# Pen-and-Paper Brain Teasers: Math for Econometrics

## Easy Exercises (5-10 minutes each)

### Linear Algebra - Vectors & Matrices

1. **Vector Addition** - Given vectors **a** = [2, -3, 5] and **b** = [1, 4, -2], compute **a** + **b** and **a** - **b**.
2. **Dot Product** - Calculate the dot product of **u** = [3, 1, -2] and **v** = [4, -1, 5]. Is the angle between them acute or obtuse?
3. **Matrix Multiplication (2×2)** - Multiply matrices:
   $$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$
4. **Matrix Transpose** - Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find $A^T$ and verify that $(A^T)^T = A$.
5. **Scalar Multiplication** - If $C = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, compute 3C and -2C.
6. **Diagonal Matrix** - Create a 3×3 diagonal matrix D with diagonal elements [2, 5, -1]. Compute $D^2$.
7. **Identity Matrix** - Verify that $I_2 \times A = A$ where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 7 \\ -2 & 4 \end{bmatrix}$.

### Calculus - Derivatives

8. **Power Rule** - Find the derivative of $f(x) = 3x^4 - 2x^3 + 5x - 7$.
9. **Product Rule** - Differentiate $g(x) = (2x + 1)(x^2 - 3)$.
10. **Chain Rule (Basic)** - Find $\frac{d}{dx}(3x + 2)^5$.
11. **Exponential Function** - Compute $\frac{d}{dx}e^{2x}$ and $\frac{d}{dx}e^{-x}$.
12. **Natural Logarithm** - Find the derivative of $h(x) = \ln(x^2 + 1)$.

### Calculus - Integrals

13. **Basic Integration** - Evaluate $\int(3x^2 + 4x - 1)dx$.
14. **Definite Integral** - Calculate $\int_0^2(x^2 + 1)dx$.
15. **Exponential Integral** - Find $\int e^{3x}dx$.

## Statistics - Descriptive Statistics

16. **Mean & Median** - Data:. Calculate mean and median.[1] [2] [3] [4] [5] [6] [7]
17. **Variance** - For data , compute the sample variance.[8] [9] [10] [11]
18. **Standard Deviation** - Given variance = 16, find standard deviation. If all values are multiplied by 2, what's the new standard deviation?
19. **Range & IQR** - Data:. Find range, Q1, Q3, and IQR.[2] [4] [10] [12] [13] [14] [15] [1]
20. **Z-score** - A test score is 85 with mean = 75 and σ = 10. Calculate the z-score. Is this score above or below average?

## Probability Basics

21. **Basic Probability** - A bag has 3 red, 5 blue, and 2 green balls. What's P(red)? P(not blue)?
22. **Complementary Events** - If P(rain) = 0.3, what's P(no rain)?
23. **Multiplication Rule** - Two independent coin flips. What's P(both heads)?
24. **Expected Value (Discrete)** - A simple game: win $10 with probability 0.3, lose $5 with probability 0.7. What's the expected value?
25. **Variance of Distribution** - Random variable X: P(X=1) = 0.5, P(X=3) = 0.5. Find E[X] and Var(X).

## Medium Exercises (10-30 minutes each)

## Linear Algebra - Matrix Operations

1. **Determinant (2×2)** - Calculate the determinant of $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$. Is A invertible?
2. **Matrix Inverse (2×2)** - Find the inverse of $B = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$. Verify by computing $B \times B^{-1}$.
3. **System of Linear Equations** - Solve using substitution or elimination:
   $\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$
4. **Matrix Form of System** - Write the system as $A\mathbf{x} = \mathbf{b}$ and solve using matrix inverse:
   $\begin{cases} 3x + 2y = 8 \\ x + y = 3 \end{cases}$
5. **Eigenvalues (2×2)** - Find eigenvalues of $C = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ by solving $\det(C - \lambda I) = 0$.
6. **Quadratic Form** - Given $Q = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, compute $\mathbf{x}^T Q \mathbf{x}$.
7. **Projection Matrix** - For vector $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find the projection onto $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

## Calculus - Optimization

8. **Critical Points** - Find critical points of $f(x) = x^3 - 6x^2 + 9x + 1$. Classify them as max, min, or inflection.
9. **Maxima/Minima** - A farmer has 100m of fence. What rectangular dimensions maximize area?
10. **Second Derivative Test** - For $g(x) = 2x^3 - 3x^2 - 12x + 5$, find local maxima and minima using the second derivative test.
11. **Partial Derivatives** - Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x, y) = x^2y + 3xy^2 - 4$.
12. **Chain Rule (Multivariate)** - If $z = x^2 + y^2$, $x = 2t$, $y = 3t$, find $\frac{dz}{dt}$.
13. **Gradient Vector** - For $f(x, y) = 3x^2 + 2xy + y^2$, compute $\nabla f$ at point (1, 2).

## Statistics - Distributions

14. **Normal Distribution** - X ~ N(100, 15). Using z-table, find P(X < 115) and P(85 < X < 115).
15. **Standardization** - Convert X ~ N(50, 10) to standard normal. What's the z-score for X = 65?
16. **Binomial Probability** - Flip fair coin 5 times. Calculate P(exactly 3 heads) and P(at least 3 heads).
17. **Covariance** - Given paired data: X = , Y =. Calculate Cov(X,Y).[12] [16] [17] [18] [19] [8]
18. **Correlation** - From the covariance above, calculate the correlation coefficient ρ(X,Y).
19. **Law of Total Probability** - Events: $A_1$ (disease, 1%), $A_2$ (no disease, 99%). Test: P(+|disease) = 0.95, P(+|no disease) = 0.05. Find P(+).
20. **Bayes' Theorem** - Using the above, find P(disease|+).

## Econometrics Basics

21. **Simple Regression (By Hand)** - Data: (x,y) = {(1,2), (2,4), (3,5), (4,7)}. Calculate slope $\beta_1$ and intercept $\beta_0$ using formulas:
    $\beta_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$
22. **Fitted Values & Residuals** - Using regression from above, calculate fitted values $\hat{y}_i$ and residuals $e_i = y_i - \hat{y}_i$.
23. **R-squared** - Calculate $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$ where $SS_{res} = \sum e_i^2$ and $SS_{tot} = \sum(y_i - \bar{y})^2$.
24. **Elasticity** - If demand function is $Q = 100 - 2P$, calculate price elasticity at P = 20.
25. **Log Returns** - Stock prices: Day 1: \$100, Day 2: \$105, Day 3: \$103. Calculate log returns for each day.

## Hard Exercises (1-2 hours each)

### Linear Algebra - Advanced

1. **Eigenvalue Decomposition** - For matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$:
   - Find eigenvalues $\lambda_1$, $\lambda_2$
   - Find corresponding eigenvectors
   - Write A as $A = PDP^{-1}$ where D is diagonal
   - Verify by computing $PDP^{-1}$
2. **Least Squares (Matrix Method)** - Data points: (1,1), (2,3), (3,2), (4,5). Fit line $y = \beta_0 + \beta_1 x$ using normal equations:
   $(X^T X)\boldsymbol{\beta} = X^T \mathbf{y}$
   Set up matrices, compute $(X^T X)^{-1}$, solve for β.
3. **Positive Definite Matrix** - Prove that $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ is positive definite by:
   - Checking all eigenvalues > 0
   - Checking leading principal minors > 0
   - Showing $\mathbf{x}^T A \mathbf{x} > 0$ for any non-zero **x**
4. **Diagonalization Application** - Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, compute $A^{10}$ using diagonalization (much easier than multiplying A by itself 10 times).
5. **Singular Value Decomposition (2×2)** - For $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, find the SVD: $B = U\Sigma V^T$. (Hint: this is already diagonal!)

### Calculus - Multivariate Optimization

6. **Unconstrained Optimization** - Find the minimum of $f(x, y) = x^2 + 2y^2 - 2xy + 2x - 4y + 5$. Use:
   - Find critical points (set ∇f = 0)
   - Compute Hessian matrix
   - Use second derivative test (check if Hessian is positive definite)
7. **Lagrange Multipliers** - Maximize $f(x, y) = xy$ subject to $x + y = 10$. Set up Lagrangian, find critical points, solve for x, y, λ.
8. **Constrained Optimization (Inequality)** - Minimize $f(x, y) = x^2 + y^2$ subject to $x + 2y \geq 4$. Use KKT conditions or graphical analysis.
9. **Taylor Approximation** - Approximate $e^{0.1}$ using Taylor series up to 3rd degree term. Calculate error compared to actual value.
10. **Integration by Parts** - Evaluate $\int xe^x dx$ using integration by parts formula.

## Statistics - Hypothesis Testing

11. **T-test (By Hand)** - Sample 1: (n=5). Sample 2: (n=5). [4] [5] [6] [7] [11] [20] [21] [22] [1] [2]
    - Calculate means and pooled variance
    - Compute t-statistic
    - With df=8, critical value (95%) ≈ 2.306. Reject $H_0: \mu_1 = \mu_2$?
12. **Chi-Square Test** - Observed frequencies:. Expected (all equal):. Calculate $\chi^2$ statistic: [6] [14] [15]
    $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$
    With df=3, critical value (95%) ≈ 7.815. Significant?
13. **Confidence Interval** - Sample mean = 50, s = 8, n = 16. Construct 95% CI for population mean (use t = 2.131 for df=15).
14. **Type I and Type II Errors** - A court trial analogy: Define null hypothesis $H_0$ (defendant innocent). What is Type I error? Type II error? Which is worse in this context?
15. **Power Calculation** - For testing $H_0: \mu = 100$ vs $H_1: \mu = 105$, with $\sigma = 10$, n = 25, $\alpha = 0.05$. Calculate the power (probability of rejecting $H_0$ when $\mu = 105$).

## Econometrics - Regression Analysis

16. **Multiple Regression Setup** - You have data on house prices (Y), square footage ($X_1$), and number of bedrooms ($X_2$). Write out:
    - The regression model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$
    - Matrix form: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$
    - Normal equations: $(X^T X)\boldsymbol{\beta} = X^T\mathbf{y}$
    - For small dataset {(Y, $X_1$, $X_2$)}: (200, 1000, 2), (250, 1200, 3), (300, 1500, 3), set up and solve
17. **Omitted Variable Bias** - True model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$. You mistakenly estimate: $Y = \alpha_0 + \alpha_1 X_1 + u$.
    - Show that $\alpha_1$ is biased if $X_1$ and $X_2$ are correlated
    - Derive the bias formula: $\text{Bias} = \beta_2 \times \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_1)}$
    - Give economic interpretation
18. **Heteroskedasticity Test** - Given residuals from regression: [2, -1, 4, -3, 5, -2, 6, -4]. Plot $e_i^2$ against fitted values. Does variance appear constant? What problem does this indicate?
19. **Autocorrelation (Durbin-Watson)** - Residuals (in time order): [1, 2, -1, -2, 0, 1, -1, 2]. Calculate Durbin-Watson statistic:
    $DW = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$
    Is there evidence of autocorrelation? (DW ≈ 2 means no autocorrelation)
20. **Instrumental Variables** - Regression: $\text{Education} = \beta_0 + \beta_1 \text{Income} + \epsilon$. Problem: Income is endogenous (reverse causality). Propose valid instrument (e.g., parental education). What conditions must instrument satisfy?

## Advanced Exercises (2-4 hours each)

### Linear Algebra - Higher Dimensions

1. **3×3 Determinant & Inverse** - Calculate the determinant of matrix A using cofactor expansion:
   $$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$
   Then find $A^{-1}$ using the adjugate method: $A^{-1} = \frac{1}{\det A}\mathrm{adj}(A)$. Verify your answer.

2. **4D Vector Operations** - Given four-dimensional vectors:
   - $\mathbf{a} = [1, -2, 3, 4]$, $\mathbf{b} = [2, 1, -1, 2]$, $\mathbf{c} = [0, 3, 2, -1]$
   - Calculate: $\mathbf{a} \cdot \mathbf{b}$, $\|\mathbf{a}\|$, angle between **a** and **b**
   - Find vector $\mathbf{d}$ orthogonal to both **a** and **b** (use Gram-Schmidt on 4D space)
   - Compute projection of **c** onto **a**

3. **3×3 Eigenvalue Problem** - For matrix:
   $$B = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$
   - Find all three eigenvalues by solving the characteristic polynomial (cubic equation)
   - Find corresponding eigenvectors for each eigenvalue
   - Verify orthogonality of eigenvectors (B is symmetric)

4. **3×3 System of Equations** - Solve using Gaussian elimination with back-substitution:
   $$\begin{cases} 2x + y - z = 8 \\ -3x - y + 2z = -11 \\ -2x + y + 2z = -3 \end{cases}$$
   Show all row operations step-by-step to reach row echelon form.

5. **Gram-Schmidt Orthogonalization (3D)** - Given basis vectors:
   $$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
   Apply Gram-Schmidt to produce orthonormal basis {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$}. Verify orthonormality.

6. **QR Decomposition (3×3)** - Using the result from exercise 5, write the original matrix with columns [$\mathbf{v}_1$ $\mathbf{v}_2$ $\mathbf{v}_3$] as $A = QR$ where Q is orthogonal and R is upper triangular.

### Calculus - Higher-Order Polynomials

7. **Fifth-Degree Polynomial** - Find all critical points and classify them for:
   $f(x) = x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$
   Use first and second derivative tests. Identify intervals of increase/decrease and concavity.

8. **Three-Variable Optimization** - Find and classify critical points of:
   $f(x, y, z) = x^2 + y^2 + z^2 - xy - xz - yz$
   - Compute gradient and set to zero
   - Compute 3×3 Hessian matrix
   - Use eigenvalue test for classification
9. **Multivariate Taylor Series** - Find the second-order Taylor expansion of $f(x, y) = e^{x+y}$ around point (0,0):
   $f(x, y) \approx f(0, 0) + \nabla f \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} [x \quad y] H \begin{bmatrix} x \\ y \end{bmatrix}$
   where H is the Hessian. Evaluate at (0.1, 0.2) and compare with actual value.
10. **Lagrange Multipliers (3 Variables)** - Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to constraint $x + 2y + 3z = 6$.
    - Set up Lagrangian with one constraint
    - Solve system of 4 equations in 4 unknowns
    - Verify solution using geometric interpretation (closest point on plane to origin)
11. **Polynomial Long Division** - Divide $p(x) = 2x^4 + 3x^3 - x^2 + 5x - 2$ by $q(x) = x^2 + x - 1$.
    Find quotient and remainder, verify $p(x) = q(x) \cdot \text{quotient} + \text{remainder}$.
12. **Partial Fraction Decomposition** - Decompose the rational function:
    $\frac{2x^3+3x^2+4x+5}{(x+1)(x-1)(x^2+1)}$
    into partial fractions. Then integrate the result.

## Statistics - Multivariate Analysis

13. **3×3 Covariance Matrix** - Given data for three variables X, Y, Z (5 observations each):
    - X: [16] [17] [18] [8] [12]
    - Y: [17] [8] [12]
    - Z: [9] [18] [8] [12]

    Calculate the 3×3 covariance matrix and 3×3 correlation matrix by hand.
14. **Multinomial Distribution** - A die is rolled 12 times. Calculate the probability of getting exactly 2 ones, 3 twos, 3 threes, 2 fours, 1 five, and 1 six using the multinomial formula:
    $P = \frac{n!}{n_1! n_2! \cdots n_k!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$
15. **Principal Component Analysis (By Hand)** - Using the 3×3 covariance matrix from exercise 13:
    - Find eigenvalues and eigenvectors
    - Order eigenvectors by eigenvalue magnitude (these are principal components)
    - Calculate proportion of variance explained by first two components
    - Transform one data point into PC space

## Econometrics - Advanced Topics

16. **Three-Variable Regression (Matrix Form)** - Fit model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon$ to data:

| Y | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| 10 | 1 | 2 | 1 |
| 15 | 2 | 3 | 2 |
| 12 | 1 | 4 | 1 |
| 18 | 3 | 5 | 2 |
| 20 | 4 | 6 | 3 |

   - Construct design matrix X (5×4, including intercept column)
   - Compute $X^T X$ (4×4 matrix)
   - Compute $(X^T X)^{-1}$ (requires 4×4 inversion)
   - Solve for $\boldsymbol{\beta} = (X^T X)^{-1} X^T \mathbf{y}$
   - Calculate fitted values and $R^2$

17. **F-Test for Joint Significance** - Using regression from exercise 16:
   - Test $H_0$: $\beta_2 = \beta_3 = 0$ (last two coefficients jointly zero)
   - Calculate SSR_restricted (regress Y on $X_1$ only) and SSR_unrestricted (full model)
   - Compute F-statistic: $F = \frac{(SSR_r - SSR_u)/q}{SSR_u/(n-k)}$
   - Compare with critical value F(2, n-4) at 5% level

18. **Polynomial Regression** - Fit cubic model $Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \epsilon$ to data:
   - (X, Y): (1, 2), (2, 3), (3, 8), (4, 15), (5, 27)
   - Set up design matrix with columns [1, X, $X^2$, $X^3$]
   - Solve normal equations
   - Evaluate model at X = 3.5

19. **Variance-Covariance Matrix of Estimators** - For the regression in exercise 16:
   - Estimate error variance: $\hat{\sigma}^2 = \frac{SSR}{n-k}$
   - Calculate $\text{Var}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2 (X^T X)^{-1}$
   - Extract standard errors (square roots of diagonal elements)
   - Compute t-statistics for each coefficient

20. **Interaction Terms** - Model with interaction: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \times X_2) + \epsilon$
Data:

| Y | $X_1$ | $X_2$ |
|---|---|---|
| 5 | 1 | 2 |

| Y | $X_1$ | $X_2$ |
|---|---|---|
| 10 | 2 | 3 |
| 8 | 1 | 4 |
| 15 | 3 | 3 |
| 12 | 2 | 4 |

- Create interaction variable $X_1 \times X_2$
- Solve for coefficients
- Interpret $\beta_3$: how does the effect of $X_1$ on Y depend on $X_2$?

## Solution Methods Reference (Check Yourself!)

## Linear Algebra Methods

**Easy Level:**

1. **Vector Addition** - *Component-wise addition/subtraction*
2. **Dot Product** - *Dot product formula: $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$; Sign test: positive → acute, negative → obtuse*
3. **Matrix Multiplication (2×2)** - *Row-by-column multiplication rule*
4. **Matrix Transpose** - *Swap rows and columns (element $a_{ij}$ becomes $a_{ji}$)*
5. **Scalar Multiplication** - *Multiply each element by the scalar*
6. **Diagonal Matrix** - *For diagonal matrices, $(D^2)_{ii} = (D_{ii})^2$, off-diagonal elements remain 0*
7. **Identity Matrix** - *Matrix multiplication; identity matrix property*

**Medium Level:**

1. **Determinant (2×2)** - *2×2 determinant formula: $\det(A) = ad - bc$; Invertibility test: $\det(A) \neq 0$*
2. **Matrix Inverse (2×2)** - *2×2 inverse formula: $A^{-1} = (1/\det(A)) \cdot [d, -b; -c, a]$ where $A = [a, b; c, d]$*
3. **System of Linear Equations** - *Substitution method; Elimination (addition) method; Graphical method*
4. **Matrix Form of System** - *Matrix equation: $x = A^{-1}b$; Cramer's rule using determinants*
5. **Eigenvalues (2×2)** - *Characteristic equation: $\det(A - \lambda I) = 0$, yields quadratic equation in λ*
6. **Quadratic Form** - *Matrix multiplication: compute Qx first, then $x^T(Qx)$*
7. **Projection Matrix** - *Vector projection formula: $proj\_u(v) = [(v \cdot u)/(u \cdot u)]u$*

**Hard Level:**

1. **Eigenvalue Decomposition** - *Characteristic polynomial: $\det(A - \lambda I) = 0$; Eigenvector: solve $(A - \lambda I)v = 0$; Diagonalization: $P = [v_1\ v_2]$, $D = \mathrm{diag}(\lambda_1, \lambda_2)$*

2. **Least Squares (Matrix Method)** - *Normal equations: $X^TX\beta = X^Ty$; Matrix solution: $\beta = (X^TX)^{-1}X^Ty$; QR decomposition (advanced); Geometric interpretation: projection onto column space of X*
3. **Positive Definite Matrix** - *Eigenvalue test; Sylvester's criterion (principal minors); Quadratic form test; Cholesky decomposition (advanced)*
4. **Diagonalization Application** - *Diagonalization: $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$; For diagonal D, $D^n$ is easy (just raise diagonal elements to power n)*
5. **Singular Value Decomposition (2×2)** - *SVD algorithm: compute eigenvalues of $B^TB$ and $BB^T$; For diagonal matrices, SVD is trivial; Relationship to eigenvalue decomposition*

**Advanced Level:**

1. **3×3 Determinant & Inverse** - *Cofactor expansion (Laplace expansion) along any row/column; Rule of Sarrus (for 3×3 only); Adjugate method: $A^{-1} = adj(A)/det(A)$; Gauss-Jordan elimination*
2. **4D Vector Operations** - *Standard dot product in n dimensions; Magnitude: $||v|| = \sqrt{(v \cdot v)}$; Angle: $\cos(\theta) = (a \cdot b)/(||a||\ ||b||)$; Gram-Schmidt process for orthogonalization*
3. **3×3 Eigenvalue Problem** - *Characteristic polynomial is cubic; Cardano's formula or numerical approximation; For symmetric matrices: real eigenvalues, orthogonal eigenvectors (Spectral theorem)*
4. **3×3 System of Equations** - *Gaussian elimination; Row echelon form; Back substitution; Gauss-Jordan for reduced row echelon form; LU decomposition*
5. **Gram-Schmidt Orthogonalization (3D)** - *Gram-Schmidt algorithm: $u_1 = v_1/||v_1||$, $u_2 = (v_2 - proj\_\{u_1\}v_2)/||v_2 - proj\_\{u_1\}v_2||$, etc.; QR decomposition is built on this*
6. **QR Decomposition (3×3)** - *A = QR where Q is orthogonal (Q from Gram-Schmidt), R is upper triangular (inner products)*

## Calculus Methods

**Easy Level:**
8. **Power Rule** - *Power rule: $d/dx(x^n) = nx^{n-1}$; Linearity of derivatives*
9. **Product Rule** - *Product rule: $(uv)' = u'v + uv'$; Alternative: expand first then differentiate*
10. **Chain Rule (Basic)** - *Chain rule: $d/dx[f(g(x))] = f'(g(x))\cdot g'(x)$*
11. **Exponential Function** - *Exponential rule with chain rule: $d/dx(e^{kx}) = ke^{kx}$*
12. **Natural Logarithm** - *Logarithmic differentiation with chain rule: $d/dx[\ln(u)] = u'/u$*
13. **Basic Integration** - *Power rule for integration: $\int x^n dx = x^{n+1}/(n+1) + C$; Linearity*
14. **Definite Integral** - *Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a)$*
15. **Exponential Integral** - *Exponential integration rule: $\int e^{kx}dx = (1/k)e^{kx} + C$*

**Medium Level:**
8. **Critical Points** - *First derivative test: set $f'(x) = 0$; Second derivative test: $f''(x) > 0$ (min), $f''(x) < 0$ (max)*
9. **Maxima/Minima** - *Constraint optimization: express area A(x) subject to perimeter constraint; Calculus: find $dA/dx = 0$*
10. **Second Derivative Test** - *Second derivative test: compute $g'(x) = 0$ for critical points, then*

*evaluate g''(x) at each*

11. **Partial Derivatives** - *Partial differentiation: treat other variables as constants*
12. **Chain Rule (Multivariate)** - *Substitution then differentiate; Multivariate chain rule: $dz/dt = (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt)$*
13. **Gradient Vector** - *Gradient vector: $\nabla f = [\partial f/\partial x, \partial f/\partial y]$*

**Hard Level:**

6. **Unconstrained Optimization** - *First-order condition: $\nabla f = 0$ (system of equations); Hessian matrix $H = [\partial^2 f/\partial x_i \partial x_j]$; Second-order condition: check eigenvalues of H or determinant tests; Bordered Hessian (for constrained)*
7. **Lagrange Multipliers** - *Lagrangian: $L(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$; First-order conditions: $\nabla L = 0$; Geometric interpretation: gradient of f parallel to gradient of constraint*
8. **Constrained Optimization (Inequality)** - *KKT (Karush-Kuhn-Tucker) conditions for inequality constraints; Complementary slackness; Graphical method: sketch feasible region; Check boundary vs interior solutions*
9. **Taylor Approximation** - *Taylor series expansion: $f(x) \approx f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \ldots$; Remainder term (Lagrange form) for error estimation*
10. **Integration by Parts** - *Integration by parts: $\int u\,dv = uv - \int v\,du$; Choice of u and dv is critical (LIATE rule: Logarithmic, Inverse trig, Algebraic, Trig, Exponential)*

**Advanced Level:**

7. **Fifth-Degree Polynomial** - *Derivative of polynomial is one degree lower; Rational root theorem for finding roots; Synthetic division; Newton's method for numerical solutions*
8. **Three-Variable Optimization** - *Gradient in 3D: $\nabla f = [\partial f/\partial x, \partial f/\partial y, \partial f/\partial z]$; 3×3 Hessian matrix; Eigenvalue test for definiteness; Sylvester's criterion*
9. **Multivariate Taylor Series** - *Multivariate Taylor: $f(x) \approx f(a) + \nabla f(a)\cdot(x-a) + \frac{1}{2}(x-a)^T H(a)(x-a)$; Hessian contains all second partials*
10. **Lagrange Multipliers (3 Variables)** - *Lagrangian: $L = f - \lambda g$; $\nabla L = 0$ gives n+1 equations; Geometric: level sets of f tangent to constraint surface*
11. **Polynomial Long Division** - *Polynomial long division algorithm; Remainder theorem; Factor theorem*
12. **Partial Fraction Decomposition** - *Method of partial fractions; Cover-up method; Heaviside's method; Integration of rational functions*

## Statistics Methods

**Easy Level:**

16. **Mean & Median** - *Mean = sum/n; Median = middle value after sorting (for odd n)*
17. **Variance** - *Sample variance formula: $s^2 = \Sigma(x_i - \bar{x})^2/(n-1)$*
18. **Standard Deviation** - *SD = √variance; Scaling property: $SD(kX) = |k|\cdot SD(X)$*
19. **Range & IQR** - *Range = max - min; Quartiles: Q1 = 25th percentile, Q3 = 75th percentile; IQR = Q3 - Q1*
20. **Z-score** - *Z-score standardization: $z = (x - \mu)/\sigma$*
21. **Basic Probability** - *Classical probability: P(A) = favorable/total; Complement rule: P(not A) = 1 - P(A)*
22. **Complementary Events** - *Complement rule: $P(A^c) = 1 - P(A)$*
23. **Multiplication Rule** - *Multiplication rule for independent events: $P(A\cap B) = P(A)\cdot P(B)$*

24. **Expected Value (Discrete)** - *Expected value formula:* $E[X] = \Sigma x_i \cdot P(x_i)$
25. **Variance of Distribution** - $E[X] = \Sigma x_i \cdot P(x_i)$; $Var(X) = E[X^2] - (E[X])^2$

**Medium Level:**
14. **Normal Distribution** - *Z-score transformation:* $z = (x - \mu)/\sigma$; *Standard normal table lookup; Symmetry properties of normal distribution*
15. **Standardization** - *Standardization formula:* $Z = (X - \mu)/\sigma$, *where* $Z \sim N(0, 1)$
16. **Binomial Probability** - *Binomial probability formula:* $P(X=k) = C(n,k) \cdot p^{k \cdot (1-p)}(n-k)$; *Binomial coefficient:* $C(n,k) = n!/(k!(n-k)!)$
17. **Covariance** - *Sample covariance formula:* $Cov(X,Y) = \Sigma(x_i - \bar{x})(y_i - \bar{y})/(n-1)$
18. **Correlation** - *Pearson correlation:* $\rho = Cov(X,Y)/(SD(X) \cdot SD(Y))$
19. **Law of Total Probability** - *Law of total probability:* $P(B) = \Sigma P(B|A_i)P(A_i)$
20. **Bayes' Theorem** - *Bayes' theorem:* $P(A|B) = P(B|A)P(A)/P(B)$

**Hard Level:**
11. **T-test (By Hand)** - *Two-sample t-test with pooled variance:* $t = (\bar{x}_1 - \bar{x}_2)/\sqrt{(s^2_p(1/n_1 + 1/n_2))}$; *Pooled variance:* $s^2_p = [(n_1-1)s_1^2 + (n_2-1)s_2^2]/(n_1+n_2-2)$; *Welch's t-test (unequal variances); Paired t-test (if data is paired)*
12. **Chi-Square Test** - *Chi-square goodness-of-fit test; Degrees of freedom:* $df = k - 1 - p$ *(k categories, p estimated parameters); Chi-square test of independence (for contingency tables)*
13. **Confidence Interval** - *t-distribution CI:* $\bar{x} \pm t \cdot (s/\sqrt{n})$; *Z-distribution CI (for large n or known* $\sigma$*); Bootstrap CI (computational method)*
14. **Type I and Type II Errors** - *Type I error (*$\alpha$*): reject true* $H_0$ *(false positive); Type II error (*$\beta$*): fail to reject false* $H_0$ *(false negative); Power =* $1 - \beta$*; Trade-off between errors*
15. **Power Calculation** - *Power =* $P(\text{reject } H_0 \mid H_1 \text{ true})$*; Calculate critical value under* $H_0$*; Find probability of exceeding critical value under* $H_1$*; Use z-distribution or t-distribution*

**Advanced Level:**
13. **3×3 Covariance Matrix** - *Covariance matrix:* $\Sigma_{ij} = Cov(X_i,X_j)$*; Symmetric matrix; Diagonal contains variances; Correlation matrix: standardize by dividing by standard deviations*
14. **Multinomial Distribution** - *Multinomial probability:* $P = n!/(n_1!n_2!...n_k!) \cdot p_1\text{^}n_1 \cdot p_2\text{^}n_2 \cdot ... \cdot p_k\text{^}n_k$*; Generalization of binomial*
15. **Principal Component Analysis (By Hand)** - *PCA steps: (1) Compute covariance matrix, (2) Find eigenvalues/eigenvectors, (3) Sort by eigenvalue, (4) Transform data using eigenvector matrix; Variance explained = eigenvalue/sum of eigenvalues*

## Econometrics Methods

**Medium Level:**
21. **Simple Regression (By Hand)** - *OLS (Ordinary Least Squares) formulas for* $\beta_1$ *and* $\beta_0$*; Method of moments; Geometric interpretation: minimize sum of squared residuals*
22. **Fitted Values & Residuals** - *Fitted value formula:* $\hat{y}_i = \beta_0 + \beta_1 x_i$*; Residual:* $e_i = y_i - \hat{y}_i$
23. **R-squared** - $R^2$ *formula using sum of squares; Alternative:* $R^2 = (\text{correlation coefficient})^2$
24. **Elasticity** - *Point elasticity formula:* $\varepsilon = (dQ/dP) \cdot (P/Q)$
25. **Log Returns** - *Log return formula:* $r_t = \ln(P_t/P_{t-1})$

**Hard Level:**
16. **Multiple Regression Setup** - *OLS in matrix form; Interpretation of coefficients (partial*

*derivatives); Multicollinearity concerns; Adjusted $R^2$*
17. **Omitted Variable Bias** - *Omitted variable bias formula derivation; Auxiliary regression: regress $X_2$ on $X_1$; Direction of bias depends on sign of $\beta_2$ and correlation; Frisch-Waugh-Lovell theorem*
18. **Heteroskedasticity Test** - *Visual inspection: plot residuals vs fitted values or vs X; Breusch-Pagan test; White test; Goldfeld-Quandt test; Consequences: OLS still unbiased but inefficient, standard errors wrong*
19. **Autocorrelation (Durbin-Watson)** - *Durbin-Watson test: $DW \approx 2(1 - \rho)$ where $\rho$ is first-order autocorrelation; Breusch-Godfrey test (higher-order); Ljung-Box test; Consequences: OLS inefficient, standard errors biased*
20. **Instrumental Variables** - *Two-Stage Least Squares (2SLS); Instrument relevance: $Cov(Z, X) \neq 0$; Instrument exogeneity: $Cov(Z, \varepsilon) = 0$; First-stage regression: regress X on Z; Second-stage: regress Y on $\hat{X}$; Weak instruments problem; Overidentification tests (if multiple instruments)*

**Advanced Level:**
16. **Three-Variable Regression (Matrix Form)** - *Design matrix construction (include column of 1s for intercept); $X^TX$ is k×k symmetric matrix; $(X^TX)^{-1}$ requires matrix inversion; $\hat{\beta} = (X^TX)^{-1}X^Ty$; $R^2 = 1 - SSR/SST$*
17. **F-Test for Joint Significance** - *F-test for multiple restrictions; F-statistic: $F = [(SSR_r - SSR_u)/q]/[SSR_u/(n-k)]$; q = number of restrictions; F-distribution with (q, n-k) degrees of freedom; Relationship to $R^2$*
18. **Polynomial Regression** - *Polynomial regression as special case of multiple regression; Design matrix includes powers of X; Multicollinearity issues with high-degree polynomials; Centered polynomials reduce collinearity*
19. **Variance-Covariance Matrix of Estimators** - *$Var(\hat{\beta}) = \sigma^2(X^TX)^{-1}$; Estimate $\sigma^2$: residual variance $s^2 = SSR/(n-k)$; Standard errors: $SE(\hat{\beta}_j) = \sqrt{[(X^TX)^{-1}]_{jj}} \cdot s$; t-statistics for hypothesis testing*
20. **Interaction Terms** - *Interaction: effect of $X_1$ depends on level of $X_2$; Coefficient interpretation: $\partial Y/\partial X_1 = \beta_1 + \beta_3 X_2$; Centering variables before creating interactions; Marginal effects at different levels*

## Research Keywords by Topic

## Linear Algebra

- **Basic operations**: Vector spaces, linear combinations, span, linear independence
- **Matrix theory**: Rank, nullity, row space, column space, null space
- **Determinants**: Properties, cofactor expansion, Cramer's rule
- **Eigentheory**: Characteristic polynomial, algebraic/geometric multiplicity, diagonalization, Jordan normal form
- **Orthogonality**: Inner products, orthogonal projections, Gram-Schmidt, QR decomposition
- **Spectral theory**: Spectral theorem, positive definiteness, Cholesky decomposition
- **SVD**: Singular value decomposition, pseudoinverse, low-rank approximation, matrix norms

- **Advanced**: Tensor products, exterior algebra, matrix exponential, Cayley-Hamilton theorem

### Calculus

- **Single variable**: Limits, continuity, mean value theorem, L'Hôpital's rule
- **Differentiation**: Power rule, product rule, quotient rule, chain rule, implicit differentiation, logarithmic differentiation
- **Integration**: Substitution, integration by parts, partial fractions, trigonometric substitution, improper integrals
- **Series**: Taylor series, Maclaurin series, convergence tests, power series
- **Multivariate**: Partial derivatives, gradient, directional derivatives, Jacobian matrix, implicit function theorem
- **Optimization**: Lagrange multipliers, KKT conditions, convexity, Jensen's inequality, envelope theorem
- **Vector calculus**: Divergence, curl, line integrals, Green's theorem, Stokes' theorem
- **Advanced**: Differential forms, calculus of variations, Legendre transform, Hamiltonian mechanics

### Statistics & Probability

- **Foundations**: Sample space, events, probability axioms, conditional probability
- **Distributions**: Discrete (binomial, Poisson, geometric), Continuous (normal, exponential, chi-square, t, F)
- **Central Limit Theorem**: Sampling distributions, standard error, law of large numbers
- **Estimation**: Point estimation, MLE (Maximum Likelihood), method of moments, bias-variance tradeoff
- **Hypothesis testing**: Null hypothesis, p-value, Type I/II errors, power, multiple testing
- **Confidence intervals**: Coverage probability, pivotal quantities, bootstrap
- **Regression**: Least squares, Gauss-Markov theorem, residual analysis
- **Multivariate**: Joint distributions, marginal distributions, independence, copulas
- **Advanced**: Sufficient statistics, Rao-Blackwell theorem, exponential families, decision theory

### Econometrics

- **OLS foundations**: BLUE, Gauss-Markov theorem, assumptions (linearity, exogeneity, homoskedasticity, no autocorrelation)
- **Violations & remedies**:
  - Heteroskedasticity: White standard errors, WLS (Weighted Least Squares), FGLS
  - Autocorrelation: Newey-West standard errors, Cochrane-Orcutt, AR models
  - Multicollinearity: VIF, ridge regression, PCA

    - Endogeneity: IV (Instrumental Variables), 2SLS, GMM (Generalized Method of Moments)
- **Model selection**: AIC, BIC, adjusted $R^2$, cross-validation, information criteria
- **Causality**: Randomized experiments, natural experiments, difference-in-differences, regression discontinuity, matching, propensity scores
- **Time series**: Stationarity, unit root tests (ADF, PP), ARIMA, VAR, cointegration, ECM (Error Correction Model), GARCH
- **Panel data**: Fixed effects, random effects, Hausman test, clustered standard errors, difference-in-differences
- **Limited dependent variables**: Logit, probit, Tobit, selection models (Heckman), count data (Poisson, negative binomial)
- **Advanced**: Quantile regression, nonparametric regression, structural models, dynamic models, spatial econometrics

### Numerical & Computational Methods

- **Linear systems**: Gaussian elimination, LU decomposition, iterative methods (Jacobi, Gauss-Seidel)
- **Eigenproblems**: Power iteration, QR algorithm, Lanczos method
- **Optimization**: Gradient descent, Newton-Raphson, quasi-Newton (BFGS), conjugate gradient, simplex method
- **Root finding**: Bisection, Newton's method, secant method
- **Integration**: Trapezoidal rule, Simpson's rule, Gaussian quadrature, Monte Carlo
- **Differential equations**: Euler's method, Runge-Kutta, finite differences

This comprehensive reference guide provides all the mathematical methods and research directions needed to tackle exercises at every level!

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