Calculus, Volume 1, 2nd Edition - Tom M. Apostal

Iain Wong

November 25th, 2021

Contents

Chapter 1

Introduction

1.1 Historical Introduction

1.1.4 Excercises

1.2 Some Basic Concepts of the Theory of Sets

1.2.5 Excercises

Question 1. Use the roster notation to designate the following sets of real numbers.

a. $A = \{x | x^2 - 1 = 0\}$ can be designated as $\{-1, 1\}$ in roster notation.

Proof.

$$\begin{aligned} A &= \{x | x^2 - 1 = 0\} \\ &= \{x | (x-1)(x+1) = 0\} \\ &\therefore \{-1, 1\} \end{aligned} \tag{1.1}$$

QED

b. $B = \{x | (x-1)^2 = 0\}$ can be designated as $\{1\}$ in roster notation.

Proof.

$$\begin{aligned} B &= \{x|(x-1)^2 = 0\} \\ &= \{x|x-1 = \sqrt{0}\} \\ &= \{x|x = 1\} \\ &\therefore \{1\} \end{aligned} \tag{1.2}$$

QED

c. $C = \{x|x+8=9\}$ can be designated as $\{1\}$ in roster notation.

Proof.

$$\begin{aligned} C &= \{x|x+8 = 9\} \\ &= \{x|x = 9-8\} \\ &= \{x|x = 1\} \\ &\therefore \{1\} \end{aligned} \tag{1.3}$$

QED

d. $D = \{x|x^3 - 2x^2 + x = 2\}$ can be designated as $\{2\}$ in roster notation.

Proof.

$$\begin{aligned} D &= \{x|x^3 - 2x^2 + x = 2\} \\ &= \{x|x^3 - 2x^2 + x - 2 = 0\} \\ &= \{x|x^2(x-2) + (x-2) = 0\} \\ &= \{x|(x^2+1)(x-2) = 0\} \\ &\therefore \{2\} \end{aligned} \tag{1.4}$$

QED

e. $E = \{x|(x+8)^2 = 9^2\}$ can be designated as $\{-17, 1\}$ in roster notation.

Proof.

$$\begin{aligned} E &= \{x|(x+8)^2 = 9^2\} \\ &= \{x|x+8 = \pm 9\} \\ &= \{x|x = \pm 9 - 8\} \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.5}$$

QED

f. $F = \{x|(x^2+16x)^2 = 17^2\}$ can be designated as $\{-17, 1, -8-\sqrt{47}, -8+\sqrt{47}\}$ in roster notation.

Proof.

$$\begin{aligned} F &= \{x|(x^2+16x)^2 = 17^2\} \\ &= \{x|x^2+16x = \pm 17\} \\ &= \{x|x^2+16x \pm 17 = 0\} \\ &= \{x|x^2+16x \pm 17 = 0\} \end{aligned} \tag{1.6}$$

Using the quadratic formula:

Definition 1.2.1. Quadratic Equation, analytical method for calculating the roots of a quadratic polynomial.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ where } ax^2+bx+c = 0 \tag{1.7}$$

Solving when the last term is $+17$:

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{16^2 - 4(1)17}}{2(1)} \\ &= -8 \pm \frac{\sqrt{188}}{2} \\ &= -8 \pm \frac{\sqrt{188}}{\sqrt{2^2}} \\ &= -8 \pm \sqrt{188/4} \\ &= -8 \pm \sqrt{47} \\ &\therefore \{-8-\sqrt{47}, -8+\sqrt{47}\} \end{aligned} \tag{1.8}$$

Solving when the last term is -17:

$$\begin{aligned} 0 &= x^2 + 16x - 17 \\ &= (x+17)(x-1) \\ &\therefore \{-17, 1\} \end{aligned} \tag{1.9}$$

QED

Question 2. For the sets in Exercise 1, note that $B \subseteq A$. List all the inclusion relations $\subseteq$ that hold among the sets A, B, C, D, E, F.

1. $A \subseteq A$
2. $B \subseteq A$
3. $B \subseteq B$
4. $B \subseteq C$
5. $B \subseteq E$
6. $B \subseteq F$
7. $C \subseteq A$
8. $C \subseteq B$
9. $C \subseteq C$
10. $C \subseteq E$
11. $C \subseteq F$
12. $D \subseteq D$
13. $E \subseteq E$
14. $E \subseteq F$
15. $F \subseteq F$

Question 3. Let $A = \{1\}$, $B = \{1, 2\}$. Discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

Definition 1.2.2. Set Equality Two sets A and B are said to be equal (or identical) if they consisit of exactly the same elements, in which case we write $A = B$. If one of the sets contains an element not in the other, we say the sets are unequal and we write $A \neq B$.

Definition 1.2.3. Subset A set A is said to be a subset of a set B, and we write $A \subseteq B$ whenever every element of A also belongs to B. We also say that A is contained in B or that B contains A. The relation $\subseteq$ is referred to as set inclusion.

a. $A \subset B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\} \tag{1.10}$$

QED

b. $A \subseteq B$

Proof.

$$\{x \in A | \exists y \in B(x = y)\}, \text{ by the definition of a subset 1.2.3} \tag{1.11}$$

QED

c. $A \in B$

Proof.

$$\begin{aligned} &\forall x \in B : x \neq A \\ &\therefore A \notin B \end{aligned} \tag{1.12}$$

QED

d. $1 \in A$

Proof.

$$\exists x \in A(x = 1) \tag{1.13}$$

QED

e. $1 \subseteq A$

Proof.

$$\begin{aligned} &\forall x \in \mathcal{P}(\mathcal{A}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{A}) \text{ is the powerset of A and x each subset} \\ &\therefore 1 \not\subset A \end{aligned} \tag{1.14}$$

QED

f. $1 \subset B$

Proof.

$$\forall x \in \mathcal{P}(\mathcal{B}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{B}) \text{ is the powerset of B and x each subset}$$
$$\therefore 1 \not\subset B \tag{1.15}$$

QED

Question 4. Solve the previous exercise if $A = \{1\}$ and $B = \{\{1\}, 1\}$.

a. $A \subset B$

Proof.

$$(\emptyset \neq (A \cap B)) \wedge ((A \cap B) \subset B) \tag{1.16}$$

QED

Question 5. Given the set $S = \{1, 2, 3, 4\}$. Display all subsets of S. There are 16 altogether, counting $\emptyset$ and S.

$$\mathcal{P}(\mathcal{S}) = \bigcup_{i=1}^{|S|} \bigcup_{j=1}^{|S|-i+1} \{S_i, ..., s_j\} \cup \{\emptyset\} \tag{1.17}$$

Question 6. x Given the following four sets $A = \{1, 2\}, B = \{\{1\}, \{2\}\}, C = \{\{1\}, \{1, 2\}\}, D = \{\{1\}, \{2\}, \{1, 2\}\}$ discuss the validty of the following statements (prove the ones that are true and explain why the others are not true).

a. A = B

Proof.

$$\begin{aligned} &\exists x \in A : x \notin B \\ &\therefore A \neq B \end{aligned} \tag{1.18}$$

QED

b. $A \subseteq B$

Proof.

$$\begin{aligned}&\forall x \in A : x \notin B\\ &\therefore A \nsubseteq B\end{aligned} \tag{1.19}$$

QED

c. $A \subset C$

Proof.

$$\begin{aligned}&\forall x \in A : x \notin C\\ &\therefore A \not\subset C\end{aligned} \tag{1.20}$$

QED

d. $A \in C$

Proof.

$$\begin{aligned}&\emptyset \neq (\{A\} \cap C)\\ &\therefore A \in C\end{aligned} \tag{1.21}$$

QED

e. $A \subset D$

Proof.

$$\begin{aligned}&\exists x \in A(x \notin D)\\ &\therefore A \not\subset D\end{aligned} \tag{1.22}$$

QED

f. $B \subset C$

Proof.

$$\begin{aligned}&\exists x \in B(x \notin C)\\ &\therefore B \not\subset C\end{aligned}$$

QED

g. $B \subset D$

Proof.

$$\forall x \in B(x \in D)$$
$$\therefore B \subset D$$

QED

h. $B \in D$

Proof.

$$\forall x \in D(x \neq B)$$
$$\therefore B \notin D$$

QED

i. $A \in D$

Proof.

$$\exists x \in D(x = A)$$
$$\therefore A \in D$$

QED

Question 7. Prove the following properties of set equality.

a. $\{a, a\} = \{a\}$.

Proof.

Every idiosyncracy is shared which can only be true of equivalent objects,
$\forall x \in \{a, a\} \cup \{a\}[x \in (\{a, a\} \cap \{a\})]$
Since no one set contains an element not in the other
these sets can only be equal; by the
Definition of Set Equality 1.2.2
$\therefore \{a, a\} = \{a\}$

QED

b. $\{a, b\} = \{b, a\}$.

Proof.

$\forall x \in \{a, b\}(x \in \{b, a\}) \wedge \forall x \in \{b, a\}(x \in \{a, b\})$
Since no one set contains an element not in the other
these sets can only be equal; by the
Definition of Set Equality 1.2.2
$\therefore \{a, b\} = \{b, a\}$

QED

c. $\{a\} = \{b, c\}$ if and only if $a = b = c$

Proof.

Let $A = \{a\}$ and $B = \{b, c\}$
$\exists x \in B(x \notin A) \implies A \neq B$
Thus
$b \neq a \vee c \neq a \implies A \neq B$
Else
$b = a = c \implies \forall x \in B(x \in A) \wedge \forall x \in A(x \in B) \implies A = B$

QED

Question 8. Commutative laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$.

Proof.

$\forall x \in A \cup B(x \in (B \cup A)) \wedge \forall x \in B \cup A(x \in (A \cup B))$
$\therefore \cup$ is commutative.
The same proof can be applied to $\cap$

QED

Proof. Second variant. From Calculus, Volume 1, 2nd Edition.

Let $X = A \cup B$, $Y = B \cup A$.
To prove that $X = Y$ we prove that $X \subseteq Y$ and $Y \subseteq X$.
Let $x \in X$.Then x is in at least one of A or B.
Hence, x is in at least one of B or A; so $x \in Y$.
Thus, every element of X is also in Yso $X \subseteq Y$.
The same can be demonstrated of $Y \subseteq X$.
$\therefore X = Y$.

QED

Question 9. Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof.

To prove the associative property of $\cup$.
We will prove the binary relation is associative
by consequence this means that any recursive composition of the operator is also associative.
Take the relationship $B \cup C$. Then the relation is a superset of the members
$\forall x \in B(x \in (B \cup C)) \wedge \forall x \in C(x \in (B \cup C))$
The statement above applies to any union of sets,
or union of sets formed from unions
That is this property is maintained across
autoregressive applications of the union operator.
$\therefore A \cup (B \cup C) = (A \cup B) \cup C$
A similar proof ca be applied to $\cap$

QED

Proof. Second variant. From Mathonline.

Suppose that $x \in (A \cup B) \cup C$.
Then it follows that $x \in (A \cup B)$ or $x \in C$.
Then $x \in A$ or $x \in B$ or $x \in C$.
Hence, $x \in A \cup (B \cup C)$.
Thus, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.
The same argument can be used to show that the RHS $\subseteq$ LHS.
$\therefore (A \cup B) \cup C = A \cup (B \cup C)$

QED

Question 10. Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

For a simple set Z of atomic elements, the intersection can be computed pointwise
$Z \cap Z' = \bigcup_{x \in Z} x \cap Z'$
Let $Z = B \cup C$ and $Z' = A$ then
This pointwise algorithm can be partitioned
due to the fact the operations are pointwise/local/isolated to begin with
$Z \cap Z' = \bigcup_{x \in B \cup C} x \cap A$
$Z \cap Z' = (\bigcup_{x \in C} x \cap A) \cup (\bigcup_{x \in B} x \cap A)$
$Z \cap Z' = (B \cap A) \cup (C \cap A)$
$(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$

QED

Proof. Second variant. From Pamini Thangarajah.

Let $x \in A \cap (B \cup C)$.
Then $x \in A$ and $x \in (B \cup C)$.
Then $x \in A$ and $x \in B$ or $x \in C$.
Which implies $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$.
Hence $x \in (A \cap B) \cup (A \cap C)$.
Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
The same argument can be made that the RHS $\subseteq$ LHS.
$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

QED

Question 11. $A \cup A = A, A \cap A = A$

Proof. The proof below can also be used for intersections by swapping out the or(s) for and(s).

Let $x \in A \cup A'$.
Then $x \in A$ or $x \in A'$.
If A' = A, then $x \in A$ or $x \in A$.
Hence, $(A \cup A) \subseteq A$.
The same process can be used to prove $A \subseteq (A \cup A)$.
$\therefore (A \cup A) = A$.

QED

Question 12. $A \subseteq A \cup B, A \cap B \subseteq A$.

Proof. We will prove $A \subseteq A \cup B$.

Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
Hence $x \in A \implies x \in A \cup B$.
$\therefore A \subseteq A \cup B$.

QED

Proof. We will prove $A \cap B \subseteq A$.

Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$.
Hence, $x \in A \cap B \implies x \in A$.
$\therefore A \cap B \subseteq A$.

QED

Question 13. $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$

Proof. Proving $A \cup \emptyset = A$

Let $x \in A \cup \emptyset$.
Then $x \in A$ or $x \in \emptyset$.
Since $x \in \emptyset$ is always false: $x \in A \cup \emptyset \implies x \in A$.
Hence, both LHS and RHS are identical by definition.
$\therefore A \cup \emptyset = A$.

QED

Proof. Proving $A \cap \emptyset = \emptyset$

Let $x \in A \cap \emptyset$.
Then $x \in A$ and $x \in \emptyset$.
By definition the empty set $\emptyset$ is empty.
Hence, $x \in \emptyset \implies false$.
Hence, $x \in A$ and $x \in \emptyset \implies$ false.
Thus, $A \cap \emptyset$ has no elements and is always empty.
$\therefore A \cap \emptyset = \emptyset$

QED

Question 14. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Proof. Proving $A \cup (A \cap B) = A$. The same approach can be used for the second equivalence class.

By the distributive property of sets, $A \cup (A \cap B) \implies (A \cup A) \cap (A \cup B)$.
Let $x \in (A \cup A) \cap (A \cup B)$.
Then $x \in (A \cup A)$ and $x \in (A \cup B)$.
Then $x \in A$ or $x \in A$ and $x \in A$ or $x \in B$.
Hence, $x \notin A \implies x \notin A \cup (A \cap B)$.
Thus, $A \cup (A \cap B) \subseteq A$.
By the simplified definition above $A \subseteq A \cup (A \cap B)$.
$\therefore A \cup (A \cap B) = A$.

QED

Question 15. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Proof.

Let $x \in A \cup B$.
Then $x \in A$ or $x \in B$.
Then $x \in A$ or $x \in B \implies x \in C$.
Hence $x \in A \implies x \in C$ and $x \in B \implies x \in C$.
Hence $A \subseteq C$ and $B \subseteq C$.
$\therefore A \subseteq C \wedge B \subseteq C \implies A \cup B \subseteq C$.

QED

Question 16. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

Proof.

Let $x \in C$.
If $C \subseteq A$ and $C \subseteq B \implies x \in A \wedge x \in B$.
$\therefore C \subseteq A \cap B$.

QED

Question 17.

Theorem 1.2.1. Subset Transitivity. If $B \subset C$ and $A \subset B$ then $A \subset C$.

Proof.

Let A, B, C be sets with the relationship $A \subset B$ and $B \subset C$.
Then $A \subset B$ and $B \subset C$ can be restated $A \subset B \subset C$.
Hence, $x \in A \implies x \in B \implies x \in C$.
$\therefore A \subset C$.
This generalizes to improper subsets by relaxing binary "proper subset" relation to allow equivalence.

QED

a. If $A \subset B$ and $B \subset C$, prove that $A \subset C$.

Proof. By the Subset Transitivity Theorem this is true. QED

b. If $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.

Proof. By the Subset Transitivity Theorem this is true. QED

c. What can you conclude if $A \subset B$ and $B \subseteq C$?

1. $A \subset C$
2. $A \neq C$

d. If $x \in A$ and $A \subseteq B$, is it necessarily true that $x \in B$?

Proof.

$A \subseteq B$ restated: $A \subset B$ or $A = B$.
Hence, $x \in A \implies x \in B$.
$\therefore x \in B$.

QED

e. If $x \in A$ and $A \in B$, is it necessarily true that $x \in B$?

Proof.

Let $B = \{A\}$. and $A = \{\emptyset.\}$
Then $A \in B$.
Hence, $\emptyset \notin B$. But $\{\emptyset\} \in B$.
$\therefore x \in A \not\Longrightarrow x \in B$.

QED

Question 18. $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof.

Let $x \in A - (B \cap C)$.
Then $x \in A$ and $x \notin B \cap C$.
Then $x \in A$ and $x \notin B$ and $x \notin C$.
Then $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$.
Hence $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.
A similar argument can be made for $(A - B) \cup (A - C)$.
$\therefore A - (B \cap C) = (A - B) \cup (A - C)$.

QED

Question 19. Let F be a class of sets. Then $B - \bigcup\limits_{A \in F} A = \bigcap\limits_{A \in F} (B - A)$ and $B - \bigcap\limits_{A \in F} A = \bigcup\limits_{A \in F} B - A$.

Proof.

Let $x \in (B - \bigcup_{A \in F} A)$.

Then $x \in B$ and $x \notin \bigcup_{A \in F} A$.

Let $C \in F$.

Then $x \in B$ and $x \notin C$ and $x \in B$ and $x \notin \bigcup_{A \in (F-C)} A$.

Then $x \in (B - C)$ and $x \in (B - \bigcup_{A \in (F-C)} A)$.

Then $x \in (B - C) \cap x \in (B - \bigcup_{A \in (F-C)} A)$.

Hence, $x \in \bigcap_{A \in F} (B - A)$.

Thus, $B - \bigcup_{A \in F} A \subseteq \bigcap_{A \in F} (B - A)$.

A similar argument can be made for $\bigcap_{A \in F} (B - A) \subseteq B - \bigcup_{A \in F} A$.

$\therefore B - \bigcup_{A \in F} A = \bigcap_{A \in F} (B - A)$.

QED

Question 20.

a. Prove that one of the following two formulas is always right and the other one is sometimes wrong:

1. $A - (B - C) = (A - B) \cup C$,

   Proof. We will prove this theorem is not always true.

   Let $x \in A - (B - C)$ and C = {B, Ø}.
   Then $x \in A - (\emptyset)$.
   Then $x \in A$.
   But the RHS evaluates to $A \cup \emptyset$.
   $\therefore A - (B - C) \neq (A - B) \cup C$.

QED

2. $A - (B \cup C) = (A - B) - C$.

   Proof. We will prove this is always true.

   Let $x \in A - (B \cup C)$.
   Then $x \in A$ and $x \notin B \cup C$.
   Then $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$.
   Then $x \in A - B - C$.
   Thus, $A - (B \cup C) \subseteq (A - B) - C$.
   A similar argument can be made for $(A - B) - C \subseteq A - (B \cup C)$.
   $\therefore A - (B \cup C) = (A - B) - C$.

   QED

b. State an additional necessary and sufficient condition for the formula which is sometimes incorrect to be always right.

$B \subseteq C$.