A Quantum Algorithm for Assessing Node Importance in the st-Connectivity Attack

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Abstract. Problems in distributed security often map naturally to graphs. The centrality of nodes assesses the importance of nodes in a graph. It is used in various applications. Cooperative game theory has been used to create nuanced and flexible notions of node centrality. However, the approach is often computationally complex to implement classically. This work describes a quantum approach to approximating the importance of nodes that maintain a target connection. Additionally, we detail a method for quickly identifying high-importance nodes. The approximation method relies on quantum subroutines for st-connectivity, approximating Shapley values, and finding the maximum of a list. Finding important nodes relies on a quantum algorithm for finding the maximum. We consider st-connectivity attack scenarios in which a malicious actor disrupts a subset of nodes to perturb the system functionality. Our methods identify the nodes that are most important with the aim of minimizing the impact of the attack. The node centrality metric identifies where more redundancy is required and can be used to enhance network resiliency. Finally, we explore the potential complexity benefits of our quantum approach in contrast to classical random sampling.

1 Introduction

Quantum science and related technologies hold significant potential for global innovation in several domains, quantum-enhanced information networks being only one of them. With recent promising results in quantum computing for combinatorial optimization problems, quantum-enhanced information networks are a promising evolution of classical distributed systems where the use of quantum technologies is expected to foster significant new paradigms [14]. This includes the development of quantum sensor networks and the enhancement of Quantum Key Distribution (QKD) technologies [9]. The integration of quantum computing under these new environments must face traditional security problems, including defense and resilience.

In the realm of graph analytics, node centrality metrics quantify properties such as the utility of a node, whether a node is critical in keeping the graph connected, or if the node is vulnerable to attack. These metrics help to determine whether a network is secure and resilient. They can guide structural changes to improve these properties. Traditional node centrality metrics look at individual nodes; however, some properties cannot be easily measured without considering coalitions of nodes.

This paper builds upon the flexible notion of game theoretic node centrality measures. Specifically, we describe a node centrality metric based on connecting two critical

nodes. We use the metric to handle the following two properties (relevant to distributed systems' security): resilience and remediation degree. The former refers to a communication network's capacity to maintain functionality and accomplish its mission, even in the face of adversarial events. An adversarial event can either occur naturally or result from deliberate actions. The latter can be used to quantify the capacity to provide restoration and mitigation capabilities after an attack to the system occurs.

We aim at addressing the aforementioned properties under the presence of adversaries in a distributed system perpetrating a given type of attacks (the st-connectivity attack). We build a methodological solution to assess the node importance. Quantifying the importance of nodes can be used to guide modifications to network topology such that the level of resilience is improved. We also explore the advantage of a quantum version of our solution, compared to a baseline classical computing solution. We present a practical implementation of our approach and provide a rough estimation and comparison of the complexity of our solution w.r.t. classical Monte Carlo methods. The code is available at our companion Github repository [3].

Paper organization — Section 2 provides motivation and preliminaries. Section 3 presents our contribution and its complexity analysis. Section 4 surveys related work. Section 5 provides the conclusion and perspectives for future work.

2 Motivation

We assume a quantum distributed system offering, e.g., quantum key expansion, entanglement swapping, and error mitigation services [9, 14], in which an adversary aims to disrupt service connectivity from node s to node t. By assuming a classical abstraction of the problem, and by focusing only on the information-gathering stage of the attack, we aim at anticipating ways for the adversary to identify the best strategies to disconnect t from s (i.e., we assume that the adversary can successfully sabotage the services in those intermediate nodes from s to t, hence avoiding any possible functionality between both nodes). Our goal is to identify the most important nodes in keeping node t accessing the services of s, allowing us to increase the resilience of the network. We accomplish this by using the game-theoretic concept of node centrality as a metric to quantify the remediation degree associated to the attack scenario.

Before moving forward, we provide first some needed preliminaries and definitions on the use of cooperative games on graphs. We start with some background concepts on which our approach is founded.

Definition 1 (Network Graph). *Define a graph $H = (N, E)$ to be a pair of the set of network nodes N and set of edges $(u, w) \in E$, with $u, w \in N$ and $u \neq w$.*

Remark 1 (Graph Representation). Let us index each node by some integer in $\mathbb{Z}_{|N|}$. Each edge is indexed by an integer in $\mathbb{Z}_{\binom{|N|}{2}}$ that is mapped to the set of pairs $\{(a, b) : a, b \in \mathbb{Z}_{|N|}, a \neq b\}$, with a bijection. We write the index of edge (u, w) as uw. We may represent the adjacency matrix of the graph with a binary string $x \in \{0, 1\}^{\binom{|N|}{2}}$, where x_{uw} is one if $(u, w) \in E$, otherwise x_{uw} is zero.

[3] `https://github.com/iain-burge/quantum_st-attack`

Definition 2 (Cooperative Games on a Network Graph [18]). *We define a cooperative game on graph $H = (N, E)$ to be the pair $G_H = (F, V)$, where $F \subseteq N$ and V is a valuation function from the subsets of F to the reals, i.e., $V : \mathcal{P}(F) \to \mathbb{R}$. With the restriction that $V(\emptyset) = 0$.*

This definition allows us to treat nodes in F as players of a game. Given a subset of nodes $R \subseteq F$, we can treat it as a binary graph coloring where the colors correspond to the inclusion (or exclusion) of the node in R. $V(R)$ represents the value of that particular graph coloring.

Though it is useful to have a value for coalitions of nodes, or their colorings, the number of combinations grows exponentially with respect to graph size. Thus, it is useful to have a metric that can condense this vast amount of information into a utility for each node. We adapt the Shapley value solution concept to our current situation.

Definition 3 (Node Shapley Value [16]). *Given a game $G_H = (F, V)$ on graph $H = (N, E)$, with $F \subseteq N$. The ith node's Shapley value Φ_i is,*

$$\sum_{R \subseteq F \setminus \{i\}} \gamma\left(|F \setminus \{i\}|, |R|\right) \cdot \left(V(R \cup \{i\}) - V(R)\right)$$

where $\gamma(n, m) = \left(\binom{n}{m}(n+1)\right)^{-1}$.

In this work, we proceed with a narrow concept of graph coloring. If node $a \in F$ is in R, it is considered *enabled*, otherwise, if a is not in R, a is considered *disabled*.

Definition 4 (Sub-graph H_Q). *We define the sub-graph $H_Q = (Q, E_Q)$ of the graph $H = (N, E)$, such that $Q \subseteq N$. $E_Q \subseteq E$ is the subset of all edges $(a, b) \in E$ where $a, b \in Q$.*

In the context of node centrality, we consider the value function $V(R)$ that indicates whether H_R maintains a particular property.

2.1 The st-Connectivity Attack

Definition 5 (st-connectivity). *Consider a graph $H = (N, E)$, with nodes $s, t \in N$. The graph H is st-connected if there exists a path from node s to node t. Formally, H is st-connected if there exists a sequence of nodes $s = u_0, u_1, u_2, \ldots, u_{r-1}, u_m = t$ such that $(u_k, u_{k+1}) \in E$ for $k \in \{0, \ldots, m-1\}$. We define the value function $V_{st} : \mathcal{P}(F) \to \mathbb{R}$,*

$$V_{st}(R) = \begin{cases} 1 & \text{if } H_{R \cup \{s,t\}} \text{ is } st\text{-connected,} \\ 0 & \text{otherwise,} \end{cases}$$

where $R \subseteq F = N \setminus \{s, t\}$, and H_R is described in Definition 4.

In the context of our scenario, the adversary aims to remove st-connectivity (source-target-connectivity). The value function returns 0 when the set of enabled nodes H_R is no longer able to keep the target connected to the source, and 1 when it maintains that property. Hence, the Shapley values (Definition 3) of each node reflect how critical it is to maintain that connection. A high Shapley value means that the node is a valuable target, while a low Shapley value means that the node is not of interest.

Definition 6 (st-connectivity attack). *Given a graph $H = (N, E)$, an st-connectivity attack is a malicious action perpetrated by an adversary. The adversary can turn off a subset of nodes $Q \subseteq F = N \setminus \{s, t\}$. The adversary's goal is to transform the graph H into a sub-graph $H_{N \setminus Q}$ that is not st-connected. Equivalently, the adversary's goal is to minimize $V_{st}(F \setminus Q)$.*

3 Quantum Approach

We present in this section our quantum algorithm for st-connectivity assessment. To begin, we define a simplified version of span programs, detailed in [4, 7].

Definition 7 (Span Program Decision Problem). *A* span program $P\left(|\tau\rangle, \mathcal{W}, x\right)$ *takes as input a unit* target vector $|\tau\rangle \in \mathbb{C}^d$*, a set of* input vectors $\mathcal{W} = \{|\mu_{k,0}\rangle : k \in \mathbb{Z}_r\} \cup \{|\mu_{k,1}\rangle : k \in \mathbb{Z}_r\} \subset \mathbb{C}^d$*, and a binary vector selection string $x = x_{r-1} \cdots x_0 \in \{0,1\}^r$. Given x, the* available vectors *are $A = \{|\mu_{k,x_k}\rangle : k \in \mathbb{Z}_r\}$. The span program P outputs 1 if the target $|\tau\rangle$ is in the span of the available vectors $\mathtt{Span}(A)$. Equivalently, P outputs 1 if there exists a complex vector $c \in \mathbb{C}^r$ such that,*

$$|\tau\rangle = \sum_{k=0}^{r-1} c_k \left|\mu_{k,x_k}\right\rangle .$$

Otherwise, the program returns 0.

We now reformulate the problem of st-connectivity as a span program decision problem [4].

Theorem 1 (Span Program for st-Connectivity). *Consider graph $H = (N, E)$.We detail a span program that determines, given $s, t \in N$, if H is st-connected. Let $|v\rangle$ be a basis vector that represents a node $v \in N$. Define $P\left(|\tau\rangle, \mathcal{W}, x\right)$, where $|\tau\rangle \in \mathbb{C}^{|N|}$,*

$$\mathcal{W} = \left\{|\mu_{uw,0}\rangle : uw \in \mathbb{Z}_{\binom{|N|}{2}}\right\} \cup \left\{|\mu_{uw,1}\rangle : uw \in \mathbb{Z}_{\binom{|N|}{2}}\right\} \subset \mathbb{C}^{|N|},$$

and x is the binary string representation of the adjacency matrix for H (Remark 1). So, x_{uw} is 1 if $(u, w) \in E$, otherwise, x_{uw} is 0. The target vector is,

$$|\tau\rangle = \frac{|t\rangle - |s\rangle}{\sqrt{2}}, \quad s, t \in N.$$

The input vectors are $|\mu_{uw,0}\rangle = 0$, and, $|\mu_{uw,1}\rangle = (|u\rangle - |w\rangle)/\sqrt{2}$, for all $u, w \in N$, and edge indices $uw \in \mathbb{Z}_{\binom{|N|}{2}}$. Thus, our available vector span is,

$$\mathtt{Span}(A) = \mathtt{Span}\left\{\frac{|u\rangle - |w\rangle}{\sqrt{2}} : x_{uw} = 1, uw \in \mathbb{Z}_{\binom{|N|}{2}}\right\}.$$

If the span program outputs 1, H is st-connected, otherwise, H is not st-connected.

Proof. Suppose $H = (N, E)$ is st-connected, then there exists a sequence of nodes $s = u_0, \ldots, u_{r-1}, u_m = t$, such that $(u_k, u_{k+1}) \in E$, $k \in \{0, \ldots, m-1\}$. As a result, for our span program $P(|\tau\rangle, \mathcal{W}, x)$, the set of available vectors A includes every

$$\frac{|u_{k+1}\rangle - |u_k\rangle}{\sqrt{2}}, with \quad k \in \mathbb{Z}_r.$$

We have,

$$|\tau\rangle = \sum_{k=0}^{r-1} \frac{|u_{k+1}\rangle - |u_k\rangle}{\sqrt{2}},$$

since the right-hand side is a telescoping sequence. As a result, the span program accepts the input as expected. A proof to show the span program rejects H when it is not st-connected is demonstrated in [7]. □

Theorem 2 (Quantum st-Connectivity Algorithm [4,7]). *There exists a quantum algorithm to decide whether a graph $H = (N, E)$, with nodes $s, t \in N$, is st-connected. The algorithm uses $\mathcal{O}(\log |N|)$ space, takes $\mathcal{O}\left(|N|^{\frac{3}{2}}\right)$ queries to the adjacency matrix up to polylogarithmic factors. The routine succeeds with probability at least $9/10$. The best possible classical algorithm takes at least $\Omega\left(|N|^2\right)$ time.*

Formally, we have a unitary quantum transformation U_{st} which acts on an auxiliary register of $\mathcal{O}(\log |N|)$ qubits $\texttt{aux}$ and an output register of one qubit $\texttt{out}$. Performing the algorithm and tracing out the auxiliary register results in,

$$tr_{aux}\left(U_{st} |0\rangle_{aux}^{\otimes \mathcal{O}(\log |N|)} |0\rangle_{out}\right) = \left((1-p) |\neg y\rangle \langle \neg y| + p |y\rangle \langle y|\right)_{out}$$

where y is one if H is st-connected and zero otherwise, and p is in range $[9/10, 1)$. Measuring the output bit returns the correct output with probability p.

Proof. We proceed with a rough sketch of the algorithm. A full algorithm and proof are provided in [7]. The algorithm is based on the span program for st-connectivity. We perform phase estimation on the unitary matrix $U = (2\Lambda - I)(2\Pi_x - I)$ with the input vector $|0\rangle$ using precision $\mathcal{O}(|N|^{3/2})$. Thus U is queried $\mathcal{O}(|N|^{3/2})$ times. If the phase estimation outputs zero, the algorithm claims that the graph H is st-connected and outputs 1. Otherwise, if the phase estimation outputs a non-zero answer, the algorithm claims that H is not st-connected, and outputs 0. It is correct with probability $9/10$. We assume, for the sake of simplicity, that $(s, t) \notin E$, this can be checked in $\mathcal{O}(|N|)$ time. We also give edge (s, t) the index $st = 0$.

U is the product of two reflections, a reflection about Λ, and a reflection about Π_x. Λ represents a projection onto the kernel of,

$$\tilde{M} = \mathcal{O}\left(\frac{1}{\sqrt{|N|}}\right) |\tau\rangle \langle 0| + \sum_{uw \in \mathbb{Z}_{\binom{|N|}{2}} \setminus \{0\}} |\mu_{uw,1}\rangle \langle uw| .$$

$\tilde{M}$ represents a transformation from the indices of edges to their respective vectors in the span program for st-connectivity. The reflection, $(2\Lambda - I)$, is implemented using a

Szegedy-type quantum walk [7, 17]. The walk is implemented in logarithmic space and time with respect to $|N|$, and is input independent. Π_x is the projection onto available vector indices and onto the target vector index,

$$\Pi_x = |0\rangle\langle 0| + \sum_{(u,w)\in E} |uw\rangle\langle uw| . \tag{1}$$

Thus, $(2\Pi_x - I)$ represents a reflection where all the indices of unavailable edges are negated. This reflection can be performed with a single query to the adjacency matrix.

Intuitively, the quantum phase estimation extracts the spectral qualities of U. The reflections $(2\Lambda - I)$ and $(2\Pi_x - I)$ are constructed such that the spectral qualities of U correspond to whether $|\tau\rangle$ is linearly independent of the available vectors. □

Remark 2 (Span Program for st-Connectivity Node Centrality). Consider the graph $H = (N, E)$. Suppose we wish to ascertain the st-connectivity of a sub-graph $H_R = (R, E_R)$, $R \subseteq N$. Equivalently, we wish to compute $V(R)$. We proceed similarly as in Theorem 1. Define the span program $P(|\tau\rangle, \mathcal{W}, x^R)$, where $|\tau\rangle$ and $\mathcal{W}$ are described in Theorem 1. Let x^R_{uw} be one if $uw \in E_R$, otherwise x^R_{uw} is zero. Equivalently, we can define x^R_{uw} to equal one if and only if x_{uw} is one and nodes $u, w \in R$.

Definition 8 (Majority Vote). *We define the majority function* $\texttt{MAJ} : \{0,1\}^n \to 0, 1$, *where* n *is odd, as,*

$$\texttt{MAJ}(z) = \begin{cases} 1 & \text{if } \sum_{k=0}^{n} z_k > n/2, \\ 0 & \text{otherwise.} \end{cases}$$

Where $z = z_{n-1} \cdots z_0 \in \{0,1\}^n$. *We also describe the quantum version of this function* $U_{\texttt{MAJ}}$ *which operates on an* n*-qubit register* `in` *and a one-qubit register* `maj`,

$$U_{\texttt{MAJ}} |z\rangle_{\texttt{in}} |0\rangle_{\texttt{maj}} = |z\rangle_{\texttt{in}} |\texttt{MAJ}(z)\rangle_{\texttt{maj}} .$$

Lemma 1 (Majority Vote Powering). *Suppose we have a quantum algorithm* U *which outputs a binary value with fixed success probability* $p > 0.5$. *Let the correct value be* $y \in \{0,1\}$. *We can augment the probability of success by repeatedly performing the algorithm and taking the majority output. In particular, suppose our repeated quantum subroutine gave an* n*-qubit output of,*

$$((1-p)|\neg y\rangle\langle\neg y| + p|y\rangle\langle y|)^{\otimes n} .$$

Then, adding an extra qubit in the form of a `maj` *register, the majority vote unitary* $U_{\texttt{MAJ}}$ *can be applied. Given a desired final failure probability bound* κ, *the* `maj` *register stores the correct answer with probability* $1 - \kappa$ *if* n *is of order* $\mathcal{O}\left(\log \kappa^{-1}\right)$. *In other words, we have failure chance* κ *given* $\mathcal{O}(\log \kappa^{-1})$ *applications of the* U *algorithm.*

Proof. Suppose we perform our quantum algorithm n times, where $n \geq 3$ is odd. This outputs a list of n bits. The probability that k bits are correct is,

$$\binom{n}{k} p^k (1-p)^{n-k}. \tag{2}$$

The threshold for a majority is $t = (n-1)/2$. So, the probability that the majority fails is $\sum_{k=0}^{t} \binom{n}{k} p^k (1-p)^{n-k}$. In Equation (2), for $k \in [0, (n-1)/2]$, the probability is increasing with respect to k. Thus, the probability of majority failure is bounded by,

$$t\binom{n}{t} p^t (1-p)^{n-t}. \tag{3}$$

By an improved version of Stirling's formula [15],

$$\binom{n}{t} < \sqrt{\frac{n}{2\pi t(n-t)}} \frac{n^n}{t^t (n-t)^{n-t}} < \sqrt{\frac{2}{\pi n}} 2^n,$$

where the latter inequality is the result of replacing t with $n/2$. Plugging the inequality into Equation (3) and once again replacing t with $n/2$ yields the new bound $\sqrt{\frac{n}{2\pi}} 2^n (p(1-p))^{n/2}$. So long as $\sqrt{p(1-p)} < 1/2$, which holds for $p > 0.5$, the upper bound for majority failure chance shrinks exponentially with respect to n. □

3.1 Quantum Algorithm for Shapley Value Approximation

The quantum algorithm for Shapley value approximation takes an approach inspired by classical random sampling [10]. Each subset of nodes is is given a probability amplitude proportional their γ coefficient in the Shapley equation (Definition 3). Classically, we would randomly sample from the distribution of node subsets, and record how much our target node increases the value of the subset. After many samples, we take the average increase in value and use it as an approximation. By Chebyshev's inequality the number of samples required scales quandraticaly with respect to desired error. The quantum approach can provide a quadratic improvement.

Theorem 3 (Quantum Algorithm for Shapley Value Approximation [5, 6]). *Take cooperative game on graph $H = (N, E)$ to be the pair $G_H = (F, V)$ where $F \subseteq N$ and V is the value function. Suppose we have a quantum implementation of V, U_V, and that we wish to find the Shapley value Φ_i of node i. Then, given a fixed desired probability for success, there exists a quantum algorithm that produces approximation $\tilde{\Phi}_i$ in,*

$$\mathcal{O}\left(\frac{\sqrt{(V_{\max} - V_{\min})(\Phi_i - V_{\min})}}{\epsilon}\right),$$

queries to the value funciton U_V. Where $V_{\max}, V_{\min}$ are respectively an upper and lower bound for the value function V, and the desired error bound is $\epsilon \geq |\Phi_i - \tilde{\Phi}_i|$.

Proof. We now give a sketch of the algorithm; a complete proof and error analysis is provided in [6]. We can uniquely encode a sub-graph H_R, $R \subseteq F$, as a binary string of the form: $b^R = b_0^R b_1^R \cdots b_{|F|-1}^R \in \{0,1\}^{|F|}$, where $b_j^R = 1$ if $k \in R$ else $b_j^R = 0$. We define quantum implementation U_V of V as,

$$U_V \left|b^R\right\rangle_{\texttt{Pl}} \left|0\right\rangle_{\texttt{Ut}} = \left|b^R\right\rangle_{\texttt{Pl}} \left(\sqrt{1 - \frac{V(R)}{V_{\max} - V_{\min}}} \left|0\right\rangle + \sqrt{\frac{V(R)}{V_{\max} - V_{\min}}} \left|1\right\rangle\right)_{\texttt{Ut}}.$$

We begin with a quantum state made of three registers: $\texttt{Pt}$, the partition register, which helps to prepare the γ probability amplitude distribution (Definition 3); $\texttt{Pl}$, the player register, which stores the sub-graph encodings; and $\texttt{Ut}$, the utility register, which stores the value of a sub-graph. We begin with the quantum state, $|0\rangle_{\texttt{Pt}}^{\otimes \ell} |0\rangle_{\texttt{Pl}}^{\otimes |F|} |0\rangle_{\texttt{Ut}}^{\otimes 1}$, where $\ell = \mathcal{O}(\log((V_{\max} - V_{\min}) \cdot \sqrt{n}/\epsilon))$. Next, prepare the $\texttt{Pt}$ register as follows,

$$\frac{1}{\sqrt{2^\ell}} \sum_{k=0}^{2^\ell - 1} |\theta_k\rangle_{\texttt{Pt}} |0\rangle_{\texttt{Pl}}^{\otimes |F|} |0\rangle_{\texttt{Ut}}^{\otimes 1},$$

where θ_k is an ℓ bit binary approximaiton of $\arcsin \sqrt{2^{-\ell} k}$. For notational simplicity, we suppose $i = |F| - 1$. Using the partition register as a control, it is efficient to transform the state to,

$$\frac{1}{\sqrt{2^\ell}} \sum_{k=0}^{2^\ell - 1} |\theta_k\rangle_{\texttt{Pt}} \left(\left(\sqrt{1 - 2^{-\ell} k} \, |0\rangle + \sqrt{2^{-\ell} k} \, |1\rangle \right)^{\otimes |F| - 1} \otimes |0\rangle \right)_{\texttt{Pl}} |0\rangle_{\texttt{Ut}}^{\otimes 1}, \quad (4)$$

Note that the bit corresponding to node i is zero. Switching to a density matrix representation and tracing out the partition register gives an approximation for the state,

$$\sum_{R \subseteq F \setminus \{i\}} \gamma\left(|F \setminus \{i\}|, |R|\right) |b^R\rangle_{\texttt{Pl}} |0\rangle_{\texttt{Ut}} \langle b^R|_{\texttt{Pl}} \langle 0|_{\texttt{Ut}}.$$

This results from the fact that $\int_0^1 (1-t)^{n-m} t^m dt = \gamma(n, m)$ for integer $n \geq 2$, and $m \in \{0, 1, \ldots, m\}$. Now, applying U_V and measuring the utility bit gives an expected value of,

$$\frac{1}{V_{\max} - V_{\min}} \sum_{R \subseteq F \setminus \{i\}} \gamma\left(|F \setminus \{i\}|, |R|\right) V(R). \quad (5)$$

Using the quantum speedup for Monte Carlo methods [13], the expected value can be approximated quadratically faster than with classical methods.

We can repeat the process with a simple modification, prepare Equation (4) where the bit corresponding to node i is one, then proceed identically to above. This yeilds the expected value,

$$\frac{1}{V_{\max} - V_{\min}} \sum_{R \subseteq F \setminus \{i\}} \gamma\left(|F \setminus \{i\}|, |R|\right) V(R \cup \{i\}). \quad (6)$$

Subtracting Equation (5) from Equation (6), then multiplying the result by $(V_{\max} - V_{\min})$ gives an approximation for the ith player's Shapley value. Note that we can compute Equation (5), Equation (6), and thus the entire Shapley approximation without measurement. As a result, we can approximately perform the transformation,

$$|i\rangle |0\rangle \rightarrow |i\rangle |\tilde{\Phi}_i\rangle. \quad (7)$$

□

Lemma 2 (Shapley Values and Unreliable Value Functions). *Consider the cooperative game $G_H = (F, V)$ on graph $H = (N, E)$ where $F \subseteq N$. We wish to find the Shapley value Φ_i of node i. Suppose $V : \mathcal{P}(F) \to \{0, 1\}$ is a binary classifier, and that V is monotonic, if $Q, R \subseteq F$ then $V(Q \cup R) \geq V(Q)$. We define $\hat{V}$, which, given $Q \subseteq F$, fails and outputs $1 - V(Q)$ with probability $\kappa \in [0, 1]$, or succeeds and outputs $V(Q)$ with probability $1 - \kappa$. Note, for simplicity, we assume a perfect implementation of the γ distribution, in reality, the implementation is an exponentially accurate approximation. Applying the Shapley value approximation using $\hat{V}$ as a substitute for V has expected value*

$$\Phi_i + \xi$$

where ξ is bounded, $|\xi| \leq 2\kappa$.

Proof. We must find the expected value of the following equation,

$$\sum_{R \in F \setminus \{i\}} \gamma(|F \setminus \{i\}|, |R|) \left(\hat{V}(R \cup \{i\}) - \hat{V}(R) \right), \tag{8}$$

By definition, the expected value of $\hat{V}(Q)$, $Q \subseteq F$, is $\kappa \cdot (1 - V(Q)) + (1 - \kappa) \cdot V(Q)$. Rearranging gives, $\mathbb{E}\left[\hat{V}(Q)\right] = V(Q) + \kappa - 2\kappa V(Q)$. Thus, Equation (8) has expected value,

$$\sum_{R \in F \setminus \{i\}} \gamma(|F \setminus \{i\}|, |R|) \left[(V(R \cup \{i\}) - V(R))(1 - 2\kappa) + 2\kappa \right].$$

Applying Definition 3 and Lemma 1 from [6], the expected value is equal to, $\Phi_i + 2\kappa(1 - \Phi_i)$. Since V is monotonic and outputs in range $\{0, 1\}$, Φ_i is in range $[0, 1]$. □

3.2 Combining the Algorithms

In this section, we describe a quantum approach for finding the st-connectivity based node centrality. Consider the cooperative game $G_H = (F, V_{st})$ on graph $H = (N, E)$, where $s, t \in N$ and $F = N \setminus \{s, t\}$. Suppose we wish to find the Shapley value Φ_i of node $i \in F$. We can represent each subset $Q \subseteq F$ with a binary string $b^Q = b_0^Q \cdots b_{|N|-1}^Q$ where b_j^Q is equal to 1 if $j \in Q$ else b_j^Q is 0. Note that, $V_{st}(Q)$ is either 0 or 1. Hence, we can take $V_{\max} = 1$ and $V_{\min} = 0$.

Consider a modified quantum algorithm for st-connectivity algorithm based on Remark 2. We define $U_{st}(Q)$, $Q \subseteq F$ to be the quantum st-connectivity algorithm for graph $H_{Q \cup \{s,t\}}$. This requires a small alteration to the projection Π_x, Equation (1). We replace Π_x with,

$$\Pi_x^Q = |0\rangle \langle 0| + \sum_{(u,w) \in E_Q} |uw\rangle \langle uw| .$$

This can be done efficiently. Instead of directly using the adjacency bit x_{uw}, we use the binary value $x_{uw} \wedge b_u^Q \wedge b_w^Q$. Note that this implementation allows us to perform the calculation for all $Q \subseteq F$ in superposition. The modification makes the algorithm easily compatible with the Shapley value algorithm.

The base quantum algorithm for st-connectivity only has a success probability of $9/10$ (Theorem 2). This is insufficient, as is demonstrated in Lemma 2. However, we can improve our accuracy with logarithmic factor more time and space complexity by repeatedly performing the quantum st-connectivity algorithm and taking the majority answer (Lemma 1). In particular, assuming a desired error κ, we can apply $U_{st}(Q)$ (Remark 2) $n \in \mathcal{O}(\log \kappa^{-1})$ times independently and take the majority vote. We begin with,

$$U_{st}(Q)^{\otimes n} \bigotimes_{k=0}^{n-1} |0\rangle_{\mathtt{aux}_k}^{\otimes \mathcal{O}(\log |N|)} |0\rangle_{\mathtt{out}_k}.$$

Tracing out the auxiliary registers gives us a state of the form required in Lemma 1. Thus, we can take the majority vote $U_{\mathtt{MAJ}}$ and output it to a new one qubit register. If we consider this our utility register `Ut` described in Theorem 3, we can apply the logic from Lemma 2. Specifically, for each the `Ut` quantum basis vector in the player register `Pl`, $|b^Q\rangle$, the utility register holds the correct output $V(Q)$ with probability $1 - \kappa$. As a result, we can define U_V as the product of repeatedly computing $U_{st}(Q)$ order $\mathcal{O}(\log \kappa^{-1})$ times, followed by a $U_{\mathtt{MAJ}}$ operation on the outputs. Thus, by Lemma 2, the expected value we are extracting, Φ_i, is shifted to $\Phi_i + \xi$, $\xi \leq 2\kappa$. Applying the quantum Monte-Carlo speed-up routine extracts the value $\Phi_i + \epsilon + \xi$. Since both ϵ and ξ can be bounded to arbitrarily small values, the algorithm is asymptotically correct.

3.3 Finding Important Nodes

Suppose we wish to find the index of a node with a large Shapley value. Let node m have the largest Shapley value Φ_m. We find node j such that their Shapley value Φ_j is greater than or equal to $\Phi_m - \epsilon$.

Lemma 3. *Consider a game $G_H = (F, V)$, where F is a subset of nodes in the graph H, and $V : \mathcal{P}(F) \to \mathbb{R}$ is the value function. Suppose player i has the largest Shapley value Φ_i, $\Phi_i \geq \Phi_j$ for all $j \in F$. Then, player i's Shapley value has the following lower bound,*

$$\Phi_i \geq \frac{V(F)}{|F|}.$$

Proof. By the property of efficiency [6], we have that, $\sum_{k=0}^{|F|-1} \Phi_k = V(F)$. Suppose that Φ_i is the maximum Shapley value. We proceed by contradiction, let $\Phi_i = (V(F)/|F|) - \epsilon$ for $\epsilon > 0$. It follows that, for all k, $\Phi_k \leq (V(F)/|F|) - \epsilon$. Thus,

$$V(F) = \sum_{k=0}^{|F|-1} \Phi_k \leq \sum_{k=0}^{|F|-1} ((V(F)/|F|) - \epsilon) = V(F) - |F|\epsilon. \tag{9}$$

A contradiction, thus Φ_i can't be less than $V(F)/|F|$. □

As a result, when searching for an important node, at worst, we need precision proportional to $V(F)/|F|$. Thus, to find our importance nodes, we create a uniform superposition of nodes stored in the `Ind` register, where each is given equal probability,

$(1/|F|)\sum_{k\in F}|k\rangle_{\texttt{Ind}}$. We perform our combined algorithm to assess the Shapley values in the st-connectivity game, storing the results $\tilde{\Phi}_k \approx \Phi_k$ in a new `Shp` register,

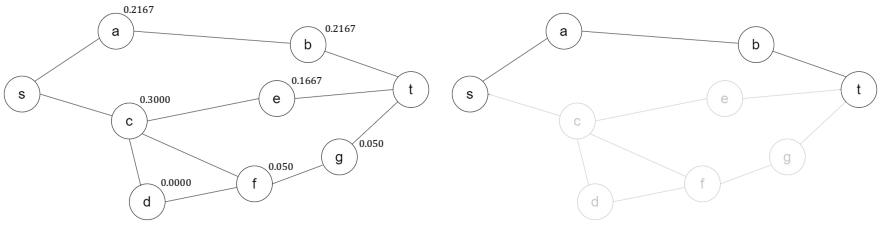
$$\frac{1}{|F|}\sum_{k\in F}|k\rangle_{\texttt{Ind}}\,|\tilde{\Phi}_k\rangle_{\texttt{Shp}}\,,$$

where $|\tilde{\Phi}_k - \Phi_k| \leq \mathcal{O}(V(F)/|F|)$. We can find the k such that $\tilde{\Phi}_k$ is minimized in $\mathcal{O}(\sqrt{|F|})$ applications of the combined algorithm using a quantum algorithm for finding the maximum [1]. By excluding players who have already been assessed, this algorithm can be repeated to find multiple high value players.

3.4 Practical Example

Let $H = (N, E)$ be the graph as shown in Figure 1a. We define cooperative game $G_H = (F, V_{st})$, with $s, t \in N$ and $F = N \setminus \{s, t\}$. Suppose we wish to find the Shapley value Φ_a of node $a \in F$. We can represent each subset $R \in F$ with binary string $b^R = b^R_a \cdots b^R_g$ where b^R_j is equal to 1 if $j \in R$ else b^R_j is 0. Note that, $V(R)$ is either 0 or 1, so we can take $V_{\max} = 1$ and $V_{\min} = 0$. We define U_V as we did in the previous section. For example, suppose we apply U_V with input string $|1100000\rangle_{\texttt{Pl}}$, this represents the subset in Figure 1b. Clearly, this subset is st-connected, since the path s, a, b, t is valid. As a result, if we perform $U_V\,|1100000\rangle_{\texttt{Pl}}\,|0\rangle_{\texttt{Ut}}$, the `Ut` register stores the correct answer, 1, with probability $1 - \kappa$. However, if we removed node a, the graph would no longer be st-connected. So, the state $U_V\,|0100000\rangle_{\texttt{Pl}}\,|0\rangle_{\texttt{Ut}}$ has the answer 0 stored in the `Ut` register with probability $1 - \kappa$.

To find Shapley value Φ_a, we proceed as follows: (i) craft a quantum state that encodes every possible subset of nodes, that does not include node a, with correct amplitude probability weights corresponding to γ; (ii) perform the unitary U_V outputting to `Ut`, i.e., repeatedly check for st-connectivity leveraging Theorem 2 and take the majority answer; (iii) extract the expected value of the utility register `Ut` using the Monte-Carlo speed-up [13]; (iv) repeat the previous steps where each subset that includes node



(a) (b)

Fig. 1: Practical example (cf. our companion Github repository for further details). (a) Shapley Values. (b) Coalitions of Nodes (in which coalitions of nodes represented by binary string 1100000).

a is considered and compare outputs. Using this strategy, we can approximate the Shapley values of each node to arbitrary accuracy (Figure 1a). As a result, we can also leverage the techniques described in Section 3.3, to quickly identify which nodes have the highest Shapley values.

3.5 Complexity Analysis

Baseline Classical Complexity – We now describe a reasonable, though not necessarily optimal method to approximate st-connectivity based node-centrality through classical methods. Let $G_H = (F, V_{st})$ be a cooperative game on $H = (N, E)$, where $s, t \in N$ and $F = N \setminus \{s, t\}$. Let us discuss the complexity of approximating player i's Shapley value. The st-connectivity can be assessed using breadth first search, with a time complexity of $\mathcal{O}(|N|^2)$. By Chebyshev's inequality, we need to query the st-connectivity algorithm $\mathcal{O}(\sigma^2/\epsilon^2)$ times, where ϵ is the desired error, and σ^2 is the variance of V_{st} over the distribution matching the Shapley value Definition 3. Since the only outputs of V_{st} are zero and one, we effectively have a Bernoulli distribution with expected value Φ_i. Thus, the variance is $\Phi_i(1 - \Phi_i)$. Since non-trivial situations do not allow for Φ_i to be close to one, we effectively have a variance of $\mathcal{O}(\Phi_i)$. Thus, given a fixed likelihood of success, the time complexity of approximating the Shapley value Φ_i with error bounded by ϵ is $\mathcal{O}\left(\Phi_i \epsilon^{-2} |N|^2\right)$.

Next, we briefly consider a method to extract important nodes. In the worst case, the largest Shapley value is of size $\mathcal{O}(V(F)/|F|) = \mathcal{O}(1/|N|)$, and in this case, most values are close together. So, an error bound $\epsilon \in \mathcal{O}(1/|N|)$ and Shapley value $\Phi_i \in \mathcal{O}(1/|N|)$ are appropriate values. Thus, we require $\mathcal{O}(|N|^3)$ operations for sufficient accuracy. Finally, we must find the Shapley value for each node, thus, naively, the worst case scenario involves about $\mathcal{O}(|N|^4)$ operations.

Quantum Complexity – Finally, let us address the complexity of our quantum approach. Note that, in this section, we drop logarithmic factors for notational simplicity. We now describe the complexity of approximating player i's Shapley value with quantum methods. U_V involves repeating the algorithm from Theorem 2 a logarithmic number of times. Thus, U_V has a time complexity of $\tilde{\mathcal{O}}\left(|N|^{3/2}\right)$. Note that Theorem 2 implicitly requires an easily addressable form of adjacency matrix, this can be implemented with `qRAM`, or possibly though native interactions with a quantum network. In this context, the Shapley value algorithm has complexity $\tilde{\mathcal{O}}\left(\sqrt{\Phi_i}/\epsilon\right)$ (Theorem 3). Thus, the complexity for finding node i's Shapley value is $\tilde{\mathcal{O}}\left(\sqrt{\Phi_i}\epsilon^{-1}|N|^{3/2}\right)$.

Applying the same rational as above, we consider the problem of extracting important nodes. Suppose the largest Shapley value is of $\mathcal{O}(1/|N|)$ and that we as a result want $\epsilon \in \mathcal{O}(1/|N|)$. Thus, to compute Shapley values to the required precision takes $\tilde{\mathcal{O}}(|N|^2)$ time. As discussed in Section 3.3 can approximate all Shapley values in superposition, then extract the maximum in $\tilde{\mathcal{O}}\left(\sqrt{|N|}\right)$ queries. Thus, our total complexity for finding important nodes takes $\tilde{\mathcal{O}}(|N|^{5/2})$ operations.

4 Related Work

The work presented in this paper combines quantum computing together with distributed systems security. Some existing research directions related to our work include (i) the study of potential advantage or speed up optimizations of quantum computing associated to probing, control, and planning of cyber-physical systems [2], as well as formally verifying properties and providing explainability of the related processes [11]; (ii) use of quantum technologies to secure quantum data communications (e.g., protecting the authenticity of quantum signals when in transit, detection of adversaries maliciously modifying quantum messages, and analysis of any other threat models affecting the security of entanglement rates to endanger applications built upon distributed quantum networks [3]); (iii) advantages of quantun technologies to build more secure ways to protect classical data with key expansion protocols like QKD, any of its flavors [14]; (iv) risks and threats posed by quantum science to contemporary information security, including the use of quantum annealers or any other quantum-inspired metaheuristics paving the way for new cracking strategies against classical or post-quantum cryptography [8].

Compared to previous work, we provide in this paper a formal approach built upon game theoretic node centrality following in line with [12, 18]. Game theoretic node centrality provides a more flexible and nuanced concept of node centrality. The st-connectivity attack, in the context of game theoretic node centrality, relies on novel methods to quantify the security properties of a graph. As previously shown [5], the Shapley values necessary for our node centrality can be approximated with quadratically fewer value function queries using quantum methods, up to polylogarithmic factors. Simultaneously, our value function, based on st-connectivity, can be assessed faster on a quantum computer by leveraging [4]. The combination of these two factors allows for a faster calculation than is possible with a classical Monte Carlo approach to solving the problem. Finally, to find high-importance nodes, we can calculate each node's Shapley value simultaneously using superposition, which yields a database of Shapley values. We can search through this database of nodes to find the node with the largest Shapley values quadratically faster than a standard search would allow [1].

5 Conclusion

We have described a quantum approach to approximating the importance of nodes that maintain a target connection, as well as how to quickly identify high-importance nodes. Our methods are built upon multiple subroutines: one for st-connectivity, another for Shapley value approximation, and a final subroutine for finding the maximum of a list. We have considered a formal attack scenario denoted as the st-connectivity attack, in which a malicious actor disrupts a subset of nodes with the goal of perturbing the system functionality. Combining our methods, we can identify the nodes that are most important, and use this information to guide topological adjustments to increase resilience. Our solution can be used as a security metric to guide remediation strategies. Perspectives for future work shall include extended subroutines to best leverage the node values needed to improve resilience.

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