

Math Booklet ¹

Iago Mendes ²

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¹A booklet with notes of Math.

²Oberlin College; double major in Physics (Astrophysics) and Computer Science.

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Algebra

1.1 Linear Algebra

1.1.1 Matrices

- Notation

$$A = [a_{ij}]$$

- Matrix Addition

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- Scalar multiplication

$$c[a_{ij}] = [ca_{ij}]$$

- Transpose

$$(aT)_{ij} = a_{ji}$$

- Matrix Multiplication

$$c_{ij} = (\text{ith row of A})(\text{jth column of B}) = \sum_{k=1}^n a_{ik}b_{kj}$$

Geometry

2.1 Analytic Geometry

2.1.1 Coordinate systems

- Cartesian coordinates (\mathbb{R}^2 and \mathbb{R}^3)

$$(x, y) \quad (x, y, z)$$

- Polar coordinates (\mathbb{R}^2)

$$(r, \theta)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Polar/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates (\mathbb{R}^3)

$$(r, \theta, z)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Cylindrical/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates (\mathbb{R}^3)

$$(\rho, \phi, \theta)$$

- Typical restrictions

$$\begin{aligned} \rho &\geq 0 \\ 0 &\leq \phi \leq \pi \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Spherical/cylindrical conversions

$$\begin{cases} r = \rho \sin \phi \\ \theta = \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan \phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

- Spherical/rectangular conversions

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan \phi = \frac{\sqrt{x^2 + y^2}}{z} \\ \tan \theta = \frac{y}{x} \end{cases}$$

Calculus

3.1 Multivariable Calculus

$$\mathbf{f}: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$

3.1.1 Partial Derivatives

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

- Gradient

$$\nabla f = (f_{x_1}, \dots, f_{x_n})$$

$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}), \dots, f_{x_n}(\mathbf{a}))$$

- Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

- Tangent plane

$$z = h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

- Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

- Hyperplane

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

- Differentiability

1. $D\mathbf{f}(\mathbf{a})$ exists
- 2.

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \dots \frac{\partial}{\partial x_{i_1}} f(x_1, \dots, x_n)$$

- Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$$

- Chain rule

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0)D\mathbf{x}(\mathbf{t}_0)$$

$$f'(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)$$

- Directional derivative

$$D_{\hat{\mathbf{u}}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = \|\nabla f(\mathbf{a})\| \cos \theta$$

3.1.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems

1. If f is a scalar-valued function of class C^2 , then

$$\nabla \times (\nabla f) = \mathbf{0}$$

2. If \mathbf{F} is a vector-valued function of class C^2 on $X \subseteq \mathbb{R}^3$, then

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

3.1.3 Maxima and Minima

- Taylor Polynomials

- First-order

$$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i)$$

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Second-order

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$

$$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion

- Hessian matrix

$$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1 x_1}(\mathbf{a}) & \cdots & f_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(\mathbf{a}) & \cdots & f_{x_n x_n}(\mathbf{a}) \end{bmatrix}$$

- Principal minor

d_k = determinant of the upperleftmost $k \times k$ submatrix of $Hf(\mathbf{a})$

1. If all $d_k > 0$, then the critical point \mathbf{a} gives a local minimum.
2. If $d_1 < 0$, $d_2 > 0$, $d_3 < 0$, \dots , then the critical point \mathbf{a} gives a local maximum.
3. If neither case 1 nor case 2 occurs, then \mathbf{a} is a saddle point.

If $d_n = 0$, the critical point \mathbf{a} is degenerate and the test fails.

- Extrema Value Theorem

If D is a compact region in \mathbb{R}^n and $f : D \rightarrow \mathbb{R}$ is continuous, then f must have a (global) maximum and minimum values on D .

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

- Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

3.1.4 Multiple Integration

- Double Integrals

$$\iint_R f \, dA = \lim_{\text{all } \Delta x_i, \Delta y_j \rightarrow 0} \sum_{i,j=1}^n f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem

$$\iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx dy = \int_a^b \int_c^d f(x, y) \, dy dx$$

- Elementary Regions

- Type 1

- * Boundaries

$$\begin{array}{ll} x = a & x = b \\ y = \gamma(x) & y = \delta(x) \end{array}$$

- * Theorem

$$\iint_D f \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y) \, dy dx$$

- Type 2

- * Boundaries

$$\begin{array}{ll} x = \alpha(y) & x = \beta(y) \\ y = c & y = d \end{array}$$

- * Theorem

$$\iint_D f \, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x, y) \, dx dy$$

- Type 3

Both type 1 and type 2.