<u>9.1b Roots of complex numbers</u>

Quote. "Knowledge must first wander through the circuitous paths of innumerable false interpretations and distorted applications; it must overcome the attempts to unite it with old errors, and thus live in conflict, until a new and unpredejuiced generation grows up to meet it "
Arthur Schopenhauer, German philosopher (1788-1860).

Vocabulary.

5th root of 1+i

- n^{th} root of z - a number w such that $w^n = z$. Also written as $w = \sqrt[n]{z}$
- Modulus and arguments of a complex number - $|z|$, θ

1. Recall:

   Euler's formula;
   $$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

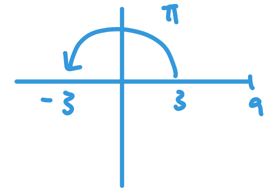
   Polar form of a complex number;
   $$z = re^{i\theta}, \quad r = |z|$$

   Demoivre's formula;
   $$z^n = r^n e^{in\theta}$$

2. **Definition:** Let n be a positive integer. A complex number w **is an** n^{th} **root of the complex number** z **if** $w^n = z$. We write this as $w = \sqrt[n]{z}$.

3. Examples:

   $w_1 = -3,\ w_2 = 3$ are square roots of 9; $w_i^2 = 9,\ w_i = \sqrt{9}$

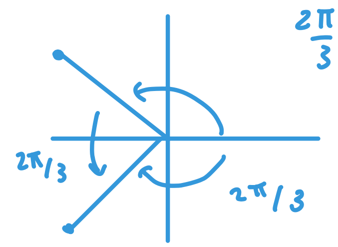


$x^2 - 9 = 0$

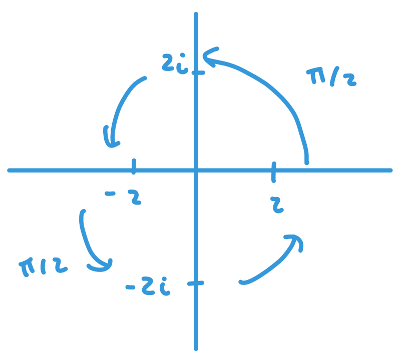
   $w_1 = 2,\ w_2 = 2e^{\frac{2\pi}{3}i},\ w_3 = 2e^{\frac{4\pi}{3}i}$ are cube roots of 8; $w_i^3 = 8,\ w_i = \sqrt[3]{8}$.

$(w_1)^2 = 4$
$(w_2)^2 = 4e^{4\pi/5 i}$
$(w_3)^2 = 4e^{8\pi/3 i}$

$x^3 - 8 = 0$



   $w_1 = 2,\ w_2 = 2i,\ w_3 = -2,\ w_4 = -2i$ are all 4^{th} roots of 16;
   $w_i^4 = 16,\ w_i = \sqrt[4]{16}$



$\frac{2\pi}{4} = \pi/2$

$x^4 - 16 = 0$

4. **Theorem**: For $n = 1, 2, 3, \ldots$, every complex number z has n, n^{th} roots

5. **Finding the n^{th} roots of z**

Given a complex number z, we write it in polar form; $\boxed{z = re^{i\theta}.}$ Note that both $r = |z|$ and θ are known. $|z|$ is the modulus of z and θ is an argument of z. Recall that an argument of a complex number is only determined up to a multiple of 2π (but the modulus is unique) - this is crucial here!

We are looking for a complex number w such that $w^n = z$. Write w in polar form; $w = se^{i\phi}$ We need to determine $s = |w|$ and ϕ in order to find w.

According to Demoivre's formula; $w^n = s^n e^{i(n\phi)}$

We equate this to z; $s^n e^{i(n\phi)} = re^{i\theta}$

Now, it is important to take into account the many ways we can express z in polar form; $z = re^{i(\theta+2\pi k)}$, $k \in \mathbb{Z}$. So we actually have the equalit**<u>ies</u>**;

$$s^n e^{i(n\phi)} = \boxed{re^{i(\theta+2\pi k)}} \quad k \in \mathbb{Z}$$

Now recall that two complex numbers, when expressed in polar form, are equal if their modululi are equal, and their arguments are equal up to a difference that is a multiple of 2π.

Applying this to the expression above, we deduce that;

$$\begin{aligned} s^n &= r \longrightarrow s = \sqrt[n]{r}, \quad \text{the unique, positive, real } n^{th} \text{ root of } r \\ n\phi &= \theta + 2\pi k \longrightarrow \phi = \frac{\theta}{n} + \frac{2\pi k}{n}, \ k \in \mathbb{Z} \end{aligned}$$

This second equation, $\phi = \frac{\theta}{n} + \frac{2\pi k}{n}$, $k \in \mathbb{Z}$, has infinitely many solutions, but <u>only n of them are unique</u>;

$$\phi_1 = \frac{\theta}{n}, \ \phi_2 = \frac{\theta + 2\pi}{n}, \ \phi_3 = \frac{\theta + 4\pi}{n}, \cdots, \ \phi_n = \frac{\theta + 2(n-1)\pi}{n}$$

For example, if n=5 this would be

$$\phi_1 = \frac{\theta}{5}, \quad \phi_2 = \frac{\theta + 2\pi}{5}, \quad \phi_3 = \frac{\theta + 4\pi}{5}, \quad \phi_4 = \frac{\theta + 6\pi}{5}, \quad \phi_5 = \frac{\theta + 8\pi}{5}$$

or, written differently,

$$\phi_1 = \frac{\theta}{5}, \quad \phi_2 = \frac{\theta}{5} + \frac{2\pi}{5}, \quad \phi_3 = \frac{\theta}{5} + \frac{4\pi}{5}, \quad \phi_4 = \frac{\theta}{5} + \frac{6\pi}{5}, \quad \phi_5 = \frac{\theta}{5} + \frac{8\pi}{5}$$

Notice that these 5 angles are equally spaced out ($2\pi/5$ radians between them) around the unit circle in the complex plane, starting from the 'fiducial' angle $\theta/5$.

- cube roots of 7
- square root of 3 + i

6. **Example** Find the 3, cubic roots of 7. Sketch them all on the complex plane, along with 7.

☆ self practice

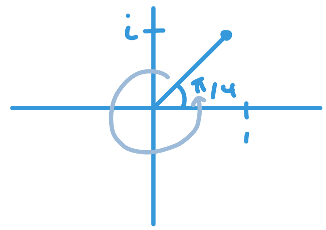
7. **Example** Find the 4, fourth roots of $z = 1 + i$. Sketch them all on the complex plane along with z.

$$z = 1 + i = \sqrt{2}\, e^{i(\pi/4)}$$

$$= \sqrt{2}\, e^{i(\pi/4 + 2k\pi)}$$

$$z^{1/4} = \left[\sqrt{2}\, e^{i(\pi/4 + 2k\pi)}\right]^{1/4}$$

$$= 2^{1/8}\, e^{i(\pi/16 + k\pi/2)}$$



$$\left[2^{1/2}\right]^{1/4} = 2^{1/8}$$

$$k = 0 \Rightarrow 2^{1/8} e^{i(\pi/16)} = w_1$$

90°

$$k = 1 \Rightarrow 2^{1/8} e^{i(\pi/16 + \pi/2)} = w_2$$

90°

$$k = 2 \Rightarrow 2^{1/8} e^{i(\pi/16 + \pi)} = w_3$$

90°

$$k = 3 \Rightarrow \underbrace{2^{1/8} e^{i(\pi/16 + 3\pi/2)}}_{\text{polar form}} = w_4$$