MAT137 Test 3 Notebook

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Definition.

This document is a collection of notes taken from MAT137 videos as well as examples adapted from their corresponding lecture slides. It is written more comprehensively than typical lecture notes, but not with the same educational rigor as a textbook. For this reason, we call this document a **notebook**.

The source code for this notebook as well as others can be found on GitHub

`https://github.com/iahuang/uoft-notebooks`

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2.1 Sigma Notation

The notation for sums uses the capital Greek letter Σ

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

From a programming perspective, the above statement is equivalent to the following code:

```
sum = 0;

for (n=1; n<=i; n++) {
    sum += a[i];
}
```

Useful Formulas

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} &&= 1 + 2 + \cdots + n \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} &&= 1^2 + 2^2 + \cdots + n^2 \\
\sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} &&= 1^3 + 2^3 + \cdots + n^3
\end{aligned}$$

Properties

Let $i_0, n \in \mathbb{Z}$.

Distributive Property

$$\sum_{i=i_0}^{n} (c \cdot a_i) = c \left(\sum_{i=i_0}^{n} a_i \right)$$

Associative/Commutative Property

$$\sum_{i=i_0}^{n} (a_i + b_i) = \sum_{i=i_0}^{n} a_i + \sum_{i=i_0}^{n} b_i$$

2.2 Supremum and Infimum

Definitions

Definition.

Let $A \subseteq \mathbb{R}$. Let $c \in \mathbb{R}$.

1. c is an **upper bound** of A when
$$\forall x \in A,\ x \leq c$$
2. c is the **least upper bound** or **supremum** of A when
   (a) c is an upper bound of A
   (b) If $b \in \mathbb{R}$ is an upper bound of A, then $c \leq b$. In other words,
$$\forall b \in \mathbb{R}, x \in A,\ x \leq b \implies c \leq b$$
   We denote this value "$\sup A$".
3. If $\sup A \in A$, then $\sup A = \max A$.
4. A is **bounded above** when A has at least one upper bound.

This definition can be extended to the definition of infimum. c is a *lower bound* of A when $\forall x \in A,\ x \geq c$; c is the *infimum* of A when it is the *greatest upper bound* and so on.

Definition. (Supremum of a Function)

The **supremum of a function f on a domain** I, denoted $\sup\limits_{x \in I} f(x)$, is equal to
$$\sup\{f(x) : x \in I\}$$

Theorems

Theorem. (Lowest Upper Bound Principle)

Let $A \subseteq \mathbb{R}$.
IF

1. A is bounded above
2. A is not empty

THEN A has a least upper bound.

Theorem.

Let f be a function defined on a domain $I \neq \emptyset$.
If f is bounded above on I, then f has a supremum on I.

Note the similarity between the above theorem and the Extreme Value Theorem that states that for a continuous function f defined on $[a, b]$, there exists both a maximum and minimum on $[a, b]$.

Supremum and the Empty Set

The following are true about the empty set, $\emptyset$:

1. **Every real number is an upper bound of $\emptyset$.**
   Remember that for some statement
   $$\forall n \in S,\ C(n)$$
   If S is empty, then the condition $C(n)$ is irrelevant; the statement is always (vacuously) true.
2. **$\emptyset$ does not have a supremum.**
   From statement (1), we see that there is no least upper bound for $\emptyset$.
3. **$\emptyset$ has no maximum.**
   The definition of maximum uses an existential quantifier; there are no elements of $\emptyset$.
4. **$\emptyset$ is bounded above.**

Examples

Let $A, B \subseteq \mathbb{R}$. Which of the following are true?

1. If $B \subseteq A$ and A is bounded above, then B is bounded above.

   This statement is **TRUE**.
   Proof. We have that A is bounded above; therefore, there exists some upper bound x, for which all elements of A are less than or equal to x. Because, B is a subset of A, all elements of B are also less than or equal to x. ■

2. If $B \subseteq A$ and B is bounded above, then A is bounded above.

   This statement is **FALSE**. Take $A = \mathbb{R}$ and $B = [0, 1]$ as a counterexample.

3. If $B \subseteq A$ and A is bounded above, then $\sup B \leq \sup A$.

   This statement is **TRUE**, assuming both A and B are non-empty. Otherwise, take $B = \emptyset$ and $\sup B$ does not exist.
   Proof. From statement (1), we have that B is also bounded above; then, by the lowest upper bound principle, we have both A and B have a supremum.
   Imagine that $\sup B > \sup A$; there must be some value $x \in B$ such that $x > \sup A$. Because $B \subseteq A$, x must then be simultaneously an element of A and also greater than every element in A, which is impossible, we have $\sup B \leq \sup A$. ■

4. If A and B are bounded above, then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

   This statement is **FALSE**. As a counterexample, consider
   $$A = \{0, 1, 2, 3\} \text{ and } B = \{2, 4\}$$
   We have $\sup A = 3$ and $\sup B = 4$, the minimum of which is 3. However,
   $$\sup(A \cap B) = \sup\{2\} = 2$$

Properties of Supremums of Functions

1. Let f and g be bounded functions on $[a, b]$. Then
   $$\sup_{x\in[a,b]} (f(x) + g(x)) \leq \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x)$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\sup_{x\in[a,c]} f(x) = \max\left\{ \sup_{x\in[a,b]} f(x), \quad \sup_{x\in[b,c]} f(x) \right\}$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then

$$\sup_{x\in[a,b]} (c \cdot f(x)) = |c| \left(\sup_{x\in[a,b]} f(x) \right)$$

2.3 The Darboux Definition of Integral

Definitions

Definition. (Partition)

A **partition** of the interval $[a, b]$ is a set P such that

1. P is finite
2. $P \subseteq [a, b]$
3. $a \in P$ and $b \in P$

Definition. (Upper and Lower Sums)

Let f be a bounded function on $[a, b]$.
Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$.
For each **subinterval** $[x_{i-1}, x_i]$:

1. Let m_i be the infimum of f on $[x_{i-1}, x_i]$

$$m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$$

2. Let M_i be the supremum of f on $[x_{i-1}, x_i]$

$$M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$$

3. Let $\Delta x_i = x_i - x_{i-1}$

The **P-lower sum** of f is

$$L_P(f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$$

The **P-upper sum** of f is

$$U_P(f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$

Definition. (Upper and Lower Integral)

Let f be a bounded function on $[a, b]$.
The **lower integral of** f **from** a **to** b, denoted $\underline{\int_a^b}(f)$, is

$$\underline{\int_a^b}(f) = \sup\{L_P(f) : P \text{ is a partition of } [a, b]\}$$

The **upper integral of** f **from** a **to** b, denoted $\overline{\int_a^b}(f)$, is

$$\overline{\int_a^b}(f) = \inf\{U_P(f) : P \text{ is a partition of } [a, b]\}$$

Definition. (Darboux Integrability)

Let f be a bounded function on $[a, b]$.
When $\underline{\int_a^b}(f) = \overline{\int_a^b}(f)$, we say that f is **integrable** on $[a, b]$ and that

$$\int_a^b f(x)dx = \underline{\int_a^b}(f) = \overline{\int_a^b}(f)$$

Continuous functions are always integrable by this definition of integral.

Theorem.

IF f is a continuous function on $[a, b]$, THEN f is integrable on $[a, b]$.

Properties of Lower and Upper Sums

Definition. (Fine Partitions)

Let P and Q be partitions of the interval $[a, b]$.
We say Q is **finer than** P when $P \subseteq Q$.

1. For every partition P of $[a, b]$, we have

$$L_P(f) \leq U_P(f)$$

2. Let P and Q be partitions of $[a, b]$.
   If $P \subseteq Q$, that is, Q is *finer* than P, then

$$L_P \leq L_Q(f)$$

   and

$$U_Q(f) \leq U_P(f)$$

   In other words, finer partitions are more "accurate".

3. Let P and Q be partitions of $[a, b]$.

$$L_P(f) \leq U_Q(f)$$

Property (3) is a more general version of property (1). Property (1) states that for a given *single* partition P, its lower sum will always be less than or equal to its upper sum. Property (3) says that for *any* two partitions of the same interval, the lower sum of either partition will always be less than or equal to the upper some of the other. This property can be proven using the other two properties.

Proof. Call $R = P \cup Q$. Then $P \subseteq R$ and $Q \subseteq R$.
By property (1), we have

$$L_P(f) \leq L_R(f) \quad \text{and} \quad U_R(f) \leq U_Q(f)$$

By property (2), we have

$$L_R(f) \leq U_R(f)$$

Therefore, we have

$$L_P(f) \leq U_Q(f)$$

■

2.4 Integral Definition Examples

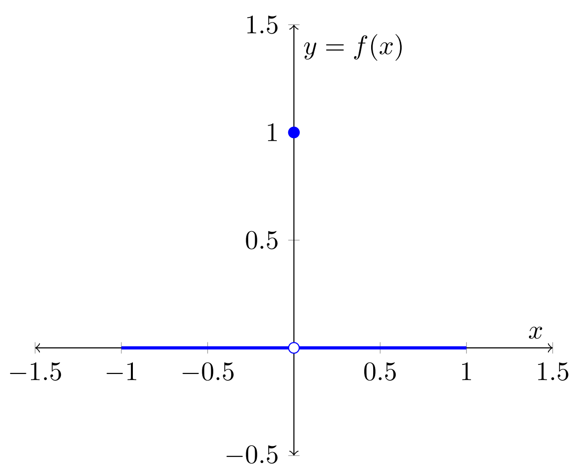
The following examples are dedicated to showing how to prove whether a function is integrable and computing simple integrals using the Darboux definition of integral.

A Simple Integrable Function

Consider the following function:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 1 \end{cases}$$

And here is its graph.



Remember that $f(x)$ is integrable if

$$\underline{\int_{-1}^{1}}(f) = \overline{\int_{-1}^{1}}(f)$$

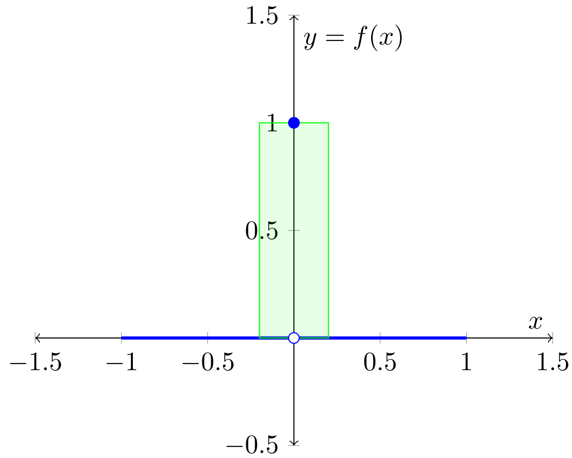
Take any partition P. On every subinterval, the infimum, $m_i = 0$, whether or not the subinterval contains $x = 0$ or not. Thus, we have the lower sums $L_P(f)$ as

$$L_P(f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$$

Furthermore, we have that

$$\underline{\int_{-1}^{1}}(f) = \sup\{\text{all lower sums } L_P(f)\} = \sup\{0\} = 0$$

Take any partition P. There will be exactly one or two subintervals that contain $x = 0$. There will be one subinterval if there is a subinterval that contains $x = 0$, and there will be two if two subintervals share a boundary at $x = 0$. In this case, a partition with two subintervals that share $x = 0$ can be thought of as one partition.



The above diagram depicts the upper sum $L_P(f)$ for some partition P, in which the subinterval containing $x = 0$ is represented using a green rectangle of arbitrary width. Because this subinterval contains $x = 0$, for which $f(x) = 1$, its height is this supremal value of 1. All other subintervals, therefore, have a height of zero, and are not included in this diagram. For this partition P, we have that the upper sum is simply the area of this rectangle

$$U_P(f) = (\text{area of the green rectangle}) = 1 \cdot (\text{width})$$

Depending on the chosen partition P, the width of the green rectangle can be as large or as small as desired; it can be as large as 2, the size of the domain of our integral, or arbitrarily small, but not zero. In other words, the set of all upper sums is simply the set of all areas for some width $0 < w \leq 2$.

$$\{\text{upper sums}\} = \{1 \cdot w : w \in (0, 2]\} = (0, 2]$$

Furthermore, the *upper integral*, the infimum of the set of all upper sums is

$$\overline{\int_{-1}^{1}}(f) = \inf\{\text{upper sums}\} = \inf(0, 2] = 0$$

Therefore, we have that

$$\underline{\int_{-1}^{1}}(f) = \overline{\int_{-1}^{1}}(f) = 0$$

and that f is integrable on $[-1, 1]$, and $\displaystyle\int_{-1}^{1} f(x)dx = 0$.

A Non-Integrable Function

There are many non-integrable functions, one of the most well known of which is called the Dirichlet function.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

In other words, $f(x)$ is 1 if x is a rational number, otherwise, $f(x) = 0$. Let us consider the integral of this function on the interval $[0, 1]$. To see why this function is not integrable, consider the following theorem.

Theorem. ("Rational Number Theorem")

On every interval $[a, b]$ (where $a, b \in \mathbb{R}$ and $a < b$),

- There exists some $x \in [a, b]$ such that x is a rational number ($x \in \mathbb{Q}$)
- There exists some $x \in [a, b]$ such that x is an irrational number ($x \notin \mathbb{Q}$)

In other words, every interval $[a, b]$ contains both rational and irrational numbers.

A simple proof for this theorem can be found in the Appendix.

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be *any* partition of $[0, 1]$. For every subinterval $I = [x_{i-1}, x_i]$, we have

- $m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x) = 0$
- $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x) = 1$

The reason for this is that every interval will have both rational and irrational numbers. Therefore, picking any subinterval and evaluating it on the function f will yield a set S for which at least one element is 0 and at least element is 1. Therefore, $\sup S = 1$ and $\inf S = 0$. As a result, our upper sum (again, for any partition P), will consist of only rectangles with height $M_i = 1$, the widths of which add up to 1. Thus, we have

$$U_P(f) = 1 \cdot 1 = 1$$

Similarly, we also have

$$L_P(f) = 1 \cdot 0 = 0$$

Finally, we have

$$\overline{\int_0^1}(f) = \inf\{1\} = 1$$

and

$$\underline{\int_0^1}(f) = \sup\{0\} = 0$$

As we can see, the upper and lower sums are not equal, and thus, $f(x)$ is not integrable on the domain $[0, 1]$. It is worth noting that the Dirichlet function *is* integrable using the Lebesgue definition of integral; however, that is outside the scope of this course.

2.5 Integrals as Limits

Introduction

As stated before in the previous section regarding, *finer* partitions produce lower upper sums and higher lower sums. Because the upper integral is the infimum of all possible upper sums, in a sense, the upper integral is the upper sum for "the finest" partition. Similarly, the lower integral is the lower sum for "the finest" partition. From this, it follows that if we were to have a limit as "fineness approaches infinity", we would be able to define integrals using limits.

Definitions

Recall the definition of *fine* partitions.

Definition. (Fine Partitions)

Let P and Q be partitions of the interval $[a, b]$.
We say Q is **finer than** P when $P \subseteq Q$.

Definition. (Norm of a Partition)

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$.
For each $i = 1, 2, \dots, n$, let $\Delta x_i = x_i - x_{i-1}$.
The **norm** of P is

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$$

Integrals as Limits

From the above two definitions, we have the following theorem:

Theorem.

Let f be a bounded function on $[a, b]$.

$$\underline{\int_a^b}(f) = \lim_{\|P\| \to 0} L_P(f)$$

and

$$\overline{\int_a^b}(f) = \lim_{\|P\| \to 0} U_P(f)$$

What this theorem states is that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \text{ such that } \ \forall \text{partition } P \text{ of } [a, b],\ \ \|P\| < \delta \implies \left| \underline{\int_a^b}(f) - L_P(f) \right| < \varepsilon$$

and

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \text{ such that } \ \forall \text{partition } P \text{ of } [a, b],\ \ \|P\| < \delta \implies \left| \overline{\int_a^b}(f) - U_P(f) \right| < \varepsilon$$

In other words, as the norm of a partition P approaches zero, the difference between its lower sum and the actual lower integral also approaches zero. The same is true for upper sums.

Because this theorem involves the domain of all possible partitions for a given interval $[a, b]$, it does not simplify computation. By defining a specific way in which the norm of our partitions approaches zero, we can simplify this definition.

Theorem.

Pick a sequence of partitions $P_1, P_2, P_3, \ldots$ for the interval $[a,b]$ satisfying

$$\lim_{n\to\infty} \|P_n\| = 0$$

Then,

$$\underline{\int_a^b}(f) = \lim_{n\to\infty} L_{P_n}(f)$$

and

$$\overline{\int_a^b}(f) = \lim_{n\to\infty} U_{P_n}(f)$$

The simplest example of such a sequence is the sequence that breaks the interval $[a,b]$ into n equally sized subintervals:

$$P_n = \left\{a + \frac{t(b-a)}{n} : t \in \{0,1,2,\ldots,n\}\right\}$$

This definition of integral using limits is more useful than the previous one, since it takes the more intuitive concept of some number approaching infinity, rather than the norm of a partition approaching zero across the domain of all partitions.

The "ε-Characterization" of Integrability

Theorem.

Let f be a bounded function on $[a,b]$.
IF f is integrable on $[a,b]$, THEN

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b] \text{ such that } U_P(f) - L_P(f) < \varepsilon$$

The above theorem represents a way to show that integrable functions can be characterized using a definition similar to that of a limit. One way to prove this theorem is by using the delta-epsilon definition of limit to construct an interval on which $f(x)$ is strictly positive. From there, we can prove that there exists at least one non-zero lower sum. By the properties of lower and upper sums then, all upper sums are also non-zero.

Proof. From the definition of integral as a limit, we have

$$\lim_{n\to\infty} L_{P_n}(f) = \underline{\int_a^b}(f) \quad \text{and} \quad \lim_{n\to\infty} U_{P_n}(f) = \overline{\int_a^b}(f)$$

For a sequence of partitions $P_1, P_2, P_3, \dots$ for the interval $[a, b]$ for which $\lim_{n\to\infty} \|P_n\| = 0$. From the limit definition, then, we have

$$\forall \varepsilon > 0,\ \exists M > 0 \text{ s.t. } n > M \implies \underline{\int_a^b}(f) - L_{P_n}(f) < \varepsilon$$

Because the lower integral is always greater than or equal to all lower sums, we can order the terms as such without need for taking an absolute value. From this statement, we more loosely say that

$$\forall \varepsilon > 0,\ \exists \text{partition } P_1 \text{ s.t. } \underline{\int_a^b}(f) - L_{P_1}(f) < \varepsilon$$

By a similar reasoning for upper sums, we have that

$$\forall \varepsilon > 0,\ \exists \text{partition } P_2 \text{ s.t. } U_{P_2}(f) - \overline{\int_a^b}(f) < \varepsilon$$

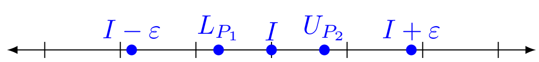
Because we also have that f is integrable on $[a, b]$, then we have that the lower and upper integrals are equal. Thus, let $I = \int_a^b f(x)dx = \underline{\int_a^b}(f) = \overline{\int_a^b}(f)$. Combining this fact with the above two statements, we get that

$$\forall \varepsilon > 0,\ \exists \text{partitions } P_1, P_2 \text{ s.t. } I - L_{P_1} < \varepsilon \text{ and } U_{P_2} - I < \varepsilon$$

Because we also know that $L_{P_1} \le I \le U_{P_2}$, we have that

$$I - \varepsilon < L_{P_1} \le I \le U_{P_2} < I + \varepsilon$$

Here is a diagram of this inequality:



The distance between $I + \varepsilon$ and $I - \varepsilon$ is 2ε. Because L_{P_1} and U_{P_2} are bounded by these two endpoints, their distance cannot exceed 2ε. Thus, we have

$$U_{P_2} - L_{P_2} < 2\varepsilon$$

Let $P_3 = P_1 \cup P_2$. Because $P_2 \subseteq P_3$ and $P_1 \subseteq P_3$, we say that P_3 is *finer* than both P_1 and P_2, and we have the inequality

$$L_{P_1} \le L_{P_3} \le I \le U_{P_3} \le U_{P_2}$$

As L_{P_3}, U_{P_3} are bounded by L_{P_1} and U_{P_2} respectively, we have also that $U_{P_3} - L_{P_3} < 2\varepsilon$. Having shown that the original statement is true for twice the value of any arbitrary value of $\varepsilon > 0$, we have also shown this to be true for all ε. ■

2.6 Riemann Sums & Riemann Integration

Definition

Definition.

Let f be a bounded function on the interval $[a, b]$.
Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$.
For each $i = 1, 2, \ldots, n$:

- Let $\Delta x_i = x_i - x_{i-1}$
- Choose a number $x_i^* \in [x_{i-1}, x_i]$ (for simplicity, we can take $x_i^* = x_{i-1}$ or $x_i^* = x_i$)

Then

$$S_P^*(f) = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$

is called a **Riemann sum** for f and P.

Notice that we say *a* Riemann sum rather than *the* Riemann sum. Because we arbitrarily choose x_i^*, there are infinitely many Riemann sums. One common choice is to take $x_i^* = x_{i-1}$. This is called a *left Riemann sum*, because it uses the left-hand value of each subinterval. To denote this choice of x_i^*, we denote the Riemann sum S_P^*.

Riemann Integrals

The relationship between Riemann sums and integrals can be summarized in the following theorem:

Theorem. ("Riemann Integral Theorem")

Let f be a bounded function on the interval $[a, b]$. Assume f is integrable on $[a, b]$.

- Pick a sequence of partitions $P_1, P_2, P_3, \ldots$ for the interval $[a, b]$ satisfying

$$\lim_{n\to\infty} \|P_n\| = 0$$

- On each subinterval of each partition, pick $x_i^* \in [x_{i-1}, x_i]$.

Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_{P_n}^*(f)$$

- The simplest example of such a sequence is the sequence that breaks the interval $[a, b]$ into n equally sized subintervals:

$$P_n = \left\{ a + \frac{t(b-a)}{n} : t \in \{0, 1, 2, \ldots, n\} \right\}$$

- The simplest choice of x_i^* would be $x_i^* = x_i$

(A proof for the above theorem is included in the Appendix)

Sometimes, this theorem is represented using the single formula:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\cdot \Delta x_i\right]^1$$

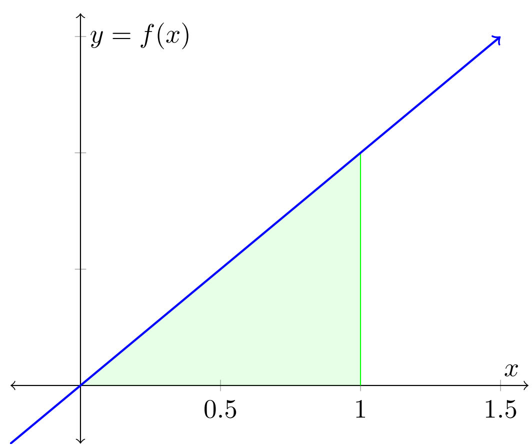
In a sense, the integral and the sum, labeled in blue represent the same concept, as does dx compared to Δx_i in red.

Computing Definite Integrals using Riemann Sums - Example 1

Calculate

$$\int_0^1 f(x)dx \quad \text{where } f(x) = x$$

using Riemann sums. It should be immediately apparent, without any calculations, that the integral is one-half.



Therefore, if the answer reached is also one-half, we know it is correct. Let us begin by choosing P_n by breaking $[0,1]$ into n equal subintervals:

$$P_n = \left\{\frac{t}{n} : t \in \{0,1,2,\ldots,n\}\right\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, 1\right\}$$

Let us also define x_i^* to be the right endpoint of every subinterval. If we have x_i^* corresponding to the subinterval $[x_{i-1}, x_i]$, then the right endpoint is x_i. Since we have

$$P_n = \{x_0, x_1, x_2, \ldots, x_n\} = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, 1\right\}$$

notice that each element of P, denoted x_i, corresponds to the element $\frac{i}{n}$. Therefore, for every subinterval, we have

$$x_i^* = \frac{i}{n}$$

Remember that a Riemann sum is calculated as

$$S_P^*(f) = \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x_i$$

Each interval is evenly spaced. Subinterval x_i corresponds to $\frac{i}{n}$, so we have the width of any subinterval, Δx_i computed as

$$\Delta x_i = x_{i+1} - x_i = \frac{i+1}{n} - \frac{i}{n} = \frac{1}{n}$$

[1]Strictly speaking, x_i^* and x_i should also depend on n, but this is standard notation.

Again, we defined x_i^* to be $\frac{i}{n}$, so substituting values of x_i^* and Δx_i gives us

$$S_P^*(f) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$$

Since we have $f(x) = x$, we have

$$S_P^*(f) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n}$$

Then, to compute the integral from the Riemann sums, we have

$$\begin{aligned}\int_0^1 f(x)dx &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n}\right] \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} i \cdot \frac{1}{n^2}\right]\end{aligned}$$

Because $\frac{1}{n^2}$ is a constant in the context of $\lim\limits_{n\to\infty}$, we then have

$$\begin{aligned}\lim_{n\to\infty}\left[\sum_{i=1}^{n} i \cdot \frac{1}{n^2}\right] &= \lim_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^{n} i \cdot\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{n^2} \cdot \frac{n(n+1)}{2}\right] \\ &= \lim_{n\to\infty}\left[\frac{n(n+1)}{2n^2}\right] \\ &= \lim_{n\to\infty}\left[\frac{n+1}{2n}\right] \\ &= \frac{1}{2}\end{aligned}$$

Finally, we have

$$\int_0^1 f(x)dx = \frac{1}{2}$$

Computing Definite Integrals using Riemann Sums - Example 2

Let $f(x) = x^2 - x + 1$. Compute

$$\int_0^2 f(x)dx$$

Following a similar process as before, let us define our sequence of partitions P_n to divide $[0,2]$ into n equally-sized subintervals as so:

$$P_n = \left\{\frac{2t}{n} : t \in \{0,1,2,\ldots,n\}\right\} = \left\{\frac{0}{n}, \frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n}{n}\right\}$$

let x_i^* be the right endpoint,

$$x_i^* = x_i = \frac{2i}{n}$$

and let

$$\Delta x_i = x_{i+1} - x_i = \frac{2(i+1)}{n} - \frac{2i}{n} = \frac{2i+2-2i}{n} = \frac{2}{n}$$

Then we have

$$
\begin{aligned}
S_P^*(f) &= \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i \\
&= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} \\
&= \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n} + 1\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8i^2}{n^3} - \frac{4i}{n^2} + \frac{2}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{8i^2}{n^3}\right) - \sum_{i=1}^{n} \left(\frac{4i}{n^2}\right) + \sum_{i=1}^{n} \left(\frac{2}{n}\right)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\int_0^2 f(x)dx = \lim_{n\to\infty} \left[S_{P_n}^*(f)\right] &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{8i^2}{n^3}\right) - \sum_{i=1}^{n} \left(\frac{4i}{n^2}\right) + \sum_{i=1}^{n} \left(\frac{2}{n}\right)\right] \\
&= \lim_{n\to\infty} \left[\left(\frac{8}{n^3}\sum_{i=1}^{n} i^2\right) - \left(\frac{4}{n^2}\sum_{i=1}^{n} i\right) + \left(\frac{2}{n}\sum_{i=1}^{n} 1\right)\right] \\
&= \lim_{n\to\infty} \left[\left(\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right) + \left(\frac{2}{n} \cdot n\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{8(n+1)(2n+1)}{6n^2}\right] - \lim_{n\to\infty} \left[\frac{4(n+1)}{2n}\right] + 2 \\
&= \frac{16}{6} - 2 + 2 \\
&= \frac{8}{3} \approx 2.667
\end{aligned}
$$

2.7 Properties of Definite Integrals

Arithemetic Properties

Definite integrals follow similar properties to sums (Σ). Assuming f and g are integrable on $[a, b]$, we have

$$\int_a^b \big[f(x) + g(x)\big]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

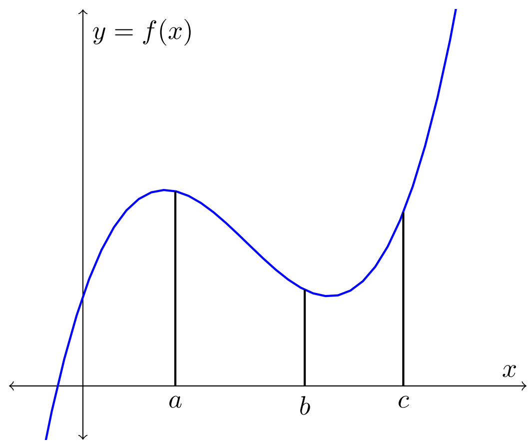
and

$$\int_a^b \big[cf(x)\big]dx = c \cdot \int_a^b f(x)dx$$

where c is some constant.

Definite Integrals over Different Intervals

Let f be some function integrable on $[a, b]$ and on $[b, c]$.



$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Note that this property is true, regardless of the order of a, b, and c.

Backwards and Zero Integrals

Reversing the bounds of a definite integral reverses the sign of its value.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

A definite integral over a zero-width interval is zero, even if the function is not integrable.

$$\int_a^a f(x)dx = 0$$

Integrals of Smaller Functions

If for all values of x on the interval $[a, b]$, we have that $f(x) \leq g(x)$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

3 Unit 8: The Fundamental Theorem of Calculus

3.1 Antiderivatives and Indefinite Integrals

Introduction

So far, we have only been dealing with definite integrals, expressions of the form

$$\int_a^b f(x)dx$$

The value of this expression is a single number. Geometrically, it measures the "area under the curve", more specifically, the area of the function above the x-axis minus the area below the x-axis. This unit will focus on the notion of indefinite integrals, written the same way, but without the notion of "from a to b":

$$\int f(x)dx$$

Definitions

Definition.

Let f be a function defined on an interval.

- An **antiderivative** of f is any function F such that $F' = f$
- The collection of all antiderivatives of f is denoted $\displaystyle\int f(x)dx$

As an example, consider $\displaystyle\int f(x)dx$ where $f(x) = x^2$. Intuitively, by reversing the process involved in the power rule, we see that $F(x) = \frac{1}{3}x^3$ is one antiderivative for f. We can verify that this is correct by taking its derivative.

$$\frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$

However, because the derivative of any constant is zero, we can also add any constant C to F. In other words,

$$\forall C \in \mathbb{R},\ \frac{d}{dx}F(x) = f(x) \quad \text{where } F(x) = \frac{1}{3}x^3 + C$$

Examples - Initial Value

Consider a function f such that $f'(x) = 2x + 1$. Such a function would follow the form

$$f(x) = x^2 + x + C$$

Say we are given that $f(0) = 1$. Then, we can find an exact value for our constant C.

$$f(x) = x^2 + c + 1$$

In general, for any functions $f(x)$ and $g(x)$ such that $f'(x) = g(x)$, we will have

$$f(x) = \cdots + C$$

If we have $C = 0$, $f(0)$ will always equal zero. Therefore, if we are given a fixed value $a = f(0)$, then we have explicitly that

$$f(x) = \cdots + a$$

If we are given a point on f where $x \neq 0$, we can still determine an explicit expression for f. Imagine we are given $f'(x) = 2x + 1$ and $f(2) = 8$. We know that

$$f(x) = x^2 + x + C$$

To resolve an explicit expression for f, we can simply solve for C. Knowing that $f(2) = 8$, we can substitute x for 2 and $f(x)$ for 8 to get the equation

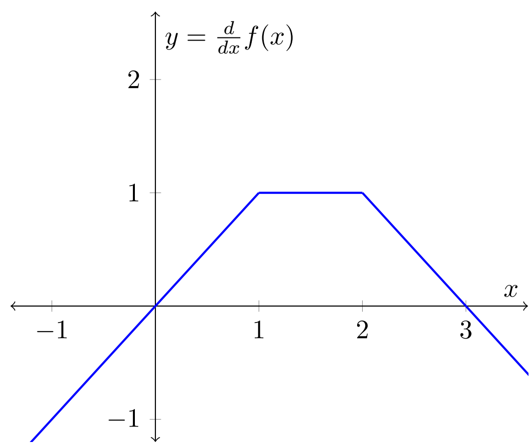
$$8 = (2)^2 + (2) + C$$

Solving this equation gives us $C = 2$. Therefore, we have

$$f(x) = x^2 + x + 2$$

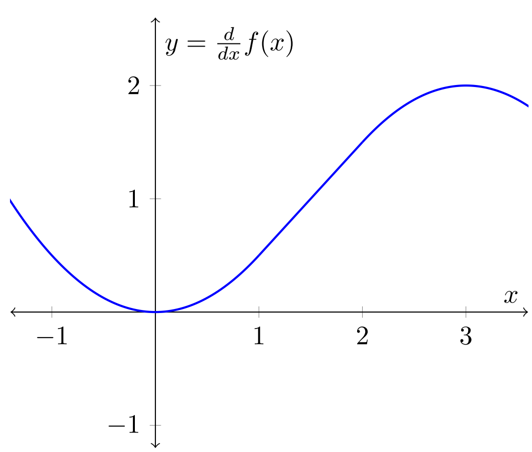
Examples - Antiderivatives of Graphed Functions

Here is an example of a common question asked in calculus classes.
Given the graph of the function $f'(x)$ below, sketch a graph of $f(x)$ where $f(0) = 0$.



As an analogy, consider the graph of $f'(x)$ to be a graph of the velocity of a car with respect to time. The graph of $f(x)$, then, its a graph of its position. From $x \in [-1, 1]$, we have that the car is moving in the negative direction, but it is accelerating in the positive direction. At $x = 0$, it briefly comes to a complete stop, and its position is zero. But then, it keeps accelerating until $x = 1$, at which point, it maintains its current velocity until $x = 2$, at which point it begins to slow down, eventually moving again in the negative direction.

Here is the "answer" to this question:



Notice the straight line ("constant velocity") segment from $x = 1$ to $x = 2$ and the "acceleration" elsewhere. Notice also, that when the graph of $f'(x)$ touches zero, the graph of $f(x)$ "stops moving" briefly (f has a critical point).

Another way to look at these types of problems is by evaluating the type of curvature of $f'(x)$ along segment. For instance, from $x \in [-1, 1]$, we have that the graph of $f'(x)$ is linear. Intuitively, then, we know that its antiderivative along that interval should be a quadratic. Similarly, constant segments of $f'(x)$ should correspond to linear antiderivatives, and quadratic or parabolic-seeming segments of $f'(x)$ should roughly correspond to cubic antiderivatives.

4 Appendix

4.1 Proof of the "Rational Number Theorem"

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$.
Consider a natural number n such that

$$\frac{\sqrt{2}}{n} < b - a$$

(Such a number can be found by taking $\left\lceil \frac{\sqrt{2}}{b-a} \right\rceil$).
Let $w = \frac{\sqrt{2}}{n}$. Notice that w is an irrational number, since $\sqrt{2}$ is irrational. Additionally, $w + a$ is irrational. We also have $a < a + w < b$, since $w < b - a$. Thus, $w + a \notin \mathbb{Q}$ and $w + a \in [a, b]$. To instead find a *rational* number, take $\frac{1}{n}$ instead of $\frac{\sqrt{2}}{n}$. ■

4.2 Proof of the "Riemann Integral Theorem"

Proof. We start by assuming the following statements for any bounded function f:

- $\lim_{n\to\infty} L_{P_n}(f) = \underline{\int_a^b}(f)$
- $\lim_{n\to\infty} U_{P_n}(f) = \overline{\int_a^b}(f)$

However, from our hypothesis, we have that f is integrable, we also have that the upper and lower integrals are equal. Thus,

$$\lim_{n\to\infty} L_{P_n}(f) = \lim_{n\to\infty} U_{P_n}(f) = \int_a^b f(x)dx$$

Additionally, we have

$$L_{P_n}(f) \le S^*_{P_n}(f) \le U_{P_n}(f)$$

Then, from the Squeeze Theorem,

$$\lim_{n\to\infty} S^*_{P_n}(f) = \int_a^b f(x)dx$$

■