

Calculus, Volume 1, 2nd Edition - Tom M.  
Apostal

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Historical Introduction . . . . .	3
1.1.4	Excercises . . . . .	3
1.2	Some Basic Concepts of the Theory of Sets . . . . .	3
1.2.5	Excercises . . . . .	3

# Chapter 1

## Introduction

### 1.1 Historical Introduction

#### 1.1.4 Exercises

### 1.2 Some Basic Concepts of the Theory of Sets

#### 1.2.5 Exercises

**Question 1.** Use the roster notation to designate the following sets of real numbers.

**a.**  $A = \{x|x^2 - 1 = 0\}$  can be designated as  $\{-1, 1\}$  in roster notation.

*Proof.*

$$\begin{aligned} A &= \{x|x^2 - 1 = 0\} \\ &= \{x|(x - 1)(x + 1) = 0\} \\ &\therefore \{-1, 1\} \end{aligned} \tag{1.1}$$

QED

**b.**  $B = \{x|(x - 1)^2 = 0\}$  can be designated as  $\{1\}$  in roster notation.

*Proof.*

$$\begin{aligned}
 B &= \{x|(x-1)^2 = 0\} \\
 &= \{x|x-1 = \sqrt{0}\} \\
 &= \{x|x = 1\} \\
 &\therefore \{1\}
 \end{aligned} \tag{1.2}$$

QED

c.  $C = \{x|x+8=9\}$  can be designated as  $\{1\}$  in roster notation.

*Proof.*

$$\begin{aligned}
 C &= \{x|x+8=9\} \\
 &= \{x|x=9-8\} \\
 &= \{x|x=1\} \\
 &\therefore \{1\}
 \end{aligned} \tag{1.3}$$

QED

d.  $D = \{x|x^3 - 2x^2 + x = 2\}$  can be designated as  $\{2\}$  in roster notation.

*Proof.*

$$\begin{aligned}
 D &= \{x|x^3 - 2x^2 + x = 2\} \\
 &= \{x|x^3 - 2x^2 + x - 2 = 0\} \\
 &= \{x|x^2(x-2) + (x-2) = 0\} \\
 &= \{x|(x^2+1)(x-2) = 0\} \\
 &\therefore \{2\}
 \end{aligned} \tag{1.4}$$

QED

e.  $E = \{x|(x+8)^2 = 9^2\}$  can be designated as  $\{-17, 1\}$  in roster notation.

*Proof.*

$$\begin{aligned}
 E &= \{x|(x+8)^2 = 9^2\} \\
 &= \{x|x+8 = \pm 9\} \\
 &= \{x|x = \pm 9 - 8\} \\
 &\therefore \{-17, 1\}
 \end{aligned} \tag{1.5}$$

QED

**f.**  $F = \{x | (x^2 + 16x)^2 = 17^2\}$  can be designated as  $\{-17, 1, -8 - \sqrt{47}, -8 + \sqrt{47}\}$  in roster notation.

*Proof.*

$$\begin{aligned}
 F &= \{x | (x^2 + 16x)^2 = 17^2\} \\
 &= \{x | x^2 + 16x = \pm 17\} \\
 &= \{x | x^2 + 16x \pm 17 = 0\} \\
 &= \{x | x^2 + 16x \pm 17 = 0\}
 \end{aligned} \tag{1.6}$$

Using the quadratic formula:

**Definition 1.2.1.** Quadratic Equation, analytical method for calculating the roots of a quadratic polynomial.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } ax^2 + bx + c = 0 \tag{1.7}$$

Solving when the last term is +17:

$$\begin{aligned}
 x &= \frac{-16 \pm \sqrt{16^2 - 4(1)17}}{2(1)} \\
 &= -8 \pm \frac{\sqrt{188}}{2} \\
 &= -8 \pm \frac{\sqrt{188}}{\sqrt{2^2}} \\
 &= -8 \pm \sqrt{188/4} \\
 &= -8 \pm \sqrt{47} \\
 \therefore &\{-8 - \sqrt{47}, -8 + \sqrt{47}\}
 \end{aligned} \tag{1.8}$$

Solving when the last term is -17:

$$\begin{aligned}
 0 &= x^2 + 16x - 17 \\
 &= (x + 17)(x - 1) \\
 \therefore &\{-17, 1\}
 \end{aligned} \tag{1.9}$$

QED

**Question 2.** For the sets in Exercise 1, note that  $B \subseteq A$ . List all the inclusion relations  $\subseteq$  that hold among the sets  $A, B, C, D, E, F$ .

1.  $A \subseteq A$
2.  $B \subseteq A$
3.  $B \subseteq B$
4.  $B \subseteq C$
5.  $B \subseteq E$
6.  $B \subseteq F$
7.  $C \subseteq A$
8.  $C \subseteq B$
9.  $C \subseteq C$
10.  $C \subseteq E$
11.  $C \subseteq F$
12.  $D \subseteq D$
13.  $E \subseteq E$
14.  $E \subseteq F$
15.  $F \subseteq F$

**Question 3.** Let  $A = \{1\}$ ,  $B = \{1, 2\}$ . Discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

**Definition 1.2.2.** Set Equality Two sets  $A$  and  $B$  are said to be equal (or identical) if they consist of exactly the same elements, in which case we write  $A = B$ . If one of the sets contains an element not in the other, we say the sets are unequal and we write  $A \neq B$ .

**Definition 1.2.3.** Subset A set  $A$  is said to be a subset of a set  $B$ , and we write  $A \subseteq B$  whenever every element of  $A$  also belongs to  $B$ . We also say that  $A$  is contained in  $B$  or that  $B$  contains  $A$ . The relation  $\subseteq$  is referred to as set inclusion.

**a.**  $A \subset B$

*Proof.*

$$\{x \in A | \exists y \in B(x = y)\} \quad (1.10)$$

QED

**b.**  $A \subseteq B$

*Proof.*

$$\{x \in A | \exists y \in B(x = y)\}, \text{ by the definition of a subset 1.2.3} \quad (1.11)$$

QED

**c.**  $A \in B$

*Proof.*

$$\begin{aligned} \forall x \in B : x \neq A \\ \therefore A \notin B \end{aligned} \quad (1.12)$$

QED

**d.**  $1 \in A$

*Proof.*

$$\exists x \in A(x = 1) \quad (1.13)$$

QED

**e.**  $1 \subseteq A$

*Proof.*

$\forall x \in \mathcal{P}(A) : 1 \neq x$ , where  $\mathcal{P}(A)$  is the powerset of A and x each subset

$$\therefore 1 \not\subseteq A \quad (1.14)$$

QED

f.  $1 \subset B$

*Proof.*

$$\begin{aligned} \forall x \in \mathcal{P}(\mathcal{B}) : 1 \neq x, \text{ where } \mathcal{P}(\mathcal{B}) \text{ is the powerset of } B \text{ and } x \text{ each subset} \\ \therefore 1 \not\subset B \end{aligned} \tag{1.15}$$

QED

**Question 4.** Solve the previous exercise if  $A = \{1\}$  and  $B = \{\{1\}, 1\}$ .

a.  $A \subset B$

*Proof.*

$$(\emptyset \neq (A \cap B)) \wedge ((A \cap B) \subset B) \tag{1.16}$$

QED

**Question 5.** Given the set  $S = \{1, 2, 3, 4\}$ . Display all subsets of  $S$ . There are 16 altogether, counting  $\emptyset$  and  $S$ .

$$\mathcal{P}(S) = \bigcup_{i=1}^{|S|} \bigcup_{j=1}^{|S|-i+1} \{S_i, \dots, s_j\} \cup \{\emptyset\} \tag{1.17}$$

**Question 6.** x Given the following four sets  $A = \{1, 2\}$ ,  $B = \{\{1\}, \{2\}\}$ ,  $C = \{\{1\}, \{1, 2\}\}$ ,  $D = \{\{1\}, \{2\}, \{1, 2\}\}$  discuss the validity of the following statements (prove the ones that are true and explain why the others are not true).

a.  $A = B$

*Proof.*

$$\begin{aligned} \exists x \in A : x \notin B \\ \therefore A \neq B \end{aligned} \tag{1.18}$$

QED

b.  $A \subseteq B$



*Proof.*

$$\begin{aligned} \forall x \in A : x \notin B \\ \therefore A \not\subseteq B \end{aligned} \tag{1.19}$$

QED

**c.**  $A \subset C$

*Proof.*

$$\begin{aligned} \forall x \in A : x \notin C \\ \therefore A \not\subset C \end{aligned} \tag{1.20}$$

QED

**d.**  $A \in C$

*Proof.*

$$\begin{aligned} \emptyset \neq (\{A\} \cap C) \\ \therefore A \in C \end{aligned} \tag{1.21}$$

QED

**e.**  $A \subset D$

*Proof.*

$$\begin{aligned} \exists x \in A (x \notin D) \\ \therefore A \not\subset D \end{aligned} \tag{1.22}$$

QED

**f.**  $B \subset C$

*Proof.*

$$\begin{aligned} \exists x \in B (x \notin C) \\ \therefore B \not\subset C \end{aligned}$$

QED

**g.**  $B \subset D$

*Proof.*

$$\begin{aligned} \forall x \in B (x \in D) \\ \therefore B \subset D \end{aligned}$$

QED

**h.**  $B \in D$

*Proof.*

$$\begin{aligned} \forall x \in D (x \neq B) \\ \therefore B \notin D \end{aligned}$$

QED

**i.**  $A \in D$

*Proof.*

$$\begin{aligned} \exists x \in D (x = A) \\ \therefore A \in D \end{aligned}$$

QED

**Question 7.** Prove the following properties of set equality.

**a.**  $\{a, a\} = \{a\}$ .

*Proof.*

Every idiosyncrasy is shared which can only be true of equivalent objects,

$$\forall x \in \{a, a\} \cup \{a\} [x \in (\{a, a\} \cap \{a\})]$$

Since no one set contains an element not in the other

these sets can only be equal; by the

Definition of Set Equality 1.2.2

$$\therefore \{a, a\} = \{a\}$$

QED

**b.**  $\{a, b\} = \{b, a\}$ .

*Proof.*

$$\forall x \in \{a, b\}(x \in \{b, a\}) \wedge \forall x \in \{b, a\}(x \in \{a, b\})$$

Since no one set contains an element not in the other these sets can only be equal; by the Definition of Set Equality 1.2.2

$$\therefore \{a, b\} = \{b, a\}$$

QED

**c.**  $\{a\} = \{b, c\}$  if and only if  $a = b = c$

*Proof.*

Let  $A = \{a\}$  and  $B = \{b, c\}$

$$\exists x \in B(x \notin A) \implies A \neq B$$

Thus

$$b \neq a \vee c \neq a \implies A \neq B$$

Else

$$b = a = c \implies \forall x \in B(x \in A) \wedge \forall x \in A(x \in B) \implies A = B$$

QED

**Question 8.** Commutative laws:  $A \cup B = B \cup A$ ,  $A \cap B = B \cap A$ .

*Proof.*

$$\forall x \in A \cup B(x \in (B \cup A)) \wedge \forall x \in B \cup A(x \in (A \cup B))$$

$\therefore \cup$  is commutative.

The same proof can be applied to  $\cap$

QED

**Question 9.** Associative laws:  $A \cup (B \cup C) = (A \cup B) \cup C$ ,  $A \cap (B \cap C) = (A \cap B) \cap C$ .

*Proof.*

$$\forall x \in B(x \in (B \cup C)) \wedge \forall x \in C(x \in (B \cup C))$$

The statement above applies to any union of sets,  
or union of sets formed from unions

That is this property is maintained across  
autoregressive applications of the union operator.

$$\therefore A \cup (B \cup C) = (A \cup B) \cup C$$

A similar proof can be applied to  $\cap$

QED

**Question 10.** Distributive laws:  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

*Proof.*

For a simple set  $Z$  of atomic elements, the intersection can be computed pointwise

$$Z \cap Z' = \bigcup_{x \in Z} x \cap Z'$$

Let  $Z = B \cup C$  and  $Z' = A$  then

This pointwise algorithm can be partitioned

due to the fact the operations are pointwise/local/isolated to begin with

$$Z \cap Z' = \bigcup_{x \in B \cup C} x \cap A$$

$$Z \cap Z' = \left( \bigcup_{x \in C} x \cap A \right) \cup \left( \bigcup_{x \in B} x \cap A \right)$$

$$Z \cap Z' = (B \cap A) \cup (C \cap A)$$

$$(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$$

QED

**Question 11.**

**Question 12.**

**Question 13.**

Question 14.

Question 15.

Question 16.

Question 17.

a.

b.

c.

d.

e.

Question 18.

Question 19.

Question 20.

a.

b.