Math Booklet [1]

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[1] A booklet with notes of Math.

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Contents

Algebra

1.1 Linear Algebra

1.1.1 Matrices

- Notation

$$A = [a_{ij}]$$

- Matrix Addition

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- Scalar multiplication

$$c[a_{ij}] = [ca_{ij}]$$

- Transpose

$$(aT)_{ij} = a_{ji}$$

- Matrix Multiplication

$$c_{ij} = (\text{ith row of A})(\text{jth column of B}) = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Geometry

2.1 Analytic Geometry

2.1.1 Coordinate systems

- Cartesian coordinates ($\mathbb{R}^2$ and $\mathbb{R}^3$)

$$(x, y) \qquad (x, y, z)$$

- Polar coordinates ($\mathbb{R}^2$)

$$(r, \theta)$$

  - Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

  - Polar/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates ($\mathbb{R}^3$)

$$(r, \theta, z)$$

  - Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

  - Cylindrical/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates ($\mathbb{R}^3$)

$$(\rho, \phi, \theta)$$

  - Typical restrictions

$$\rho \geq 0$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

  - Spherical/cylindrical conversions

$$\begin{cases} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan\phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

– Spherical/rectangular conversions

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \frac{y}{x} \end{cases}$$

Calculus

3.1 Single Variable Calculus

3.1.1 Limits

- Squeeze Theorem

$$\begin{cases} g(x) \leq f(x) \leq h(x) \\ \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \end{cases}$$
$$\therefore \quad \lim_{x\to a} f(x) = L$$

- Fundamental Trigonometric Limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

- Fundamental Exponential Limit

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

3.1.2 Differentiation

- Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

  - Constant function

$$\frac{d}{dx}c = 0$$

- Derivative of Transcendent Functions
  - Sine function

$$\frac{d}{dx}\sin x = \cos x$$

  - Cosine function

$$\frac{d}{dx}\cos x = -\sin x$$

  - Logarithm function

$$\frac{d}{dx}\log x = \frac{1}{x}$$

  - Exponential function

$$\frac{d}{dx}e^x = e^x$$

- Properties

- Sum and difference
$$(u+v)' = u' + v'$$
- Product
$$(u \cdot v)' = u' \cdot v + u \cdot v'$$
- Produc with a constant
$$(c \cdot u)' = c \cdot u'$$
- Quotient
$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$
  * Polynomial function
$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = nx^{n-1}$$
  * Tangent function
$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \sec^2 x$$
  * Secant function
$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \sec x \cdot \tan x$$

- Chain Rule
$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$
  - Exponential function (not natural)
$$\frac{d}{dx}a^x = a^x \cdot \log a$$
  - Logarithm of a function
$$\frac{d}{dx}\log g(x) = \frac{g'(x)}{g(x)}$$
- Derivative of The Inverse Function
$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$
  - Arcsine function
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
  - Arctangent function
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$
- Mean Value Theorem
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$
  in which $a < c < b$

3.1.3 Applications of Differentiation

- L'Hospital Rule
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f'(a)}{g'(a)}$$
  Cases: $\frac{0}{0}$ and $\frac{\infty}{\infty}$
- Infinity Hierarchy
$$\log x < \sqrt[n]{x} < \sqrt{x} < x < x^2 < x^n < e^x < x! < x^x \to \infty$$

- Curve Sketching

$$f' > 0 \Rightarrow f \text{ is increasing}$$
$$f' < 0 \Rightarrow f \text{ is decreasing}$$
$$f'' > 0 \Rightarrow f' \text{ is increasing} \Rightarrow f \text{ is concave up}$$
$$f'' < 0 \Rightarrow f' \text{ is decreasing} \Rightarrow f \text{ is concave down}$$

  - General Strategy
    1. Plot discontinuities (especially infinite), endpoints (or $x \to \pm\infty$), and easy points (optional);
    2. Solve $f'(x) = 0$, and plot critical points and values;
    3. Decide whether $f' > 0$ or $f' < 0$ on each interval between critical points;
    4. Analyse when the curve is concave up ($f'' > 0$) or down ($f'' < 0$), and what is/are the inflection point(s) ($f''(x_0) = 0$); and
    5. Combine everything.

- Linear Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x \qquad (x \approx 0)$$
  2. Cosine
$$\cos x \approx 1 \qquad (x \approx 0)$$
  3. Exponential
$$e^x \approx 1 + x \qquad (x \approx 0)$$
  4. Logarithm
$$\log(1 + x) \approx x \qquad (x \approx 0)$$
  5. Sum to the power of n
$$(1 + x)^n \approx 1 + n \cdot x \qquad (x \approx 0)$$

- Quadratic Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} \cdot (x - a)^2 \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x \qquad (x \approx 0)$$
  2. Cosine
$$\cos x \approx 1 - \frac{x^2}{2} \qquad (x \approx 0)$$
  3. Exponential
$$e^x \approx 1 + x + \frac{x^2}{2} \qquad (x \approx 0)$$
  4. Logarithm
$$\log(1 + x) \approx x - \frac{x^2}{2} \qquad (x \approx 0)$$
  5. Sum to the power of n
$$(1 + x)^n \approx 1 + n \cdot x + \frac{n(n-1)}{2} x^2 \qquad (x \approx 0)$$

- Taylor's Series

$$f(x) \approx f(a) + \sum_{n=1}^{\infty} \frac{(x - a)^n}{n!} f^{(n)}(a) \qquad (x \approx a)$$

  1. Sine
$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \qquad (x \approx 0)$$

4. Logarithm

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \qquad (x \approx 0)$$

$$\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ... \qquad (x \approx 1)$$

5. Arctangent

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... \qquad (x \approx 0)$$

- Power Series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = \sum_{n=0}^{\infty} a_n x^n$$

OBS.: converges when $|x| < R$, where R is the radius of convergence.

1. Geometric Series ($R = 1$)

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

3.1.4 Integration

- Definition

$$\int_a^b f(x)\,dx = \text{Area under the curve}$$

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

- Properties

1.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

2.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

3.

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \quad \leftrightarrow \quad f(x) \leq g(x)$$

4.

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

5.

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

6.

$$\int_a^a f(x)dx = 0$$

- Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$F(x) = \int_a^x f(t)\,dt$$
$$F'(x) = f(x)$$

- Antiderivatives (Indefinite Integral)

1. Powers (Polynomials)

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C \quad \leftrightarrow \quad n \neq -1$$

2. Trigonometric Functions

$$\int \sin x\ dx = -\cos x + C$$
$$\int \cos x\ dx = \sin x + C$$
$$\int \sec^2 x\ dx = \tan x + C$$

3. Important Fractions

$$\int \frac{dx}{x} = \log|x| + C$$
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$
$$\int \frac{dx}{1-x^2} = \tan^{-1} x + C$$

4. Others

$$\int \log x\ dx = x(\log x - 1) + C$$

- Properties of Some Transcendental Functions

1. $L(x) = \int_1^x \frac{dt}{t}$

$$L(ab) = L(a) + L(b)$$

2. $F(x) = \int_0^x e^{-t^2} dt$

$$\lim_{x\to\infty} F(x) = \frac{\sqrt{\pi}}{2}$$

3. $Li(x) = \int_2^x \frac{dt}{ln\ t}$

$$Li(x) \approx \text{number of primes} < x$$

- Improper Integrals $\left(f(x) = \frac{1}{x^p}\right)$

$$\int_1^\infty \frac{dx}{x^p} \to \infty \qquad \text{(diverges if } p \leq 1)$$
$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \qquad \text{(converges if } p > 1)$$
$$\int_0^1 \frac{dx}{x^p} \to +\infty \qquad \text{(diverges if } p \geq 1)$$
$$\int_0^1 \frac{dx}{x^p} = \frac{1}{p-1} \qquad \text{(converges if } p < 1)$$

3.1.5 Techniques of Integration

3.1.6 Applications of Integration

3.2 Multivariable Calculus

$\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$

3.2.1 Partial Derivatives

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_n)}{h}$$

- Gradient

$$\nabla f = (f_{x_1},\ ...,\ f_{x_n})$$
$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}),\ ...,\ f_{x_n}(\mathbf{a}))$$

- Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

- Tangent plane

$$z = h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

  – Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

  – Hyperplane

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$
$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

- Differentiability

  1. $D\mathbf{f}(\mathbf{a})$ exists
  2. $$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{||\mathbf{x} - \mathbf{a}||} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \ldots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_1}} f(x_1, \ldots, x_n)$$

  – Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \ldots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \ldots \partial x_{j_k}}$$

- Chain rule

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0) D\mathbf{x}(\mathbf{t}_0)$$
$$f'(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)$$

- Directional derivative

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = ||\nabla f(\mathbf{a})|| \cos\theta$$

3.2.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \cdots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems
  1. If f is a scalar-valued function of class C^2, then
  $$\nabla \times (\nabla f) = \mathbf{0}$$
  2. If $\mathbf{F}$ is a vector-valued function of class C^2 on $X \subseteq \mathbb{R}^3$, then
  $$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

3.2.3 Maxima and Minima

- Taylor Polynomials
  - First-order
  $$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i)$$
  $$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$
  - Second-order
  $$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$
  $$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$
- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion
  - Hessian matrix
  $$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1 x_1}(\mathbf{a}) & \cdots & f_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(\mathbf{a}) & \cdots & f_{x_n x_n}(\mathbf{a}) \end{bmatrix}$$

- Principal minor

$$d_k = \text{determinant of the upperleftmost } k \times k \text{ submatrix of } Hf(\mathbf{a})$$

1. If all $d_k > 0$, then the critical point $\mathbf{a}$ gives a local minimum.
2. If $d_1 < 0,\ d_2 > 0,\ d_3 < 0,\ \ldots$, then the critical point $\mathbf{a}$ gives a local maximum.
3. If neither case 1 nor case 2 occurs, then $\mathbf{a}$ is a saddle point.

If $d_n = 0$, the critical point $\mathbf{a}$ is degenerate and the test fails.

- Extrema Value Theorem

If D is a compact region in $\mathbb{R}^n$ and $f : D \to R$ is continuous, then f must have a (global) maximum and minimum values on D.

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

- Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

3.2.4 Multiple Integration

- Double Integrals

$$\iint_R f\ dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem ($\mathbb{R}^2$)

$$\iint_R f\ dA = \int_c^d \int_a^b f(x,y)\ dxdy = \int_a^b \int_c^d f(x,y)\ dydx$$

- Elementary Regions ($\mathbb{R}^2$)
  - Type 1
    - Boundaries

$$\begin{aligned} x &= a & x &= b \\ y &= \gamma(x) & y &= \delta(x) \end{aligned}$$

    - Theorem

$$\iint_D f\ dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\ dydx$$

  - Type 2
    - Boundaries

$$\begin{aligned} x &= \alpha(y) & x &= \beta(y) \\ y &= c & y &= d \end{aligned}$$

    - Theorem

$$\iint_D f\ dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\ dxdy$$

  - Type 3

Simultaneously of type 1 and type 2.

- Triple Integrals

$$\iiint_B f\ dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i,j,k=1}^{n} f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

- Fubini's Theorem ($\mathbb{R}^3$)

$$\iiint_B f\ dV = \int_a^b \int_c^d \int_p^q f(x, y, z)\ dzdydx = \text{other orders}$$

- Elementary Regions ($\mathbb{R}^3$)
  - Type 1
    * Boundaries

$$z = \phi(x, y) \qquad z = \psi(x, y)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\phi(x,y)}^{\psi(x,y)} f(x, y, z)\ dzdydx$$

  - Type 2
    * Boundaries

$$x = \alpha(y, z) \qquad z = \beta(y, z)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\alpha(y,z)}^{\beta(y,z)} f(x, y, z)\ dxdydz$$

  - Type 3
    * Boundaries

$$y = \gamma(x, z) \qquad y = \delta(x, z)$$

    * Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\gamma(x,z)}^{\delta(x,z)} f(x, y, z)\ dydxdz$$

  - Type 4

Simultaneously of types 1, 2, and 3.

- The Jacobian
  $\mathbf{T} : D^* \to D$
  $\mathbf{T}(u, v) = (x(u, v),\ y(u, v))$

$$\frac{\partial(x, y)}{\partial(u, v)} = \det D\mathbf{T}(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

- Change of Integration

$$\iint_D f(x, y)\ dxdy = \iint_{D^*} f(x(u, v),\ y(u, v)) \left(\frac{\partial(x, y)}{\partial(u, v)}\right)\ dudv$$