Testing Bell's inequality with entangled photons

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Bell's inequality, $|S| \leq 2$, must hold true for any hidden local variable theory. In this experiment, we performed a test of quantum mechanics by evaluating this inequality with polarization-entangled photon pairs generated through spontaneous parametric down-conversion in a nonlinear birefringent crystal. An analysis of our prepared state revealed a deviation from the ideal maximally entangled state, with the polarization angle θ_l estimated at approximately 25°, rather than the expected 45°. Consequently, our measured value of $S = 1.8(1)$ fails to violate Bell's inequality. A "corrected" model assuming ideal entanglement predicts $S = 2.0(1)$, but the high uncertainty prevents a definitive conclusion. Despite this, the experiment provided valuable hands-on experience with foundational tests of quantum mechanics. Future work would benefit from better preparation of the state and increased data collection time to reduce statistical noise.

I. INTRODUCTION

The field of quantum optics, which explores the quantum nature of light and its interaction with matter, has its roots in the early 20th century with the development of quantum mechanics itself. While light had been successfully described as an electromagnetic wave by Maxwell's theory, phenomena such as the photoelectric effect and blackbody radiation needed a quantized description, in which light is composed of discrete packets of energy called photons [1]. This revolutionary idea, introduced by Albert Einstein, laid the foundation for understanding the fundamental quantum properties of light.

Another pivotal moment in the development of quantum optics was the theoretical work of John Bell addressing a long-standing debate about the completeness of quantum mechanics [2], sparked by the Einstein-Podolsky-Rosen (EPR) paradox [3]. EPR argued that quantum mechanics might be an incomplete theory because it seemed to allow for "spooky action at a distance", where measurements on one entangled particle instantaneously affect the state of another, regardless of the distance separating them. EPR then proposed the existence of "local hidden variables" that would predetermine the outcomes of measurements. Bell's work showed that any local hidden variable theory must satisfy a statistical inequality, which is violated in quantum mechanics. This provided an experimentally testable way to distinguish between quantum mechanics and local hidden variable theories.

The first experimental test of Bell's inequality was done by Clauser, Horne, Shimony, and Holt (CHSH) [4], which provided evidence against local hidden variable theories and supported the predictions of quantum mechanics. Subsequent experiments solidified these findings [5, 6], having improved precision and addressing potential loopholes.

In this experiment, we use a pair of entangled photons, generated by a nonlinear birefringent crystal, to test Bell's inequality. In Section II, we provide an overview of the CHSH form of Bell's inequality. Section III describes the experimental setup and methods used to generate and measure the photon pairs. In Section IV, we present the results of our measurements, including an analysis of the prepared state and an evaluation of Bell's inequality. Finally, we conclude in Section V with a summary of our findings and possible areas of improvement.

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II. THEORY

We use non-linear optic properties of birefringent crystals to generate entangled photon pairs, which are then sent to a couple of detectors that are preceded by polarizers. If α and β represent the angles from the detector polarizers to the vertical, it can be shown that the probability that both detections come out as "vertical" is given by

$$\begin{aligned} P_{VV}(\alpha, \beta) = &\sin^2\alpha \sin^2\beta \cos^2\theta_l \\ &+ \cos^2\alpha \cos^2\beta \sin^2\theta_l \\ &+ \frac{1}{4}\sin 2\alpha \sin 2\beta \sin 2\theta_l \cos\phi_m \end{aligned} \tag{1}$$

[7], where θ_l represents the initial polarization of the laser beam and ϕ_m is an averaged accumulated phase shift.

In the maximally entangled state ($\theta_l = 45°$ and $\phi = 0°$), Eq. (1) reduces to

$$P_{VV}^{\mathrm{QM}}(\alpha, \beta) = \frac{1}{2}\cos^2(\beta - \alpha) \tag{2}$$

[7]. However, if we consider a local realistic hidden variable theory (HVT), it can be shown that such maximally entangled state would lead to a probability of

$$P_{VV}^{\mathrm{HVT}}(\alpha, \beta) = \frac{1}{2} - \frac{|\beta - \alpha|}{\pi} \tag{3}$$

[7].

In studying the feasibility of a HVT, John Bell derived an inequality that must be true for any HVT [2]. In the

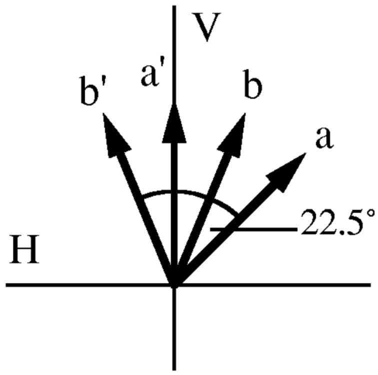

FIG. 1: Schematic showing the angles used in Eq. (5). Here, $a = -45°$, $b = -22.5°$, $a = 0°$, and $b' = 22.5°$. This figure is reproduced from [7].

context of Eq. (3), it was found in [4] that this inequality can be expressed as

$$|S| \leq 2 \tag{4}$$

[7], where

$$S \equiv E(a,b) - E(a,b') + E(a',b) + E(a',b') \tag{5}$$

$$\begin{aligned} E(\alpha,\beta) \equiv\ & P_{VV}(\alpha,\beta) + P_{VV}(\alpha_\perp,\beta_\perp) \\ & - P_{VV}(\alpha,\beta_\perp) - P_{VV}(\alpha_\perp,\beta), \end{aligned} \tag{6}$$

and the angles a, b, a', and b' are defined as in Fig. 1.

This means that if a HVT exists, then Eq. (4) must be satisfied. However, if the quantum mechanical prediction from Eq. (2) is correct, then the value of S will be greater than 2, violating Eq. (4).

It is important to note that Quantum Mechanics does not necessarily violate Eq. (4) in all cases. However, we have chosen the angles in Fig. 1 such that the quantum mechanical prediction for S is maximized (and greater than 2) in the maximally entangled state [7], thus violating Eq. (4).

III. METHODS

Our experimental setup is shown in Fig. 2. A vertically polarized 407 nm laser beam is sent into a couple of wave plates. The half-wave plate (HWP) is used to rotate the polarization of the photons to be as close as possible to the desired angle $\theta_l = 45°$, while the quarter-wave plate (QWP) corrects the phase shift from the crystal so that $\phi_m \approx 0$. We will discuss how we have calibrated the angles of these wave plates in the next section. If done correctly, the photons emerging from these wave plates will be in the maximally entangled state,

$$\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle) \tag{7}$$

[8], where $|H\rangle$ and $|V\rangle$ represent states of photons polarized horizontally and vertically, respectively. If not done correctly, the photons will not be in the maximally entangled state, and the value of S might not be help us in evaluating Bell's inequality.

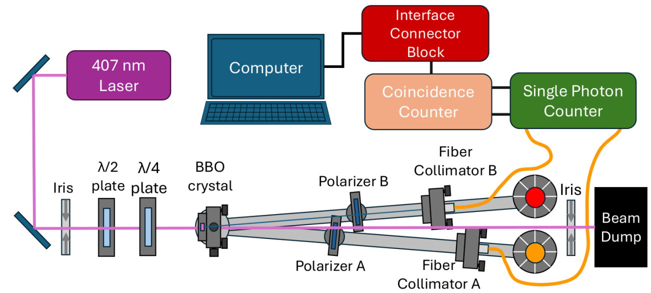

FIG. 2: Experimental setup. This figure is reproduced from [8].

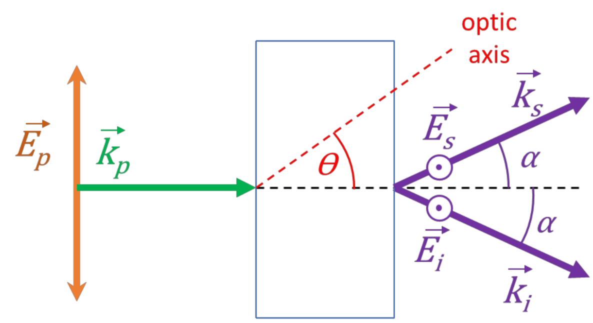

FIG. 3: Representation of the photon conversion in the BBO crystal. This figure is reproduced from [8].

The entangled photons are then sent to a pair of beta barium borate (BBO) crystals, where they undergo spontaneous parametric down-conversion, producing two orthogonally polarized beams [7]. This conversion is represented in Fig. 3. The angles θ and α are related to the laser frequency ω_p by

$$\frac{\sin^2\theta}{\tilde{n}_e(\omega_p)^2} + \frac{\cos^2\theta}{n_o(\omega_p)^2} = \frac{\sec^2\theta}{n_o(\frac{1}{2}\omega_p)^2} \tag{8}$$

[8], where $\tilde{n}_e$ and n_o are the extraordinary and ordinary refractive indices of the BBO crystal, respectively. For our BBO crystal, these refractive indices are specified by

$$n_o(\lambda)^2 = 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2, \tag{9}$$

$$\tilde{n}_e(\lambda)^2 = 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01515\lambda^2 \tag{10}$$

[8], where λ is the wavelength measured in microns. Knowing that our BBO crystal was cut so that its optic axis is $\theta = 30°$, we can use Eqs. (8)–(10) to find $\alpha \approx 3.21°$. This angle tells us how to orient the polarizers relative to the BBO crystal so that we can detect the two converted beams. It is essential to check this in the alignment process.

The two beams are then sent to polarizers at angles α and β from the vertical. The photons are then detected by a couple of single-photon detectors. The detectors are connected to a coincidence counter, which records the number of times both detectors detect a photon at the same time in a coincidence window of $\tau = 25$ ns [8].

The coincidence counter gives us $N(\alpha, \beta)$, the number of coincidences for a given configuration of the polarizers. We can re-write Eq. (6) in terms of $N(\alpha, \beta)$ as

$$E(\alpha,\beta) \equiv \frac{N(\alpha,\beta)+N(\alpha_\perp,\beta_\perp)-N(\alpha,\beta_\perp)-N(\alpha_\perp,\beta)}{N(\alpha,\beta)+N(\alpha_\perp,\beta_\perp)+N(\alpha,\beta_\perp)+N(\alpha_\perp,\beta)} \quad (11)$$

[7]. With this, we can calculate the value of S using Eq. (5).

It's important to note that we need to subtract background single-photon counts from our measurements. By blocking the beam path, we measured the background single-photon count rate in detector A $\dot{N}_A^{\text{background}} \approx 923$ Hz and in detector B $\dot{N}_B^{\text{background}} \approx 624$ Hz. After measuring the raw counts in both detectors (N_A^{raw} and N_B^{raw}), we define

$$N_A \equiv N_A^{\text{raw}} - T\dot{N}_A^{\text{background}}, \quad (12)$$

$$N_B \equiv N_B^{\text{raw}} - T\dot{N}_B^{\text{background}}, \quad (13)$$

where T is the duration of the measurement.

Additionally, we know from [8] that there is an expected number of accidental coincidences given by

$$N^{\text{ac}} = N_A N_B \frac{\tau}{T}. \quad (14)$$

After measuring the raw coincidences N^{raw}, we can then define the corrected coincidences as

$$N \equiv N^{\text{raw}} - N^{\text{ac}}. \quad (15)$$

IV. RESULTS

Our first set of measurements (shown in Table I) was taken with $T = 60$ s in order to analyze our prepared state. Our second set of measurements (shown in Table II) was taken with $T = 8$ minutes to be used in to evaluate S.

This section is divided into three parts. First, we look at the calibration of the wave plates, which is essential to prepare the photons in the maximally entangled state. Then, we analyze the state of the photons we have been able to create. Finally, we calculate the value of S in an attempt to evaluate Bell's inequality.

A. Calibration of wave plates

To calibrate the HWP, we varied its angle from the vertical λ_H and recorded $N(\alpha, \beta)$ in two configurations: $(\alpha, \beta) = (0°, 0°)$ and $(\alpha, \beta) = (90°, 90°)$. We then fitted the data to the expected shape of $\sin^2(\lambda_{\text{HWP}})$ using SciPy's `curve_fit` procedure [9]. The results are shown in the left panel of Fig. 4. Seeking the state in Eq. (7), we want to equalize $N(0°, 0°)$ and $N(90°, 90°)$, which is satisfied when $\lambda_H \approx 62°$.

We calibrated the QWP similarly. We varied its angle from the vertical λ_Q and recorded $N(45°, 45°)$, which we want to maximize. The resulting data and fit are shown in the right panel of Fig. 4. From this, we see that the maximum of $N(45°, 45°)$ is satisfied when $\lambda_Q \approx 140°$.

All of these measurements had a duration of $T = 30$ s.

α	β	N_A^{raw}	N_B^{raw}	N^{raw}	N
0°	0°	66195	64087	10	10
90°	90°	76322	97093	17	17
0°	90°	64369	96829	4	4
45°	45°	74406	86618	19	19
45°	−45°	73555	76268	6	6

TABLE I: Set of 5 measurements taken with $T = 60$ s.

α	β	N_A^{raw}	N_B^{raw}	N^{raw}	N
−45°	−22.5°	548510	565065	76	75
−45°	22.5°	546175	626254	24	23
−45°	67.5°	546660	807829	63	61
−45°	112.5°	545895	738278	120	118
0°	−22.5°	522904	522904	48	48
0°	22.5°	520821	625289	37	36
0°	67.5°	519373	808650	27	26
0°	112.5°	522048	744202	19	18
45°	−22.5°	605777	578021	41	40
45°	22.5°	597014	632914	75	73
45°	67.5°	592238	810973	141	138
45°	112.5°	593777	745974	66	64
90°	−22.5°	623314	568441	39	38
90°	22.5°	617312	620594	45	43
90°	67.5°	619769	817720	175	172
90°	112.5°	621931	746304	139	136

TABLE II: Set of 16 measurements taken with $T = 8$ minutes.

B. Analysis of state

From [8], we know that the expected number of coincidences can be modelled by

$$\begin{aligned} N(\alpha,\beta) = A(&\sin^2\alpha\sin^2\beta\cos^2\theta_l \\ &+\cos^2\alpha\cos^2\beta\sin^2\theta_l \\ &+\frac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m) + C \end{aligned} \quad (16)$$

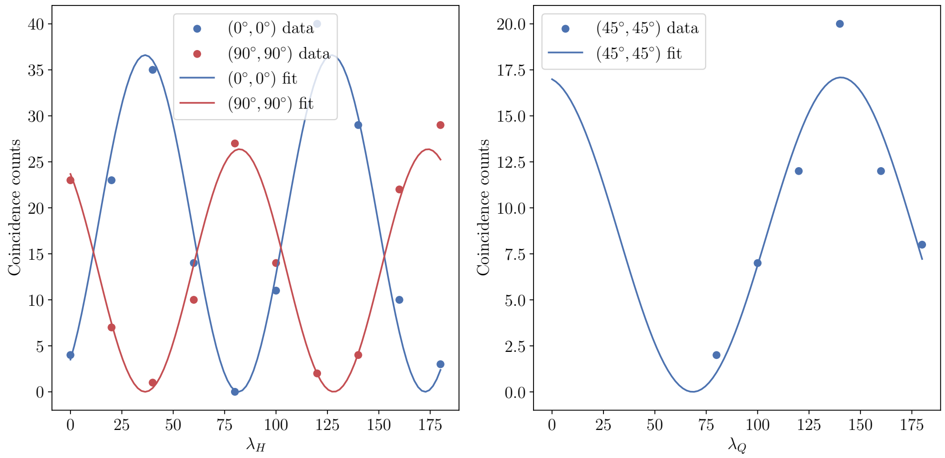

FIG. 4: Calibration of the HWP (left) and the QWP (right).

and that the model parameters can be estimated with

$$C = N(0°, 90°), \tag{17}$$

$$A = N(0°, 0°) + N(90°, 90°) - 2C, \tag{18}$$

$$\tan^2 \theta_l = \frac{N(90°, 90°) - C}{N(0°, 0°) - C}, \tag{19}$$

$$\cos \phi_m = \frac{1}{\sin 2\theta_l} \left(4 \frac{N(45°, 45°) - C}{A} - 1 \right). \tag{20}$$

Using the results from Table I, we can use the analytical expressions in Eqs. (17)–(20) as a quick check of the state we have prepared. Doing so, we obtain

$$A = 19(6), \tag{21}$$

$$\theta_l = 55(8)°, \tag{22}$$

$$\cos \phi_m = 2(1), \tag{23}$$

$$C = 4(2). \tag{24}$$

Eq. (22) clearly deviates from the desired $\theta_l = 45°$, which is likely due to improper calibration of the wave plates or misalignment of the other optical components. Additionally, note that we should have $|\cos \phi_m| \leq 1$, but this is violated by Eq. (23). This suggests that Eqs. (17)–(20) might not be the proper way to describe our prepared state. A comparison between the measurements from Table I and the model with the analytic check parameters in Eqs. (21)–(24) is shown in Fig. 5(a).

In order to get a better estimate of the prepared state used in the data of actual interest, we can fit the measurements from Table II into a model like Eq. (16). We performed such fit using `SciPy`'s `least_squares` procedure [9]. This fit gives us

$$A = 200(20), \tag{25}$$

$$\theta_l = 25(4)°, \tag{26}$$

$$\phi_m = -0.003° \pm 300000°, \tag{27}$$

$$C = 20(7) \tag{28}$$

Unfortunately, Eq. (26) is again far from the desired $\theta_l = 45°$. Also, note that the high error in Eq. (27) indicates that the least squares fit did not see any changes in the numerical residuals when varying ϕ_m. A comparison between the measurements from Table II and the model with the fit parameters in Eqs. (25)–(28) is shown in Fig. 5(b).

The significant difference between Eq. (22) and Eq. (26) is concerning and should be further investigated in future experiments. One possible source for this discrepancy is in how the two approaches handle the difficulty in estimating ϕ_m, leading to the strange results in Eq. (23) and Eq. (27).

C. Bell's inequality

Using the measurements in Table II, we can calculate the value of S using Eq. (5) and Eq. (11). The resulting value is

$$S = 1.8(1). \tag{29}$$

Unfortunately, we do not get the expected violation of Bell's inequality, as this value is less than 2. This is likely due to the fact that we did not properly prepare the photons in the maximally entangled state, as discussed in the previous section.

That said, we can try to estimate what S would be if we use our model from Eq. (16) with the parameters found in Eqs. (25)–(28), except that we enforce $\theta_l = 45°$. The result of this "corrected" model is shown in Fig. 5(c). The contrast between Fig. 5(b) and Fig. 5(c) highlights that our state's value of θ_l has a significant impact in the distribution of our coincidence measurements. With this, we can re-calculate S using Eq. (5) and Eq. (11) to find

$$S = 2.0(1). \tag{30}$$

While this brings us closer to the expected violation of

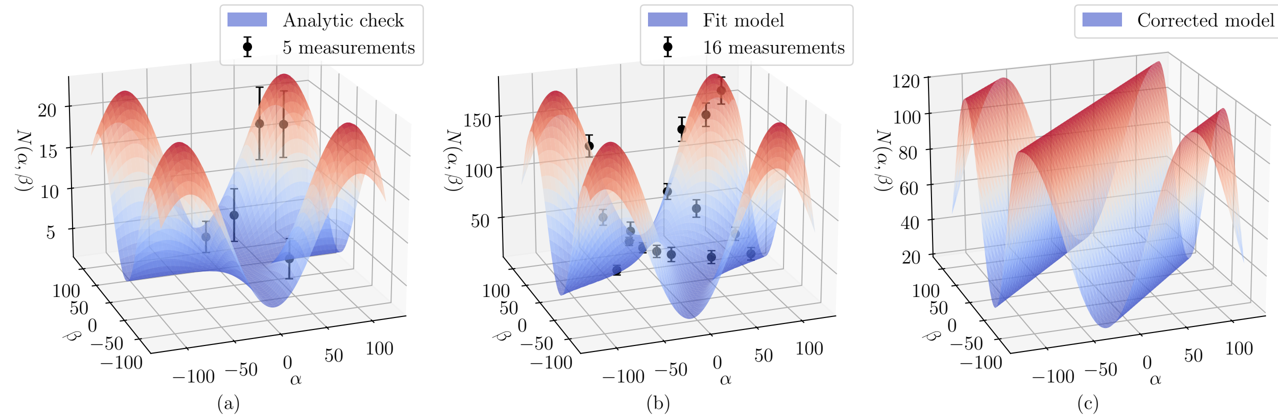

FIG. 5: Visualizations of $N(\alpha, \beta)$ based on the model in Eq. (16). (a) uses the analytic parameters from Eqs. (21)–(24). (b) uses the fit parameters from Eqs. (25)–(28). (c) uses the fit parameters from Eqs. (25)–(28) but enforces $\theta_l = 45°$.

Bell's inequality, it is still not enough to conclude that we have violated Eq. (4) because of our uncertainty.

V. CONCLUSION

Using polarization-entangled photon pairs generated through spontaneous parametric down-conversion in a birefringent crystal, we conducted an experiment to test Bell's inequality. Our analysis of the prepared state revealed deviations from the ideal maximally entangled state, as indicated by the estimated value of $\theta_l \approx 25°$, which differs significantly from the desired $\theta_l = 45°$. Therefore, the calculated value of $S = 1.8(1)$ did not show a violation of the Bell inequality ($|S| \leq 2$). This non-ideal entangled state preparation is likely due to imperfect calibration of the wave plates or misalignment in the optical setup. While a "corrected" model assuming a maximally entangled state ($\theta_l = 45°$) gave us a value of $S = 2.0(1)$, the high uncertainty still prevents a definitive conclusion regarding the violation of Bell's inequality. That being said, this experiment provided a great opportunity to gain hands-on experience with quantum optics and the details of testing fundamental aspects of quantum mechanics.

Future work should focus on refining the preparation of the maximally entangled state. This could involve a more precise calibration of the half-wave and quarter-wave plates, as well as a thorough re-alignment of the entire optical path. It is also essential to investigate the discrepancy observed in the estimated θ_l between the analytical expressions in Eqs. (17)–(20) and the fit according to the model in Eq. (16). Furthermore, increasing the data acquisition time for each measurement point could reduce statistical uncertainties and provide a more robust evaluation of S.

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