Calculus, Volume 1, 2nd Edition - Tom M. Apostal

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November 25th, 2021

Contents

Chapter 1

Introduction

1.1 Historical Introduction

1.1.4 Excercises

1.2 Some Basic Concepts of the Theory of Sets

1.2.5 Excercises

Question 1. Use the roster notation to designate the following sets of real numbers.

a. $A = \{x|x^2 - 1 = 0\}$ can be designated as $\{-1, 1\}$ in roster notation.

Proof.

$$\begin{aligned} A &= \{x|x^2 - 1 = 0\} \\ &= \{x|(x-1)(x+1) = 0\} \\ &\therefore \{-1, 1\} \end{aligned} \tag{1.1}$$

QED

b. $B = \{x|(x-1)^2 = 0\}$ can be designated as $\{1\}$ in roster notation.

Proof.

$$\begin{aligned} B &= \{x|(x-1)^2 = 0\} \\ &= \{x|x-1 = \sqrt{0}\} \\ &= \{x|x = 1\} \\ &\therefore \{1\} \end{aligned} \tag{1.2}$$

QED

c. $C = \{x|x+8=9\}$ can be designated as $\{1\}$ in roster notation.

Proof.

$$\begin{aligned} C &= \{x|x+8=9\} \\ &= \{x|x = 9-8\} \\ &= \{x|x = 1\} \\ &\therefore \{1\} \end{aligned} \tag{1.3}$$

QED

d. $D = \{x|x^3 - 2x^2 + x = 2\}$ can be designated as $\{2\}$ in roster notation.

Proof.

$$\begin{aligned} D &= \{x|x^3 - 2x^2 + x = 2\} \\ &= \{x|x^3 - 2x^2 + x - 2 = 0\} \\ &= \{x|x^2(x-2) + (x-2) = 0\} \\ &= \{x|(x^2+1)(x-2) = 0\} \\ &\therefore \{2\} \end{aligned} \tag{1.4}$$

QED

e. $E = \{x|(x+8)^2 = 9^2\}$ can be designated as $\{-17, 1\}$ in roster notation.

Proof.

$$\begin{aligned} E &= \{x|(x+8)^2 = 9^2\} \\ &= \{x|x+8 = \pm 9\} \\ &= \{x|x = \pm 9 - 8\} \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.5}$$

QED

f. $F = \{x|(x^2+16x)^2 = 17^2\}$ can be designated as $\{-17, 1, -8-\sqrt{47}, -8+\sqrt{47}\}$ in roster notation.

Proof.

$$\begin{aligned} F &= \{x|(x^2+16x)^2 = 17^2\} \\ &= \{x|x^2+16x = \pm 17\} \\ &= \{x|x^2+16x \pm 17 = 0\} \\ &= \{x|x^2+16x \pm 17 = 0\} \end{aligned} \tag{1.6}$$

Using the quadratic formula:

Definition 1.2.1. Quadratic Equation, analytical method for calculating the roots of a quadratic polynomial.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ where } ax^2+bx+c = 0 \tag{1.7}$$

Solving when the last term is +17:

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{16^2-4(1)17}}{2(1)} \\ &= -8 \pm \frac{\sqrt{188}}{2} \\ &= -8 \pm \frac{\sqrt{188}}{\sqrt{2^2}} \\ &= -8 \pm \sqrt{188/4} \\ &= -8 \pm \sqrt{47} \\ &\therefore \{-8-\sqrt{47}, -8+\sqrt{47}\} \end{aligned} \tag{1.8}$$

Solving when the last term is −17:

$$\begin{aligned} 0 &= x^2+16x-17 \\ &= (x+17)(x-1) \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.9}$$

QED

Question 2. For the sets in Exercise 1, note that $B \subseteq A$. List all the inclusion relations $\subseteq$ that hold among the sets A, B, C, D, E, F.

1. $A \subseteq A$
2. $B \subseteq A$
3. $B \subseteq B$
4. $B \subseteq C$
5. $B \subseteq E$
6. $B \subseteq F$
7. $C \subseteq A$
8. $C \subseteq B$
9. $C \subseteq C$
10. $C \subseteq E$
11. $C \subseteq F$
12. $D \subseteq D$
13. $E \subseteq E$
14. $E \subseteq F$
15. $F \subseteq F$

Question 3. Let $A = \{1\}$, $B = \{1, 2\}$. Discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

Definition 1.2.2. Set Equality Two sets A and B are said to be equal (or identical) if they consisit of exactly the same elements, in which case we write $A = B$. If one of the sets contains an element not in the other, we say the sets are unequal and we write $A \neq B$.

Definition 1.2.3. Subset A set A is said to be a subset of a set B, and we write $A \subseteq B$ whenever every element of A also belongs to B. We also say that A is contained in B or that B contains A. The relation $\subseteq$ is referred to as set inclusion.

a. $A \subset B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\} \tag{1.10}$$

QED

b. $A \subseteq B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\}, \text{ by the definition of a subset 1.2.3} \tag{1.11}$$

QED

c. $A \in B$

Proof.

$$\begin{aligned} &\forall x \in B : x \neq A \\ &\therefore A \notin B \end{aligned} \tag{1.12}$$

QED

d. $1 \in A$

Proof.

$$\exists x \in A(x = 1) \tag{1.13}$$

QED

e. $1 \subseteq A$

Proof.

$$\begin{aligned} &\forall x \in \mathcal{P}(\mathcal{A}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{A}) \text{ is the powerset of A and x each subset} \\ &\therefore 1 \not\subset A \end{aligned} \tag{1.14}$$

QED

f. $1 \subset B$

Proof.

$$\begin{aligned}
&\forall x \in \mathcal{P}(\mathcal{B}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{B}) \text{ is the powerset of B and x each subset} \\
&\therefore 1 \not\subset B
\end{aligned} \tag{1.15}$$

QED

Question 4. Solve the previous exercise if $A = \{1\}$ and $B = \{\{1\}, 1\}$.

a. $A \subset B$

Proof.

$$(\emptyset \neq (A \cap B)) \wedge ((A \cap B) \subset B) \tag{1.16}$$

QED

Question 5. Given the set $S = \{1, 2, 3, 4\}$. Display all subsets of S. There are 16 altogether, counting $\emptyset$ and S.

$$\mathcal{P}(\mathcal{S}) = \bigcup_{i=1}^{|S|} \bigcup_{j=1}^{|S|-i+1} \{S_i, ..., s_j\} \cup \{\emptyset\} \tag{1.17}$$

Question 6. x Given the following four sets $A = \{1, 2\}, B = \{\{1\}, \{2\}\}, C = \{\{1\}, \{1, 2\}\}, D = \{\{1\}, \{2\}, \{1, 2\}\}$ discuss the validty of the following statements (prove the ones that are true and explain why the others are not true).

a. A = B

Proof.

$$\begin{aligned}
&\exists x \in A : x \notin B \\
&\therefore A \neq B
\end{aligned} \tag{1.18}$$

QED

b. $A \subseteq B$

Proof.

$$\begin{aligned} &\forall x \in A : x \notin B \\ &\therefore A \nsubseteq B \end{aligned} \tag{1.19}$$

QED

c. $A \subset C$

Proof.

$$\begin{aligned} &\forall x \in A : x \notin C \\ &\therefore A \not\subset C \end{aligned} \tag{1.20}$$

QED

d. $A \in C$

Proof.

$$\begin{aligned} &\emptyset \neq (\{A\} \cap C) \\ &\therefore A \in C \end{aligned} \tag{1.21}$$

QED

e. $A \subset D$

Proof.

$$\begin{aligned} &\exists x \in A(x \notin D) \\ &\therefore A \not\subset D \end{aligned} \tag{1.22}$$

QED

f. $B \subset C$

Proof.

$$\begin{aligned} &\exists x \in B(x \notin C) \\ &\therefore B \not\subset C \end{aligned}$$

QED

g. $B \subset D$

Proof.

$$\forall x \in B(x \in D)$$
$$\therefore B \subset D$$

QED

h. $B \in D$

Proof.

$$\forall x \in D(x \neq B)$$
$$\therefore B \notin D$$

QED

i. $A \in D$

Proof.

$$\exists x \in D(x = A)$$
$$\therefore A \in D$$

QED

Question 7. Prove the following properties of set equality.

a. $\{a, a\} = \{a\}$.

Proof.

Every idiosyncracy is shared which can only be true of equivalent objects,
$\forall x \in \{a, a\} \cup \{a\}[x \in (\{a, a\} \cap \{a\})]$
Since no one set contains an element not in the other
these sets can only be equal; by the
Definition of Set Equality 1.2.2
$\therefore \{a, a\} = \{a\}$

QED

b. $\{a, b\} = \{b, a\}$.

c. $\{a\} = \{b, c\}$ if and only if $a = b = c$

Question 8.

Question 9.

Question 10.

Question 11.

Question 12.

Question 13.

Question 14.

Question 15.

Question 16.

Question 17.

a.

b.

c.

d.

e.

Question 18.

Question 19.

Question 20.

a.

b.