Recurrence Relations

Niloufar Shafiei

Review

A recursive definition of a sequence specifies

- one or more initial terms
- a rule for determining subsequent terms from those that precede them.

Example:

$a_0=3$ $a_1=5$ $a_n= a_{n-1} - a_{n-2}$

$a_2 = a_1 - a_0 = 5 - 3 = 2$

Review

- A rule to determining subsequent terms from those that precede them is called a **recurrence relation**.

- The terms that precede the first term where the recurrence relation takes effect are called **initial conditions**.

- The recurrence relation and initial conditions uniquely determine a sequence.

Recurrence relation

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

Example

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$

Assume $a_0=0$ and $a_1=3$.

Solution:

- Check initial conditions:

  $a_0 = 3(0)=0$ $\qquad$ $a_1 = 3(1) = 3$

- Check if $\{3n\}$ satisfies $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-1} - a_{n-2} = 2[3(n-1)] - 3(n-2) = 6n - 6 - 3n + 6 = 3n = a_n$

- So, $\{3n\}$ is a solution for $a_n = 2a_{n-1} - a_{n-2}$

Example

Determine whether the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n = 2,3,4,….

Assume $a_0=1$ and $a_1=2$.

Solution:

- Check initial conditions

  $a_0= 2^0 = 1 \quad a_1= 2^1 = 2$
- Check if $\{2^n\}$ satisfies $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-1} - a_{n-2} = 2(2^{n-1}) - 2^{n-2} = 2^{n-2}(4-1) = 3(2^{n-2}) \neq 2^n$

- So, $\{2^n\}$ is not a solution for $a_n = 2a_{n-1} - a_{n-2}$
- You can also disprove it by a counterexample

  $(a_2 = 2^2 = 4$

  $a_2 = 2a_1 - a_0 = 2(2) - 1 = 3 \neq 4)$

Example

Determine whether the sequence $\{a_n\}$, where $a_n = 5$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for n = 2,3,4,….

Assume $a_0=5$ and $a_1=5$.

Solution:

- Check initial conditions

  $a_0= 5 \qquad a_1= 5$

- Check if {5} satisfies $a_n = 2a_{n-1} - a_{n-2}$

$2a_{n-1} - a_{n-2} = 2(5) - 5 = 10 - 5 = 5 = a_n$

- So, {5} is a solution for $a_n = 2a_{n-1} - a_{n-2}$

Modeling with recurrence relations

Recurrence relations can be used to model a wide variety of problems.

Solving recurrence relations

- Solving recurrence relations
  - Guess a solution
    - Do not make random guess, make educated guess
    - Solving a recurrence often takes some creativity.
    - If you are solving a recurrence and you have seen a similar one before, then you might be able to use the same technique.
  - Verify your guess
    - It is usually pretty easy if you guessed right.
    - It is usually a straightforward argument using mathematical induction

Example

Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually.

How much will be is the account after 30 years?

Solution:

- Determine a_n
  - a_n is “the amount in the account after n years”.
- Find the recurrence relation that a_n satisfies and the initial condition
  - $a_n = a_{n-1} + (0.11)\, a_{n-1} = (1.11)\, a_{n-1}$
  - $a_0 = 10,000$

Example

Solution:

- $a_n = (1.11)\ a_{n-1}$
- $a_0 = 10{,}000$

- Guess a solution (formula) for a_n

$a_1 = (1.11)\ a_0 = (1.11)\ (10{,}000)$

$a_2 = (1.11)\ a_1 = (1.11)^2\ (10{,}000)$

$a_3 = (1.11)\ a_2 = (1.11)^3\ (10{,}000)$

:

$a_n = (1.11)\ a_{n-1} = (1.11)^n\ (10{,}000)$

- Verify your guess using induction
  - Basis step:

    $a_0 = (1.11)^0\ (10{,}000) = 10{,}000$

Example

Solution:

- Inductive step: $(\forall k\ (a_k \rightarrow a_{k+1}))$
  Assume $a_k = (1.11)^k\ (10{,}000)$.
  $a_{k+1} = (1.11)\ a_k$ (by recurrence relation)
  $= (1.11)(1.11)^k\ (10{,}000)$ (by inductive hypothesis)
  $= (1.11)^{k+1}\ (10{,}000)$
- So, the solution is valid.

- $a_{30} = (1.11)^{30}\ (10{,}000) = 228{,}922.97$

Example

A young pair of rabbits is placed on an island.

A pair of rabbits does not breed until they are 2 months old.

After they are 2 months old, each pair of rabbits produces another pair each month.

Find a recurrence relation for the number of pairs of rabbits on the island after n months.

Solution:

- Determine a_n
  - a_n is “the number of pairs of rabbits after n months”.
- Find the recurrence relation that a_n satisfies and the initial condition
  - $a_1 = 1$ and $a_2 = 1$
  - The number of pairs after n months is equal to the number of rabbits in the previous month plus the number of newborns (which is equal to the number of pairs in two months ago)
  - $a_n = a_{n-1} + a_{n-2}$

Example (identifying arithmetic sequences)

Find a Solution for the following recurrence.

$a_0 = 12$

$a_1 = 17$

$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} - 12$

Solution:

- Guess a solution (formula) for a_n

  $a_0 = 12 \qquad a_1 = 17$

  $a_2 = 17 + 17 - 12 = 22$

  $a_3 = 17 + 22 - 12 = 27$

  $a_4 = 22 + 22 - 12 = 32$

  :

  - To find a formula for an arithmetic sequence check the differences of successive terms.

n	a_n	$a_n - a_{n-1}$
0	12	-
1	17	5
2	22	5
3	27	5
4	32	5

Example (identifying arithmetic sequences)

Find a Solution for the following recurrence.

$a_0 = 12$

$a_1 = 17$

$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} - 12$

Solution:

- So, $a_n = 5n + d$.
- Now we need to find d.
  - $a_2 = 10 + d = 22$, so $d = 12$.
- $a_n = 5n + 12$.
- Prove the solution using induction (exercise)

Example (identifying polynomial sequences)

Find a Solution for the following recurrence.

$a_0 = 7$

$a_1 = 12$

$a_n = a_{n-2} + 8n - 2$

Solution:

- Guess a solution (formula) for a_n

$a_0 = 7$ $\quad$ $a_1 = 12$

$a_2 = 7 + 16 - 2 = 21$

$a_3 = 12 + 24 - 2 = 34$

$a_4 = 21 + 32 - 2 = 51$

$a_5 = 34 + 40 - 2 = 72$

$a_6 = 51 + 48 - 2 = 97$

:

Example (identifying polynomial sequences)

Solution: :

- To find a formula for a polynomial sequence check the differences of successive terms and then check the differences of those and so on till you see that these differences form an arithmetic sequence.
  - So, $a_n = bn^2 + cn + d$.
    $a_0 = d = 7$
    $a_1 = b + c + 7 = 12$
    $a_2 = 4b + 2c + 7 = 21$
    So, b=2 and c=3.
  - $a_n = 2n^2 + 3n + 7$
- Prove the solution using induction (exercise)

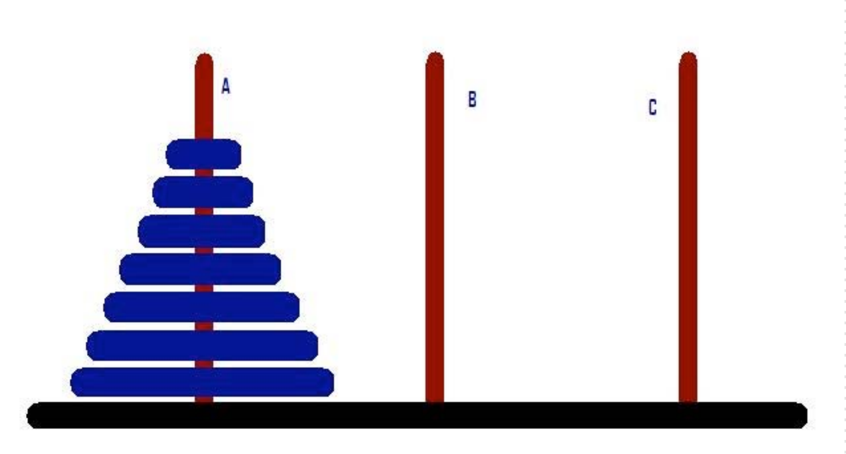
n	a_n	$b_n = a_n - a_{n-1}$	$b_n - b_{n-1}$
0	7	-	-
1	12	5	-
2	21	9	4
3	34	13	4
4	51	17	4
5	72	21	4
6	97	25	4

Example (identifying geometric sequences)

The Tower of Hanoi consists of 3 pegs mounted on a board together with disks of different sizes.

Initially these disks are placed on the first peg in order of size, with the largest on the bottom.

Disks can be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.

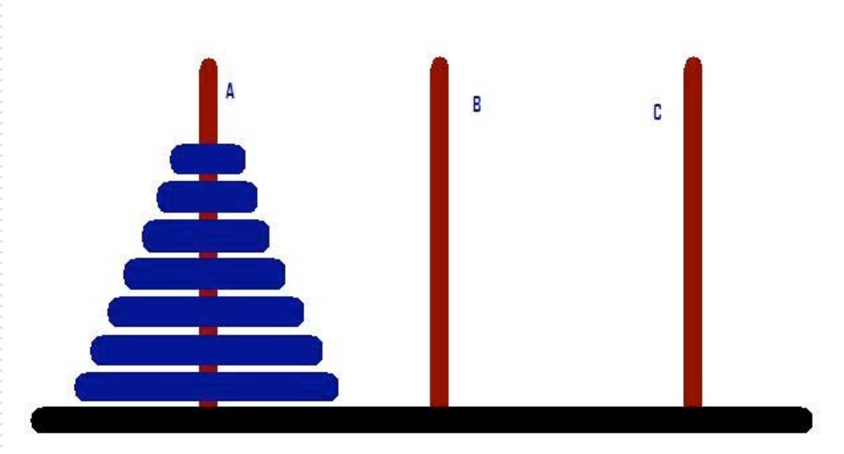


Example (identifying geometric sequences)

Goal: have all disks on the second peg in order of size, with the largest on the bottom.

H_n is the required number of moves needed to solve the Tower of Hanoi with n disks.

Find a recurrence relation and a solution for the sequence $\{H_n\}$.



Example (identifying geometric sequences)

Solution:

- H_n is "the number of moves needed to solve the Tower of Hanoi with n disks".
- Find the recurrence relation that H_n satisfies and the initial condition
  - $H_1 = 1$ (One disk can be transfer from peg 1 to peg 2 in one move.)
  - Determine recurrence relation of H_n
    - To transfer n disks from peg 1 to peg 2
      - We use H_{n-1} moves to transfer n-1 disks from peg 1 to peg 3
      - We transfer the largest disk from peg 1 to peg 2 in one move
      - Finally, we use H_{n-1} moves to transfer n-1 disks from peg 3 to peg 2
    - $H_n = 2H_{n-1} + 1$

      Since H_{n-1} might not be the minimum number of moves, $H_n \leq 2H_{n-1} + 1$.

      $H_n = 2H_{n-1} + 1$ requires some additional proof.

      (We do not go through this additional proof here.)

Example (identifying geometric sequences)

Solution:

- Guess a solution (formula) for a_n

  $H_1 = 1$

  $H_2 = 2H_1 + 1 = 3$

  $H_3 = 2H_2 + 1 = 7$

  $H_4 = 2H_3 + 1 = 15$

  $H_5 = 2H_4 + 1 = 31$

  :

  - To find a formula for a geometric sequence check the ratios between successive terms.
  - So, $H_n = x2^n + y$.
  - $H_1 = 2x + y = 1$
  - $H_2 = 4x + y = 3$
  - So, x=1 and y=-1 and $H_n = 2^n - 1$.

n	H_n	H_n/H_{n-1}
1	1	-
2	3	3
3	7	2.33
4	15	2.14
5	31	2.06

Example (identifying geometric sequences)

Solution:

- Verify your guess using induction
  - P(n) is $H_n = 2^n - 1$.
  - Basis step:

    $H_1 = 2^1 - 1 = 1$
  - Inductive step: $(\forall k\, (H_k \rightarrow H_{k+1}))$

    Assume $H_k = 2^k - 1$.

    $H_{k+1} = 2H_k + 1$ (by recurrence relation)

    $= 2(2^k - 1) + 1$ (by inductive hypothesis)

    $= 2^{k+1} - 1$
  - So, the solution is valid.

Example

Find a recurrence relation and initial conditions for the number of bit strings of length n that do not have two consecutive 0s.

Solution:

- a_n is “the number of of bit strings of length n that do not have two consecutive 0s”.
- Find the recurrence relation that a_n satisfies and the initial conditions
  - Determine recurrence relation of a_n
    - Bit strings of length n that do not have two consecutive 0s can be constructed in two ways.
      - Adding 1 to such bit strings of length n-1
      - Adding 10 to such bit strings of length n-2
    - $a_n = a_{n-1} + a_{n-2}$
  - $a_1 = 2 \qquad a_2 = 3$

Example

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits.

Let a_n be the number of valid n-digit codewords.

Find a recurrence relation for a_n.

Solution:

- Determine recurrence relation of a_n
  - n-digit codewords can be constructed in two ways.
    - Adding a digit other than 0 to (n-1)-digit codewords
    - Adding 0 to a string of length n-1 that is not valid
  - $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$
- $a_1 = 9 \quad a_2 = 82$

Example

Find a recurrence relation for C_n, the number of ways to parenthesize the product of n+1 numbers $x_0, x_1, \ldots, x_n$, to specify the order of multiplication.

For example:

$C_3 = 5$, because there are 5 ways to parenthesize x_0, x_1, x_2, x_3.

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$ $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$ $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$

$x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$ $x_0 \cdot (x_1 \cdot (x_2 \cdot X_3))$

Example

Find a recurrence relation for C_n, the number of ways to parenthesize the product of n+1 numbers x_0, x_1, …, x_n, to specify the order of multiplication.

Solution:

- Determine recurrence relation of C_n
  - Parenthesizing of multiplication of n+1 numbers can be constructed in different ways.
    - Parenthesizing of first number and parenthesizing n numbers
    - Parenthesizing of first two number and parenthesizing n-1 numbers
    - Parenthesizing of first three numbers and parenthesizing n-2 numbers
    - ….

Example

Find a recurrence relation for C_n, the number of ways to parenthesize the product of n+1 numbers $x_0, x_1, \ldots, x_n$, to specify the order of multiplication.

Solution:

- $C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n-2}C_1 + C_{n-1}C_0$
- $C_n = \sum_{K=0}^{n-1} C_kC_{n-k-1}$

Recommended exercises

1,8,9,11,13,15,17,19,27,29,35

Eric Ruppert’s Notes about Solving Recurrences

(http://www.cse.yorku.ca/course_archive/2007-08/F/1019/A/recurrence.pdf)