

Math Booklet ¹

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¹A booklet with notes of Math.

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Contents

| | | |
|----------|---|----------|
| 1 | Algebra | 3 |
| 1.1 | Linear Algebra | 3 |
| 1.1.1 | Matrices | 3 |
| 2 | Geometry | 4 |
| 2.1 | Analytic Geometry | 4 |
| 2.1.1 | Quadric Surfaces | 4 |
| 2.1.2 | Coordinate systems | 4 |
| 3 | Calculus | 6 |
| 3.1 | Single Variable Calculus | 6 |
| 3.1.1 | Limits | 6 |
| 3.1.2 | Differentiation | 6 |
| 3.1.3 | Applications of Differentiation | 7 |
| 3.1.4 | Integration | 9 |
| 3.1.5 | Techniques of Integration | 11 |
| 3.1.6 | Applications of Integration | 12 |
| 3.2 | Multivariable Calculus | 13 |
| 3.2.1 | Partial Derivatives | 13 |
| 3.2.2 | Vector-valued Functions | 14 |
| 3.2.3 | Maxima and Minima | 14 |
| 3.2.4 | Multiple Integration | 15 |
| 3.2.5 | Applications of Integration | 17 |

Algebra

1.1 Linear Algebra

1.1.1 Matrices

- Notation

$$A = [a_{ij}]$$

- Matrix Addition

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- Scalar multiplication

$$c[a_{ij}] = [ca_{ij}]$$

- Transpose

$$(aT)_{ij} = a_{ji}$$

- Matrix Multiplication

$$c_{ij} = (\text{ith row of A})(\text{jth column of B}) = \sum_{k=1}^n a_{ik}b_{kj}$$

Geometry

2.1 Analytic Geometry

2.1.1 Quadric Surfaces

- Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- Elliptic paraboloid

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperbolic paraboloid

$$\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

- Elliptic cone

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Hyperboloid of two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

2.1.2 Coordinate systems

- Cartesian coordinates (\mathbb{R}^2 and \mathbb{R}^3)

$$(x, y) \quad (x, y, z)$$

- Polar coordinates (\mathbb{R}^2)

$$(r, \theta)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Polar/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates (\mathbb{R}^3)

$$(r, \theta, z)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Cylindrical/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates (\mathbb{R}^3)

$$(\rho, \phi, \theta)$$

- Typical restrictions

$$\begin{aligned} \rho &\geq 0 \\ 0 &\leq \phi \leq \pi \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Spherical/cylindrical conversions

$$\begin{cases} r = \rho \sin \phi \\ \theta = \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan \phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

- Spherical/rectangular conversions

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan \phi = \frac{\sqrt{x^2 + y^2}}{z} \\ \tan \theta = \frac{y}{x} \end{cases}$$

Calculus

3.1 Single Variable Calculus

3.1.1 Limits

- Squeeze Theorem

$$\begin{cases} g(x) \leq f(x) \leq h(x) \\ \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L \end{cases} \quad \therefore \quad \lim_{x \rightarrow a} f(x) = L$$

- Fundamental Trigonometric Limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- Fundamental Exponential Limit

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

3.1.2 Differentiation

- Definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

- Constant function

$$\frac{d}{dx} c = 0$$

- Derivative of Transcendent Functions

- Sine function

$$\frac{d}{dx} \sin x = \cos x$$

- Cosine function

$$\frac{d}{dx} \cos x = -\sin x$$

- Logarithm function

$$\frac{d}{dx} \log x = \frac{1}{x}$$

- Exponential function

$$\frac{d}{dx} e^x = e^x$$

- Properties

- Sum and difference

$$(u + v)' = u' + v'$$

- Product

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

- Product with a constant

$$(c \cdot u)' = c \cdot u'$$

- Quotient

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

- * Polynomial function

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{-n} = nx^{n-1}$$

- * Tangent function

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \sec^2 x$$

- * Secant function

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \sec x \cdot \tan x$$

- Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

- Exponential function (not natural)

$$\frac{d}{dx} a^x = a^x \cdot \log a$$

- Logarithm of a function

$$\frac{d}{dx} \log g(x) = \frac{g'(x)}{g(x)}$$

- Derivative of The Inverse Function

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

- Arcsine function

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

- Arctangent function

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

- Mean Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in which $a < c < b$

3.1.3 Applications of Differentiation

- L'Hospital Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x \rightarrow a} \frac{f'(a)}{g'(a)}$$

Cases: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

- Infinity Hierarchy

$$\log x < \sqrt[n]{x} < \sqrt{x} < x < x^2 < x^n < e^x < x! < x^x \rightarrow \infty$$

- Curve Sketching

$$f' > 0 \Rightarrow f \text{ is increasing}$$

$$f' < 0 \Rightarrow f \text{ is decreasing}$$

$$f'' > 0 \Rightarrow f' \text{ is increasing} \Rightarrow f \text{ is concave up}$$

$$f'' < 0 \Rightarrow f' \text{ is decreasing} \Rightarrow f \text{ is concave down}$$

- General Strategy

1. Plot discontinuities (especially infinite), endpoints (or $x \rightarrow \pm\infty$), and easy points (optional);
2. Solve $f'(x) = 0$, and plot critical points and values;
3. Decide whether $f' > 0$ or $f' < 0$ on each interval between critical points;
4. Analyse when the curve is concave up ($f'' > 0$) or down ($f'' < 0$), and what is/are the inflection point(s) ($f''(x_0) = 0$); and
5. Combine everything.

- Linear Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) \quad (x \approx a)$$

1. Sine

$$\sin x \approx x \quad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 \quad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x \quad (x \approx 0)$$

4. Logarithm

$$\log(1 + x) \approx x \quad (x \approx 0)$$

5. Sum to the power of n

$$(1 + x)^n \approx 1 + n \cdot x \quad (x \approx 0)$$

- Quadratic Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} \cdot (x - a)^2 \quad (x \approx a)$$

1. Sine

$$\sin x \approx x \quad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2} \quad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2} \quad (x \approx 0)$$

4. Logarithm

$$\log(1 + x) \approx x - \frac{x^2}{2} \quad (x \approx 0)$$

5. Sum to the power of n

$$(1 + x)^n \approx 1 + n \cdot x + \frac{n(n-1)}{2} x^2 \quad (x \approx 0)$$

- Taylor's Series

$$f(x) \approx f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) \quad (x \approx a)$$

1. Sine

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (x \approx 0)$$

4. Logarithm

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (x \approx 0)$$

$$\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \quad (x \approx 1)$$

5. Arctangent

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad (x \approx 0)$$

• Power Series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

OBS.: converges when $|x| < R$, where R is the radius of convergence.

1. Geometric Series ($R = 1$)

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

3.1.4 Integration

• Definition

$$\int_a^b f(x) dx = \text{Area under the curve}$$

$$\int_a^b f(x) dx = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x_i$$

• Properties

1.

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

2.

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

3.

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx \quad \leftrightarrow \quad f(x) \leq g(x)$$

4.

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

5.

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

6.

$$\int_a^a f(x) dx = 0$$

- Fundamental Theorem of Calculus

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$F(x) = \int_a^x f(t) dt$$

$$F'(x) = f(x)$$

- Antiderivatives (Indefinite Integral)

1. Powers (Polynomials)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \leftrightarrow \quad n \neq -1$$

2. Trigonometric Functions

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

3. Important Fractions

$$\int \frac{dx}{x} = \log |x| + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{1-x^2} = \tan^{-1} x + C$$

4. Others

$$\int \log x dx = x(\log x - 1) + C$$

- Properties of Some Transcendental Functions

1. $L(x) = \int_1^x \frac{dt}{t}$

$$L(ab) = L(a) + L(b)$$

2. $F(x) = \int_0^x e^{-t^2} dt$

$$\lim_{x \rightarrow \infty} F(x) = \frac{\sqrt{\pi}}{2}$$

3. $Li(x) = \int_2^x \frac{dt}{\ln t}$

$$Li(x) \approx \text{number of primes } < x$$

- Improper Integrals ($f(x) = \frac{1}{x^p}$)

$$\int_1^\infty \frac{dx}{x^p} \rightarrow \infty \quad (\text{diverges if } p \leq 1)$$

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad (\text{converges if } p > 1)$$

$$\int_0^1 \frac{dx}{x^p} \rightarrow +\infty \quad (\text{diverges if } p \geq 1)$$

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{p-1} \quad (\text{converges if } p < 1)$$

3.1.5 Techniques of Integration

- Integration by Substitution

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(t) dt$$

- Integration by Parts

$$\int_a^b uv' dx = (uv)|_a^b - \int_a^b u'v dx$$

- Trigonometric Integration

1. Powers - Easy case (at least one odd exponent) \rightsquigarrow use the fundamental formula and substitution

$$\sin^2 x + \cos^2 x = 1$$

2. Powers - Hard case (only even exponents) \rightsquigarrow use half-angle formulas or other trigonometric identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

– products \rightarrow sums

$$2 \cdot \sin m \cdot \cos n = \sin(m+n) + \sin(m-n)$$

$$2 \cdot \cos m \cdot \cos n = \cos(m+n) + \cos(m-n)$$

$$2 \cdot \sin m \cdot \cos n = \cos(m-n) - \cos(m+n)$$

3. Tangent

$$\int \tan x dx = -\ln(\cos x) + C$$

4. Secant

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

- Summary of Trigonometric Substitutions

| If integrand contains | make substitution | to get |
|-----------------------|--|------------------------------------|
| $\sqrt{a^2 - x^2}$ | $x = a \cos \theta$ or $x = a \sin \theta$ | $a \sin \theta$ or $a \cos \theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a \tan \theta$ | $a \sec \theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a \sec \theta$ | $a \tan \theta$ |

- Partial Fractions (common cases)

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\frac{p(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{p(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

$$\frac{p(x)}{x^2 + a} = \frac{Ax + B}{x^2 + a}$$

$$\frac{p(x)}{(x^2 + a)^2} = \frac{Ax + B}{x^2 + a} + \frac{Cx + D}{(x^2 + a)^2}$$

3.1.6 Applications of Integration

- Volumes

1. General

$$V = \int A(x) dx$$

2. Solid of Revolution (Method of Disks)

- Method of Disks

$$V = \int \pi y^2 dx \quad \text{around x-axis}$$

- Method of Shells

$$V = \int (\text{circumference})(\text{height}) dx \quad \text{around y-axis}$$

- (a) Sphere

$$V = \pi \left(ax^2 - \frac{x^3}{3} \right)$$

- Average Value

$$\frac{f(1) + f(2) + \dots + f(n)}{n} = \frac{\sum_{i=1}^n f(i)}{n} \approx \frac{1}{n} \int_0^n f(x) dx$$

$$Ave(a, b) = \frac{1}{b-a} \int_a^b f(x) dx$$

$$Ave(a, b) = \frac{\int_a^b f(x) w(x) dx}{\int_a^b w(x) dx}$$

- Arc-length

$$ds = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \sqrt{1 + [f'(x)]^2} dx$$

$$S = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- Superficial Area

$$A = \int_{s_0}^{s_n} 2\pi y ds = 2\pi \int_{x_0}^{x_n} f(x) \sqrt{1 + [f'(x)]^2} dx \quad (\text{around x-axis})$$

$$A = \int_{s_0}^{s_n} 2\pi x ds = 2\pi \int_{x_0}^{x_n} f(x) \sqrt{1 + [f'(x)]^2} dx \quad (\text{around y-axis})$$

- Sphere

$$A = 2\pi a(x_1 - x_2)$$

- Numerical Integration

1. Riemann Sums

$$\int_a^b f(x) dx \approx \Delta x (y_1 + y_2 + \dots + y_n) \quad (\text{right-hand sum})$$

$$\int_a^b f(x) dx \approx \Delta x (y_0 + y_1 + \dots + y_{n-2} + y_{n-1}) \quad (\text{left-hand sum})$$

If $f(x)$ decreases, lower estimation = right-hand sum, and higher estimation = left-hand sum.

If $f(x)$ increases, higher estimation = right-hand sum, and lower estimation = left-hand sum.

2. Trapezoidal Rule

$$\int_a^b f(x) dx \approx \Delta x \left(\frac{y_0}{2} + y_1 + \dots + y_{n-1} + \frac{y_n}{2} \right)$$

3. Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

$$\text{OBS.: } |\text{simpson's value} - \text{exact value}| \approx (\Delta x)^2$$

3.2 Multivariable Calculus

$$\mathbf{f}: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$

3.2.1 Partial Derivatives

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

- Gradient

$$\begin{aligned}\nabla f &= (f_{x_1}, \dots, f_{x_n}) \\ \nabla f(\mathbf{a}) &= (f_{x_1}(\mathbf{a}), \dots, f_{x_n}(\mathbf{a}))\end{aligned}$$

- Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

- Tangent plane

$$\begin{aligned}z &= h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) &= 0\end{aligned}$$

- Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

- Hyperplane

$$\begin{aligned}\mathbf{h}(\mathbf{x}) &= \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}) \\ \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) &= 0\end{aligned}$$

- Differentiability

1. $D\mathbf{f}(\mathbf{a})$ exists
- 2.

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \dots \frac{\partial}{\partial x_{i_1}} f(x_1, \dots, x_n)$$

- Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$$

- Chain rule

$$\begin{aligned}D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) &= D\mathbf{f}(\mathbf{x}_0)D\mathbf{x}(\mathbf{t}_0) \\ f'(\mathbf{x}(t)) &= \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)\end{aligned}$$

- Directional derivative

$$D_{\hat{\mathbf{u}}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = \|\nabla f(\mathbf{a})\| \cos \theta$$

3.2.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems

1. If f is a scalar-valued function of class C^2 , then

$$\nabla \times (\nabla f) = \mathbf{0}$$

2. If \mathbf{F} is a vector-valued function of class C^2 on $X \subseteq \mathbb{R}^3$, then

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

3.2.3 Maxima and Minima

- Taylor Polynomials

- First-order

$$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i)$$

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Second-order

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$

$$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion

- Hessian matrix

$$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1 x_1}(\mathbf{a}) & \cdots & f_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(\mathbf{a}) & \cdots & f_{x_n x_n}(\mathbf{a}) \end{bmatrix}$$

- Principal minor

d_k = determinant of the upperleftmost $k \times k$ submatrix of $Hf(\mathbf{a})$

1. If all $d_k > 0$, then the critical point \mathbf{a} gives a local minimum.
2. If $d_1 < 0$, $d_2 > 0$, $d_3 < 0$, \dots , then the critical point \mathbf{a} gives a local maximum.
3. If neither case 1 nor case 2 occurs, then \mathbf{a} is a saddle point.

If $d_n = 0$, the critical point \mathbf{a} is degenerate and the test fails.

- Extrema Value Theorem

If D is a compact region in \mathbb{R}^n and $f : D \rightarrow \mathbb{R}$ is continuous, then f must have a (global) maximum and minimum values on D .

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

- Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

3.2.4 Multiple Integration

- Double Integrals

$$\iint_R f \, dA = \lim_{\text{all } \Delta x_i, \Delta y_j \rightarrow 0} \sum_{i,j=1}^n f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem (\mathbb{R}^2)

$$\iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx dy = \int_a^b \int_c^d f(x, y) \, dy dx$$

- Elementary Regions (\mathbb{R}^2)

- Type 1

- * Boundaries

$$\begin{aligned} x &= a & x &= b \\ y &= \gamma(x) & y &= \delta(x) \end{aligned}$$

- * Theorem

$$\iint_D f \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y) \, dy dx$$

- Type 2

- * Boundaries

$$\begin{aligned} x &= \alpha(y) & x &= \beta(y) \\ y &= c & y &= d \end{aligned}$$

- * Theorem

$$\iint_D f \, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x, y) \, dx dy$$

- Type 3

Simultaneously of type 1 and type 2.

- Triple Integrals

$$\iiint_B f \, dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \rightarrow 0} \sum_{i,j,k=1}^n f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

- Fubini's Theorem (\mathbb{R}^3)

$$\iiint_B f \, dV = \int_a^b \int_c^d \int_p^q f(x, y, z) \, dz dy dx = \text{other orders}$$

- Elementary Regions (\mathbb{R}^3)

– Type 1

* Boundaries

$$z = \phi(x, y) \quad z = \psi(x, y)$$

* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\phi(x, y)}^{\psi(x, y)} f(x, y, z) \, dz dy dx$$

– Type 2

* Boundaries

$$x = \alpha(y, z) \quad z = \beta(y, z)$$

* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\alpha(y, z)}^{\beta(y, z)} f(x, y, z) \, dx dy dz$$

– Type 3

* Boundaries

$$y = \gamma(x, z) \quad y = \delta(x, z)$$

* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\gamma(x, z)}^{\delta(x, z)} f(x, y, z) \, dy dx dz$$

– Type 4

Simultaneously of types 1, 2, and 3.

- The Jacobian

$$\mathbf{T} : D^* \rightarrow D$$

$$\mathbf{T}(u, v) = (x(u, v), y(u, v))$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \det D\mathbf{T}(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\mathbf{U} : W^* \rightarrow W$$

$$\mathbf{U}(u, v, w) = (x(u, v, w), y(u, v, w))$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det D\mathbf{U}(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

– Polar coordinates

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r$$

– Cylindrical coordinates

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

- Spherical coordinates

$$\frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = \rho^2 \sin \phi$$

- Change of Integration

$$\iint_D f(x, y) \, dx dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$\iiint_W f(x, y, z) \, dx dy dz = \iiint_{W^*} f(x(u, v, w), y(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

3.2.5 Applications of Integration

- Average Value

- $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f_{avg} = \frac{1}{A_D} \iint_D f(x, y) \, dA = \frac{\iint_D f(x, y) \, dA}{\iint_D dA}$$

- $f : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f_{avg} = \frac{1}{V_W} \iiint_W f(x, y, z) \, dV = \frac{\iiint_W f(x, y, z) \, dV}{\iiint_W dV}$$

- Center of mass

- $\delta : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\bar{x} = \frac{\iint_D x \delta(x, y) \, dA}{\iint_D \delta(x, y) \, dA} \quad \bar{y} = \frac{\iint_D y \delta(x, y) \, dA}{\iint_D \delta(x, y) \, dA}$$

- * Centroid (constant density)

$$\bar{x} = \frac{1}{A_D} \iint_D x \, dA = \frac{\iint_D x \, dA}{\iint_D dA}$$

$$\bar{y} = \frac{1}{A_D} \iint_D y \, dA = \frac{\iint_D y \, dA}{\iint_D dA}$$

- $\delta : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}$

$$\bar{x} = \frac{\iiint_W x \delta(x, y, z) \, dV}{\iiint_W \delta(x, y, z) \, dV} \quad \bar{y} = \frac{\iiint_W y \delta(x, y, z) \, dV}{\iiint_W \delta(x, y, z) \, dV}$$

$$\bar{z} = \frac{\iiint_W z \delta(x, y, z) \, dV}{\iiint_W \delta(x, y, z) \, dV}$$

- * Centroid (constant density)

$$\bar{x} = \frac{1}{V_W} \iiint_W x \, dV = \frac{\iiint_W x \, dV}{\iiint_W dV}$$

$$\bar{y} = \frac{1}{V_W} \iiint_W y \, dV = \frac{\iiint_W y \, dV}{\iiint_W dV}$$

$$\bar{z} = \frac{1}{V_W} \iiint_W z \, dV = \frac{\iiint_W z \, dV}{\iiint_W dV}$$