Math Booklet [1]

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[1] A booklet with notes of Math.
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Contents

Algebra

1.1 Linear Algebra

1.1.1 Matrices

- Notation
$$A = [a_{ij}]$$
- Matrix Addition
$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
- Scalar multiplication
$$c[a_{ij}] = [ca_{ij}]$$
- Transpose
$$(aT)_{ij} = a_{ji}$$
- Matrix Multiplication
$$c_{ij} = \text{(ith row of A)(jth column of B)} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Geometry

2.1 Analytic Geometry

2.1.1 Coordinate systems

- Cartesian coordinates ($\mathbb{R}^2$ and $\mathbb{R}^3$)

$$(x, y) \qquad (x, y, z)$$

- Polar coordinates ($\mathbb{R}^2$)

$$(r, \theta)$$

  – Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

  – Polar/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

- Cylindrical coordinates ($\mathbb{R}^3$)

$$(r, \theta, z)$$

  – Typical restrictions

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi$$

  – Cylindrical/rectangular conversions

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \\ z = z \end{cases}$$

- Spherical coordinates ($\mathbb{R}^3$)

$$(\rho, \phi, \theta)$$

  – Typical restrictions

$$\rho \geq 0$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$

  – Spherical/cylindrical conversions

$$\begin{cases} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} \rho^2 = r^2 + z^2 \\ \tan\phi = \frac{r}{z} \\ \theta = \theta \end{cases}$$

– Spherical/rectangular conversions

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \frac{y}{x} \end{cases}$$

Calculus

3.1 Multivariable Calculus

$\mathbf{f}: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
$f: X \subseteq \mathbb{R}^n \to \mathbb{R}$

3.1.1 Partial Derivatives

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_n)}{h}$$

- Gradient

$$\nabla f = (f_{x_1},\ ...,\ f_{x_n})$$
$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}),\ ...,\ f_{x_n}(\mathbf{a}))$$

- Derivative matrix

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

- Tangent plane

$$z = h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

  - Normal vector

$$\mathbf{n} = -f_x(a, b)\hat{\mathbf{i}} - f_y(a, b)\hat{\mathbf{j}} + \hat{\mathbf{k}} = (-f_x(a, b), -f_y(a, b), 1)$$

  - Hyperplane

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$
$$\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

- Differentiability

  1. $D\mathbf{f}(\mathbf{a})$ exists
  2. $$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})}{||\mathbf{x} - \mathbf{a}||} = 0$$

- Higher-order partial derivative

$$\frac{\partial^k f}{\partial x_{i_k} \ldots \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \cdots \frac{\partial}{\partial x_{i_1}} f(x_1, \ldots, x_n)$$

- Clairaut's Theorem

$$\frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}}$$

- Chain rule

$$D(\mathbf{f} \circ \mathbf{x})(\mathbf{t}_0) = D\mathbf{f}(\mathbf{x}_0)D\mathbf{x}(\mathbf{t}_0)$$
$$f'(\mathbf{x}(t)) = \nabla f(\mathbf{x}) \bullet \mathbf{x}'(t)$$

- Directional derivative

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \hat{\mathbf{u}} = ||\nabla f(\mathbf{a})|| \cos\theta$$

3.1.2 Vector-valued Functions

- Arclength
- Vector fields
- Del operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

- Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- Divergence

$$\nabla \bullet \mathbf{F} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \dots + \frac{\partial f}{\partial x_n}$$

- Curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- Theorems
  1. If f is a scalar-valued function of class C^2, then

$$\nabla \times (\nabla f) = \mathbf{0}$$

  2. If $\mathbf{F}$ is a vector-valued function of class C^2 on $X \subseteq \mathbb{R}^3$, then

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0$$

3.1.3 Maxima and Minima

- Taylor Polynomials
  - First-order

$$p_1(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i)$$
$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

  - Second-order

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$
$$p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})^T Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

- Differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Hessian Criterion
  - Hessian matrix

$$Hf(\mathbf{a}) = \begin{bmatrix} f_{x_1x_1}(\mathbf{a}) & \cdots & f_{x_1x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{a}) & \cdots & f_{x_nx_n}(\mathbf{a}) \end{bmatrix}$$

  - Principal minor

$$d_k = \text{determinant of the upperleftmost } k \times k \text{ submatrix of } Hf(\mathbf{a})$$

  1. If all $d_k > 0$, then the critical point $\mathbf{a}$ gives a local minimum.
  2. If $d_1 < 0,\ d_2 > 0,\ d_3 < 0,\ \ldots$, then the critical point $\mathbf{a}$ gives a local maximum.
  3. If neither case 1 nor case 2 occurs, then $\mathbf{a}$ is a saddle point.

  If $d_n = 0$, the critical point $\mathbf{a}$ is degenerate and the test fails.

- Extrema Value Theorem

  If D is a compact region in $\mathbb{R}^n$ and $f : D \to R$ is continuous, then f must have a (global) maximum and minimum values on D.

- Lagrange Multiplier Theorem

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

  - Constraint

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid g(\mathbf{x}) = c\}$$

3.1.4 Multiple Integration

- Double Integrals

$$\iint_R f\ dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c}_{ij}) \Delta x_i \Delta y_j$$

- Fubini's Theorem ($\mathbb{R}^2$)

$$\iint_R f\ dA = \int_c^d \int_a^b f(x,y)\ dxdy = \int_a^b \int_c^d f(x,y)\ dydx$$

- Elementary Regions ($\mathbb{R}^2$)
  - Type 1
    - Boundaries

$$x = a \qquad x = b$$
$$y = \gamma(x) \qquad y = \delta(x)$$

    - Theorem

$$\iint_D f\ dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y)\ dydx$$

  - Type 2
    - Boundaries

$$x = \alpha(y) \qquad x = \beta(y)$$
$$y = c \qquad y = d$$

  - Theorem

$$\iint_D f\ dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\ dxdy$$

- Type 3

Simultaneously of type 1 and type 2.

- Triple Integrals

$$\iiint_B f\ dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i,j,k=1}^{n} f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

- Fubini's Theorem ($\mathbb{R}^3$)

$$\iiint_B f\ dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\ dzdydx = \text{other orders}$$

- Elementary Regions ($\mathbb{R}^3$)
  - Type 1
    - Boundaries

$$z = \phi(x,y) \qquad z = \psi(x,y)$$

    - Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\ dzdydx$$

  - Type 2
    - Boundaries

$$x = \alpha(y,z) \qquad z = \beta(y,z)$$

    - Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z)\ dxdydz$$

  - Type 3
    - Boundaries

$$y = \gamma(x,z) \qquad y = \delta(x,z)$$

    - Theorem

$$\iiint_B f\ dV = \iint_{\text{shadow}} \int_{\gamma(x,z)}^{\delta(x,z)} f(x,y,z)\ dydxdz$$

  - Type 4

Simultaneously of types 1, 2, and 3.

- The Jacobian
  $\mathbf{T} : D^* \to D$
  $\mathbf{T}(u,v) = (x(u,v),\ y(u,v))$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det D\mathbf{T}(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

- Change of Integration

$$\iint_D f(x,y)\ dxdy = \iint_{D^*} f(x(u,v),\ y(u,v)) \left(\frac{\partial(x,y)}{\partial(u,v)} \right)\ dudv$$