

# Math Booklet <sup>1</sup>

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<sup>1</sup>A booklet with notes of Math.

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# Contents

<b>1</b>	<b>Algebra</b>	<b>3</b>
1.1	Linear Algebra . . . . .	3
1.1.1	Matrices . . . . .	3
<b>2</b>	<b>Geometry</b>	<b>4</b>
2.1	Analytic Geometry . . . . .	4
2.1.1	Coordinate systems . . . . .	4
<b>3</b>	<b>Calculus</b>	<b>6</b>
3.1	Single Variable Calculus . . . . .	6
3.1.1	Limits . . . . .	6
3.1.2	Differentiation . . . . .	6
3.1.3	Applications of Differentiation . . . . .	7
3.1.4	Integration . . . . .	9
3.1.5	Techniques of Integration . . . . .	9
3.1.6	Applications of Integration . . . . .	9
3.2	Multivariable Calculus . . . . .	9
3.2.1	Partial Derivatives . . . . .	9
3.2.2	Vector-valued Functions . . . . .	10
3.2.3	Maxima and Minima . . . . .	11
3.2.4	Multiple Integration . . . . .	12

# Algebra

## 1.1 Linear Algebra

### 1.1.1 Matrices

- Notation

$$A = [a_{ij}]$$

- Matrix Addition

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

- Scalar multiplication

$$c[a_{ij}] = [ca_{ij}]$$

- Transpose

$$(aT)_{ij} = a_{ji}$$

- Matrix Multiplication

$$c_{ij} = (\text{ith row of A})(\text{jth column of B}) = \sum_{k=1}^n a_{ik}b_{kj}$$

# Geometry

## 2.1 Analytic Geometry

### 2.1.1 Coordinate systems

- Cartesian coordinates ( $\mathbb{R}^2$  and  $\mathbb{R}^3$ )

$$(x, y) \quad (x, y, z)$$

- Polar coordinates ( $\mathbb{R}^2$ )

$$(r, \theta)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Polar/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates ( $\mathbb{R}^3$ )

$$(r, \theta, z)$$

- Typical restrictions

$$\begin{aligned} r &\geq 0 \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Cylindrical/rectangular conversions

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates ( $\mathbb{R}^3$ )

$$(\rho, \phi, \theta)$$

- Typical restrictions

$$\begin{aligned} \rho &\geq 0 \\ 0 &\leq \phi \leq \pi \\ 0 &\leq \theta \leq 2\pi \end{aligned}$$

- Spherical/cylindrical conversions

$$\begin{cases} r = \rho \sin \phi \\ \theta = \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan \phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

– Spherical/rectangular conversions

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan \phi = \frac{\sqrt{x^2 + y^2}}{z} \\ \tan \theta = \frac{y}{x} \end{cases}$$

# Calculus

## 3.1 Single Variable Calculus

### 3.1.1 Limits

- Squeeze Theorem

$$\begin{cases} g(x) \leq f(x) \leq h(x) \\ \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L \end{cases} \quad \therefore \quad \lim_{x \rightarrow a} f(x) = L$$

- Fundamental Trigonometric Limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- Fundamental Exponential Limit

$$\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

### 3.1.2 Differentiation

- Definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

- Constant function

$$\frac{d}{dx} c = 0$$

- Derivative of Transcendent Functions

- Sine function

$$\frac{d}{dx} \sin x = \cos x$$

- Cosine function

$$\frac{d}{dx} \cos x = -\sin x$$

- Logarithm function

$$\frac{d}{dx} \log x = \frac{1}{x}$$

- Exponential function

$$\frac{d}{dx} e^x = e^x$$

- Properties

- Sum and difference

$$(u + v)' = u' + v'$$

- Product

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

- Product with a constant

$$(c \cdot u)' = c \cdot u'$$

- Quotient

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

- \* Polynomial function

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{-n} = nx^{n-1}$$

- \* Tangent function

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \sec^2 x$$

- \* Secant function

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \sec x \cdot \tan x$$

- Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

- Exponential function (not natural)

$$\frac{d}{dx} a^x = a^x \cdot \log a$$

- Logarithm of a function

$$\frac{d}{dx} \log g(x) = \frac{g'(x)}{g(x)}$$

- Derivative of The Inverse Function

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

- Arcsine function

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

- Arctangent function

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

- Mean Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in which  $a < c < b$

### 3.1.3 Applications of Differentiation

- L'Hospital Rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x \rightarrow a} \frac{f'(a)}{g'(a)}$$

Cases:  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$

- Infinity Hierarchy

$$\log x < \sqrt[n]{x} < \sqrt{x} < x < x^2 < x^n < e^x < x! < x^x \rightarrow \infty$$

- Curve Sketching

$$f' > 0 \Rightarrow f \text{ is increasing}$$

$$f' < 0 \Rightarrow f \text{ is decreasing}$$

$$f'' > 0 \Rightarrow f' \text{ is increasing} \Rightarrow f \text{ is concave up}$$

$$f'' < 0 \Rightarrow f' \text{ is decreasing} \Rightarrow f \text{ is concave down}$$

- General Strategy

1. Plot discontinuities (especially infinite), endpoints (or  $x \rightarrow \pm\infty$ ), and easy points (optional);
2. Solve  $f'(x) = 0$ , and plot critical points and values;
3. Decide whether  $f' > 0$  or  $f' < 0$  on each interval between critical points;
4. Analyse when the curve is concave up ( $f'' > 0$ ) or down ( $f'' < 0$ ), and what is/are the inflection point(s) ( $f''(x_0) = 0$ ); and
5. Combine everything.

- Linear Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) \quad (x \approx a)$$

1. Sine

$$\sin x \approx x \quad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 \quad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x \quad (x \approx 0)$$

4. Logarithm

$$\log(1 + x) \approx x \quad (x \approx 0)$$

5. Sum to the power of  $n$

$$(1 + x)^n \approx 1 + n \cdot x \quad (x \approx 0)$$

- Quadratic Approximation

$$f(x) \approx f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} \cdot (x - a)^2 \quad (x \approx a)$$

1. Sine

$$\sin x \approx x \quad (x \approx 0)$$

2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2} \quad (x \approx 0)$$

3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2} \quad (x \approx 0)$$

4. Logarithm

$$\log(1 + x) \approx x - \frac{x^2}{2} \quad (x \approx 0)$$

5. Sum to the power of  $n$

$$(1 + x)^n \approx 1 + n \cdot x + \frac{n(n-1)}{2} x^2 \quad (x \approx 0)$$

- Taylor's Series

$$f(x) \approx f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) \quad (x \approx a)$$

1. Sine

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (x \approx 0)$$



## 2. Cosine

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (x \approx 0)$$

## 3. Exponential

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (x \approx 0)$$

## 4. Logarithm

$$\log(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (x \approx 0)$$

$$\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \quad (x \approx 1)$$

## 5. Arctangent

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad (x \approx 0)$$

## • Power Series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

OBS.: converges when  $|x| < R$ , where  $R$  is the radius of convergence.

1. Geometric Series ( $R = 1$ )

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

**3.1.4 Integration****3.1.5 Techniques of Integration****3.1.6 Applications of Integration****3.2 Multivariable Calculus**

$$\mathbf{f}: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f: X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$

**3.2.1 Partial Derivatives**

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

## • Gradient

$$\begin{aligned} \nabla f &= (f_{x_1}, \dots, f_{x_n}) \\ \nabla f(\mathbf{a}) &= (f_{x_1}(\mathbf{a}), \dots, f_{x_n}(\mathbf{a})) \end{aligned}$$

## • Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

## • Tangent plane

$$\begin{aligned} z &= h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) &= 0 \end{aligned}$$

- Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

- Hyperplane

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

- Differentiability

1.  $D\mathbf{f}(\mathbf{a})$  exists

- 2.

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \dots \frac{\partial}{\partial x_{i_1}} f(x_1, \dots, x_n)$$

- Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$$

- Chain rule

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0)D\mathbf{x}(\mathbf{t}_0)$$

$$f'(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)$$

- Directional derivative

$$D_{\hat{\mathbf{u}}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = \|\nabla f(\mathbf{a})\| \cos \theta$$

### 3.2.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems

1. If  $f$  is a scalar-valued function of class  $C^2$ , then

$$\nabla \times (\nabla f) = \mathbf{0}$$

2. If  $\mathbf{F}$  is a vector-valued function of class  $C^2$  on  $X \subseteq \mathbb{R}^3$ , then

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

### 3.2.3 Maxima and Minima

- Taylor Polynomials

- First-order

$$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i)$$

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Second-order

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^n f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$

$$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion

- Hessian matrix

$$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1 x_1}(\mathbf{a}) & \cdots & f_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(\mathbf{a}) & \cdots & f_{x_n x_n}(\mathbf{a}) \end{bmatrix}$$

- Principal minor

$d_k$  = determinant of the upperleftmost  $k \times k$  submatrix of  $Hf(\mathbf{a})$

1. If all  $d_k > 0$ , then the critical point  $\mathbf{a}$  gives a local minimum.
2. If  $d_1 < 0$ ,  $d_2 > 0$ ,  $d_3 < 0$ ,  $\dots$ , then the critical point  $\mathbf{a}$  gives a local maximum.
3. If neither case 1 nor case 2 occurs, then  $\mathbf{a}$  is a saddle point.

If  $d_n = 0$ , the critical point  $\mathbf{a}$  is degenerate and the test fails.

- Extrema Value Theorem

If  $D$  is a compact region in  $\mathbb{R}^n$  and  $f : D \rightarrow \mathbb{R}$  is continuous, then  $f$  must have a (global) maximum and minimum values on  $D$ .

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

- Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

### 3.2.4 Multiple Integration

- Double Integrals

$$\iint_R f \, dA = \lim_{\text{all } \Delta x_i, \Delta y_j \rightarrow 0} \sum_{i,j=1}^n f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem ( $\mathbb{R}^2$ )

$$\iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx dy = \int_a^b \int_c^d f(x, y) \, dy dx$$

- Elementary Regions ( $\mathbb{R}^2$ )

- Type 1

- \* Boundaries

$$\begin{aligned} x &= a & x &= b \\ y &= \gamma(x) & y &= \delta(x) \end{aligned}$$

- \* Theorem

$$\iint_D f \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y) \, dy \, dx$$

- Type 2

- \* Boundaries

$$\begin{aligned} x &= \alpha(y) & x &= \beta(y) \\ y &= c & y &= d \end{aligned}$$

- \* Theorem

$$\iint_D f \, dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x, y) \, dx \, dy$$

- Type 3

Simultaneously of type 1 and type 2.

- Triple Integrals

$$\iiint_B f \, dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \rightarrow 0} \sum_{i,j,k=1}^n f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

- Fubini's Theorem ( $\mathbb{R}^3$ )

$$\iiint_B f \, dV = \int_a^b \int_c^d \int_p^q f(x, y, z) \, dz \, dy \, dx = \text{other orders}$$

- Elementary Regions ( $\mathbb{R}^3$ )

- Type 1

- \* Boundaries

$$z = \phi(x, y) \quad z = \psi(x, y)$$

- \* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\phi(x,y)}^{\psi(x,y)} f(x, y, z) \, dz \, dy \, dx$$

- Type 2

- \* Boundaries

$$x = \alpha(y, z) \quad z = \beta(y, z)$$

- \* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\alpha(y,z)}^{\beta(y,z)} f(x, y, z) \, dx \, dy \, dz$$

- Type 3

- \* Boundaries

$$y = \gamma(x, z) \quad y = \delta(x, z)$$

- \* Theorem

$$\iiint_B f \, dV = \iint_{\text{shadow}} \int_{\gamma(x,z)}^{\delta(x,z)} f(x, y, z) \, dy \, dx \, dz$$

- Type 4

Simultaneously of types 1, 2, and 3.

- The Jacobian

$$\mathbf{T} : D^* \rightarrow D$$

$$\mathbf{T}(u, v) = (x(u, v), y(u, v))$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \det D\mathbf{T}(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- Change of Integration

$$\iint_D f(x, y) \, dx dy = \iint_{D^*} f(x(u, v), y(u, v)) \left( \frac{\partial(x, y)}{\partial(u, v)} \right) \, du dv$$