    - Boundaries

$$\begin{aligned} x &= \alpha(y) & x &= \beta(y) \\ y &= c & y &= d \end{aligned}$$

    - Theorem

$$\iint_D f\, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\, dxdy$$

  - Type 3

Simultaneously of type 1 and type 2.

- Triple Integrals

$$\iiint_B f\, dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i,j,k=1}^{n} f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

- Fubini's Theorem ($\mathbb{R}^3$)

$$\iiint_B f\ dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\ dzdydx = \text{other orders}$$

- Elementary Regions ($\mathbb{R}^3$)
  - Type 1
    * Boundaries

$$z = \phi(x,y) \qquad z = \psi(x,y)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\ dzdydx$$

  - Type 2
    * Boundaries

$$x = \alpha(y,z) \qquad z = \beta(y,z)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z)\ dxdydz$$

  - Type 3
    * Boundaries

$$y = \gamma(x,z) \qquad y = \delta(x,z)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\gamma(x,z)}^{\delta(x,z)} f(x,y,z)\ dydxdz$$

  - Type 4

Simultaneously of types 1, 2, and 3.

- The Jacobian
$\mathbf{T} : D^* \to D$
$\mathbf{T}(u,v) = (x(u,v),\ y(u,v))$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det D\mathbf{T}(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$\mathbf{U} : W^* \to W$
$\mathbf{U}(u,v,w) = (x(u,v,w),\ y(u,v,w))$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det D\mathbf{U}(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

  - Polar coordinates

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

  - Cylindrical coordinates

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

- Spherical coordinates

$$\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \rho^2 \sin\phi$$

- Change of Integration

$$\iint_D f(x,y)\ dxdy = \iint_{D^*} f(x(u,v),\ y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ dudv$$

$$\iiint_W f(x,y,z)\ dxdyz = \iiint_{W^*} f(x(u,v,w),\ y(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|\ dudvdw$$

3.2.5 Applications of Integration

- Average Value
  - $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

$$f_{avg} = \frac{1}{A_D} \iint_D f(x,y)\ dA = \frac{\iint_D f(x,y)\ dA}{\iint_D dA}$$

  - $f : W \subset \mathbb{R}^3 \to \mathbb{R}$

$$f_{avg} = \frac{1}{V_W} \iiint_W f(x,y,z)\ dV = \frac{\iiint_W f(x,y,z)\ dV}{\iiint_W dV}$$

- Center of mass
  - $\delta : D \subset \mathbb{R}^2 \to \mathbb{R}$

$$\overline{x} = \frac{\iint_D x\delta(x,y)\ dA}{\iint_D \delta(x,y)\ dA} \qquad \overline{y} = \frac{\iint_D y\delta(x,y)\ dA}{\iint_D \delta(x,y)\ dA}$$

    * Centroid (constant density)

$$\overline{x} = \frac{1}{A_D} \iint_D x\ dA = \frac{\iint_D x\ dA}{\iint_D\ dA}$$

$$\overline{y} = \frac{1}{A_D} \iint_D y\ dA = \frac{\iint_D y\ dA}{\iint_D\ dA}$$

  - $\delta : W \subset \mathbb{R}^3 \to \mathbb{R}$

$$\overline{x} = \frac{\iiint_W x\delta(x,y,z)\ dV}{\iiint_W \delta(x,y,z)\ dV} \qquad \overline{y} = \frac{\iiint_W y\delta(x,y,z)\ dV}{\iiint_W \delta(x,y,z)\ dV}$$

$$\overline{z} = \frac{\iiint_W z\delta(x,y,z)\ dV}{\iiint_W \delta(x,y,z)\ dV}$$

    * Centroid (constant density)

$$\overline{x} = \frac{1}{V_W} \iiint_W x\ dV = \frac{\iiint_W x\ dV}{\iiint_W\ dV}$$

$$\overline{y} = \frac{1}{V_W} \iiint_W y\ dV = \frac{\iiint_W y\ dV}{\iiint_W\ dV}$$

$$\overline{z} = \frac{1}{V_W} \iiint_W z\ dV = \frac{\iiint_W z\ dV}{\iiint_W\ dV}$$

3.2.6 Line Integrals

- Scalar Line Integrals in $\mathbb{R}^n$

$$\int_{\mathbf{x}} f\ ds = \int_a^b f(\mathbf{x})\ ||\mathbf{x}'(t)||\ \ dt$$

- Vector Line Integrals in $\mathbb{R}^n$

$$\int_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}) \bullet ||\mathbf{x}'(t)||\ \ dt$$

- Differential form

$$\int_{\mathbf{x}} \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{x}} M\ dx + N\ dy + P\ dz$$

- Green's Theorem

$$\oint_C \mathbf{F} \bullet d\mathbf{s} = \int_{\mathbf{x}} M\ dx + N\ dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\ dxdy$$

- Vector reformulation

$$\oint_C \mathbf{F} \bullet d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \bullet \hat{\mathbf{k}}\, dA$$

- Divergence Theorem

$$\oint_C \mathbf{F} \bullet \hat{\mathbf{n}}\ ds = \iint_D \nabla \bullet \mathbf{F}\, dA$$

- Conservative Vector Fields
  - Path independence

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{s} = \int_{C_2} \mathbf{F} \bullet d\mathbf{s}$$

  - Theorems

1. A continuous vector field **F** has path-independent line integrals if and only if

$$\oint_C \mathbf{F} \bullet d\mathbf{s} = 0$$

   for all simple closed curves C.
2. Suppose that **F** is continuous on a *connected* region $R \subseteq \mathbb{R}^n$; then,

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \mathbf{F} \text{ has path-independent line integrals over curves in } R$$

3. Suppose that **F** is of class C^1 on a *simply-connected* domain R in $\mathbb{R}^2$ or $\mathbb{R}^3$; then,

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \nabla \times \mathbf{F} = \mathbf{0}$$