2

In One Line and Anywhere. Permutations as Linear Orders. Inversions.

2.1 Inversions

2.1.1 The Generating Function of Permutations by Inversions

In Section 1.3, we looked at descents of permutations. That is, we studied instances in which an entry in a permutation was larger than *the entry directly following it.* A more comprehensive permutation statistic is that of *inversions.* This statistic will look for instances in which an entry of a permutation is smaller than *some entry following it* (not necessarily directly).

DEFINITION 2.1 *Let $p = p_1p_2\cdots p_n$ be a permutation. We say that (p_i, p_j) is an inversion of p if $i < j$ but $p_i > p_j$.*

Example 2.2
Permutation 31524 has four inversions, namely $(3,1)$, $(3,2)$, $(5,2)$, and $(5,4)$.
□

This line of research started as early as 1901 [206]. In this section, we survey some of the most interesting results in this area. The number of inversions of p will be denoted by $i(p)$, though some authors prefer $inv(p)$. It is clear that $0 \leq i(p) \leq \binom{n}{2}$ for all n-permutations, and that the two extreme values are attained by permutations $12\cdots n$ and $n(n-1)\cdots 1$, respectively. It is relatively easy to find the generating function enumerating all permutations of length n with respect to their number of inversions.

THEOREM 2.3
For all positive integers $n \geq 2$, we have

$$\sum_{p\in S_n} x^{i(p)} = I_n(x) = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}).$$

PROOF We prove the statement by induction on n. In fact, we prove that each of the $n!$ expansion terms of the product $I_n(x)$ corresponds to exactly one permutation in S_n. Moreover, the expansion term $x^{a_1}x^{a_2}\cdots x^{a_{n-1}}$ will correspond to the unique permutation in which, for $i \in [n]$, the entry $i+1$ precedes exactly a_i entries that are smaller than itself.

If $n = 2$, then there are two permutations to count, $p = 12$ has no inversions, and $p' = 21$ has one inversion. So $\sum_{p\in S_2} x^{i(p)} = 1 + x$ as claimed. Furthermore, $p = 12$ is represented by the expansion term 1, and $p' = 21$ is represented by the expansion term x.

Now let us assume that we know that the statement is true for $n-1$, and prove it for n. Let p be a permutation of length $n-1$. Insert the entry n into p to get the new permutation q. If we insert n into the last position, we create no new inversions. If we insert n into the next-to-last position of p, we create one new inversion as n will be larger than the last element of q. In general, if we insert n into p so that it precedes exactly i entries of p, we create i new inversions as n will form an inversion with each entry on its right, and with no entry on its left. Therefore, depending on where we inserted n, the new permutation q has 0 or 1 or 2, etc., or $n-1$ more inversions than p did. If p was represented by the expansion term $x^{a_1}x^{a_2}\cdots x^{a_{n-2}}$, and n is inserted so that it precedes i entries, then q is represented by the new expansion term $x^{a_1}x^{a_2}\cdots x^{a_{n-2}}x^i$. This argument works for all p, proving that

$$I_n(x) = (1+x+\cdots+x^{n-1})I_{n-1}(x) = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

∎

Later in this chapter we will have the techniques to write Theorem 2.3 in a much more compact form.

Therefore, the number $b(n,k)$ of n-permutations with k inversions is the coefficient of x^k in $I_n(x)$. The fact that the polynomial $I_n(x)$ can be decomposed into a product of factors enables us to prove the following result on these numbers.

COROLLARY 2.4
For any fixed n, the sequence $b(n,0), b(n,1), \cdots, b(n, \binom{n}{2})$ is log-concave.

PROOF Let us call a polynomial log-concave if its coefficients form a log-concave sequence. It is then not hard to prove (see Exercise 22 of Chapter 1) that the product of log-concave polynomials is log-concave. The previous theorem shows that the generating function of our sequence is the product of several log-concave polynomials (of the form $1 + x + x^2 + \cdots + x^t$), therefore our sequence itself is log-concave. ∎

The first few values of the numbers $b(n,k)$ are shown in Figure 2.1.

n=1	1
n=2	1 1
n=3	1 2 2 1
n=4	1 3 5 6 5 3 1
n=5	1 4 9 15 20 22 20 15 9 4 1

FIGURE 2.1
The values of $b(n,k)$ for $n \leq 5$. Row n starts with $b(n,0)$.

We would like to point out that it is not true that $I_n(x)$ has real roots only. Indeed, if $n \geq 3$, then $I_n(x)$ is a multiple of $1+x+x^2$, and therefore has some complex roots.

We have found, with not much effort, the generating function of the numbers $b(n,k)$ for fixed n. An enumerative combinatorialist will certainly ask next whether it is possible to find a recursive, or even better, an explicit formula for these numbers, just as we did for the numbers $A(n,k)$ in Section 1.1. As we will see, the latter is a somewhat more difficult task.

To start, we prove a recurrence relation.

LEMMA 2.5
Let $n \geq k$. Then we have

$$b(n+1,k) = b(n+1,k-1) + b(n,k). \tag{2.1}$$

PROOF Let $p = p_1 p_2 \cdots p_{n+1}$ be an $(n+1)$-permutation with k inversions, where $k \leq n$. If $p_{n+1} = n+1$, then we can remove the entry $n+1$ from the end of p and get an n-permutation with k inversions. If $p_i = n+1$ for $i \leq n$, then let us interchange $n+1$ and the entry immediately following it. This results in an $(n+1)$-permutation with $k-1$ inversions in which the entry $n+1$ is not in the first position. However, all $(n+1)$-permutations with $k-1$ inversions have that property (that $n+1$ is not in the first position) as putting $n+1$ to the first position would result in at least $n \geq k > k-1$ inversions. This completes the proof, and also shows why the condition $n \geq k$ is needed. ∎

Even without going into details, it is obvious that (2.1) does not hold in general, that is, when $k > n$. For instance, if $k > \binom{n}{2}/2$, then $b(n+1,k) < b(n+1,k-1)$, which makes it impossible for (2.1) to hold as $b(n,k) \geq 0$. See Exercise 28 for a recurrence relation for the case when $k > n$.

Finding an explicit formula for the numbers $b(n,k)$ is significantly more difficult, even if we assume $n \geq k$. A little examination of the polynomial

$I_n(x) = \sum_{k=0}^{\binom{n}{2}} b(n,k)x^k$ shows that

$$\begin{aligned}
b(n,0) &= 1 = \binom{n}{0}, & \\
b(n,1) &= n-1 = \binom{n}{1} - \binom{n}{0} & n \geq 1, \\
b(n,2) &= \binom{n}{2} - \binom{n}{0}, & n \geq 2, \\
b(n,3) &= \binom{n+1}{3} - \binom{n}{1} & n \geq 3, \\
b(n,4) &= \binom{n+2}{4} - \binom{n+1}{2} & n \geq 4.
\end{aligned}$$

In order to see how these results are obtained, and to obtain a general formula, we need some notions that most readers are probably familiar with.

DEFINITION 2.6 *Let n be a positive integer. If $a_1 + a_2 + \cdots + a_k = n$, and the a_i are all* positive *integers, then we say that the k-tuple $(a_1, a_2, \cdots, a_k)$ is a* composition *of n into k parts. If the a_i are all* nonnegative *integers, then we say that the k-tuple $(a_1, a_2, \cdots, a_k)$ is a* weak composition *of n into k parts.*

In the unlikely event that the reader has not met compositions before, the reader should take a moment to prove that the number of compositions of n into k parts is $\binom{n-1}{k-1}$, whereas the number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$.

For instance, to get $b(n,2)$ as the coefficient of x^2 in $I_n(x)$, one has to count the weak compositions of 2 into $n-1$ nonnegative parts, the first of which is at most 1. Indeed, one needs to find the coefficient of x^2 in the generating function $I_n(x) = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1})$. The number of all weak compositions of 2 into $n-1$ parts is $\binom{2+n-1-1}{n-1} = \binom{n}{n-1} = n$, one of which consists of a first part equal to 2. This proves that $b(n,2) = n-1$. See Exercises 3 and 4 for proofs in the cases of $k=3$ and $k=4$.

This line of formulae suggests that maybe the formula for $b(n,k)$ will be obtained by taking the difference of two suitably chosen binomial coefficients. However, this conjecture is false as we have

$$b(n,5) = \binom{n+3}{5} - \binom{n+2}{3} + 1.$$

Further conjectures claiming $b(n,k)$ to satisfy a formula of comparable simplicity also turn out to be false. The truth is a bit more complicated than that.

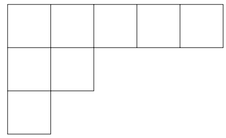

FIGURE 2.2
The Ferrers shape of $p = (5, 2, 1)$.

Our main tool in finding the correct formula comes, remarkably, from the theory of *integer partitions*. Many readers are probably familiar with the following definition.

DEFINITION 2.7 *Let $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ be integers so that $a_1 + a_2 + \cdots + a_m = n$. Then the array $a = (a_1, a_2, \cdots, a_m)$ is called a* partition *of the* integer *n, and the numbers a_i are called the* parts *of the partition a. The number of all partitions of n is denoted by $p(n)$.*

Partitions of the integer n are not to be confused with partitions of the set $[n]$. If there is a danger of confusion, we may refer to the objects we have just defined as *integer partitions* and to the objects we defined in Section 1.1 as *set partitions.*

Example 2.8
The integer 5 has seven partitions, namely (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$. Therefore, $p(5) = 7$. □

The topic of integer partitions has been extensively researched for several centuries, from combinatorial, number theoretical, and analytic aspects. See [10] for a survey.

We will use the following simple, but extremely useful, representation of partitions by diagrams. A *Ferrers shape* of a partition $p = (a_1, a_2, \cdots, a_k)$ is a set of n square boxes with sides parallel to the coordinate axes so that in the ith row we have a_i boxes and all rows start at the same vertical line. The Ferrers shape of the partition $p = (5, 2, 1)$ is shown in Figure 2.2. Clearly, there is an obvious bijection between partitions of n and Ferrers shapes of size n.

We will need some basic facts about the generating functions of various partitions.

PROPOSITION 2.9
The ordinary generating function of the numbers $p(n)$ is

$$\sum_{n\geq 0} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}. \tag{2.2}$$

PROOF We can decompose the right-hand side as

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^i+x^{2i}+\cdots).$$

It is now clear that the coefficient of x^n in this product is equal to the number of vectors $(c_1, c_2, \cdots)$ with nonnegative integer coefficients for which $\sum_{i=1}^{\infty} ic_i = n$. Note that such a vector can have only a finite number of nonzero coordinates. Finally, there is a natural bijection between these vectors and the partitions of n. This bijection maps $(c_1, c_2, \cdots)$ into the partition that has c_i parts equal to i. So the coefficient of x^n on the right-hand side is $p(n)$. ∎

COROLLARY 2.10
Let $p_d(n)$ be the number of partitions of n into distinct *parts. Then we have*

$$\sum_{n\geq 0} p_d(n)x^n = \prod_{i=1}^{\infty}(1+x^i). \tag{2.3}$$

COROLLARY 2.11
Let $p(n,m)$ be the number of partitions of n into at most m *parts. Then we have*

$$\sum_{n\geq 0} p(n,m)x^n = \prod_{i=1}^{m} \frac{1}{1-x^i}. \tag{2.4}$$

A *pentagonal number* is a non-negative integer n satisfying $n = \frac{1}{2}(3j^2 \pm j)$ for some non-negative integer j. So the first few pentagonal numbers are $0, 1, 2, 5, 7, \cdots$.

The following partition identity is the most interesting one in our quest for the formula for the numbers $b(n,k)$. It is not nearly as easy as the preceding two corollaries.

LEMMA 2.12
Let $p_{o,d}(n)$ (resp. $p_{e,d}(n)$) be the number of partitions of n into distinct odd parts (resp. distinct even parts). Then we have

$$p_{e,d}(n) - p_{o,d}(n) = \begin{cases} 0 \text{ if } n \text{ is not pentagonal,} \\ (-1)^j \text{ if } n = \frac{1}{2}(3j^2 \pm j). \end{cases}$$

FIGURE 2.3
We have $b(p) = 3,\ g(p) = 1$, and $b(q) = 1,\ g(q) = 2$.

PROOF We will define a map ϕ from the set of partitions of n into distinct even parts to the set of partitions of n into distinct odd parts. We will then show that if n is not a pentagonal number, then our map is a bijection, proving the first part of our theorem. The second part of the theorem will be proved by analyzing why ϕ is not a bijection if n is pentagonal.

Let $q = (q_1, q_2, \cdots, q_k)$ be any partition of n. We define two parameters for q. The simpler one, $g(q)$ (for gray) is just the size of the smallest part of q. In other words, $g(q) = q_k$. The other one, $b(q)$ (for black) is the length of the longest strictly decreasing subsequence $(q_1, q_2, \cdots)$ of parts of q in which each part is exactly one less than the part immediately preceding it. Note that by definition, this strictly decreasing subsequence *must* start with q_1. So in particular, if $q_2 \neq q_1 - 1$, then $b(q) = 1$.

See Figure 2.3 for an illustration.

The set of black boxes will be called the *outer rim* of a Ferrers shape.

Now let p be a partition of n into an even number of distinct parts. We distinguish two cases.

1. If $b(q) \geq g(q)$, then we remove the last part of q, and add one to each of the first $g(q)$ parts of q, to get the partition $\phi(q)$. This operation decreases the number of parts by one, and always keeps all parts distinct. The procedure can always be carried out *except* in the case when each part of q, including the last one, is one less then the preceding one, and $b(q) = g(q)$, such as in the case of $q = (3, 2)$. We do not define ϕ in that exceptional case.

   Note that at the end of this procedure, $b(\phi(q)) < g(\phi(q))$ holds.

2. If $b(q) < g(q)$, then we decrease each of the first $b(q)$ parts of q by one, then affix a last part of size $b(q)$ to the end of q to get the partition $\phi(q)$. This operation increases the number of parts by one, and keeps the parts all distinct, *except* in the case when each part of q, including the last one, is one less than the preceding one, and $b(q) = g(q) - 1$, such as in the case of $q = (4, 3, 2)$. We do not define ϕ in that exceptional case.

   Note that at the end of this procedure, $b(\phi(q)) \geq g(\phi(q))$ holds.

So the only times when $\phi(q)$ is not defined are as follows.

(A) When $b(q) = g(q)$, and the last part of q and the outer rim of its Ferrers shape both consist of j boxes. Therefore, the partition q is of the form $q = (2j-1, 2j-2, \cdots, j)$, so $n = \frac{(3j-1)j}{2}$.

(B) When $b(q) = g(q) - 1$, then the outer rim of q consists of j boxes, and the last part of q is $j+1$. Therefore, we have $q = (2j, 2j-1, \cdots, j+1)$, so $n = \frac{(3j+1)j}{2}$.

Note that in both of these exceptional cases, the integer j had to be even, to assure that q had an even number of parts.

So in all cases but the exceptional cases (A) and (B), our function ϕ maps into the set of partitions of n into an odd number of distinct parts. It is clear that ϕ is one-to-one since it is one-to-one on both parts of its domain, and the images of the two parts are disjoint, as explained in the last sentence of the definition of $\phi(q)$ in each case. Let us examine whether ϕ is surjective.

Let r be a partition of n into an odd number of distinct parts, and let us try to find the preimage of r under ϕ. If $g(r) \leq b(r)$, then this preimage can be found by removing the last part of r and adding one to each of the first $g(r)$ parts of r. This can always be done, unless $g(r) = b(r)$, and each part of r, including the last one, is one less than the preceding one. If $g(r) > b(r)$, then this preimage can be found by decreasing each of the first $b(r)$ parts of r by one, and creating a new, last part of r that is of size $b(r)$. This procedure does not yield a partition into distinct parts only if $b(r) = g(r) - 1$, and each part of r is one less than the preceding one. So we can find the unique preimage $\phi^{-1}(r)$ of r unless

(A') $b(r) = g(r)$, so $r = (2j-1, 2j-2, \cdots, j)$, and therefore, $n = \frac{(3j-1)j}{2}$, or

(B') $b(r) = g(r) - 1$, so $r = (2j, 2j-1, \cdots, j+1)$, and therefore, $n = \frac{(3j+1)j}{2}$.

Also note that in cases (A') and (B'), the number j has to be odd to ensure that r has an odd number of parts.

In other words, if n is not a pentagonal number, then ϕ is a bijection from the set of partitions enumerated by $p_{e,d}(n)$ onto the set of partitions enumerated by $p_{o,d}(n)$.

In exceptional cases (A) and (B), (which occur when $n = \frac{(3j+1)j}{2}$ for some even positive integer j), there is one partition in the domain of ϕ that does not get mapped into a partition consisting of an odd number of parts, showing that $p_{e,d}(n) - p_{o,d}(n) = 1 = (-1)^j$ as j is even.

In exceptional cases (A') and (B'), (which occur when $n = \frac{(3j-1)j}{2}$ for some odd positive integer j), there is one partition of n into a distinct number of odd parts that does not have a preimage under ϕ, proving that $p_{e,d}(n) - p_{o,d}(n) = -1 = (-1)^j$ as j is odd. ∎

Now let $p_d(n,m)$ be the number of partitions of n into m distinct parts. As a consequence of the previous lemma, note that if n is not of the form $(3j^2+j)/2$ or $(3j^2-j)/2$ for some nonnegative integer j, then we have

$$\sum_{m=1}^{n}(-1)^m p_d(n,m) = 0.$$

Otherwise, we have

$$\sum_{m=1}^{n}(-1)^m p_d(n,m) = (-1)^j.$$

The following Corollary links the pentagonal numbers to the enumeration of permutations according to their number of inversions.

COROLLARY 2.13

[Euler's formula.] We have

$$f(x) = (1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - \cdots$$
$$= \sum_{j\in\mathbf{Z}}(-1)^j x^{(3j^2+j)/2}.$$

PROOF The left-hand side is similar to the generating function of the numbers $p_d(n)$ as given in (2.3), except for the negative sign within each term. This implies that the coefficient of x^n on the left-hand side is not simply the sum of all the numbers $p_d(n,m)$, but their *signed sum* $\sum_m(-1)^m p_d(n,m)$. We know from Lemma 2.12 that this sum is 0, except when n is of the form $(3j^2+j)/2$ or $(3j^2-j)/2$, in which case this sum is equal to $(-1)^j$. This completes the proof. ∎

We mention that the rather unusual summation $\sum_{j\in\mathbf{Z}}$ in Euler's formula is used to include pentagonal numbers of the form $(3j^2+j)/2$ and $(3j^2-j)/2$ in the same sum. One can think of the sum $\sum_{j\in\mathbf{Z}}(-1)^j x^{(3j^2+j)/2}$ as the sum in which j ranges through all integers in order $0,-1,1,-2,2,-3,3,\cdots$. For $j\in\mathbf{Z}$, let us set $d_j=(3j^2+j)/2$.

Recall that by Theorem 2.3, the polynomial $I_n(x)$ can be rearranged as

$$I_n(x) = \prod_{i=1}^{n}(1+x+\cdots x^{i-1}) = \prod_{i=1}^{n}\frac{1-x^i}{1-x}.$$

Let $k\leq n$. While $I_n(x)$ is a polynomial and $\frac{f(x)}{(1-x)^n}$ is an infinite product, their factors of degree at most k agree, therefore their coefficients for terms of degree at most k also agree. So our task is reduced to finding the coefficient of x^k in

$$f(x)\cdot(1-x)^{-n} = f(x)\cdot\sum_{h\geq 0}\binom{n+h-1}{h}x^h,$$

j	0	1		2	
		−	+	−	+
d_j	0	1	2	5	7

FIGURE 2.4
The first five pentagonal numbers.

where we set $\binom{-1}{0} = 1$. In order to get a term with coefficient k in the product $f(x) \cdot (1-x)^{-n}$, we have to multiply the term $(-1)^j x^{(3j^2+j)/2} = (-1)^j x^{d_j}$ of $f(x)$ by the term of $(1-x)^{-n}$ that has exponent $k - d_j$, that is, in which $h = k - d_j$. Therefore, we have proved the following theorem.

THEOREM 2.14
Let $n \geq k$. Then the coefficient of x^k in $I_n(x)$, or, equivalently, the number of n-permutations with k inversions, is

$$b(n,k) = \sum_j (-1)^j \binom{n+k-d_j-1}{k-d_j} \tag{2.5}$$

where j is such that the pentagonal number d_j is at most as large as k.

The first few pentagonal numbers are shown in Figure 2.4.

Expanding (2.5), we see that if $n \geq k$, then the formula for $b(n,k)$ starts as follows.

$$b(n,k) = \binom{n+k-1}{k} - \binom{n+k-2}{k-1} - \binom{n+k-3}{k-2} + \binom{n+k-6}{k-5} + \binom{n+k-8}{k-7} - \cdots.$$

2.1.2 Major Index

There are other permutation statistics that are *equidistributed* with the number of inversions. That is, there exist other permutation statistics *stat* so that for all non-negative integers n and k, the number of n-permutations p satisfying $stat(p) = k$ is equal to $b(n,k)$. The most famous of these statistics is the major index, which was named after the rank of its inventor, Percy MacMahon, in the British Army.

DEFINITION 2.15 *Let $p = p_1 p_2 \cdots p_n$ be a permutation, and define* *the* major index *or* greater index $maj(p)$ *of* p *to be the sum of the descents of* p*. That is,* $maj(p) = \sum_{i \in D(p)} i$*.*

Example 2.16
If $p = 352461$, then $D(p) = \{2, 5\}$, therefore $maj(p) = 7$. □

In 1916, MacMahon showed [201] the following surprising theorem by proving that the two relevant generating functions were identical. It was not until 1968 that a bijective proof was found by Dominique Foata [129], who worked in a more general setup. Another proof that can be turned into a bijective proof is given in Exercises 31 and 32. We present Foata's proof in the simplified language of permutations.

THEOREM 2.17
For all positive integers n *and all nonnegative integers* k*, there are as many* n*-permutations with* k *inversions as there are* n*-permutations with major index* k*.*

In other words, the permutation statistics "number of inversions," which we denoted by i, and "major index," which we denoted by maj, are *equidistributed* on S_n. If a permutation statistic s has the same distribution on S_n as i, then s is called *Mahonian.*

PROOF (of Theorem 2.17) For any permutation $p = p_1 p_2 \cdots p_n$, we call the entry p_i large if $p_i > p_n$, and we call p_i small if $p_i < p_n$.

We are going to prove our statement by recursively defining a bijection $\phi : S_n \to S_n$ so that for all $p \in S_n$, the equality $maj(p) = i(\phi(p))$ holds. Our map ϕ will have the additional feature of keeping the last element of p fixed.

It will not surprise the reader that we define $\phi(1) = 1$ for the initial case of $n = 1$, and $\phi(12) = 12$ and $\phi(21) = 21$ for the case of $n = 2$.

Now let us assume that we have defined ϕ for all $(n-1)$-permutations. In order to define ϕ for all n-permutations, we distinguish two cases. Let $p = p_1 p_2 \cdots p_n$ be any n-permutation.

1. First we consider the case when p_{n-1} is a small entry. In this case, take $w_p = \phi(p_1 p_2 \cdots p_{n-1}) = q_1 q_2 \cdots q_{n-1}$. Let $q_{i_1}, q_{i_2}, \cdots, q_{i_j}$ be the small entries of p in w_p, that is, those that are less than p_n. Set $i_0 = 0$. Let $Q_j = q_{i_{j-1}+1} \cdots q_{i_j}$. In other words, the Q_j provide the unique decomposition of w into subwords that contain exactly one small entry, and contain that small entry in the last position. For instance, if $q_1 \cdots q_6 = 425613$, then there are two small entries, 1 and 2, and

therefore, $Q_1 = 42$, and $Q_2 = 561$. Now define

$$f(Q_j) = \begin{cases} Q_j \text{ if } Q_j \text{ is of length at most 1,} \\ \\ x_m x_1 x_2 \cdots x_{m-1} \text{ if } Q_j = x_1 x_2 \cdots x_m, \text{ with } m \geq 2. \end{cases}$$

Finally, define

$$f(w_p) = f(Q_1) f(Q_2) \cdots f(Q_k),$$

and

$$\phi(p) = f(w_p) p_n.$$

Example 2.18

Let $n = 5$, and $p = 54213$. Then we have $w_p = \phi(5421) = 5421$, and $Q_1 = 542$, $Q_2 = 1$. Therefore, $f(w_p) = 2541$, and so $\phi(p) = 25413$. ▯

2. When p_{n-1} is a large entry, the procedure is very similar. The only difference is in the definition of the strings Q_j. In this case, the Q_j provide the unique decomposition of w into subwords that contain exactly one *large* entry, and contain that *large* entry in the last position.

Example 2.19

Let $n = 5$, and let $p = 13452$. Then we have $w_p = \phi(1345) = 1345$, and $Q_1 = 13$, $Q_2 = 4$, and $Q_3 = 5$. Therefore, $f(w_p) = 3145$, and so $\phi(p) = 31452$. ▯

It is easy to see that $\phi : S_n \to S_n$ is a bijection. Indeed, verifying both cases, one sees that the first rule was used to create $\phi(p)$ if and only if the last element of $\phi(p)$ is larger than the first element of $\phi(p)$. Once we know which rule was used to create $\phi(p)$, we can recover w_p from $f(w_p)$. Indeed, if the first (resp. second) rule was used, then the $f(Q_i)$ are the subwords that contain only one small (resp. large) entry, and contain that small (resp. large) entry in the *first* position. As f is a bijection, recovering the $f(Q_i)$ this way allows us to recover the Q_i, and therefore, w_p itself. Finally, $\phi : S_{n-1} \to S_{n-1}$ is a bijection by induction, so we recover $p_1 p_2 \cdots p_{n-1}$ from $f(p_1 p_2 \cdots p_{n-1}) = w_p$.

We still need to prove that $\phi : S_n \to S_n$ has the desired property, that is, it maps a permutation with major index k into a permutation with k inversions. We accomplish this by considering the two above cases separately.

1. When p_{n-1} is a small entry, then

$$maj(p) = maj(p_1 p_2 \cdots p_{n-1}) = i(\phi(p_1 p_2 \cdots p_{n-1})) = i(w_p). \qquad (2.6)$$

How does the map f change the number of inversions of $w(p)$? It does not change the order among the small entries, or among the large entries. If a small entry belongs to the subword Q_j of length $t > 1$, then it jumps forward and passes all $t-1$ large entries of Q_j, decreasing the number of inversions by $t-1$.

As each large entry will be passed by one small entry, the total decrease in inversions is equal to the number of large entries, that is, to $n - p_n$. However, affixing p_n to the end of $f(w_p)$ will create precisely $n - p_n$ new inversions. Therefore,

$$i(\phi(p)) = i(f(w_p)p_n) = i(w_p),$$

which, compared to (2.6), shows that $maj(p) = i(\phi(p))$ as claimed.

2. When p_{n-1} is a large entry, then

$$\begin{aligned} maj(p) &= maj(p_1 p_2 \cdots p_{n-1}) + (n-1) && (2.7) \\ &= i(\phi(p_1 p_2 \cdots p_{n-1})) + n - 1 = i(w_p) + n - 1. && (2.8) \end{aligned}$$

When f is applied to w_p, each large entry belonging to a subword of length $t > 1$ jumps forward, passes all $t-1$ small entries of its subword, and increases the number of inversions by $t-1$. Each small entry is passed by one large entry, so the total increase in the number of inversions is equal to the number of small entries, that is, $p_n - 1$. On the other hand, affixing p_n to the end of $f(w_p)$ will create precisely $n - p_n$ new inversions. Therefore,

$$i(\phi(p)) = i(f(w_p)p_n = i(w_p) + (p_n - 1) + (n - p_n) = i(w_p) + n - 1,$$

which, compared to (2.7), shows that again, $maj(p) = i(\phi(p))$ as claimed.

∎

Other examples of Mahonian statistics can be found among the exercises.

2.1.3 An Application: Determinants and Graphs

2.1.3.1 The Explicit Definition of Determinants

There are several undergraduate mathematics courses and textbooks that only give a recursive definition of the *determinant* of a square matrix. That is, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be equal to $ad - bc$, and then the determinant of the $n \times n$ matrix $A = (a_{ij})$ is defined to be

$$\det A = \sum_{j=1}^{n} (-1)^{j-1} a_{1j} A_{1j} \qquad (2.9)$$

where A_{1j} is the $(n-1)\times(n-1)$ matrix obtained from A by removing the first row and the jth column.

If that is the only definition of determinants the reader has seen, he may find the following result interesting.

THEOREM 2.20
Let $A=(a_{ij})$ be an $n\times n$ matrix. Then we have

$$\det A = \sum_{p\in S_n} (-1)^{i(p)} a_{1p_1}a_{2p_2}\cdots a_{np_n}. \tag{2.10}$$

That is, $\det A$ is obtained by taking all $n!$ possible n-tuples of entries so that there is exactly one of the n entries in each row and each column, multiplying the elements of each such n-tuple together, finally taking a signed sum of these $n!$ products, where the sign is determined by the parity of $i(p)$, and p is the permutation determined by each chosen n-tuple.

In other words, the n-tuples correspond to all possible placements of n rooks on an $n\times n$ chessboard so that no two of them hit each other.

Example 2.21
Let $n=3$. Then there are three 3-permutations with an even number of inversions, namely 123, 312, and 231, and there are three 3-permutations with an odd number of inversions, namely 132, 213, and 321. Therefore, we have

$$\det A = a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}.$$

$\square$

PROOF (of Theorem 2.20). We prove the statement by induction on n, the initial cases of $n=1$ and $n=2$ being obvious. Let us assume that the statement is true for $(n-1)\times(n-1)$ matrices. That means that

$$\det A_{1j} = \sum_q (-1)^{i(q)} a_{2q_2}a_{3q_3}\cdots a_{nq_n}, \tag{2.11}$$

where $q=q_2q_3\cdots q_n$ is a *partial permutation*, that is, a list of the integers $1,2,\cdots,j-1,j+1,\cdots,n$ in some order.

Therefore, $a_{1j}\det A_{1j}$ will contribute the products of all n-tuples starting with a_{1j} to $\det A$. In other words, $a_{1j}\det A_{1j}$ will correspond to all non-hitting rook placements in which there is a rook in position j of the first row. This argument can be applied for each j. So our theorem will be proved if we can show that the signs of these products are what they should be.

Substituting the expression provided for A_{1j} by formula (2.11) into formula (2.9), we see that the sign of the n-tuple that belongs to q becomes

$(-1)^{j-1+i(q)}$. And indeed, the permutation $p = jq_2q_3 \cdots q_n$ has precisely $j-1$ more inversions than the partial permutation $q = q_2q_3 \cdots q_n$ as j is larger than $j-1$ other elements. This shows that the contribution of this n-tuple is indeed counted with sign $(-1)^{j-1}$. ∎

2.1.3.2 Perfect Matchings in Bipartite Graphs

The explicit definition of the determinant has some surprising applications in graph theory. A *perfect matching* M in G is a set of pairwise disjoint edges covering all vertices. In other words, each vertex of G belongs to exactly one edge in M. The graph G is called bipartite if the vertex set of G can be cut into two parts X and Y so that all edges of G have one vertex in X and one vertex in Y. The *truncated adjacency matrix* of a simple bipartite graph G is the matrix $B(G) = (b_{ij})$ in which $b_{ij} = 1$ if there is an edge between $i \in X$ and $j \in Y$, and $b_{ij} = 0$ otherwise. In other words, the rows of B represent the vertices of X, and the columns of B represent the vertices of Y.

Whether a bipartite graph has a perfect matching is an interesting and well-studied question. A sufficient and necessary condition for this existence problem is the well-known Marriage Theorem, which is included in most elementary graph theory books, such as [34].

The concept of truncated adjacency matrices provides us with a sufficient condition that is very easy to verify.

THEOREM 2.22
Let G be a bipartite graph with $|X| = |Y| = n$ that does not have a perfect matching. Then $\det B(G) = 0$.

In other words, if $\det B(G) \neq 0$, then G has a perfect matching.

PROOF We prove that $\det B(G) = 0$ by showing that all $n!$ summands in the explicit definition (2.10) of $B(G)$ are equal to 0. This is because the existence of a nonzero term $b_{1p_1}b_{2p_2} \cdots b_{np_n}$ would be equivalent to the existence of a perfect matching, namely the perfect matching in which $i \in X$ is matched to $p_i \in Y$. ∎

We also note that the *number* of all perfect matchings of G can be obtained by computing the *permanent* of $B(G)$ that is defined by

$$\operatorname{per} B(G) = \sum_{p \in S_n} b_{1p_1}b_{2p_2} \cdots b_{np_n}.$$

That is, $\operatorname{per} B(G)$ is defined just like $\det B(G)$, except that each term is added with a positive sign.

2.2 Inversions in Permutations of Multisets

Instead of permuting the elements of our favorite set, $[n]$, in this section we are going to permute elements of *multisets*. We will use the notation $\{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ for the multiset consisting of a_i copies of i, for all $i \in [k]$.

For our purposes, a *permutation* of a multiset is just a way of listing all its elements. It is straightforward to see, and is proved in most undergraduate textbooks on enumerative combinatorics, that the number of all permutations of the multiset $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ is

$$\frac{n!}{a_1!a_2!\cdots a_k!},$$

where $n = a_1 + a_2 + \cdots + a_k$.

An inversion of a permutation $p = p_1p_2\cdots p_n$ of a multiset is defined similarly to the way in which it was for permutations of sets, that is, (i, j) is an inversion if $i < j$, but $p_i > p_j$.

Example 2.23

The multiset-permutation 1322 has two inversions, $(2, 3)$, and $(2, 4)$. $\Box$

If we want to generalize Theorem 2.3 for permutations of multisets, that is, if we want to count permutations of multisets according to their inversions, we encounter exciting and surprising connections between the objects at hand, and a plethora of remote-looking areas of combinatorics.

Our goal is to find a closed expression for the sum

$$\sum_{p\in S_K} q^{i(p)}, \tag{2.12}$$

where S_K denotes the set of all permutations of the multiset K. We cannot reasonably expect something quite as simple as the result of Theorem 2.3 as the formula to be found will certainly depend on each of the a_i, and not just their sum n. Therefore, the reader will hopefully understand that we need some new notions before we can find the desired closed formula for (2.12).

Let $[\mathbf{n}] = 1 + q + q^2 + \cdots + q^{n-1}$, the polynomial whose importance we know from Theorem 2.3, and let $[\mathbf{n}]! = [\mathbf{1}] \cdot [\mathbf{2}] \cdot \cdots \cdot [\mathbf{n}]$. Do not confuse $[n] = \{1, 2, \cdots, n\}$, which is a set, and $[\mathbf{n}] = 1 + q + q^2 + \cdots + q^{n-1}$, which is a polynomial. One way to avoid the danger of confusion is to use the notation $[n]_q$ instead of $[\mathbf{n}]$, but that notation can result in crowded formulas. Note that if we substitute $q = 1$, then $[\mathbf{i}] = i$, and therefore $[\mathbf{n}]! = n!$, so this concept generalizes the concept of factorials. The crucial definition of this section is the following.

DEFINITION 2.24 *Let k and n be positive integers so that $k \leq n$. Then the (n, k)-*Gaussian coefficient *or q-*binomial coefficient *is denoted by $\begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}$, and is given by*

$$\begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix} = \frac{[\mathbf{n}]!}{[\mathbf{k}]![(\mathbf{n}-\mathbf{k})]!}.$$

Note that $\begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix} = \begin{bmatrix}\mathbf{n}\\ \mathbf{n-k}\end{bmatrix}$ clearly holds. Also note that substituting $q = 1$ reduces this definition to that of the usual binomial coefficients. This, and other connections between binomial and q-binomial coefficients will be further explored shortly. Finally, we can define q-multinomial coefficients accordingly.

DEFINITION 2.25 *Let $a_1, a_2, \cdots, a_k$ be positive integers satisfying $\sum_{i=1}^{k} a_i = n$. Then the $(a_1, a_2, \cdots, a_k)$-*Gaussian coefficient*, or q-*multinomial coefficient *is denoted by $\begin{bmatrix}\mathbf{n}\\ \mathbf{a_1, a_2, \cdots, a_k}\end{bmatrix}$, and is given by*

$$\begin{bmatrix}\mathbf{n}\\ \mathbf{a_1, a_2, \cdots, a_k}\end{bmatrix} = \frac{[\mathbf{n}]!}{[\mathbf{a_1}]![\mathbf{a_2}]!\cdots[\mathbf{a_k}]!}.$$

We point out that similarly to multinomial coefficients, the q-multinomial coefficients satisfy the identity

$$\begin{bmatrix}\mathbf{n}\\ \mathbf{a_1, a_2, \cdots, a_k}\end{bmatrix} = \begin{bmatrix}\mathbf{n}\\ \mathbf{a_1}\end{bmatrix}\begin{bmatrix}\mathbf{n-a_1}\\ \mathbf{a_2}\end{bmatrix}\begin{bmatrix}\mathbf{n-a_1-a_2}\\ \mathbf{a_3}\end{bmatrix}\cdots\begin{bmatrix}\mathbf{a_k}\\ \mathbf{a_k}\end{bmatrix}. \tag{2.13}$$

Note that using this terminology, Theorem 2.3 can be written as

$$\sum_{p \in S_n} q^{i(p)} = [\mathbf{n}]!.$$

The Gaussian coefficients look like rational functions of q, but, as we will soon see, it is not difficult to prove that they are in fact *polynomials* in q. Even more strongly, they are polynomials with *positive integer* coefficients. This is why sometimes they are called *Gaussian polynomials.*

2.2.1 An Application: Gaussian Polynomials and Subset Sums

Before we start applying Gaussian polynomials to obtain generating functions of multiset permutations, it seems beneficial to take a look at one of their several natural occurrences. The advantage of this will be that the reader will see in what sense the Gaussian coefficients $\begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}$ are generalizations of the binomial coefficients $\binom{n}{k}$. That, in turn, will be helpful in putting into context the recurrence relations of Gaussian coefficients that we are going to use.

THEOREM 2.26
Let n and k be fixed non-negative integers so that $k \leq n$. Let a_i denote the number of k-element subsets of $[n]$ whose elements have sum $i + \binom{k+1}{2}$, that is, i larger than the minimum. Then we have

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \sum_{i=0}^{k(n-k)} a_i q^i. \tag{2.14}$$

In other words, $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$ is the ordinary generating function of the k-element subsets of $[n]$ according to the sum of their elements.

Example 2.27
Let $n = 4$ and $k = 2$. Then, among the six 2-element subsets of [4], two, namely $\{1,4\}$ and $\{2,3\}$, have sum 5, and all other sums from 3 to 7 are attained by exactly one subset. Therefore, the right-hand side of (2.14) becomes $1 + q + 2q^2 + q^3 + q^4$, which is indeed equal to

$$\begin{bmatrix} \mathbf{4} \\ \mathbf{2} \end{bmatrix} = \frac{(q^4-1)(q^4-q)}{(q^2-1)(q^2-q)} = (q^2+1)(q^2+q+1).$$

□

PROOF (of Theorem 2.26) We prove the statement by induction on n, the initial case of $n = 1$ being obvious. Let us assume that the statement is true for $n-1$ and prove it for n. Exercise 21 shows that

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = q^{n-k} \cdot \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k-1} \end{bmatrix} + \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k} \end{bmatrix}. \tag{2.15}$$

Therefore, our induction step will be complete if we can show that the polynomials $\sum_{i=0}^{k(n-k)} a_i q^i$ satisfy the same recurrence relation. That is, let b_i be the number of k-element subsets of $[n-1]$ whose sum of elements is $i + \binom{k+1}{2}$ and let c_i be the number of $(k-1)$-element subsets of $[n-1]$ whose sum of elements is $i + \binom{k}{2}$; we then need to show that

$$\sum_{i=0}^{k(n-k)} a_i q^i = \left(\sum_{i=0}^{k(n-k-1)} b_i q^i \right) + \left(q^{n-k} \cdot \sum_{i=0}^{(k-1)(n-k)} c_i q^i \right).$$

This is the same as showing that $a_i = b_i + c_i q^{n-k}$ for all i, where undefined coefficients are to be treated as zero. However, the last equation is clearly true as a k-subset of $[n]$ either does not contain n, and then it is accounted for by b_i, or it does, and then it is accounted for by $c_{i-(n-k)}$, because of the shift in the definition of c_i. ∎

2.2.2 Inversions and Gaussian Coefficients

Now we are ready to announce and prove the result describing the generating function of multiset-permutations according to the number of their inversions.

THEOREM 2.28
Let $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ *be a multiset so that* $\sum_{i=1}^{k} a_i = n$, *and let* S_K *denote the set of all permutations of* K. *Then we have*

$$\sum_{p \in S_K} q^{i(p)} = \begin{bmatrix} \mathbf{n} \\ \mathbf{a_1, a_2, \cdots, a_k} \end{bmatrix}. \tag{2.16}$$

PROOF First we prove the statement in the special case of $k = 2$. In this case, K' is a multiset consisting of a_1 copies of 1 and a_2 copies of 2, so that $a_1 + a_2 = n$, and an inversion is an occurrence of a 2 on the left of a 1. We need to prove that in this special case, we have

$$\sum_{p \in S'_K} q^{i(p)} = \begin{bmatrix} \mathbf{n} \\ \mathbf{a_1} \end{bmatrix}. \tag{2.17}$$

We prove this statement by induction on n. For $n = 1$, the statement is trivially true as $\left[\begin{smallmatrix} \mathbf{1} \\ \mathbf{0} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{1} \\ \mathbf{1} \end{smallmatrix}\right] = 1$. Now assume the statement is true for $n-1$, and prove it for n. A multiset permutation of K' either ends in a 2, and then its last entry is not involved in any inversion, or it ends in a 1, and then its last entry is involved in exactly $a_2 = n - a_1$ inversions. By the induction hypothesis, this means that

$$\sum_{p \in S'_K} q^{i(p)} = \begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_1} \end{bmatrix} + q^{n-a_1} \cdot \begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_1 - 1} \end{bmatrix}.$$

By (2.15), it is now easy to see that the right-hand side is in fact equivalent to $\begin{bmatrix} \mathbf{n} \\ \mathbf{a_1} \end{bmatrix}$, completing the induction proof of (2.17).

We are now in a position to prove our Theorem in its general form. We will do this by induction on k, the case of $k = 1$ being trivial, and the case of $k = 2$ being solved above. Let us assume that the statement of the theorem is true for $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$, and prove that then it is also true for $K^+ = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}, (k+1)^{a_{k+1}}\}$.

Note that any permutation of K^+ is completely determined by the pair (p', p''), where p' is the multiset-permutation obtained from p by replacing all entries less than $k+1$ by 1, and p'' is the permutation obtained from p by removing all copies of $k+1$. It is then clear that

$$i(p) = i(p') + i(p''),$$

and that p' and p'' are independent of each other.

Then the problem of finding $\sum_{p'} q^{i(p')}$ is clearly equivalent to the previous special case, and therefore we get that $\sum_{p'} q^{i(p')} = \begin{bmatrix} \mathbf{n} \\ \mathbf{a_{k+1}} \end{bmatrix}$.

Now let us find $\sum_{p''} q^{i(p'')}$. If we remove all copies of $k+1$, we can apply the induction hypothesis, and see that

$$\sum_{p''} q^{i(p'')} = \begin{bmatrix} \mathbf{n - a_{k+1}} \\ \mathbf{a_1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{n - a_{k+1} - a_1} \\ \mathbf{a_2} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{a_k} \\ \mathbf{a_k} \end{bmatrix}.$$

Finally, as any p' (consisting of $a_1 + a_2 + \cdots + a_k$ copies of 1, and a_{k+1} copies of $k+1$) can be paired with any p'' (consisting of a_i copies of i for $i \in [k]$), it follows that

$$\begin{aligned} \sum_{p \in S'_K} q^{i(p)} &= \sum_{p'} q^{i(p')} \cdot \sum_{p''} q^{i(p'')} \\ &= \begin{bmatrix} \mathbf{n} \\ \mathbf{a_{k+1}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{n - a_{k+1}} \\ \mathbf{a_1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{n - a_{k+1} - a_1} \\ \mathbf{a_2} \end{bmatrix} \cdot \cdots \cdot \begin{bmatrix} \mathbf{a_k} \\ \mathbf{a_k} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{n} \\ \mathbf{a_1, a_2, \cdots, a_{k+1}} \end{bmatrix}. \end{aligned}$$

Here the last equation is immediate from (2.13). This completes our induction proof. ▮

2.2.3 Major Index and Permutations of Multisets

Recall that for permutations of the set $[n]$, we found in Theorem 2.17 that the statistics i and maj were equidistributed. We would like to see whether something similar is true for permutations of multisets. In order to be able to do that, we need to define the major index of multiset permutations. As a first step to that end, we need to define descents of multiset permutations.

Fortunately, both of these definitions are what one expects them to be. If $p = p_1 p_2 \cdots p_n$ is a permutation of a multiset, then we say that i is a descent of p if $p_i > p_{i+1}$. Similarly, the major index of the multiset permutation p is defined by $maj(p) = \sum_{i \in D(p)} i$.

Now we are ready to state the q-generalization of Theorem 2.17.

THEOREM 2.29

Let $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ be a multiset so that $a_1 + a_2 + \cdots + a_k = n$. Then the statistics i and maj are equidistributed on the set $S(K)$ of all permutations of K. In other words,

$$\sum_{p \in S(K)} q^{i(p)} = \sum_{p \in S(K)} q^{maj(p)} = \begin{bmatrix} \mathbf{n} \\ \mathbf{a_1, a_2, \cdots, a_k} \end{bmatrix}.$$

PROOF As we have proved Theorem 2.17 by a bijection, this time we present an inductive proof. Just as in the proof of Theorem 2.28, we first treat the special case when K consists of a_1 copies of 1 and a_2 copies of 2 only, with $a_1 + a_2 = n$. Denote this specific multiset by K_2. In this special case, the statement is obviously true for $n = 1$. Now let us assume that we know the statement for all positive integers less than n, and prove it for n.

Now let us partition our permutations of K_2 according to the *position of the last 2*. If the position of the last 2 is i, then we can be sure that there are no descents of p in positions $i+1, \cdots, n-1$. There is a descent at i, unless $i = n$. Finally, the string on the left of p_i consists of $a_2 - 1$ copies of 2, and $a_1 - (n-i)$ copies of 1. This implies that, by our induction hypothesis, we have

$$\sum_p q^{maj(p)} = q^i \begin{bmatrix} \mathbf{i-1} \\ \mathbf{a_2-1} \end{bmatrix}$$

where p ranges the permutations of K_2 in which the last 2 is in position $a_2 \leq i \leq n-1$, and

$$\sum_p q^{maj(p)} = \begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_2-1} \end{bmatrix} = \begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_1} \end{bmatrix},$$

where p ranges the permutations of K_2 that end in 2. Therefore, all we have to show to complete the induction step is that

$$\begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_1} \end{bmatrix} + \sum_{i=a_2}^{n-1} q^i \begin{bmatrix} \mathbf{i-1} \\ \mathbf{a_2-1} \end{bmatrix} = \begin{bmatrix} \mathbf{n} \\ \mathbf{a_1} \end{bmatrix}, \tag{2.18}$$

or, equivalently,

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{a_1} \end{bmatrix} - \begin{bmatrix} \mathbf{n-1} \\ \mathbf{a_1} \end{bmatrix} = q^{a_2} \sum_{i=a_2}^{n-1} q^{i-a_2} \begin{bmatrix} \mathbf{i-1} \\ \mathbf{a_2-1} \end{bmatrix},$$

and this last statement is true as it is clearly equivalent to the recurrence relation proved in Exercise 23. This completes our induction proof for the special case when $K = K_2$.

Finally, to prove the theorem for general K, we can proceed by induction on k, very much like in the proof of Theorem 2.28. The details are similar to that proof, and are left to the reader. ∎

As we have mentioned, there are many interesting occurrences of Gaussian coefficients in combinatorics. Perhaps the most direct one is the following.

THEOREM 2.30

Let q be a power of a prime number, and let V be an n-dimensional vector space over the q-element field. Then the number of k-dimensional subspaces of V is $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$.

PROOF First, let us choose a k-tuple of vectors in V that form an (ordered) basis for a k-dimensional subspace. For this, we have to choose k linearly independent vectors from our vector space V. For the first basis vector v_1, we can choose any vector in V except the zero vector, so we have $q^n - 1$ choices. For the second basis vector, we cannot choose any multiples of v_1, therefore we have only $q^n - q$ choices. For the third vector, we cannot choose any of the q^2 possible linear combinations of v_1 and v_2, yielding $q^n - q^2$ choices, and so on. Iterating this argument, we see that we have

$$(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1}) \tag{2.19}$$

choices for an ordered basis of a k-dimensional subspace of V. It goes without saying that any such subspace has many ordered bases. In fact, repeating the above argument with k playing the role of n shows that the number of ordered bases of a k-dimensional subspace is

$$(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1}).$$

So this is how many times each k-dimensional subset of V is counted by (2.19). Therefore, the number of such subspaces is

$$\frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})} = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}$$
$$= \frac{[\mathbf{n}][\mathbf{n-1}]\cdots[\mathbf{n-k+1}]}{[\mathbf{k}]!} = \begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}.$$

∎

We will see some alternative interpretations of the Gaussian coefficients in the exercises.

Exercises

1. (+) Let us generalize the notion of Eulerian polynomials as follows. Let

$$B_n(x) = \sum_{p \in S_n} (-1)^{i(p)} x^{1+d(p)}.$$

That is, the only difference between this definition and that of $A_n(x)$ is that here the parity of the number of inversions is taken into account. Prove that

$$B_{2n}(x) = (1-x)^n A_n(x),$$

and

$$B_{2n+1}(x) = (1-x)^n A_{n+1}(x).$$

2. Following the line of thinking found in Exercise 17 of Chapter 1, define the *r-major index* of p, denoted by $rmaj(p)$ as

$$rmaj(p) = \left(\sum_{i \in RD(p)} i \right) + |\{(i,j) : 1 \leq i < j \leq n,\ p_i > p_j > p_i - r\}|,$$

where $RD(p)$ denotes the set of all r-falls of p, as defined in the mentioned exercise.

   (a) Explain why the r-major index is a generalization of both the number of inversions and the major index.

   (b) Prove that for any positive integers r, the r-major index is a mahonian statistic.

3. Prove (without using the general formula for $b(n,k)$) that $b(n,3) = \binom{n+1}{3} - \binom{n}{1}$ if $n \geq 3$.

4. Prove (without using the general formula for $b(n,k)$) that $b(n,4) = \binom{n+2}{4} - \binom{n+1}{2}$ if $n \geq 4$.

5. Let us call a $2n$-permutation $p = p_1p_2 \cdots p_{2n}$ 2-ordered if $p_1 < p_3 < \cdots < p_{2n-1}$ and $p_2 < p_4 < \cdots < p_{2n}$. Prove that

$$\sum_p i(p) = n4^{n-1},$$

where the sum is taken over all 2-ordered $2n$-permutations p.

6. Let $m > 1$ be a positive integer, and let j be a nonnegative integer, with $j < m$. Prove that if n is large enough, then the number of n-permutations p for which $i(p) \equiv j \bmod(\mathrm{m})$ is independent of j.

7. Let T be a rooted tree with root 0 and non-root vertex set $[n]$. Define an *inversion* of T to be a pair (i,j) of vertices so that $i > j$, and the unique path from 0 to j goes through i. How many such trees have zero inversions?

8. Let $p \in S_n$ have $n-2$ descents. What is the minimal and maximal possible value of $i(p)$?

9. It follows from Lemma 2.12 that

$$\sum_{j\ even} p(n - a(j)) = \sum_{j\ odd} p(n - a(j)),$$

where $a(j) = (3j^2 + j)/2$. Find a direct bijective proof of this identity.

10. Let $p = p_1p_2\cdots p_n$ be a permutation, and let our goal be to eliminate all four-tuples of entries (p_a, p_b, p_c, p_d) in which $a < b < c < d$ and $p_a < p_c < p_b < p_d$. In order to achieve that goal, we use the following algorithm. We choose a four-tuple F with the above property at random, and interchange its two middle entries. By doing that, we took away the undesirable property of F, but we may have created new four-tuples with that property. Then pick another four-tuple with that property, and repeat the procedure.

    Prove that no matter what p is, and how we choose our four-tuples, this algorithm will always stop, that is, it will eliminate all four-tuples with the undesirable property.

11. Is it true that $b(n, k)$ is a polynomially recursive function of n for any fixed k? (Polynomially recursive functions are defined in Exercise 29 of Chapter 1.)

12. Let $p(n, k)$ be the number of partitions of n into k parts. Let $P(x) = \sum_{i=1}^{n} p(n, k)x^k$. Does there exist an integer $n > 2$ so that $P(x)$ has real zeros only?

13. Let B_n be the set of all n-tuples $(b_1, b_2, \cdots, b_n)$ of non-negative integers that satisfy $b_i \leq i - 1$ for all i. How many elements of B_n satisfy $\sum_{i=1}^{n} b_i = k$?

14. Let B_n be defined as in Exercise 13, and let $B(n, k)$ be the number of n-tuples in B_n that have exactly k different entries. Find a formula for $B(n, k)$.

15. Express $b(n, k)$ using summands of the type $b(n - 1, i)$.

16. Compute the value of $\sum_{k=0}^{\binom{n}{2}} (-1)^k b(n, k)$.

17. Let $p \in S_n$ have $n-1$ alternating runs, and let us assume that $n = 2k+1$. What is the minimal and maximal possible value of $i(p)$?

18. (a) (+) Let $A = \{1^{a_1}, 2^{a_2}\}$, and let us assume that a_1 and a_2 are relative primes to each other, with $a_1 + a_2 = n$. Let $I(A, k)$ be the number of permutations p of A so that

    $$i(p) \equiv k \mod n.$$

    Prove that $I(A, k) = \frac{1}{n}\binom{n}{a_1}$ for all k.

    (b) + What can we say about $I(A, k)$ if a_1 and a_2 have largest common divisor $d > 1$?

19. (+) The *Denert* statistic, denoted by den, is defined on S_n as follows. Let $p \in S_n$, then $den(p)$ is the number of pairs (l, k) of integers satisfying $1 \leq l < k \leq n$, and one of the conditions listed below

$$p_k < p_l \leq k,$$

$$p_l \leq k < p_k.$$

$$k < p_k < p_l.$$

So for instance, $den(132) = 2$ as the pair $(2, 3)$ satisfies the first condition, and the pair $(1, 2)$ satisfies the second condition. Prove that the Denert statistic is mahonian.

20. We know that $\binom{n}{k}$ is the number of northeastern lattice paths from $(0, 0)$ to $(k, n-k)$. Extend this correspondence to one that provides an interpretation for $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$.

21. Prove by way of computation that

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k} \end{bmatrix} + q^{n-k} \cdot \begin{bmatrix} \mathbf{n-1} \\ \mathbf{k-1} \end{bmatrix}.$$

22. Prove the identity of the previous exercise by a combinatorial argument.

23. Prove that

$$\begin{bmatrix} \mathbf{m} \\ \mathbf{k} \end{bmatrix} = q^{m-k} \cdot \begin{bmatrix} \mathbf{m-1} \\ \mathbf{k-1} \end{bmatrix} + q^{m-k-1} \cdot \begin{bmatrix} \mathbf{m-2} \\ \mathbf{k-1} \end{bmatrix} + \cdots + q \cdot \begin{bmatrix} \mathbf{k} \\ \mathbf{k-1} \end{bmatrix} + \begin{bmatrix} \mathbf{k-1} \\ \mathbf{k-1} \end{bmatrix}.$$

24. Prove by way of computation that $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$ is always a polynomial with non-negative integers as coefficients.

25. Prove that

$$\begin{bmatrix} \mathbf{i+k} \\ \mathbf{k} \end{bmatrix} = \sum_{n \geq 0} q^n p(i, k, n),$$

where $p(i, k, n)$ is the number of partitions of the integer n into at most i parts of size at most k each.

26. For what values of n and k will $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$ have log-concave coefficients?

27. Let $m \leq n$, and let

$$A_m(n) = \{1, 1, 2, 2, \cdots, m, m, m+1, m+2, \cdots, n\}.$$

Let $a_m(n)$ be the number of all permutations of the multiset $A_m(n)$ in which $12 \cdots n$ occurs as a subword. (The letters of this subword do not have to be consecutive entries of the permutation.) Prove that

$$a_{m+1}(n) = (n + 2m)a_m(n) - m(n+m)a_{m-1}(n).$$

28. Let $n < k \leq \binom{n}{2}$. Prove that
$$b(n+1,k) = b(n+1,k-1) + b(n,k) - b(n,k-n-1).$$

29. (+) Find a formula for
$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}.$$

30. (+) Consider the following refinement of the Eulerian polynomials. Let
$$A_{n,k}(q) = \sum_{p} q^{maj(p)},$$
where the sum is taken over all n-permutations having $k-1$ descents. These polynomials are often called the q-Eulerian polynomials. Prove that
$$[\mathbf{x}]^n = \sum_{k=1}^{n} A_{n,k}(q) \begin{bmatrix} \mathbf{x+n-k} \\ \mathbf{n} \end{bmatrix}.$$

31. Let p be a permutation of length $n-1$. Insert the entry n into all p in all possible ways. This yields n distinct permutations of length n. Compute the major index of each of these permutations. Prove that all these n major indices will be distinct, and that their set will be the set of integers in the interval $[maj(p), maj(p)+n-1]$.

32. Use the result of the previous exercise to give an induction proof of Theorem 2.17, that is, of the fact that i and maj are equidistributed.

33. A simple graph G on vertex set $[n]$ is called the *inversion* graph of the n-permutation p if ij is an edge of G if and only if (i,j) is an inversion of G. Find an example for an unlabeled graph U that is not an inversion graph of any permutation. Try to find an example with as few vertices as possible.

34. Let $p = p_1p_2\cdots p_n$ be an n-permutation, and let $G(p)$ be the inversion graph of p as defined in the previous exercise. Let $i < j < k$ be three elements of $[n]$, and interchange the strings $p_i\cdots p_{j-1}$ and $p_j\cdots p_{k-1}$ of p, to get the permutation
$$p' = p_1\cdots p_{i-1}p_j\cdots p_{k-1}p_i\cdots p_{j-1}p_k\cdots p_n.$$
Describe $G(p')$ in terms of $G(p)$.

35. A graph G on vertex set $\{a_1, a_2, \cdots, a_n\}$ is called a *comparability graph* if there exists a poset P on vertex set $\{a_1, a_2, \cdots, a_n\}$ so that (a_i, a_j) is an edge in G if and only if a_i and a_j are comparable elements in P. Prove that all inversion graphs are comparability graphs, but not all finite comparability graphs are inversion graphs.

Problems Plus

1. Prove that the polynomials $I_n(q)$ have log-concave coefficients without using generating functions.

2. Let
$$Exp_q(x) = \sum_{n\geq 0} q^{\binom{n}{2}} \frac{x^n}{[\mathbf{n}]!}.$$
Note that for $q = 1$, the power series $Exp_q(x)$ reduces to the power series $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$.
Express the *bivariate* generating function
$$1 + \sum_{n\geq 1} \sum_{p\in S_n} t^{d(p)} q^{i(p)} \frac{u^n}{[\mathbf{n}]!}$$
in terms of Exp_q.

3. Let $exp_q(x) = \sum_{n\geq 0} \frac{x^n}{[\mathbf{n}]!}$. Express the *bivariate* generating function
$$1 + \sum_{n\geq 1} \sum_{p\in S_n} t^{exc(p)} q^{maj(p)} \frac{u^n}{[\mathbf{n}]!}$$
in terms of exp_q.

4. Let $a(k,l)$ be the number of n-permutations having k descents and major index l. Let $d(k,l)$ be the number of n-permutations q having k excedances and satisfying $den(q) = l$. Prove that $a(k,l) = d(k,l)$. This fact can be referred to by saying that the (den, exc) statistic is *Euler-Mahonian.*

5. Find a permutation statistic $s : S_n \to \mathbf{N}$ so that the number $c(k,l)$ of n-permutations p for which $s(p) = k$ and $i(p) = l$ is equal to $a(k,l)$ of the previous problem. In other words, find a statistic s so that the joint statistic (s,i) is euler-mahonian.

6. Prove that the joint statistic (dmc, maj) is euler-mahonian. See Exercise 10 of Chapter 1 for the definition of the statistic dmc.

7. Let $1 \leq k \leq n$. Prove, using the polynomial $I_n(x)$, that the number of n-permutations p for which
$$maj(p) \equiv j \pmod{k}$$
does not depend on j.

8. Define the (q,r)-Eulerian polynomials by

$$A[n,k,r] = \sum_{p \in S_n} q^{rmaj(p)},$$

where $rmaj(p)$ is the r-major index of p. Prove that

$$A[n,k,r] = [\mathbf{r}+\mathbf{k}]A[n-1,k,r] + q^{k+r-1}[\mathbf{n}+\mathbf{1}-\mathbf{k}-\mathbf{r}]A[n-1,k-1,r].$$

9. A *parking function* is a function $f : [n] \rightarrow [n]$ so that for all $i \in [n]$, there are at least i elements $j \in [n]$ for which $f(j) \leq i$. Prove that the number of parking functions on $[n]$ satisfying $\sum_{j=1}^{n} f(j) = \binom{n}{2} - k$ is equal to the number of rooted trees with root 0 and non-root vertex set $[n]$ that have k inversions. (See Exercise 7 for a definition of inversion in a tree.)

10. Prove that the Gaussian polynomial $\begin{bmatrix}\mathbf{n}\\ \mathbf{k}\end{bmatrix}$ has unimodal coefficients.

11. Let $A = \{1^{a_1}, 2^{a_2}\}$, and let d be the largest common divisor of a_1 and a_2. Now let $J(A,k)$ be the number of permutations p of A that have first entry 1, and for which

$$i(p) \equiv k \mod a_1$$

holds. Let $0 \leq t \leq \frac{a_1}{d} - 1$. Prove that

$$J(A,k) = \frac{d}{a_1}\binom{n-1}{a_1-1}.$$

Note the difference from Exercise 18. Here we are looking at residue classes modulo a_1, not modulo n.

12. Log-concavity is a concept for sequences of *numbers*, but it can be extended to a concept for sequences of *polynomials* as follows.

    Let $p_0(q), p_1(q), \cdots, p_m(q)$ be a sequence of polynomials with nonnegative coefficients. We say that this sequence is *q-log-concave* if the polynomial $p_k^2(q) - p_{k-1}(q)p_{k+1}(q)$ has non-negative coefficients for all k. Prove that for any fixed n, the sequence of polynomials $\begin{bmatrix}\mathbf{n}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{n}\\ \mathbf{1}\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{n}\\ \mathbf{n}\end{bmatrix}$ is q-log-concave.

13. Let us extend the notion of unimodality to polynomials as follows. Let $p_0(q), p_1(q), \cdots, p_m(q)$ be a sequence of polynomials with non-negative coefficients. We say that this sequence is *q-unimodal* if there exists an index j so that $0 \leq j \leq m$ and for all i, the polynomial $p_j(q) - p_i(q)$ has non-negative coefficients. Note that a q-log-concave sequence does not have to be q-unimodal. Prove that for any fixed n, the sequence of polynomials $\begin{bmatrix}\mathbf{n}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{n}\\ \mathbf{1}\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{n}\\ \mathbf{n}\end{bmatrix}$ is q-unimodal.

14. Generalize the result of Exercise 27 to the multiset

$$A_{m,n,r} = \{1^{r+1}, 2^{r+1}, \cdots, m^{r+1}, (m+1)^r, \cdots, n^r\}$$

as follows. Let $a_{m,r}(n)$ be the number of permutations of $A_{m,n,r}$ that contain a subword consisting of r copies of 1, then r copies of 2, and so on, ending with r copies of n. Again, the letters of the subword do not have to be consecutive entries in the permutation. Prove that

$$a_{m+1,r}(n) = (rn + 2m - r + 1)a_{m,r}(n) - m(rn + m)a_{m-1,r}(n),$$

where $a_{0,r}(n) = 1$, and $a_{1,r}(n) = rn + 1 - r$.

Solutions to Problems Plus

1. This result is proved by a recursively built injection in [36].

2. Richard Stanley [234] proved that

$$1 + \sum_{n\geq 1} \sum_{p\in S_n} t^{d(p)} q^{i(p)} \frac{u^n}{[\mathbf{n}]!} = \frac{1-t}{Exp_q(u(t-1)-t)}.$$

Note that in this problem and the next, the exponent of t is the number of descents (respectively, excedances), and not that parameter plus one. Therefore, setting $q = 1$, we do not get the exponential generating function of the Eulerian polynomials $A_n(x)$, but of the polynomials $A_n(x)/x$.

3. Michelle Wachs and John Shareshian [228] proved that

$$1 + \sum_{n\geq 1} \sum_{p\in S_n} t^{exc(p)} q^{maj(p)} \frac{u^n}{[\mathbf{n}]!} = \frac{(1-tq)exp_q(x)}{exp_q(xtq) - tqexp_q(x)}.$$

We point out that generating functions in which x^n is divided by $[\mathbf{n}]!$ are sometimes called *q-exponential generating functions.*

4. This result is due to Dominique Foata and Doron Zeilberger [133], who proved it by providing alternative interpretations for the Denert statistic. In particular, they showed that

$$den(p) = i_1 + i_2 + \cdots + i_m + i(Exc\,p) + i(Nexc\,p),$$

where $i_1, i_2, \cdots, i_m$ are the excedances of p, while $Exc\,p$ is the *substring* $p_{i_1}p_{i_2}\cdots p_{i_m}$, and $Nexc\,p$ is the substring obtained from p by removing $Exc\,p$. For our example in the text, the permutation 132, we get that

$$den(132) = 2 + 0 + 0 = 2,$$

as we should.

5. Such a statistic was given by Mark Skandera in [230].

6. This result was obtained by Dominique Foata and Guo-Niu Han [134], who first proved that the joint statistics (d, i) and (d, maj) had the same distribution.

7. For $k = n$, the statement means that there are $(n-1)!$ permutations in S_n so that $maj(p) \equiv j (\bmod n)$. This result was first proved in [23], using a heavy algebraic machinery. In that same paper, the authors provided a bijective proof as well, but that still used Standard Young Tableaux and the Robinson–Schensted correspondence, which we will cover in Chapter 7. The general statement for $k \in [n]$ was given the following simple and beautiful proof in [24]. We know from Theorems 2.3 and 2.17 that

$$\sum_{p \in S_n} x^{maj(p)} = I_n(x) = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}).$$

If we count our permutations according to the remainder of the major index modulo k, then we have to take the above equation modulo the polynomial $x^k - 1$. If x is any kth root of unity other than 1, then the left-hand side vanishes as there is at least one factor $(1+x+\cdots+x^{k-1}) = \frac{x^k-1}{x-1}$ on the right-hand side, which vanishes. Therefore,

$$\sum_{p \in S_n} x^{maj(p)} = c(1+x+\cdots+x^{k-1}) \qquad \bmod(x^k-1)$$

as the left-hand side and the right-hand side have $k-1$ common roots. Setting $x = 1$, we get that $c = n!/k$, and that

$$I_n(k) = \frac{n!}{k} \cdot (1+x+\cdots+x^{k-1}) \qquad \bmod(x^k-1).$$

That proves that the number of n-permutations p satisfying

$$maj(p) \equiv j \pmod k$$

is $n!/k$, proving our claim.

8. This result is due to Don Rawlings [217].

9. This result, in a slightly different form, was found by Germain Kreweras [189].

10. There are several proofs of this fact that had first been noticed by Cayley at the end of the nineteenth century. Some of these proofs are reasonably short, but use sophisticated machinery. See [224], or [216] for such proofs. An elementary proof was given by Kathy O'Hara [210], who used the subset sum interpretation of Gaussian coefficients in her proof. Her argument was later explained in an expository article by Zeilberger [275].

11. This result was proved in [79]. The authors showed that if $p = p_1p_2\cdots p_n$, then exactly d of the n cyclic translates $p_1p_2\cdots p_n$, $p_2\cdots p_np_1$, $\cdots$, $p_np_1\cdots p_{n-1}$ have first entry 1 and inversion number k modulo a_1. As $\frac{d}{a_1}\binom{n-1}{a_1-1} = \frac{d}{n}\binom{n}{a_1}$, this proves the result. Note that the result is identical to the result of Exercise 18 (b), even if in that exercise we counted different permutations.

12. This result was proved in [81].

13. This is a special case of a more general result of Lynne Butler [82], which is of group-theoretical flavor. In her proof, Butler uses the interesting fact that the number of subgroups of order q^k of the Abelian group Z_q^n is the Gaussian polynomial $\left[\begin{smallmatrix}\mathbf{n}\\ \mathbf{k}\end{smallmatrix}\right]$, where q is a prime.

14. It is proved in [148] that the exponential generating function of the sequence $a_{m,r}(n)$ is

$$(1-x)^{-rn+1}\exp\left(\frac{-rx}{1-x}\right),$$

from which the proof of our statement follows. In [273], Lilly Yen sketches a bijective proof.

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