Calculus, Volume 1, 2nd Edition - Tom M. Apostal

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November 25th, 2021

Contents

Chapter 1

Introduction

1.1 Some Basic Concepts of the Theory of Sets

1.1.1 Excercises

1. Use the roster notation to designate the following sets of real numbers.

   Proposition 1. $A = \{x|x^2 - 1 = 0\}$ can be designated as $\{-1, 1\}$ in roster notation.

   Proof.

$$\begin{aligned} A &= \{x|x^2 - 1 = 0\} \\ &= \{x|(x-1)(x+1) = 0\} \\ &\therefore \{-1, 1\} \end{aligned} \tag{1.1}$$

QED

   Proposition 2. $B = \{x|(x-1)^2 = 0\}$ can be designated as $\{1\}$ in roster notation.

   Proof.

$$\begin{aligned} B &= \{x|(x-1)^2 = 0\} \\ &= \{x|x - 1 = \sqrt{0}\} \\ &= \{x|x = 1\} \\ &\therefore \{1\} \end{aligned} \tag{1.2}$$

QED

Proposition 3. $C = \{x|x+8=9\}$ can be designated as $\{1\}$ in roster notation.

Proof.

$$\begin{aligned} C &= \{x|x+8=9\} \\ &= \{x|x=9-8\} \\ &= \{x|x=1\} \\ &\therefore \{1\} \end{aligned} \tag{1.3}$$

QED

Proposition 4. $D = \{x|x^3 - 2x^2 + x = 2\}$ can be designated as $\{2\}$ in roster notation.

Proof.

$$\begin{aligned} D &= \{x|x^3 - 2x^2 + x = 2\} \\ &= \{x|x^3 - 2x^2 + x - 2 = 0\} \\ &= \{x|x^2(x-2) + (x-2) = 0\} \\ &= \{x|(x^2+1)(x-2) = 0\} \\ &\therefore \{2\} \end{aligned} \tag{1.4}$$

QED

Proposition 5. $E = \{x|(x+8)^2 = 9^2\}$ can be designated as $\{-17, 1\}$ in roster notation.

Proof.

$$\begin{aligned} E &= \{x|(x+8)^2 = 9^2\} \\ &= \{x|x+8 = \pm 9\} \\ &= \{x|x = \pm 9 - 8\} \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.5}$$

QED

Proposition 6. $F = \{x|(x^2 + 16x)^2 = 17^2\}$ can be designated as $\{-17, 1, -8 - \sqrt{47}, -8 + \sqrt{47}\}$ in roster notation.

Proof.

$$\begin{aligned} F &= \{x|(x^2 + 16x)^2 = 17^2\} \\ &= \{x|x^2 + 16x = \pm 17\} \\ &= \{x|x^2 + 16x \pm 17 = 0\} \\ &= \{x|x^2 + 16x \pm 17 = 0\} \end{aligned} \tag{1.6}$$

Using the quadratic formula:

Definition 1.1.1. Quadratic Equation, analytical method for calculating the roots of a quadratic polynomial.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } ax^2 + bx + c = 0 \tag{1.7}$$

Solving when the last term is $+17$:

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{16^2 - 4(1)17}}{2(1)} \\ &= -8 \pm \frac{\sqrt{188}}{2} \\ &= -8 \pm \frac{\sqrt{188}}{\sqrt{2^2}} \\ &= -8 \pm \sqrt{188/4} \\ &= -8 \pm \sqrt{47} \\ &\therefore \{-8 - \sqrt{47}, -8 + \sqrt{47}\} \end{aligned} \tag{1.8}$$

Solving when the last term is -17:

$$\begin{aligned} 0 &= x^2 + 16x - 17 \\ &= (x + 17)(x - 1) \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.9}$$

QED

2. For the sets in Exercise 1, note that $B \subseteq A$. List all the inclusion relations $\subseteq$ that hold among the sets A, B, C, D, E, F.

   (a) $A \subseteq A$

   (b) $B \subseteq A$

   (c) $B \subseteq B$

   (d) $B \subseteq C$

   (e) $B \subseteq E$

   (f) $B \subseteq F$

   (g) $C \subseteq A$

   (h) $C \subseteq B$

   (i) $C \subseteq C$

   (j) $C \subseteq E$

   (k) $C \subseteq F$

   (l) $D \subseteq D$

   (m) $E \subseteq E$

   (n) $E \subseteq F$

   (o) $F \subseteq F$

3. Let $A = \{1\}$, $B = \{1, 2\}$. Discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

   Definition 1.1.2. Set Equality Two sets A and B are said to be equal (or identical) if they consisit of exactly the same elements, in which case we write $A = B$. If one of the sets contains an element not in the other, we say the sets are unequal and we write $A \neq B$.

   Definition 1.1.3. Subset A set A is said to be a subset of a set B, and we write $A \subseteq B$ whenever every element of A also belongs to B. We also say that A is contained in B or that B contains A. The relation $\subseteq$ is referred to as set inclusion.

   (a)

Proposition 7. $A \subset B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\} \tag{1.10}$$

QED

(b)

Proposition 8. $A \subseteq B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\}, \text{ by the definition of a subset 1.1.3} \tag{1.11}$$

QED

(c)

Proposition 9. $A \in B$

Proof.

$$\begin{aligned} &\forall x \in B : x \neq A \\ &\therefore A \notin B \end{aligned} \tag{1.12}$$

QED

(d)

Proposition 10. $1 \in A$

Proof.

$$\exists x \in A(x = 1) \tag{1.13}$$

QED

(e)

Proposition 11. $1 \subseteq A$

Proof.

$$\forall x \in \mathcal{P}(\mathcal{A}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{A}) \text{ is the powerset of A and x each subset}$$
$$\therefore 1 \not\subset A \tag{1.14}$$

QED

(f)

Proposition 12. $1 \subset B$

Proof.

$$\forall x \in \mathcal{P}(\mathcal{B}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{B}) \text{ is the powerset of B and x each subset}$$
$$\therefore 1 \not\subset B \tag{1.15}$$

QED

4. Solve the previous exercise if $A = \{1\}$ and $B = \{\{1\}, 1\}$.

(a)

Proposition 13. $A \subset B$

Proof.

$$(\emptyset \neq (A \cap B)) \wedge ((A \cap B) \subset B) \tag{1.16}$$

QED

5. Given the set $S = \{1, 2, 3, 4\}$. Display all subsets of S. There are 16 altogether, counting $\emptyset$ and S.

$$\mathcal{P}(\mathcal{S}) = \bigcup_{i=1}^{|S|} \bigcup_{j=1}^{|S|-i+1} \{S_i, ..., s_j\} \cup \{\emptyset\} \tag{1.17}$$

6. Given the following four sets $A = \{1, 2\}, B = \{\{1\}, \{2\}\}, C = \{\{1\}, \{1, 2\}\}, D = \{\{1\}, \{2\}, \{1, 2\}\}$ discuss the validty of the following statements (prove the ones that are true and explain why the others are not true).

(a)

Proposition 14. A = B

Proof.

$$\begin{aligned} &\exists x \in A : x \notin B \\ &\therefore A \neq B \end{aligned} \tag{1.18}$$

QED

(b)

Proposition 15. $A \subseteq B$

Proof.

$$\begin{aligned} &\forall x \in A : x \notin B \\ &\therefore A \nsubseteq B \end{aligned} \tag{1.19}$$

QED

(c)

Proposition 16. $A \subset C$

Proof.

$$\begin{aligned} &\forall x \in A : x \notin C \\ &\therefore A \not\subset C \end{aligned} \tag{1.20}$$

QED

(d)

Proposition 17. $A \in C$

Proof.

$$\begin{aligned} &\emptyset \neq (\{A\} \cap C) \\ &\therefore A \in C \end{aligned} \tag{1.21}$$

QED