1

In One Line and Close. Permutations as Linear Orders.

1.1 Descents

The "most orderly" of all n-permutations is obviously the increasing permutation $123\cdots n$. All other permutations have at least some "disorder" in them; for instance, it happens that an entry is immediately followed by a *smaller* entry in them. This simple phenomenon is at the center of our attention in this Section.

1.1.1 The Definition of Descents

DEFINITION 1.1 *Let $p = p_1p_2\cdots p_n$ be a permutation, and let $i < n$ be a positive integer. We say that i is a* descent *of p if $p_i > p_{i+1}$. Similarly, we say that i is an* ascent *of p if $p_i < p_{i+1}$.*

Example 1.2
Let $p = 3412576$. Then 2 and 6 are descents of p, while 1, 3, 4, and 5 are ascents of p. $\square$

Note that the descents denote the *positions* within p, and not the entries of p. The set of all descents of p is called the *descent set of* p and is denoted by $D(p)$. The cardinality of $D(p)$, that is, the number of descents of p, is denoted by $d(p)$, though certain authors prefer $des(p)$.

This very natural notion of descents raises some obvious questions for the enumerative combinatorialist. How many n-permutations are there with a given number of descents? How many n-permutations are there with a given descent set? If two n-permutations have the same descent set, or same number of descents, what other properties do they share?

The answers to these questions are not always easy, but are always interesting. We start with the problem of finding the number of permutations with a given descent set S. It turns out that it is even easier to find the number of permutations whose descent set is *contained* in S.

LEMMA 1.3
Let $S = \{s_1, s_2, \cdots, s_k\} \subseteq [n-1]$, and let $\alpha(S)$ be the number of n-permutations whose descent set is contained in S. Then we have

$$\alpha(S) = \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\binom{n-s_2}{s_3-s_2}\cdots\binom{n-s_k}{n-s_k}.$$

PROOF The crucial idea of the proof is the following. We arrange our n entries into $k+1$ segments so that the first i segments together have s_i entries for each i. Then, within each segment, we put our entries in increasing order. Then the only places where the resulting permutation has a chance to have a descent is where two segments meet, that is, at $s_1, s_2, \cdots, s_k$. Therefore, the descent set of the resulting permutation is contained in S.

How many ways are there to arrange our entries in these segments? The first segment has to have length s_1, and therefore can be chosen in $\binom{n}{s_1}$ ways. The second segment has to be of length $s_2 - s_1$, and has to be disjoint from the first one. Therefore, it can be chosen in $\binom{n-s_1}{s_2-s_1}$ ways. In general, segment i must have length $s_i - s_{i-1}$ if $i < k+1$, and has to be chosen from the remaining $n - s_{i-1}$ entries, in $\binom{n-s_{i-1}}{s_i-s_{i-1}}$ ways. There is only one choice for the last segment as all remaining $n - s_k$ entries have to go there. This completes the proof. ∎

Now we are in a position to state and prove the formula for the number of n-permutations with a given descent set.

THEOREM 1.4
Let $S \subseteq [n-1]$. Then the number of n-permutations with descent set S is

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha(T). \tag{1.1}$$

PROOF This is a direct conclusion of the Principle of Inclusion and Exclusion. (See any textbook on introductory combinatorics, such as [34], for this principle.) Note that permutations with a given h-element descent set $H \subseteq S$ are counted $a_h = \sum_{i=0}^{|S-H|}(-1)^i\binom{|S-H|}{i} = (1+(-1))^{|S-H|}$ times on the right-hand side of (1.1). The value of a_h is 0 except when $|S-H| = 0$, that is, when $S = H$. So the right hand side counts precisely the permutations with descent set S. ∎

1.1.2 Eulerian Numbers

Let $A(n,k)$ be the number of n-permutations with $k-1$ descents. You may be wondering what the reason for this shift in the parameter k is. If p has $k-1$ descents, then p is the union of k increasing subsequences of consecutive entries. These are called the *ascending runs* of p. (Some authors call them just "runs," some others call something else "runs." This is why we add the adjective "ascending" to avoid confusion.) Also note that in some papers, $A(n,k)$ is used to denote the number of permutations with k descents.

Example 1.5
The three ascending runs of $p = 2415367$ are 24, 15, and 367. $\Box$

Example 1.6
There are four permutations of length three with one descent, namely 132, 213, 231, and 312. Therefore, $A(3,2) = 4$. Similarly, $A(3,3) = 1$ corresponding to the permutation 321, and $A(3,1) = 1$, corresponding to the permutation 123. $\Box$

Thus the permutations with k ascending runs are the same as permutations with $k-1$ descents, providing one answer for the notation $A(n,k)$. We note that some authors use the notation $\left\langle {n \atop k} \right\rangle$ for $A(n,k)$.

The numbers $A(n,k)$ are called the *Eulerian numbers*, and have several beautiful properties. Several authors provided extensive reviews of this field, including Carlitz [84], Foata and Schützenberger [131], Knuth [183], and Charalambides [83]. In our treatment of the Eulerian numbers, we will make an effort to be as combinatorial as possible, and avoid the analytic methods that probably represent a majority of the available literature. We start by proving a simple recursive relation.

THEOREM 1.7
For all positive integers k and n satisfying $k \leq n$, we have

$$A(n,k+1) = (k+1)A(n-1,k+1) + (n-k)A(n-1,k).$$

PROOF There are two ways we can get an n-permutation p with k descents from an $(n-1)$-permutation p' by inserting the entry n into p'. Either p' has k descents, and the insertion of n does not form a new descent, or p' has $k-1$ descents, and the insertion of n does form a new descent.

In the first case, we have to put the entry n at the end of p', or we have to insert n between two entries that form one of the k descents of p'. This means we have $k+1$ choices for the position of n. As we have $A(n-1,k+1)$ choices for p', the first term of the right-hand side is explained.

In the second case, we have to put the entry n at the front of p', or we have to insert n between two entries that form one of the $(n-2)-(k-1)$ ascents of p'. This means that we have $n-k$ choices for the position of n. As we have $A(n-1,k)$ choices for p', the second part of the right-hand side is explained, and the theorem is proved. ∎

We note that $A(n, k+1) = A(n, n-k)$; in other words, the Eulerian numbers are symmetric. Indeed, if $p = p_1p_2\cdots p_n$ has k descents, then its reverse $p^r = p_np_{n-1}\cdots p_1$ has $n-k-1$ descents.

The following theorem shows some additional significance of the Eulerian numbers. In fact, the Eulerian numbers are sometimes *defined* using this relation.

THEOREM 1.8

Set $A(0,0) = 1$, and $A(n,0) = 0$ for $n > 0$. Then for all nonnegative integers n, and for all real numbers x, we have

$$x^n = \sum_{k=1}^{n} A(n,k)\binom{x+n-k}{n}. \tag{1.2}$$

Example 1.9

Let $n = 3$. Then we have $A(3,1) = 1$, $A(3,2) = 4$, and $A(3,3) = 1$, enumerating the sets of permutations $\{123\}$, $\{132, 213, 231, 312\}$, and $\{321\}$. And indeed, we have

$$x^3 = \binom{x+2}{3} + 4\binom{x+1}{3} + \binom{x}{3}.$$

$\square$

PROOF (of Theorem 1.8) Assume first that x is a positive integer. Then the left-hand side counts the n-element sequences in which each digit comes from the set $[x]$. We will show that the right-hand side counts these same sequences. Let $a = a_1a_2\cdots a_n$ be such a sequence. Rearrange the a into a nondecreasing order $a' = a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_n}$, with the extra condition that identical digits appear in a' in the increasing order of their indices. Then $i = i_1i_2\cdots i_n$ is an n-permutation that is uniquely determined by a. Note that i_1 tells from which position of a the first entry of i comes, i_2 tells from which position of a the second entry of i comes, and so on.

For instance, if $a = 311243$, then the rearranged sequence is $a' = 112334$, leading to the permutation $i = 234165$.

If we can show that each permutation i having $k-1$ descents is obtained from exactly $\binom{x+n-k}{n}$ sequences a this way, then we will have proved the theorem.

The crucial observation is that if $a_{i_j} = a_{i_{j+1}}$ in a', then $i_j < i_{j+1}$ in i. Taking contrapositives, if j is a descent of $p(a) = i_1i_2\cdots i_n$, then $a_{i_j} < a_{i_{j+1}}$. This means that the sequence a' has to be *strictly increasing* whenever j is a descent of $p(a)$. The reader should verify that in our running example, i has descents at 3 and 5, and indeed, a' is strictly increasing in those positions.

How many sequences a can lead to the permutation $i = 234165$? It follows from the above argument that in sequences with that property, we must have

$$1 \leq a_2 \leq a_3 \leq a_4 < a_1 \leq a_6 < a_5 \leq x,$$

as strict inequality is required in the third and fifth positions. The above chain of inequalities is obviously equivalent to

$$1 \leq a_2 < a_3 + 1 < a_4 + 2 < a_1 + 2 < a_6 + 3 < a_5 + 3 \leq x + 3,$$

and therefore, the number of such sequences is clearly

$$\binom{x+3}{6}.$$

So this is the number of sequences a for which $a' = 234165$. Generalizing this argument for any n and for permutations i with $k-1$ descents, we get that each n-permutation with $k-1$ descents will be obtained from $\binom{x+(n-1)-(k-1)}{n} = \binom{x+n-k}{n}$ sequences.

If x is not a positive integer, note that the two sides of the equation to be proved can both be viewed as polynomials in the variable x. As they agree for infinitely many values (the positive integers), they must be identical. ∎

Exercise 7 gives a more mechanical proof that simply uses Theorem 1.7.

COROLLARY 1.10
For all positive integers n, we have

$$x^n = \sum_{k=0}^{n} A(n,k) \binom{x+k-1}{n}.$$

PROOF Replace x by $-x$ in the result of Theorem 1.8. We get

$$x^n(-1)^n = \sum_{k=0}^{n} A(n,k) \binom{-x+n-k}{n}.$$

Now note that

$$\binom{-x+n-k}{n} = \frac{(-x+n-k)(-x+n-k-1)\cdots(-x+1-k)}{n!}$$
$$= (-1)^n \binom{x+k-1}{n}.$$

Comparing these two identities yields the desired result. ∎

The obvious question that probably crossed the mind of the reader by now is whether there exists an *explicit formula* for the numbers $A(n,k)$. The answer to that question is in the affirmative, though the formula contains a summation sign. This formula is more difficult to prove than the previous formulae in this Section.

THEOREM 1.11
For all nonnegative integers n and k satisfying $k \leq n$, we have

$$A(n,k) = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n. \tag{1.3}$$

While this theorem is a classic (it is more than a hundred years old), we could not find an immaculately direct proof for it in the literature. Proofs we did find used generating functions, or manipulations of double sums of binomial coefficients, or inversion formulae to obtain (1.3). Therefore, we solicited simple, direct proofs at the problem session of the 15th Formal Power Series and Algebraic Combinatorics conference, which took place in Vadstena, Sweden. The proof we present here was contributed by Richard Stanley. A similar proof was proposed by Hugh Thomas.

PROOF (of Theorem 1.11) Let us write down $k-1$ bars with k compartments in between. Place each element of $[n]$ in a compartment. There are k^n ways to do this, the term in the above sum indexed by $i = 0$. Arrange the numbers in each compartment in increasing order. For example, if $k = 4$ and $n = 9$, then one arrangement is

$$237||19|4568. \tag{1.4}$$

Ignoring the bars we get a permutation (in the above example, it is 237194568) with *at most* $k-1$ descents.

There are several issues to take care of. There could be empty compartments, or there could be neighboring compartments with no descents in between. We will show how to sieve out permutations having either of these problems, and therefore, less than $k-1$ descents, at the same time.

Let us say that a bar is a *wall* if it is not immediately followed by another bar. Let us say that a wall is *extraneous* if by removing it we still get a legal arrangement, that is, an arrangement in which each compartment consist of integers in *increasing* order.

For instance, in (1.4), the second bar is an extraneous wall. Our goal is to enumerate the arrangements with *no extraneous walls*, as these are clearly in bijection with permutations with $k-1$ descents.

In order to do this, we will apply the Principle of Inclusion and Exclusion. Let us call the spaces between consecutive entries of a permutation, as well as the space preceding the first entry and the space following the last entry a *position.* So we associate $n+1$ positions to an n-permutation. Let $S \subseteq [n+1]$, and let A_S be the set of arrangements in which there is a an extraneous wall in each position belonging to S.

Let $i \leq k-1$ be the size of S. Then we claim that

$$|A_S| = (k-i)^n.$$

In order to see this, first take any legal arrangment that contains $k-i-1$ bars. There are $(k-i)^n$ such arrangements. Now insert i extra bars by inserting one to each position that belongs to S. (If there is already a bar in such a position, then put the new bar immediately on the right of that bar.) This results in an arrangement that belongs to A_S. Conversely, each arrangment belonging to A_S will be obtained exactly once in this way. Indeed, if $a \in A_S$, then removing one bar from each of the i positions that belong to S, we get the unique original arrangment with $k-i-1$ bars that leads to a.

As there are $\binom{n+1}{i}$ choices for the set S, and $A_\emptyset$ is the set of arrangments with $k-1$ bars, none of which is an extraneous wall, the proof of our Theorem is now immediate by the Principle of Inclusion and Exclusion. ∎

For the sake of completeness, we include a more computational proof that does not need a clever idea as the previous one did.

First, we recall a lemma from the theory of binomial coefficients.

LEMMA 1.12

[Cauchy's Convolution Formula] Let x and y be real numbers, and let z be a positive integer. Then we have

$$\binom{x+y}{z} = \sum_{d=0}^{z} \binom{x}{d}\binom{y}{z-d}.$$

Note that Lemma 1.12 is sometimes called *Vandermonde's Identity.*

PROOF Let us assume first that x and y are positive integers. Then the left-hand side enumerates the z-element subsets of the set $[x+y]$, while the right-hand side enumerates these same objects, according to the size of their intersection with the set $[x]$.

For general x and y, note that both sides can be viewed as polynomials in x and y, and they agree for infinitely many values (the positive integers). Therefore, they have to be identical. ∎

PROOF (of Theorem 1.11) As a first step, consider formula (1.2) with

$x = 1$, then with $x = 2$, and then for $x = i$ for $i \leq k$. We get

$$1 = A(n,1) \cdot \binom{n}{n},$$
$$2^n = A(n,2) \cdot \binom{n}{n} + A(n,1) \cdot \binom{n+1}{n},$$

and so on, the hth equation being

$$h^n = \sum_{j=0}^{h-1} A(n,k-j)\binom{n+j-1}{n}, \tag{1.5}$$

and the last equation being

$$k^n = \sum_{j=0}^{k-1} A(n,k-j)\binom{n+j-1}{n} \tag{1.6}$$

We will now add certain multiples of our equations to the last one, so that the left-hand side becomes the right-hand side of formula (1.3) that we are trying to prove.

To start, let us add $(-1)\binom{n+1}{1}$ times the $(k-1)$st equation to the last one. Then add $\binom{n+1}{2}$ times the $(k-2)$nd equation to the last one. Continue this way, that is, in step i, add $(-1)^i\binom{n+1}{i}$ times the $(k-i)$th equation to the last one. This gives us

$$\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}(k-i)^n = \sum_{j=1}^{k} A(n,j)\sum_{i=0}^{k-j}\binom{n+k-i-j}{n}\binom{n+1}{i}(-1)^i. \tag{1.7}$$

The left-hand side of (1.7) agrees with the right-hand side of (1.3). Therefore, (1.3) will be proved if we can show that the coefficient $a(n,j)$ of $A(n,j)$ on the right-hand side above is 0 for $j < k$. It is obvious that $a(n,k) = 1$ as $A(n,k)$ occurs in the last equation only.

Set $b = k - j$. Then $a(n,k)$ can be transformed as follows.

$$a(n,k) = \sum_{i=0}^{b}(-1)^i\binom{n+1}{i}\binom{n-i+b}{n}.$$

Recalling that for positive x, we have $\binom{-x}{a} = \binom{x+a-1}{a}(-1)^a$, and noting that $(-1)^b = (-1)^{b-2i}$, this yields

$$(-1)^b a(n,k) = \sum_{i=0}^{b}(-1)^{b-i}\binom{n+1}{i}\binom{n-i+b}{n}$$

$$= \sum_{i=0}^{b}(-1)^{b-i}\binom{n+1}{i}\binom{n-i+b}{b-i} = \sum_{i=0}^{b}\binom{n+1}{i}\binom{-1-n}{b-i} = \binom{0}{b} = 0,$$

where the last step holds as $b = k - j > 0$, and the next-to-last step is a direct application of Cauchy's convolution formula.

This shows that the right-hand side of (1.7) simplifies to $A(n, k)$, and proves our theorem. ∎

1.1.3 Stirling Numbers and Eulerian Numbers

DEFINITION 1.13 *A* partition *of the set* $[n]$ *into* r *blocks is a distribution of the elements of* $[n]$ *into* r *disjoint non-empty sets, called blocks, so that each element is placed into exactly one block.*

In Section 2.1, we will define the different concept of partitions of an *integer*. If there is a danger of confusion, then partitions of the set $[n]$ will be called *set partitions*, to distinguish them from partitions of the integer n.

Example 1.14
Let $n = 7$ and $r = 4$. Then $\{1, 2, 4\}, \{3, 6\}, \{5\}, \{7\}$ is a partition of $[7]$ into four blocks. □

Note that neither the order of blocks nor the order of elements within each block matters. That is, $\{4, 1, 2\}, \{6, 3\}, \{5\}, \{7\}$ and $\{4, 1, 2\}, \{6, 3\}, \{7\}, \{5\}$ are considered the same partition as the one in Example 1.14.

DEFINITION 1.15 *The number of partitions of* $[n]$ *into* k *blocks is denoted by* $S(n, k)$ *and is called a* Stirling number of the second kind.

By convention, we set $S(n, 0) = 0$ if $n > 0$, and $S(0, 0) = 1$. The next chapter will explain what the Stirling numbers of the first kind are.

Example 1.16
The set $[4]$ has six partitions into three parts, each consisting of one doubleton and two singletons. Therefore, $S(4, 3) = 6$. □

Whereas Stirling numbers of the second kind do not directly count permutations, they are inherently related to two different sets of numbers that do. One of them is the set of Eulerian numbers, and the other one is the aforementioned set of Stirling numbers of the first kind. Therefore, exploring some properties of the numbers $S(n, k)$ in this book is well-motivated. See Figure 1.1 for the values of $S(n, k)$ for $n \leq 5$.

See Exercises 8 and 14 for two simple recurrence relations satisfied by the numbers $S(n, k)$. It turns out that an explicit formula for these numbers can be proved without using the recurrence relations.

n=0	1					
n=1	0	1				
n=2	0	1	1			
n=3	0	1	3	1		
n=4	0	1	7	6	1	
n=5	0	1	15	25	10	1

FIGURE 1.1
The values of $S(n,k)$ for $n \leq 5$. Note that the Northeast–Southwest diagonals contain values of $S(n,k)$ for fixed k. Row n starts with $S(n,0)$.

LEMMA 1.17
For all positive integers n and r, we have

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n.$$

PROOF An *ordered partition* of $[n]$ into r blocks is a partition of $[n]$ into r blocks in which the set of blocks is totally ordered. So $\{1,3\},\{2,4\}$ and $\{2,4\},\{1,3\}$ are different ordered partitions of $[4]$ into two blocks. Note that an ordered partition of $[n]$ into r blocks is just the same as a surjection from $[n]$ to $[r]$. In orde to enumerate all such surjections, let A_i be the set of functions from $[n]$ into $[r]$ whose image *does not* contain i. The function $f:[n]\rightarrow[r]$ is a surjection if and only if it is not contained in $A_1 \cup A_2 \cup \cdots \cup A_r$, and our claim can be proved by a standard application of the Principle of Inclusion and Exclusion. ∎

Stirling numbers of the second kind and Eulerian numbers are closely related, as shown by the following theorem.

THEOREM 1.18
For all positive integers n and r, we have

$$S(n,r) = \frac{1}{r!}\sum_{k=0}^{r}A(n,k)\binom{n-k}{r-k}. \tag{1.8}$$

PROOF Multiplying both sides by $r!$ we get

$$r!S(n,r) = \sum_{k=0}^{r}A(n,k)\binom{n-k}{r-k}.$$

Here the left-hand side is obviously the number of ordered partitions of $[n]$ into r blocks. We will now show that the right-hand side counts the same objects. Take a permutation p counted by $A(n,k)$. The k ascending runs of p then naturally define an ordered partition of $[n]$ into k parts. If $k=r$, then there is nothing left to do. If $k<r$, then we will split up some of the ascending runs into several blocks of consecutive elements, in order to get an ordered partition of r blocks. As we currently have k blocks, we have to increase the number of blocks by $r-k$. This can be achieved by choosing $r-k$ of the $n-k$ "gap positions" (gaps between two consecutive entries within the same block).

This shows that we can generate $\sum_{k=0}^{r} A(n,k)\binom{n-k}{r-k}$ ordered partitions of $[n]$ that consist of r blocks each by the above procedure. It is straightforward to show that each such partition will be obtained exactly once. Indeed, if we write the elements within each block of the partition in increasing order, we can just read the entries of the ordered partition left to right and get the unique permutation having at most r ascending runs that led to it. We can then recover the gap positions used. This completes the proof. ∎

Inverting this result leads to a formula expressing the Eulerian numbers by the Stirling numbers of the second kind.

COROLLARY 1.19
For all positive integers n and k, we have

$$A(n,k)=\sum_{r=1}^{k} S(n,r)r!\binom{n-r}{k-r}(-1)^{k-r}. \tag{1.9}$$

PROOF Let us consider formula (1.8) for each $r\leq k$, and multiply each by $r!$. We get the equations

$$1!\cdot S(n,1)=A(n,1)\binom{n-1}{0},$$
$$2!\cdot S(n,2)=A(n,1)\binom{n-1}{1}+A(n,2)\binom{n-2}{0},$$

the equation for general r being

$$r!\cdot S(n,r)=\sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}, \tag{1.10}$$

and the last equation being

$$k!\cdot S(n,k)=\sum_{i=1}^{k} A(n,i)\binom{n-i}{r-i}. \tag{1.11}$$

Our goal is to eliminate each term from the right-hand side of (1.11), except for the term $A(n,k)\binom{n-k}{k-k} = A(n,k)$. We claim that this can be achieved by multiplying (1.10) by $(-1)^{k-r}\binom{n-r}{k-r}$, doing this for all $r \in [k-1]$, then adding these equations to (1.11).

To verify our claim, look at the obtained equation

$$\sum_{r=1}^{k} S(n,r)r!(-1)^{k-r}\binom{n-r}{k-r} = \sum_{r=1}^{k}(-1)^{k-r}\binom{n-r}{k-r}\sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}, \tag{1.12}$$

or, after changing the order of summation,

$$\sum_{r=1}^{k} S(n,r)r!(-1)^{k-r}\binom{n-r}{k-r} = \sum_{i=1}^{r} A(n,i)\binom{n-i}{r-i}\sum_{r=1}^{k}(-1)^{k-r}\binom{n-r}{k-r} \tag{1.13}$$

whose left-hand side is identical to the right-hand side of (1.9).

It is obvious that the coefficient of $A(n,k)$ on the right-hand side is $\binom{n-k}{k-k} = 1$. Therefore, our statement will be proved if we can show that the coefficient $t(n,i)$ of $A(n,i)$ in the last expression is equal to zero if $i < k$.

Note that $\binom{n-i}{r-i} = 0$ if $r < i$. Therefore, for any fixed $i < k$, we have

$$\begin{aligned} t(n,i) &= \sum_{r=i}^{k}\binom{n-i}{r-i}\binom{n-r}{k-r}(-1)^{k-r} = \sum_{r=i}^{k}\binom{n-i}{r-i}\binom{k-n-1}{k-r} \\ &= \binom{k-i-1}{k-i} = 0. \end{aligned}$$

We used Cauchy's convolution formula (Lemma 1.12) in the last step. This proves that if $i < k$, then $A(n,i)$ vanishes on the right-hand side of (1.13). We have discussed that $A(n,k)$ will have coefficient 1 there. (This can be seen again by setting $k = i$ in the last expression, leading to $t(n,i) = \binom{-1}{0} = 1$.) So (1.13) implies the claim of this corollary. ∎

1.1.4 Generating Functions and Eulerian Numbers

There are several ways one can define a generating function whose coefficients are certain Eulerian numbers. Let us start with a "horizontal" version.

DEFINITION 1.20 *For all nonnegative integers n, the polynomial*

$$A_n(x) = \sum_{k=1}^{n} A(n,k)x^k$$

is called the nth Eulerian polynomial.

The Eulerian polynomials have several interesting properties that can be proved by purely combinatorial means. We postpone the study of those properties until the next subsection. For now, we will explore the connection between these polynomials and some infinite generating functions.

THEOREM 1.21
For all positive integers n, the nth Eulerian polynomial has the alternative description

$$A_n(x) = (1-x)^{n+1} \sum_{i \geq 0} i^n x^i.$$

Note that Euler first defined the polynomials $A_n(x)$ in the above form.

Example 1.22
For $n = 1$, we have

$$A_1(x) = (1-x)^2 \sum_{i \geq 0} i x^i = (1-x)^2 \cdot \frac{x}{(1-x)^2} = x,$$

and for $n = 2$, we have

$$A_2(x) = (1-x)^3 \sum_{i \geq 0} i^2 x^i = (1-x)^3 \cdot \left(\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \right) = x + x^2.$$

□

PROOF (of Theorem 1.21) Let us use (1.3) to write the Eulerian polynomials as

$$\sum_{k=1}^{n} A(n,k) x^k = \sum_{k=1}^{n} \sum_{0 \leq i \leq k} (-1)^i \binom{n+1}{i} (k-i)^n x^k$$

$$= \sum_{k=1}^{n} \left(\sum_{0 \leq i \leq k} (-1)^{k-i} \binom{n+1}{k-i} i^n x^k \right).$$

Changing the order of summation, and noting that the sum in parentheses, being equal to $A(n,k)$, vanishes for $k > n$, we get

$$\sum_{i \geq 0} i^n x^i \cdot \sum_{k \geq i} \binom{n+1}{k-i} (-x)^{k-i} = (1-x)^{n+1} \sum_{i \geq 0} i^n x^i.$$

∎

n=1	1					
n=2	1	1				
n=3	1	4	1			
n=4	1	11	11	1		
n=5	1	26	66	26	1	
n=6	1	57	302	302	57	1

FIGURE 1.2
Eulerian numbers for $n \leq 6$. Again, the NE–SW diagonals contain the values of $A(n,k)$ for fixed k. Row n starts with $A(n,1)$.

It is often useful to collect all Eulerian numbers $A(n,k)$ for all n and all k in a master generating function. This function turns out to have the following simple form.

THEOREM 1.23
Let

$$r(t,u) = \sum_{n\geq 0}\sum_{k\geq 0} A(n,k)t^k \frac{u^n}{n!}.$$

Then we have

$$r(t,u) = \frac{1-t}{1-te^{u(1-t)}}.$$

PROOF Using the result of Theorem 1.21, we see that

$$r(t,u) = \sum_{n\geq 0}\left((1-t)^{n+1}\sum_{i\geq 0} i^n t^i\right)\frac{u^n}{n!} = (1-t)\sum_{i\geq 0} t^i \sum_{n\geq 0}\frac{(iu(1-t))^n}{n!} =$$

$$(1-t)\sum_{i\geq 0} t^i e^{iu(1-t)} = \frac{1-t}{1-te^{u(1-t)}}.$$

∎

1.1.5 The Sequence of Eulerian Numbers

Let us take a look at the numerical values of the Eulerian numbers for small n, and $k = 0, 1, \cdots, n-1$. The nth row of Figure 1.2 contains the values of $A(n,k)$, for $1 \leq k \leq n$, up to $n = 6$.

We notice several interesting properties. As we pointed out before, the sequence $A(n,k)$ is symmetric for any fixed n. Moreover, it seems that these sequences first increase steadily, then decrease steadily. This property is so important in combinatorics that it has its own name.

DEFINITION 1.24 *We say that the sequence of positive real numbers* $a_1, a_2, \cdots, a_n$ *is* unimodal *if there exists an index* k *such that* $1 \leq k \leq n$, *and* $a_1 \leq a_2 \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$.

The sequences $A(n,k)_{\{1 \leq k \leq n\}}$ seem to be unimodal for any fixed n. In fact, they seem to have a stronger property.

DEFINITION 1.25 *We say that the sequence of positive real numbers* $a_1, a_2, \cdots, a_n$ *is* log-concave *if* $a_{k-1}a_{k+1} \leq a_k^2$ *holds for all indices* k.

PROPOSITION 1.26
If the sequence $a_1, a_2, \cdots, a_n$ *of positive real numbers is log-concave, then it is also unimodal.*

PROOF The reader should find the proof first, then check the proof that we provide as a solution for Exercise 5. ■

The conjecture suggested by our observations is in fact correct. This is the content of the following theorem.

THEOREM 1.27
For any positive integer n, *the sequence* $A(n,k)_{\{1 \leq k \leq n\}}$ *of Eulerian numbers is log-concave.*

While this result has been known for a long time, it was usually shown as a corollary to a stronger, analytical result that we will discuss shortly, in Theorem 1.34. Direct combinatorial proofs of this fact are more recent. The proof we present here was given by Bóna and Ehrenborg [37] who built on an idea of Vesselin Gasharov [146].

If a path on a square grid uses steps $(1,0)$ and $(0,1)$ only, we will call it a *northeastern lattice path.*

Before proving the theorem, we need to set up some tools, which will be useful in the next section as well. We will construct a bijection from the set $\mathcal{A}(n,k)$ of n-permutations with k descents onto that of labeled northeastern lattice paths with n edges, exactly k of which are vertical. (Note the shift in parameters: $|\mathcal{A}(n,k)| = A(n,k+1)$, but this will not cause any confusion.)

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lattice paths with n edges, exactly k of which are vertical. (Note the shift in parameters: $|\mathcal{A}(n,k)| = A(n,k+1)$, but this will not cause any confusion.)

Let $\mathcal{P}(n)$ be the set of labeled northeastern lattice paths that have edges $a_1, a_2, \ldots, a_n$ and that corresponding positive integers $e_1, e_2, \ldots, e_n$ as labels, so that the following hold:

(i) the edge a_1 is horizontal and $e_1 = 1$,

(ii) if the edges a_i and a_{i+1} are both vertical, or both horizontal, then $e_i \geq e_{i+1}$,

(iii) if a_i and a_{i+1} are perpendicular to each other, then $e_i + e_{i+1} \leq i + 1$.

The starting point of a path in $\mathcal{P}(n)$ has no additional significance. Let $\mathcal{P}(n,k)$ be the set of all lattice paths in $\mathcal{P}(n)$ which have k vertical edges, and let $P(n,k) = |\mathcal{P}(n,k)|$.

PROPOSITION 1.28

The following two properties of paths in $\mathcal{P}(n)$ are immediate from the definitions.

- *For all $i \geq 2$, we have $e_i \leq i - 1$.*
- *Fix the label e_i. If e_{i+1} can take value v, then it can take all positive integer values $w \leq v$.*

Also note that all restrictions on e_{i+1} are given by e_i, independently of preceding e_j, $j < i$. Now we are going to explain how we will encode our permutations by these labeled lattice paths.

LEMMA 1.29

The following description defines a bijection from $\mathcal{S}(n)$ onto $\mathcal{P}(n)$, where $\mathcal{S}(n)$ is the set of all n-permutations. Let $p \in \mathcal{S}(n)$. To obtain the edge a_i and the label e_i for $2 \leq i \leq n$, restrict the permutation p to the i first entries and relabel the entries to obtain a permutation $q = q_1 \cdots q_i$ of $[i]$. Then proceed as follows.

1. *If the position $i-1$ is a descent of the permutation p (equivalently, of the permutation q), let the edge a_i be vertical and the label e_i be equal to q_i.*
2. *If the position $i-1$ is an ascent of the permutation p, let the edge a_i be horizontal and the label e_i be $i + 1 - q_i$.*

Moreover, this bijection restricts naturally to a bijection between $\mathcal{A}(n,k)$ and $\mathcal{P}(n,k)$ for $0 \leq k \leq n-1$.

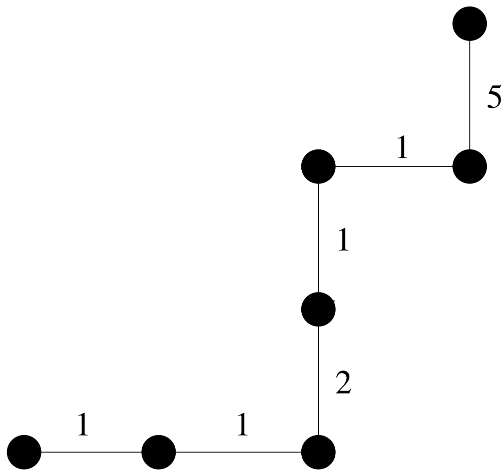

FIGURE 1.3
The image of the permutation 243165.

PROOF
The described map is clearly injective. Let us assume that $i-1$ and i are both descents of the permutation p. Let q, respectively r, be the permutation when restricted to the i, respectively $i+1$, first elements. Observe that q_i is either r_i or $r_i - 1$. Since $r_i > r_{i+1}$ we have $q_i \geq r_{i+1}$ and condition (ii) is satisfied in this case. By similar reasoning the three remaining cases (based on $i-1$ and i being ascents or descents) are shown, hence the map is into the set $\mathcal{P}(n)$.

To see that this is a bijection, we show that we can recover the permutation p from its image. To that end, it is sufficient to show that we can recover p_n, and then use induction on n for the rest of p. To recover p_n from its image, simply recall that p_n is equal to the label ℓ of the last edge if that edge is vertical, and to $n+1-\ell$ if that edge is horizontal. Conditions (ii) and (iii) assure that this way we always get a number between 1 and n for p_n. ∎

See Figure 1.3 for an example of this bijection.

Now we are in position to prove that the Eulerian numbers are log-concave.

PROOF (of Theorem 1.27). We construct an *injection*

$$\Phi : \mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1) \longrightarrow \mathcal{P}(n, k) \times \mathcal{P}(n, k).$$

This injection Φ will be defined differently on different parts of the domain.

Let $(P, Q) \in \mathcal{P}(n, k-1) \times \mathcal{P}(n, k+1)$. Place the initial points of P and Q at $(0, 0)$ and $(1, -1)$, respectively. Then the endpoints of P and Q are

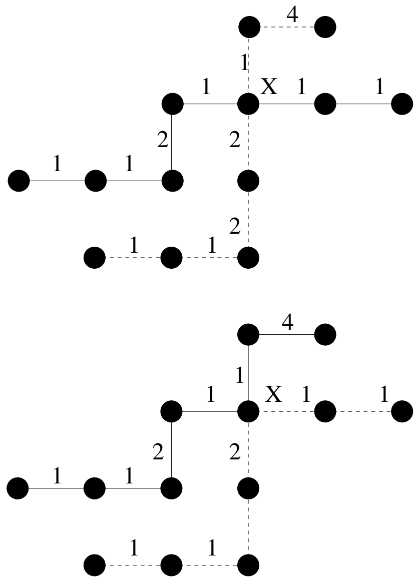

FIGURE 1.4
The new pair of paths.

$(n-k+1, k-1)$ and $(n-k, k)$, respectively, so while Q starts "below" P, it ends "above" P.

Let X be the *first* (most southwestern) common point of P and Q. It then follows that P arrives to X by an east step, and Q arrives to X by a north step. We will now show how to proceed if neither P nor Q changes directions at X, that is, P leaves X by an east step, and Q leaves X by a north step. The other cases are very similar and are left as exercises. Essentially, in all of the other cases, one of the transformations discussed below will have the desired effect when applied appropriately.

Decompose $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where P_1 is a path from $(0,0)$ to X, P_2 is a path from X to $(n-k, k)$, Q_1 is a path from $(1,-1)$ to X, and Q_2 is a path from X to $(n-k+1, k-1)$. Let a, b, c, d be the labels of the four edges adjacent to X as shown in Figure 1.5, the edges AX and XB originally belonging to P and the edges CX and XD originally belonging to Q. Then by condition (ii) we have $a \geq b$ and $c \geq d$. Let $P' = P_1 \cup Q_2$ and let $Q' = Q_1 \cup P_2$.

1. If P' and Q' are valid paths, that is, if their labeling fulfills conditions (i)–(iii), then we set $\Phi(P,Q) = (P', Q')$. See Figure 1.4 for this construction. This way we have defined Φ for pairs $(P,Q) \in \mathcal{P}(n,k) \times \mathcal{P}(n,k)$ in which $a + d \leq i$ and $b + c \leq i$, where $i - 1$ is the sum of the two coordinates of X. We also point out that we have not changed any labels, therefore in (P', Q') we still have $a \geq b$ and $c \geq d$, though that is no

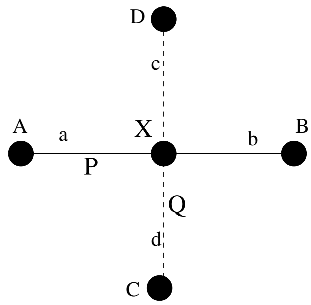

FIGURE 1.5
Labels around the point X.

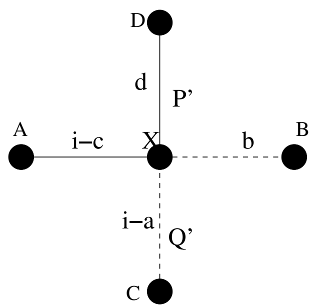

FIGURE 1.6
New labels around the point X.

longer required as the edges in question are no longer parts of the same path.

It is clear that $\Phi(P,Q) = (P',Q') \in \mathcal{P}(n,k) \times \mathcal{P}(n,k)$, (in particular, (P',Q') belongs to the subset of $\mathcal{P}(n,k) \times \mathcal{P}(n,k)$ consisting of *intersecting* pairs of paths), and that Φ is one-to-one.

2. We still have to define $\Phi(P,Q)$ for those pairs $(P,Q) \in \mathcal{P}(n,k-1) \times \mathcal{P}(n,k+1)$ for which it cannot be defined in the way it was defined in the previous case, that is, when either $a+d > i$ or $b+c > i$ holds. The reader is invited to verify that such pairs (P,Q) actually exist. One example is when P is the path belonging to the permutation 1237654, and Q is the path belonging to the permutation 4567123.

   Change the label of the edge AX to $i-c$ and change the label of the edge CX to $i-a$ as seen in Figure 1.6, then proceed as in the previous case to get $\Phi(P,Q) = (P',Q')$, where $P' = P_1 \cup Q_2$ and $Q' = Q_1 \cup P_2$.

   We claim that P' and Q' are valid paths. Indeed we had at least one of $a+d > i$ and $b+c > i$, so we must have $a+c > i$ as $a \geq b$ and $c \geq d$. Therefore, $i-a < c$ and $i-c < a$, so we have decreased the values of the labels of edges AX and CX, and that is always possible as shown in Proposition 1.28. Moreover, no constraints are violated in P' and Q' by the edges adjacent to X as $i-c+d \leq i$ and $i-a+b \leq i$. It is also clear that Φ is one-to-one on this part of the domain, too. Finally, we have to show that the image of this part of the domain is disjoint from that of the previous part. This is true because in this part of the domain we have at least one of $a+d > i$ and $b+c > i$, that is, at least one of $i-c < b$ and $i-a < d$, so in the image, at least one of the pairs of edges AX, XB and CX, XD does not have the property that the label of the first edge is at least as large as that of the second one. And, as pointed out in the previous case, all elements of the image of the previous part of the domain do have that property.

Given $\Phi(P,Q) = (P', Q')$, the vertex X can be uniquely determined as the most southwestern point of the intersection of P' and Q'.

It then follows that the map Φ we created is an injection. This proves the inequality

$$A(n,k-1)A(n,k+1) \leq A(n,k)^2,$$

so our theorem is proved. ▮

There is a property of sequences of positive real numbers that is even stronger than log-concavity.

DEFINITION 1.30 *Let $a_1, a_2, \cdots, a_n$ be a sequence of positive real numbers. We say that this sequence* has real roots only *or* real zeros only *if the polynomial $\sum_{i=1}^n a_i x^i$ has real roots only.*

We note that sometimes the sequence can be denoted $a_0, a_1, \cdots, a_n$, and sometimes it is better to look at the polynomial $\sum_{i=0}^n a_i x^i$ (which, of course, has real roots if and only if $\sum_{i=0}^n a_i x^{i+1}$ does).

Example 1.31
For all positive integers n, the sequence $a_0, a_1, \cdots, a_n$ defined by $a_i = \binom{n}{i}$ has real zeros only. ▯

SOLUTION We have $\sum_{i=0}^n a_i x^i = \sum_{i=0}^n \binom{n}{i} x^i = (1+x)^n$, so all roots of our polynomial are equal to -1. ▮

Having real zeros is a stronger property than being log-concave, as is shown by the following theorem of Newton.

THEOREM 1.32
If a sequence of positive real numbers has real roots only, then it is log-concave.

PROOF Let $a_0, a_1, \cdots, a_n$ be our sequence, and let $P(x) = \sum_{k=0}^{n} a_k x^k$. Then for all roots (x, y) of the polynomial $Q(x, y) = \sum_{k=0}^{n} a_k x^k y^{n-k}$, the ratio (x/y) must be real. (Otherwise x/y would be a non-real root of $P(x)$). Therefore, by Rolle's Theorem, this also holds for the partial derivatives $\partial Q/\partial x$ and $\partial Q/\partial y$. Iterating this argument, we see that the polynomial $\partial^{a+b} Q/\partial x^a \partial y^b$ also has real zeros, if $a + b \leq n - 1$. In particular, this is true in the special case when $a = j - 1$, and $b = n - j - 1$, for some fixed j. This implies that the quadratic polynomial $R(x, y) = \partial^{n-2} Q/\partial x^{j-1} \partial y^{n-j-1}$ has real roots only, and therefore the discriminant of $R(x, y)$ is non-negative. On the other hand, we can compute $R(x, y)$ by computing the relevant partial derivatives. Note that we only have to look at the values of k ranging from $j - 1$ to $j + 1$ as all other summands of $Q(x, y)$ vanish after derivation. We get

$$R(x,y) = a_{j-1} \cdot (j-1)! \frac{1}{2} (n-j+1)! y^2 + a_j j! (n-j)! xy + a_{j+1} (n-j-1)! \frac{1}{2} (j+1)!$$

As we said, this polynomial has to have a non-negative discriminant, meaning that

$$a_j^2 \geq \frac{j+1}{j} \cdot \frac{n-j+1}{n-j} \cdot a_{j-1} a_{j+1}, \tag{1.14}$$

which is stronger than our original claim, $a_j^2 \geq a_{j-1} a_{j+1}$. ∎

The alert reader has probably noticed that by (1.14), a log-concave sequence does not necessarily have real zeros only. For instance, the sequence 1, 1, 1 is certainly log-concave, but $1 + x + x^2$ has two complex roots.

One might ask why we would want to know whether a combinatorially defined sequence has real zeros or not. In certain cases, proving the real zeros property is the only, or the easiest, way to prove log-concavity and unimodality. In some cases, unimodality and log-concavity can be proved by other means, but that does not always tell us where the maximum or maxima of a given sequence is, or just how many maxima the sequence has. Note that a constant sequence is always log-concave, so a log-concave sequence could possibly have any number of maxima. The following Proposition shows that in a sequence with real zeros only, the situation is much simpler.

PROPOSITION 1.33
If the sequence $\{a_k\}_{0 \leq k \leq n}$ has real zeros only, then it has either one or two maximal elements.

PROOF Formula (1.14) shows that in such a sequence, the ratio a_{j+1}/a_j strictly decreases, so it can be equal to 1 for at most one index j. ∎

Theorem 3.25 will show how to find the maximum (or maxima) of a sequence with real zeros.

The following theorem shows that Eulerian numbers have this last, stronger property as well.

THEOREM 1.34

For any fixed n, the sequence $\{A(n,k)\}_k$ of Eulerian numbers has real roots only. In other words, all roots of the polynomial

$$A_n(x) = \sum_{k=1}^{n} A(n,k)x^k$$

are real.

Recall that the polynomials $A_n(x)$ of Theorem 1.34 are called the *Eulerian polynomials.* This theorem is a classic result, but surprisingly, it is not easy to find a full, self-contained proof for it in the literature. The ideas of the proof we present here are due to Herb Wilf and Aaron Robertson.

PROOF (of Theorem 1.34) Theorem 1.7 implies

$$A_n(x) = (x - x^2)A'_{n-1}(x) + nxA_{n-1}(x) \qquad (n \geq 1; A_0(x) = x).$$

Indeed, the coefficient of x^k on the left-hand side is $A(n,k)$, while the coefficient of x^k on the right-hand side is

$$kA(n-1,k) - (k-1)A(n-1,k-1) + nA(n-1,k-1) =$$

$$kA(n-1,k) + (n-k+1)A(n-1,k-1) = A(n,k).$$

Now note that the right-hand side closely resembles the derivative of a product. This suggests the following rearrangement:

$$A_n(x) = x(1-x)^{n+1}\frac{d}{dx}\left\{(1-x)^{-n}A_{n-1}(x)\right\} \tag{1.15}$$

with $n \geq 1$ and $A_0(x) = x$.

The Eulerian polynomial $A_0(x) = x$ vanishes only at $x = 0$. Suppose, inductively, that $A_{n-1}(x)$ has $n-1$ distinct real zeros, one at $x = 0$, and the others negative. From (1.15), or otherwise, $A_n(x)$ vanishes at the origin. Further, by Rolle's Theorem, (1.15) shows that $A_n(x)$ has a root between each pair of consecutive roots of $A_{n-1}(x)$. This accounts for $n-1$ of the roots of $g_n(x)$. Since we have accounted for all but one root, the remaining last root must be real since complex roots of polynomials with real coefficients come in conjugate pairs. ∎

We mention that an elementary survey of unimodal, log-concave, and real-roots-only sequences can be found in [35]. The articles [75] and [236] are high-level survey papers.

Eulerian numbers can count permutations according to properties other than descents. Let $p = p_1p_2\cdots p_n$ be a permutation. We say that i is an *excedance* of p if $p_i > i$. (Note that for this definition, it is important to require that the entries of p are the elements of $[n]$ and not some other n-element set.)

Example 1.35
The permutation 24351 has three excedances, 1, 2, and 4. Indeed, $p_1 = 2 > 1$, $p_2 = 4 > 2$, and $p_4 = 5 > 4$. □

THEOREM 1.36
The number of n-permutations with $k-1$ excedances is $A(n,k)$.

We postpone the proof of this theorem until Section 3.3.2, where it will become surprisingly easy, due to a different way of looking at permutations. However, we mention that if $f : S_n \to \mathbf{N}$ is a function associating natural numbers to permutations, then it is often called a *permutation statistic.* (Recall from the Introduction that S_n denotes the set of all n-permutations.) If a permutation statistic f has the same distribution as the statistic "number of descents", that is, if for all $k \in [n]$, we have

$$|\,\{p \in S_n : f(p) = k\}\,| = |\,\{p \in S_n : d(p) = k\}\,|, \tag{1.16}$$

then we say that f is an *Eulerian* statistic. So Theorem 1.36 says that "number of excedances," sometimes denoted by *exc*, is an Eulerian statistic. We will see further Eulerian statistics in the Exercises section.

1.2 Alternating Runs

Let us modify the notion of ascending runs that we discussed in the last section. Let $p = p_1p_2\cdots p_n$ be a permutation. We say that p changes direction at position i if either $p_{i-1} < p_i > p_{i+1}$, or $p_{i-1} > p_i < p_{i+1}$. In other words, p changes directions when p_i is either a *peak* or a *valley.*

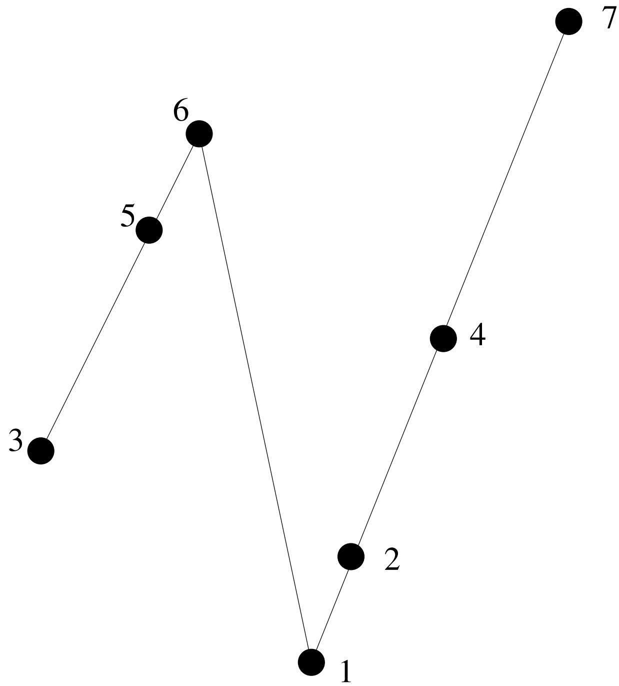

FIGURE 1.7
Permutation 3561247 has three alternating runs.

DEFINITION 1.37 *We say that p has k* alternating runs *if there are $k-1$ indices i so that p changes direction at these positions.*

For example, $p = 3561247$ has 3 alternating runs as p changes direction when $i = 3$ and when $i = 4$. A geometric way to represent a permutation and its alternating runs by a diagram is shown in Figure 1.7. The alternating runs are the line segments (or edges) between two consecutive entries where p changes direction. So a permutation has k alternating runs if it can be represented by k line segments so that the segments go "up" and "down" exactly when the entries of the permutation do.

The origins of this line of work go back to the nineteenth century. More recently, D. E. Knuth [183] has discussed the topic in connection to sorting and searching.

Let $G(n,k)$ denote the number of n-permutations having k alternating runs. There are significant similarities between these numbers and the Eulerian numbers. For instance, for fixed n, both sequences have real zeros only, and both satisfy similar recurrence relations. However, the sequence of the $G(n,k)$ is not symmetric. On the other hand, almost half of all roots of the generating function $G_n(x) = \sum_{p\in S_n} x^{r(p)} = \sum_{k\geq 1} G(n,k)x^k$ are equal to -1. Here $r(p)$

2				
2	4			
2	12	10		
2	28	58	32	
2	60	236	300	122

FIGURE 1.8
The values of $G(n,k)$ for $n \leq 6$. The first value of row n is $G(n,1)$. The NE–SW diagonals contain the values of $G(n,k)$ for fixed k.

denotes the number of alternating runs of p.

First we prove a simple recurrence relation on the numbers $G(n,k)$, which was first proved by André in 1883.

LEMMA 1.38
For positive integers n and k we have

$$G(n,k) = kG(n-1,k) + 2G(n-1,k-1) + (n-k)G(n-1,k-2), \quad (1.17)$$

where we set $G(1,0) = 1$, and $G(1,k) = 0$ for $k > 0$.

PROOF Let p be an $(n-1)$-permutation having k alternating runs, and let us try to insert n into p without increasing the number of alternating runs. We can achieve that by inserting n at one of k positions. These positions are right before the beginning of each descending run, and right after the end of each ascending run. This gives us $kG(n-1,k)$ possibilities.

Now let q be an $(n-1)$-permutation having $k-1$ alternating runs. We want to insert n into q so that it increases the number of alternating runs by 1. We can achieve this by inserting n into one of two positions. These two positions are very close to the beginning and the end of q. Namely, if q starts in an ascending run, then insert n to the front of q, and if q starts in a descending run, then insert n right after the first entry of q. Proceed dually at the end of the permutation.

Finally, let r be an $(n-1)$-permutation having $k-2$ alternating runs, and observe that by inserting n into any of the remaining $n-(k-2)-2 = n-k$ positions, we increase the number of alternating runs by two. This completes the proof. ∎

The first values of $G(n,k)$ are shown in Figure 1.8 for $n \leq 6$.

Looking at these values of $G(n,k)$, we note they are all even. This is easy to explain as p and its reverse always have the same number of alternating runs.

Taking a second look at the polynomials $G_n(x)$, we note that $G_4(x) =$

$(x+1)(10x^2+2x)$, and that

$$G_5(x) = 32x^4 + 58x^3 + 28x^2 + 2x = (x+1)(32x^3 + 26x^2 + 2x).$$

Further analysis shows that $G_6(x)$ and $G_7(x)$ are divisible by $(x+1)^2$, and that $G_8(x)$ and $G_9(x)$ are divisible by $(x+1)^3$, and so on. In general, it seems that for any positive integer $n \geq 4$, the polynomial $G_n(x)$ is divisible by $(x+1)^{\lfloor (n-2)/2 \rfloor}$.

This is an interesting observation, and one that is certainly of combinatorial flavor. For instance, if we just wanted to prove that $G_n(x)$ is divisible by $x+1$, we could proceed as follows. We could arrange our permutations into pairs, so that each pair consists of two permutations, one with r alternating runs, and one with $r+1$ alternating runs. If we could do that, that would imply that $G_n(x) = (1+x)F_n(x)$. Here $F_n(x)$ is the generating function by the number of alternating runs for the set of permutations that consists of *one* element from each pair, the one with the smaller number of alternating runs. If we can appropriately "iterate" this argument, then we will succeed in proving that $G_n(x)$ is divisible by a power of $(1+x)$.

Before we start proving the claim that -1 is a root of $G_n(x)$ with a high multiplicity, we point out that one might also wonder whether the polynomials $G_n(x)/2(x+1)^j$ have some natural combinatorial interpretation for each index $j \leq \lfloor (n-2)/2 \rfloor$. Our proof provides such an interpretation. In order to give that proof, we need the following definitions that were first introduced in [37].

DEFINITION 1.39 *For $j \leq m = \lfloor (n-2)/2 \rfloor$, we say that p is a j-*half-ascending permutation *if, for all positive integers $i \leq j$, we have $p_{n+1-2i} < p_{n+2-2i}$. If $j = m$, then we will simply say that p is a* half-ascending permutation.

So p is a 1-half-ascending permutation if $p_{n-1} < p_n$. In a j-half-ascending permutation, we have j constraints, and they involve the rightmost j disjoint pairs of entries. We call these permutations half-ascending because at least half of the involved positions are ascents.

Now we define a modified version of the polynomials $G_n(x)$ for j–half-ascending permutations. As we will see, one of these polynomials will provide the desired combinatorial interpretation for $G_n(x)/(1+x)^m$.

DEFINITION 1.40 *Let p be a $(j+1)$-half-ascending permutation. Let $r_j(p)$ be the number of alternating runs of the substring $p_1, p_2, \ldots, p_{n-2j}$, and let $s_j(p)$ be the number of descents of the substring $p_{n-2j}, p_{n+1-2j}, \ldots, p_n$. Denote $t_j(p) = r_j(p) + s_j(p)$, and define*

$$G_{n,j}(x) = \sum_{p \in S_n} x^{t_j(p)}.$$

So in other words, $G_{n,j}$ enumerates the alternating runs in the non-half-ascending part and the first two elements of the half-ascending part, and the descents in the rest of the half-ascending part.

LEMMA 1.41
For all $n \geq 4$ and $1 \leq j \leq \lfloor (n-2)/2 \rfloor$, we have

$$\frac{G_n(x)}{2(x+1)^j} = G_{n,j}(x).$$

PROOF We prove the statement by induction on j. Let $j = 1$. Clearly, we can restrict our attention to the set of permutations in which $p_{n-3} < p_{n-2}$. Indeed, if p does not satisfy that condition, then its complement p^c will, and vice versa (where p^c is the n-permutation whose ith entry is $n+1-p_i$), and p and p^c certainly have the same number of alternating runs.

Let I be the involution acting on the set of all n-permutations (that satisfy the inequality $p_{n-3} < p_{n-2}$) that swaps the last two entries of each permutation. For instance, $I(5613427) = 5613472$. It is then straightforward to verify that I either increases the number of alternating runs by one, or it decreases it by one. Therefore, I is just the involution we were looking for. Indeed, we have

$$\frac{1}{2}G_n(x) = \sum_{P(p)} x^{r(p)} + x^{r(p)+1} = \sum_{P(p)} (x+1)x^{r(p)},$$

where P ranges through all $n!/4$ pairs created by the involution I, and $P(p)$ is the permutation in P that has the *smaller* number of alternating runs. By verifying all (essentially, two, see the example below) possible cases, we see that for all these $n!/4$ permutations p, the following occurs. The number $r(p)$ equals the number $t_1(q)$ of the permutation $P(q)$ in the pair P that is in the same pair as p *and ends in an ascent.* Therefore, the last equality implies

$$\frac{1}{2}G_n(x) = \sum_{P(q)} (x+1)x^{t_1(q)} = (x+1)G_{n,1}(x),$$

where P again ranges the $n!/4$ pairs created by I. Therefore, the initial case is proved.

Figure 1.9 shows the twelve 4-permutations for which $p_1 < p_2$ holds, in pairs formed by I. The values $r(p)$ and $t_1(p)$ are shown as well. One then verifies that in each of these pairs, the permutation with the smaller number of alternating runs has a number of alternating runs equal to the $t_1(p)$-value of the element of that pair in which $p_3 < p_4$. This argument carries over for $n > 4$. Indeed, I has no effect on the number of alternating runs of the substring of the first $n-4$ entries of p.

Now let us assume that we know that the statement holds for $j-1$ and prove it for j. Apply I to the two rightmost entries of our permutations

1234 r(p)=1 t_1(p)=1	1243 r(p)=2
1324 r(p)=3 t_1(p)=2	1342 r(p)=2
1423 r(p)=3 t_1(p)=2	1432 r(p)=2
2314 r(p)=3 t_1(p)=2	2341 r(p)=2
2413 r(p)=3 t_1(p)=2	2431 r(p)=2
3412 r(p)=3 t_1(p)=2	3421 r(p)=2

FIGURE 1.9
The values of $r(p)$ and $t_1(p)$ for $n = 4$.

to get pairs as in the initial case, and apply the induction hypothesis to the leftmost $n-2$ elements. By the induction hypothesis, the string of the leftmost $n-2$ elements can be replaced by a j-half-ascending $(n-2)$-permutation, and the number of runs can be replaced by the t_{j-1}-parameter. In particular, $p_{n-3} < p_{n-2}$ will hold, and therefore we can verify that our statement holds in both cases ($p_{n-2} < p_{n-1}$ or $p_{n-2} > p_{n-1}$) exactly as we did in the proof of the initial case. ∎

So almost half of the roots of $G_n(x)$ are equal to -1; in particular, they are real numbers. This raises the question whether the other half are real numbers as well. That question has recently been answered in the affirmative by Herb Wilf [270]. In his proof, he used the rather close connections between Eulerian polynomials, and the generating functions $G_n(x) = \sum_{k\geq 1} G(n,k)x^k$. This connection, established in [101], and given in a more concise form in [183], can be described by

$$G_n(x) = \left(\frac{1+x}{2}\right)^{n-1} (1+w)^{n+1} A_n\left(\frac{1-w}{1+w}\right), \tag{1.18}$$

where $w = \sqrt{\frac{1-x}{1+x}}$. The proof of (1.18) uses the similarities between the recurrence relations for $A_n(x)$ and $G_n(x)$ to get a differential equation satisfied by certain generating functions in two variables. The details can be found in [101], pages 157–162.

THEOREM 1.42
(H. Wilf [270].) For any fixed n*, the polynomial* $G_n(x)$ *has real roots only.*

PROOF From (1.18) it follows that $G_n(x)$ can vanish only if either $x = -1$ or $x = 2y/(1+y^2)$, where y is a zero of A_n. Indeed, if y is a root of $A_n(x)$ and $y = \frac{1-w}{1+w}$, then $w = \frac{1-y}{1+y}$. Therefore,

$$\sqrt{\frac{1-x}{1+x}} = \frac{1-y}{1+y}.$$

Squaring both sides and solving for x, we get our claim. As we know that the roots of A_n are real, our statement is proved. ∎

It is possible to continue our argument involving half-ascending permutations to give a fully combinatorial proof of the weaker statement that $G_n(x)$ is always log-concave. Let $m = \lfloor (n-2)/2 \rfloor$. Then we know from Lemma 1.41 that

$$G_n(x) = 2(1+x)^m G_{n,m}(x). \tag{1.19}$$

As the polynomial $(1+x)^m$ is obviously log-concave, and the product of log-concave polynomials is log-concave (see Exercise 22), the log-concavity of $G_n(x)$ will be proved if we can prove the following Lemma.

LEMMA 1.43
For all integers $n \geq 4$*, the polynomial* $G_{n,m}(x)$ *has log-concave coefficients.*

We prove the lemma for even values of n. See Exercise 27 for the necessary modifications for odd n. The following Proposition is obvious.

PROPOSITION 1.44
Let n *be an even positive integer. Let* p *be a half-ascending* n*-permutation. Then* p *has* $2k+1$ *runs if and only if* p *has* k *descents, or, in other words, when* $t(p) = k+1$.

PROOF (of Lemma 1.43). The reader is asked to review the proof of Theorem 1.27. In that proof, the log-concavity of the Eulerian numbers was established by an injective map Φ. This map Φ acted on pairs of lattice paths that corresponded to pairs of permutations. Now note that in that lattice path representation of permutations, half-ascending permutations correspond to lattice paths in which all even-indexed steps are horizontal. Observe that Φ preserves this property; that is, the restriction of Φ to the set of pairs of *half-ascending* permutations in $\mathcal{A}(n, k-1) \times \mathcal{A}(n, k+1)$ is an injection into the set of pairs of *half-ascending* permutations in $\mathcal{A}(n,k) \times \mathcal{A}(n,k)$. This, together with Proposition 1.44, proves our claim. $\blacksquare$

1.3 Alternating Subsequences

1.3.1 Definitions and a Recurrence Relation

The concept of alternating subsequences in permutations was introduced by Richard Stanley in [243]. In this section, we describe some of the major results of this recently developed subject, and we explain the very close connection between alternating subsequences and alternating runs.

DEFINITION 1.45 *An* alternating subsequence *in a permutation* $p = p_1p_2\cdots p_n$ *is a subsequence* $p_{i_1}p_{i_2}\cdots p_{i_k}$ *so that*

$$p_{i_1} > p_{i_2} < p_{i_3} > p_{i_4} < \cdots .$$

Similarly, a reverse alternating subsequence *in* p *is a subsequence* $p_{j_1}p_{j_2}\cdots p_{j_k}$ *so that*

$$p_{j_1} < p_{i_2} > p_{i_3} < p_{i_4} > \cdots .$$

The length of the longest alternating subsequence of p is denoted by as(p).

For instance, if $p = 3416527$, then 3165, 31657, and 427 are examples of alternating subsequences of p.

Example 1.46

If $p = 35714268$, then as(p) $= 5$. Indeed, 31426 is an alternating subsequence of p of length 5. On the other hand, no alternating subsequence of p can contain more than one of the first three entries of p, and no alternating subsequence can contain more than two of the last three entries of p. Therefore, no alternating subsequence of p can be longer than five. $\square$

Interestingly, the parameter as(p) is much easier to handle in many aspects than its older and more studied brother, the length of the longest *increasing* subsequence of p. (We will study the latter in Chapter 4 and various later chapters.) The reason for this is that as(p) is more conducive to arguments using recurrences, due to the following fact.

PROPOSITION 1.47

Let p *be an* n*-permutation. Then* p *contains an alternating subsequence of length* as(p) *that contains the entry* n.

PROOF Let us assume the contrary, that is, that there exists an n-permutation p so that each maximum-length alternating subsequence of p avoids the entry n. This means that all maximum-length alternating subsequences of p start on the right of n, or end on the left of n, or "skip" n; that is, contain an entry on the left of n that is followed by an entry on the right of n. It is easy to see that each of these three kinds of subsequences can be transformed into an alternating subsequence of the same length that contains n. For instance, if s is an alternating subsequence of p so that $a < b$ are two consecutive entries in s, and a is on the left of n and b is on the right of n, then we can replace b by n in s, and obtain the alternating subsequence s' that is still of maximum length. If $b < a$, then we can replace a by n to get the maximum length alternating subsequence s'' that contains n. Similarly, if t is a maximum length alternating subsequence that starts on the right of n, then the first entry of t can simply be replaced by n to create a maximum length alternating subsequence containing n. Finally, if u is a maximum length alternating subsequence ending on the left of n, then u has to end in an ascent, otherwise n could be appended to the end of u to get a longer alternating subsequence, which would be a contradiction. So u ends in an ascent, but then the last entry of u can be replaced by n. ∎

The following definitions set the framework for our efforts to count permutations according to their alternating subsequences.

DEFINITION 1.48 *Let $a_k(n)$ be the number of permutations p of length n whose longest alternating subsequence is of length k. Set $a_0(0) = 1$. Furthemore, let $b_k(n) = \sum_{i=1}^{k} a_k(n)$ be the number of permutations of length n with no alternating subsequences longer than k.*

Note that $a_k(n) = b_k(n) - b_{k-1}(n)$.

Our first theorem on the subject of alternating subsequences turns the observation of Proposition 1.47 into an enumeration formula.

THEOREM 1.49
[243] Let $1 \leq k \leq n+1$. Then

$$a_k(n+1) = \sum_{j=0}^{n} \binom{n}{j} \sum_{2r+s=k-1} (a_{2r}(j) + a_{2r+1}(j))a_s(n-j),$$

where r and s range non-negative *integers satisfying $2r + s = k - 1$.*

PROOF We are going to build an $(n+1)$-permutation p that has a longest alternating subsequence of length k. Let us place the entry $n+1$ into the $(j+1)$st position of p; let L be the subsequence of the first j entries of p

(since these are the entries on the left of p), and let R be the subsequence of the last $n-j$ entries of p.

It goes without saying that there are $\binom{n}{j}$ ways to choose the set of entries in L.

Let us now focus on a longest alternating subsequence *long* of $p = L(n+1)R$ containing $n+1$. Then *long* intersects L in an alternating subsequence *alts* of length $2r$ and R in a reverse alternating subsequence *rev* of length $s = k-1-2r$. Clearly, no reverse alternating subsequence of R can be longer than *rev*. Therefore, as(rev) = s, so there are $a_s(n-j)$ choices for R. The situation for L is a little bit more complicated. Indeed, *alts* may or may not be the longest alternating subsequence of L. If *alts* is of even length, then it is possible that *alts* can be extended by another entry within L, but then that entry cannot be followed by an ascent (and so by the entry $n+1$) in any alternating subsequences. Therefore, as(alts) = 2r or as(alts) = 2r + 1, and so the number of choices for L is $a_{2r}(j) + a_{2r+1}(j)$, completing the proof. ∎

Theorem 1.49 can be turned into generating function identities, which in turn can be turned into explicit formulae. The interested reader can find the details in [243].

THEOREM 1.50
For all positive integers $k \leq n$ we have

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{r+2s\leq k \\ r\equiv k \pmod 2}} (-2)^s \binom{k-s}{(k+r)/2}\binom{n}{s} r^n.$$

As $a_k(n) = b_k(n) - b_{k-1}(n)$, formulae for $a_k(n)$ can be obtained from the preceding theorem.

For small values of k, Theorem 1.50 yields simple formulae such as

1. $b_1(n) = 1$,
2. $b_2(n) = 2^{n-1}$,
3. $b_3(n) = \frac{3^n-(2n-3)}{4}$,
4. $b_4(n) = \frac{4^n-2(n-2)2^n}{8}$,
5. $b_5(n) = \frac{5^n-(2n-5)3^n+2(n^2-5n+5)}{16}$,
6. $b_6(n) = \frac{6^n-2(n-3)4^n+(2n^2-12n+15)2^n}{32}$.

1.3.2 Alternating Runs and Alternating Subsequences

The length of the longest alternating subsequence of a permutation is closely connected to the number of alternating runs as shown by the following proposition.

PROPOSITION 1.51
Let $n \geq 2$. Then $a_k(n) = \frac{1}{2}(G(n,k-1) + G(n,k))$.

PROOF If an n-permutation p has i alternating runs and starts in a descent, then $\mathrm{as(p) = i + 1}$ as can be seen by considering the entries of p that are peaks or valleys, as well as the first and last entry of p. It follows from the pigeon-hole principle that p cannot contain a longer alternating subsequence. If p starts in an ascent, then $\mathrm{as(p) = i}$ by similar considerations.

Therefore, the n-permutations p satisfying $\mathrm{as(p) = k}$ are precisely the n-permutations with $k-1$ alternating runs starting in a descent and the n-permutations with k alternating runs starting in an ascent. ∎

Proposition 1.51 implies that if $T_n(x) = \sum_{p \in S_n} x^{\mathrm{as(p)}}$, then

$$T_n(x) = \frac{1}{2}(1+x)G_n(x).$$

So the polynomials $T_n(x)$ have real roots only, and $\lfloor n/2 \rfloor$ of their roots are equal to -1.

The first few polynomials $T_n(x)$ are shown below.

1. $T_1(x) = x$,
2. $T_2(x) = x + x^2$,
3. $T_3(x) = x + 3x^2 + 2x^3$,
4. $T_4(x) = x + 7x^2 + 11x^3 + 5x^4$,
5. $T_5(x) = x + 15x^2 + 43x^3 + 45x^4 + 16x^5$,
6. $T_6(x) = x + 31x^2 + 148x^3 + 268x^4 + 211x^5 + 61x^6$,
7. $T_7(x) = x + 63x^2 + 480x^3 + 1344x^4 + 1767x^5 + 1113x^6 + 272x^7$.

1.3.3 Alternating Permutations

Sometimes an entire permutation is an alternating sequence, leading to the following definition.

DEFINITION 1.52 *We say that the n-permutation p is* alternating *if the longest alternating subsequence of p is of length n. Similarly, we say that*

p is reverse alternating *if the longest reverse alternating subsequence of p is of length n.*

For instance, 312 and 5241736 are alternating permutations. Clearly, p is alternating if and only if its *complement*, that is, the n-permutation whose ith entry is $n+1-p_i$, is reverse alternating.

The number of alternating n-permutations is called an *Euler* number (not to be confused with the Eulerian numbers $A(n,k)$) and is denoted by E_n. The reader is invited to verify that $E_2 = 1$, $E_3 = 2$, $E_4 = 5$, and $E_5 = 16$. The Euler numbers have a very interesting exponential generating function. This is the content of the next theorem.

THEOREM 1.53
Set $E_0 = 1 = E_1$. Then we have

$$E(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

PROOF Let $L(n+1)R$ be an alternating or reverse alternating permutation of length $n+1$. So L is the string on the left of the maximal entry, and R is the string on the right of the maximal entry. Then R is reverse alternating, and so is L^r, that is, the reverse of L.

This observation leads to the recurrence relation

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k},$$

for $n \geq 1$. In terms of generating functions, this is equivalent to

$$2E'(x) = E^2(x) + 1,$$

with $E(0) = 1$.

The claim of the Theorem is now proved by verifying that $\sec x + \tan x$ is indeed a solution to this initial value problem, and noting that the solution of this initial value problem is unique. ∎

We point out that $\sec x$ is an even function and $\tan x$ is an odd function, so $\tan x$ is the exponential generating function for the Euler numbers with odd indices, and $\sec x$ is the exponential generating function for the Euler numbers with even indices. The numbers E_{2n} are often called the *secant numbers* and the numbers E_{2n+1} are often called the *tangent numbers*. There are various other combinatorial objects that are counted by the Euler numbers. Exercises 15 and 46 show two of them.

Alternating permutations have many fascinating properties, and we will return to those in upcoming chapters.

Exercises

1. (–) Simplify the formula obtained for $\alpha(S)$ in Lemma 1.3.

2. Let $p+1$ be a prime. What can be said about $A(p,k)$ modulo $p+1$?

3. (–) Find an alternative proof for the fact that $A(n,k+1) = A(n,n-k)$.

4. (–) What is the value of $A'_n(1)$? Here $A_n(x)$ denotes the nth Eulerian polynomial.

5. Prove Proposition 1.26.

6. We have n boxes numbered from 1 to n. We run an n-step experiment as follows. In step i, we drop one ball into a box, chosen randomly from boxes labeled 1 through i. So during the entire experiment, n balls will be dropped. Let $B(n,k)$ be the number of experiments in which at the end, $k-1$ boxes are left empty. Prove that $B(n,k) = A(n,k)$.

7. Deduce Theorem 1.8 from Theorem 1.7.

8. (–) Prove that for all positive integers $k \leq n$, we have
$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

9. Prove that
$$A(n,k) = \sum_{h=k-1}^{n} (-1)^{h-k+1} \binom{h}{k-1} S(n,n-h) \cdot (n-h)!.$$

10. Let $p = p_1p_2\cdots p_n$ be a permutation, and let b_i be the number of indices $j < i$ so that $p_j > p_i$. Find a formula for the number $C(n,k)$ of arrays $(b_1, b_2, \cdots, b_n)$ obtained this way in which exactly k different integers occur. Note that the permutation statistic defined as above is often called the *Dumont statistic,* and its value on the permutation p is denoted by $dmc(p)$.

11. Prove that
$$A_n(x) = x\sum_{k=1}^{n} k!S(n,k)(x-1)^{n-k}.$$

12. (–) Prove that
$$\sum_{i=1}^{k} A(n,k) \leq k^n.$$

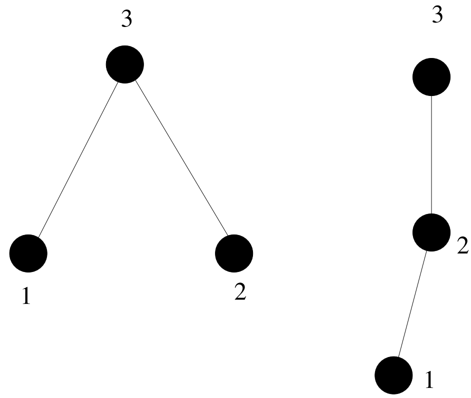

FIGURE 1.10
The two decreasing non-plane trees on vertex set [3].

13. We say that i is a *weak excedance* of $p = p_1 p_2 \cdots p_n$ if $p_i \geq i$. Assuming Theorem 1.36, prove that the number of n-permutations with k weak excedances is $A(n, k)$.

14. Prove that for all positive integers n, we have
$$S(n+1, k+1) = \sum_{m=k}^{n} \binom{n}{m} S(m, k).$$

15. A *decreasing non-plane tree* is a rooted tree on vertex set $[n]$ in which each non-leaf vertex has at most two children, and the label of each vertex is smaller than that of its parent. See Figure 1.10 for the two decreasing non-plane trees on vertex set [3]. Let T_n denote the number of decreasing non-plane trees on vertex set $[n]$. Prove that $T_n = E_n$.

16. Find a combinatorial proof for Corollary 1.19.

17. Let r be a positive integer, and let us say that $i \in [n-1]$ is an *r-fall* of the permutation $p = p_1 p_2 \cdots p_n$ if $p(i) \geq p(i+1) + r$. Let $A(n, k, r)$ denote the number of all n-permutations with $k-1$ such r-falls. The numbers $A(n, k, r)$ are called the r-Eulerian numbers. Prove that
$$A(n, k, r) = (k + r - 1)A(n-1, k, r) + (n + 2 - k - r)A(n-1, k-1, r).$$

18. Let $A(n, k, r)$ be defined as in the previous exercise. Prove that
$$A(n+r-1, k, r) = (k-1)! \sum_{i=0}^{n-k} (-1)^i \binom{n+r}{i} \binom{n+r-k-i}{r-1} (n-k-i+1)^n.$$

19. Are the r-Eulerian numbers defined in Exercise 17 and the l-Takács-Eulerian numbers defined in Problem Plus 3 identical? (Try to give a very short solution.)

20. Let k be a fixed positive integer. Find the ordinary generating function $F_k(x) = \sum_{n \geq k} S(n,k)x^n$.

21. Prove that for all positive integers n, we have

$$x^n = \sum_{m=0}^{n} S(n,m)(x)_m. \tag{1.20}$$

Recall that $(x)_m = x(x-1)\cdots(x-m+1)$.

22. Let $P(x)$ and $Q(x)$ be two polynomials with log-concave and positive coefficients. Prove that the polynomial $P(x)Q(x)$ also has log-concave coefficients.

23. Is it true that if $P(x)$ and $Q(x)$ are two polynomials with unimodal coefficients, then $P(x)Q(x)$ also has unimodal coefficients?

24. Is it true that if $P(x)$ and $Q(x)$ are two polynomials with *symmetric and unimodal* coefficients, then $P(x)Q(x)$ also has *symmetric and unimodal* coefficients?

25. (–)

    (a) Find an explicit formula for $G(n,2)$.

    (b) Find an explicit formula for $G(n,3)$.

    Here $G(n,k)$ is the number of n-permutations with k alternating runs.

26. (–) Let n be a fixed positive integer. For what pairs (k,m) does there exist an n-permutation with k descents and m alternating runs?

27. Prove Lemma 1.43 for odd values of n.

28. Prove that for n sufficiently large, we have $(k-1)^n < G(n,k) < k^n$, for all $k \geq 2$.

29. A sequence $f : \mathbf{N} \to \mathbf{C}$ is called P-recursive if there exist polynomials $P_0, P_1, \cdots, P_k \in \mathbf{Q}[n]$, with $P_k \neq 0$ so that

$$P_k(n+k)f(n+k) + P_{k-1}(n+k-1)f(n+k-1) + \cdots + P_0(n)f(n) = 0 \tag{1.21}$$

for all natural numbers n. Here P-recursive stands for "polynomially recursive". For instance, the function f defined by $f(n) = n!$ is P-recursive as $f(n+1) - (n+1)f(n) = 0$. Prove that for any fixed k, $A(n,k)$ is a P-recursive function of n.

30. Prove that for any fixed k, the function $S(n, k)$ is a polynomially recursive function of n.

31. A *decreasing binary tree* is a rooted binary plane tree that has vertex set $[n]$ and root n, and in which each vertex has 0, 1, or 2 children, and each child is smaller than its parent. Prove that the number of decreasing binary trees is $n!$.

32. Prove that the number of decreasing binary trees on $[n]$ in which $k-1$ vertices have a left child is $A(n, k)$.

33. Let $2 \leq i \leq n-1$. Recall that we say that p_i is a *peak* of the permutation $p = p_1 p_2 \cdots p_n$ if p_i is larger than both of its neighbors, that is $p_{i-1} < p_i$ and $p_i > p_{i+1}$. Let $n \geq 4$, and let $k \geq 0$. Find a formula for the number $Peak(n, k)$ of n-permutations having exactly k peaks.

34. How many decreasing binary trees are there on n vertices in which exactly one vertex has two children?

35. Use decreasing binary trees to prove that for any fixed n, the sequence of Eulerian numbers $\{A(n, k)\}_{1 \leq k \leq n}$ is symmetric and unimodal.

36. Which is the stronger requirement for two permutations, to have the same set of descents, or to have decreasing binary trees that are identical as unlabeled trees?

37. The *minmax tree* of a permutation $p_1 p_2 \cdots p_n$ is defined as follows. Let $p = umv$ where m is the *leftmost* of the minimum and maximum letters of p, u is the subword preceding m, and v is the subword following m. The *minmax tree* T_p^m has m as its root. The right subtree of T_p^m is obtained by applying the definition recursively to v. Similarly, the left subtree of T_p^m is obtained by applying the definition recursively to u. See Figure 1.11 for an example.

    (a) Prove that if $1 \leq i \leq n-2$, then there are $n!/3$ permutations p so that p_i is a leaf in T_p^m.

    (b) How many n-permutations p are there so that p_{n-1} (resp. p_n) is a leaf in T_p^m?

38. (+) Let $p = p_1 p_2 \cdots p_n$ be a permutation, and let $x \in [n]$. Define the x-factorization of p into the set of strings $u\lambda(x)x\gamma(x)v$ as follows. The string $\lambda(x)$ is the longest string of consecutive entries that are larger than x and are immediately on the left of x, and the string $\gamma(x)$ is the longest string of of consecutive entries that are larger than x and are immediately on the right of x. Finally, $u(x)$ and $v(x)$ are the leftover strings at the beginning and end of p. Note that each of $\lambda(x)$, $\gamma(x)$, u, and v can be empty.

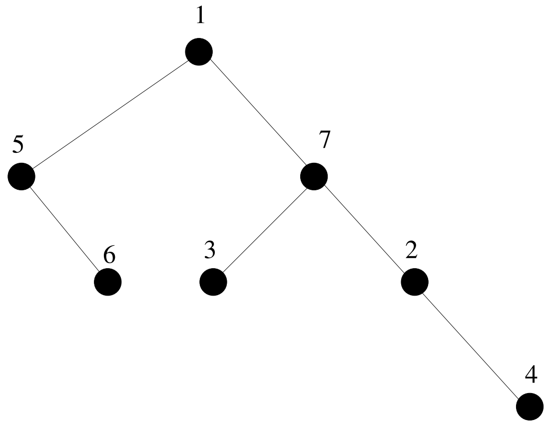

FIGURE 1.11
The minmax tree of $p = 5613724$.

For instance, if $p = 31478526$ and $x = 4$, then $u = 31$, $\lambda(x) = \emptyset$, $\gamma(x) = 785$, and $v = 26$.

The notion of *André permutations* proved to be useful in various areas of algebraic combinatorics. We say that p is an *André permutation of the first kind* if

(a) There is no i so that $p_i > p_{i+1} > p_{i+2}$, and

(b) $\gamma(x) = \emptyset \Longrightarrow \lambda(x) = \emptyset$, and

(c) if $\gamma(x)$ and $\lambda(x)$ are both nonempty, then $\max \lambda(x) < \max \gamma(x)$.

Prove that p is an André permutation of the first kind if and only if all non-leaf nodes of the minmax tree T_p^m are chosen because they are minimum (and not maximum) nodes.

39. Attach labels to the edges of a $k \times (n-k+1)$ square grid of points as shown in Figure 1.12. That is, both the edges of column i and row i get label i.

Take a northeastern lattice path s from the southwest corner to the northeast corner of the grid. This path s will consist of $n-1$ steps. Define the weight P_s of s as the product of the labels of all edges of s. Prove that

$$A(n,k) = \sum_s P_s,$$

where the sum is taken over all $\binom{n-1}{k-1}$ northeastern lattice paths s from the southwest corner to the northeast corner.

40. Prove, preferably by a combinatorial argument, that if $k < (n-1)/2$, then we have $G(n,k) \leq G(n,k+1)$. (Note that the sequence is growing

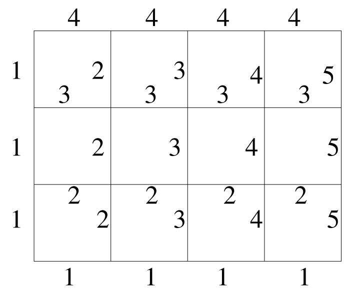

FIGURE 1.12
The labeled grid for $k = 4$ and $n = 8$.

even further than that, but we do not yet have the methods to prove it.)

41. Let r be a positive integer, and modify the labeling of the vertical edges in the previous exercise so that the label of the edges in column i is $i + r - 1$ instead of i. Prove that

$$A(n, k, r) = r! \sum_{s,r} P_s,$$

where $A(n, k, r)$ is an r-Eulerian number as defined in Exercise 17, the weight P_s of a path s is still the product of the labels of its edges, and the sum is taken on all $\binom{n-r}{k-1}$ northeastern lattice paths from $(0, 0)$ to $(k - 1, n - r - k)$.

42. (–) Let Q_n be the set of permutations of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ in which for all i, all entries between the two occurrences of i are larger than i. For instance, Q_2 has three elements, namely 1122, 1221, and 2211. Note that the elements of Q_n are called *Stirling permutations* of length $2n$. Prove that for $n \geq 2$, the set Q_n has $(2n - 1)!!$ elements.

43. Let $p = p_1 p_2 \cdots p_{2n} \in Q_n$ be a Stirling permutation as defined in the previous exercise. Let us say that $i \in \{1, \cdots, 2n\}$ is a descent of p if $a_i > a_{i+1}$ or $i = 2n$. Let us say that $i \in \{1, 2, \cdots, 2n - 1\}$ is a *plateau* of p if $a_i = a_{i+1}$.

    (a) Let $C_{n,i}$ be the number of elements of Q_n with i descents. Prove that then for all positive integers $n, i \geq 2$, the recurrence relation $C_{n,i} = iC_{n-1,i} + (2n - i)C_{n-1,i-1}$ holds.

    (b) Let $c_{n,i}$ be the number of elements of Q_n with i plateaux. Prove that then for all positive integers $n, i \geq 2$, the recurrence relation $c_{n,i} = ic_{n-1,i} + (2n - i)c_{n-1,i-1}$ holds.

    (c) Conclude that $C_{n,i} = c_{n,i}$ for all positive integers i and n, with $i \leq n$.

44. (+) Keep the notation of the previous exercise, and let

$$C_n(x) = \sum_{i=1}^{n} C_{n,i} x^i.$$

Prove that for all positive integers n, the roots of the polynomial $C_n(x)$ are all real, distinct, and non-positive.

45. (–) Find a direct combinatorial proof (no generating functions, no alternating runs) for the fact that for $n \geq 2$, the number as(p) is even for exactly half of all n-permutations p.

46. A permutation $p = p_1 p_2 \cdots p_n$ is called a *simsun* permutation if there exists no $k \leq n$ so that removing the entries larger than k from p, the remaining permutation has two descents in consecutive positions. For instance, $p = 35241$ is not simsun, since selecting $k = 3$ leads to the substring 321. Prove that the number of simsun permutations of length n is E_{n+1}.

Problems Plus

1. A *simplicial complex* is a collection Δ of subsets of a given set with the property that if $E \in \Delta$, and $F \subseteq E$, then $F \in \Delta$. The sets that belong to the collection Δ are called the *faces* of Δ. If $S \in \Delta$ has i elements, then we call S an $(i-1)$-dimensional face. The dimension of Δ is, by definition, the dimension of its maximal faces.

   Prove that there exists a simplicial complex Δ whose set of $(i-1)$-dimensional faces is in natural bijection with the set of n-permutations having exactly $i-1$ descents.

2. (a) Let T be a rooted tree with root 0 and non-root vertex set $[n]$. Define a vertex of T to be a descent if it is greater than at least one of its children. Prove that the number of forests of rooted trees on a given vertex set with $i+1$ leaves and j descents is the same as the number of forests of rooted trees with $j+1$ leaves and i descents.

   (b) Why is the above notion of descents a generalization of the notion of descents in permutations?

3. Define the l-Stirling numbers of the second kind by the recurrence

$$S(n+1, k, l) = S(n, k-1, l) + k^l S(n, k, l),$$

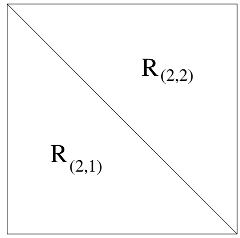

FIGURE 1.13
The regions $R_{n,k}$ for $n = 2$.

and the initial conditions $S(0,0,l) = 1$, $S(n,0,l) = 0$ for $n \geq 1$, and $S(0,k,l) = 0$ for $k \geq 1$. Note that for $l = 1$, these are just the Stirling numbers of the second kind as shown in Exercise 8. Define the l-Takács-Eulerian numbers by

$$A_t(n,k,l) = \sum_{r=k-1}^{n} (-1)^{r-k+1} \binom{r}{k-1} S(n, n-r, l)[(n-r)!]^l.$$

Note that in the special case of $l = 1$, we get the Eulerian numbers, as shown in Exercise 9. Prove that these numbers generalize Eulerian numbers in the following sense. Modify the experiment of Exercise 6 so that in each step, l balls are distributed, independently from each other. Prove that $\frac{A_t(n,k,l)}{n!^l}$ is the probability that after n steps, exactly k boxes remain empty.

4. Let $k \leq n$ be fixed positive integers. Compute the volume of the region $R_{n,k}$ of the hypercube $[0,1]^n$ contained between the two hyperplanes $\sum_{i=1}^n x_i = k-1$ and $\sum_{i=1}^n x_i = k$. See Figure 1.13 for an illustration.

5. (a) Let $p = p_1 p_2 \cdots p_n$ be a permutation, and define

$$\delta_p = \sum_{1 \leq i < j \leq n} \big| |i-j| - |p_i - p_j| \big|.$$

Prove that the smallest possible *positive* value of δ_p is $2n-4$.

(b) Which graph theoretical problem contains part (a) as a special case?

6. Let G be a graph. A *k-coloring* of G is the number of ways to color the vertices of G using only the colors $1, 2, \cdots, k$ so that adjacent vertices have different colors. Let $P(n)$ be the number of n-colorings of G. It is then well-known that $P(n)$ is a polynomial function of n, called the *chromatic polynomial* of G.

Now let

$$F_G(x) = \sum_{n \geq 0} P(n)x^n.$$

It is proved in [197] that $F_G(x) = \frac{Q(x)}{(1-x)^{m+1}}$, where $Q(x)$ is a polynomial of degree m, and with nonnegative integer coefficients.

So we can set $Q(x) = \sum_{i=k}^{m} w_i x^i$, where k is the smallest number for which G has a k-coloring, called the *chromatic number* of G.

(a) Find a combinatorial interpretation for the numbers w_p in terms of permutations.

(b) Explain why the polynomial $Q(x)$ is a generalization of the Eulerian polynomials.

7. Let n, i, and j be fixed positive integers, and set

$$S(i,j,n) = \sum_{0 \leq k \leq n} k^i (n-k)^j.$$

Prove that

$$S(i,0,n) = \sum_{r=0}^{i} \binom{n+1}{r+1} r! S(i,r).$$

8. Let us say that a permutation p contains a *very tight ascending run* of length k if it has k consecutive entries $p_i p_{i+1} \cdots p_{i+k-1}$ so that $p_{i+j} = p_i + j - 1$ for $0 \leq j \leq k-1$. In other words, the sequence $p_i p_{i+1} \cdots p_{i+k-1}$ is a sequence of *consecutive integers*.

   Find a formula for the number of permutatations of length $r + k$ containing a tight ascending run of length at least k, if $k > r$.

9. The *Bessel number* $B(n,k)$ is defined as the number of partitions of $[n]$ into k nonempty blocks of size at most two. Prove that for any fixed n, the sequence $B(n,1), B(n,2), \cdots, B(n,n)$ is unimodal.

10. (a) Prove that if $n = 2^m - 1$ for some positive integer m, then all Eulerian numbers $A(n,k)$ with $1 \leq k \leq n$ are odd.

    (b) Generalize the statement of part (a).

11. Let $A(n,k)_i$ denote the number of n-permutations with $k-1$ descents that begin with i.

    Prove that

$$A(n,k)_i = \sum_{j \geq 0} (-1)^{k-1-j} \binom{n}{k-1-j} j^{i-1} (j+1)^{n-i}.$$

12. John selects an n-permutation p at random. Jane must predict the descent set of p. What is Jane's best bet?

13. Prove that the Euler numbers E_n satisfy the formula
$$\frac{E_n}{n!} = 2\left(\frac{2}{\pi}\right)^{n+1} \sum_{k\geq 0}(-1)^{k(n+1)}\frac{1}{(2k+1)^{n+1}}.$$

14. Let P be a finite partially ordered set whose vertices are bijectively labeled with the elements of $[n]$. The *Jordan–Hölder set* of P, denoted by $L(P)$, is the set of linear extensions of P. That is, $L(P)$ is the set of permutations of $p = p_1p_2\cdots p_n$ so that if $p_i <_P p_j$, then $i < j$ as integers.

    Let $W(P,x) = \sum_{p\in L(P)} x^{d(p)}$. The *Stanley–Neggers conjecture* claimed that for all finite posets P, the polynomial $W(P,x)$ has real roots only.

    (a) Let us call a finite poset P a *forest* if each element of P is covered by at most one other element. That is, P is a forest if for each $x \in P$, there is at most one $y \in P$ so that $x < y$ but there is no z so that $x < z < y$. Prove that the Stanley–Neggers conjecture is true for forests.

    (b) Deduce that the Eulerian polynomials $A_n(x)$ have real roots only.

15. (a) Determine the polynomial $W(P,x)$ if P is the disjoint union of an m-element chain and an n-element chain.

    (b) Let $P_{m,n}$ be the poset that consists of the disjoint union of an m-element chain and an n-element chain so that the elements of the first chain are labeled $1, 2, \cdots, m$ from the bottom up, and the elements of the second chain are labeled $m+1, m+2, \cdots, m+n$ from the bottom up, with the extra relation $m+1 < m$ added. See Figure 1.14 for an illustration.
    Compute $W(P_{m,n}, x)$.

    (c) Prove that the Stanley–Neggers conjeture is false.

16. Let a_n be the number of $(n+1)$-permutations that do not contain a very tight ascending run of length two. (See Problem Plus 8 for the definition of a very tight ascending run.) Find a closed formula for the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$.

Solutions to Problems Plus

1. This result is due to Vesselin Gasharov [146], who used the same lattice path model in his solution as he used to injectively prove that the

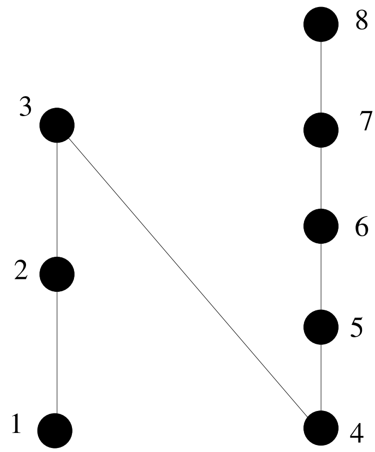

FIGURE 1.14
The poset $P_{3,5}$.

Eulerian polynomials have log-concave coefficients.

2. (a) This result is due to Ira Gessel [152]. Let $d(F)$ be the number of descents of a forest F and let $l(F)$ be the number of leaves of F. Then let $u_n(\alpha, \beta)$ be the bivariate generating function

$$u_n(\alpha, \beta) = \sum_F \alpha^{d(F)} \beta^{l(F)-1},$$

where the sum is over all rooted forests on $[n]$. Then Gessel shows that the trivariate generating function

$$U(x, \alpha, \beta) = \sum_{n \geq 1} u_n(\alpha, \beta) \frac{x^n}{n!}$$

is symmetric in α and β by proving that it satisfies the functional equation

$$1 + U = (1 + \alpha U)(1 + \beta U)e^{x(1-\alpha-\beta-\alpha\beta U)}.$$

(b) If the number of leaves is one, then the tree consists of one line, and the sequence of the vertices corresponds to an n-permutation. The notion of descents of the tree then simplifies to that of descent in this permutation.

3. This result is due to Lajos Takács [252], though note that his paper denoted the Eulerian number $A(n, k)$ by $A(n, k - 1)$. The main idea of the proof is the following. Let

$$B_r(n) = \sum_{k=r}^{n} \binom{k}{r} P(n, k),$$

where $P(n,k)$ is the probability that at the end of the trials there are k empty boxes. Then it can be proved that

$$B_r(n) = S(n, n-r, l) \left(\frac{(n-r)!}{n!} \right)^l$$

by showing that both sides satisfy the same recurrence relations. Then, by the formula $P(k,n) = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} B_r(n)$, our claim follows.

4. The volume of $R_{n,k}$ is equal to $A(n,k)/k!$. A nice combinatorial proof was given by Richard Stanley [241]; though the result was probably known by Laplace. The main element of Stanley's proof is the following measure-preserving map. It is straightforward that $A(n,k)/k!$ is the volume of the set $S_{n,k}$ of all points $(x_1, x_2, \cdots, x_n) \in [0,1]^n$ for which $x_{i-1} < x_i$ for exactly k values of i. (This includes $i = 0$, where we set $x_0 = 0$.) Let $f(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n)$ where

$$y_i = \begin{cases} x_{i-1} - x_i \text{ if } x_{i-1} > x_i, \\ \\ 1 + x_{i-1} - x_i \text{ if } x_{i-1} < x_i \end{cases}$$

Note that f is not defined on the set of points where $x_{i-1} = x_i$ for some i, but that is not a problem as the set of those points has volume zero. Apart from that, however, f maps the rest of $S_{n,k}$ into $R(n,k)$ as $k - 1 \leq \sum_{i=1}^{n} = k - x_n \leq k$. Stanley then shows that apart from a subset of volume zero of $R(n,k)$, the map f has an inverse, and that f is an affine transformation of determinant $(-1)^n$, implying that f is order-preserving.

5. (a) This result is due to W. Aitken [1]. He called δ_p the *total relative displacement* of p.

   (b) Let G be a graph with n vertices, and let $d(x,y)$ be the graph-theoretical distance (number of edges in the shortest path) between x and y. Then, for a permutation p of the vertices of G, one can define
$$\delta_{G,p} = \sum_{1 \leq i < j \leq n} |d(p_x, p_y) - d(x,y)|.$$
   Then part (a) corresponds to the special case when G is the path $12 \cdots n$. Also note that δ_p of part (a) is equal to 0 if and only if $p = 123 \cdots n$ or $p = n \cdots 321$, which is also a special case of the general fact that $\delta_{G,p} = 0$ if and only if p is an automorphism of G.

6. This result is due to I. Tomescu [255]. In that paper, various formulae are proved for the numbers w_k.

(a) Let I be an acyclic orientation of G, and let G have m vertices. The transitive closure I' is then a partial ordering of $[m]$. Let f be a *bijective* coloring of the vertices of I that is compatible with I'. In other words, if $x < y$, then $f(x) <_{I'} f(y)$. Finally, let $T(I)$ be the set of all total orders that extend I'. In other words, the $T(I)$ are all the possible choices for the bijective coloring f.

Now for any $f \in T(I)$, note that f in fact defines a permutation of $[m]$. Let $U(I)$ be the set of all these permutations. Finally, let $M(G)$ be the *multiset* obtained by taking the union of all $T(I)$, for all acyclic orientations I of G, preserving the multiplicities.

It is then proved in [255] that for any graph G, the coefficient w_k is the number of permutations in $M(G)$ that have k ascents.

(b) If G is the empty graph on m vertices, then $M(G)$ contains all $m!$ permutations of length m, and then $Q(x) = A_m(x)/x$.

7. There are several papers that are devoted to exploring connections between powers of integers and Stirling numbers of the second kind, or Eulerian numbers. See, for instance, [160] for this result.

8. It is proved in [172] that this number is $r!(r^2 + r + 1)$.

9. It is easy to prove that

$$B(n,k) = \frac{n!}{2^{n-k}(n-k)!(2k-n)!},$$

and then the result follows by checking that the sequence $B(n+1,k)/B(n,k)$ is decreasing, therefore if it dips below 1, it has to stay below 1. This result was published in [88].

10. (a) It is well-known (see, for example, [34], Exercise 14 of Chapter 4) that if t is a power of 2, then $\binom{t}{k}$ is even, except when $k = t$ or $k = 1$. In our case, this means that $\binom{n+1}{k}$ is always even, except in those special cases, and the claim follows from Theorem 1.11.

(b) Similarly, if $n = p^m - 1$ where p is a prime, then all Eulerian numbers $A(n,k)$ are congruent to 1 modulo p.

A combinatorial proof of these facts not using Theorem 1.11 can be found in [253].

11. This result is due to Mark Conger, and was published in [95].

12. Jane's best bet is that p will be alternating or reverse alternating, that is, that $D(p) = \{1, 3, \cdots\} \cap [n-1]$ or $D(p) = \{2, 4, \cdots\} \cap [n-1]$. This is a classic result that has several proofs. Richard Stanley's survey paper [244] gives a modern proof using non-commuting variables, and mentions earlier proofs.

Note that the answer is what one would intuitively expect. If $i \in D(p)$, then one would expect that $i+1$ is more likely to be an ascent than a descent, and if it is an ascent, then $i+2$ is more likely to be a descent, and so on.

13. A comprehensive treatment of the analytical tools needed to prove this can be found in *Analytic Combinatorics* by Philippe Flajolet and Robert Sedgewick [128]. In particular, a special case of Theorem IV.10 of that book states that if $f(x) = \sum_{n\geq 0} f_n x^n$ is a function all of whose singularities in the closed disc $|x| \leq R$ are simple poles, these poles are at points $\alpha_1, \alpha_2, \cdots$ in increasing order of their distance from 0, and p_i is the residue of f at α_i, then

$$f_n = \sum_{j\geq 1} p_j \alpha_j^{-n} + O(R^{-n}).$$

All singularities of $\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x$ are simple poles. They are at $\pi/2$, at $-3\pi/2$, at $5\pi/2$, and so on. The residue at each pole can be computed, and each turns out to be -2. As R goes to infinity, the sum on the right-hand side converges. This proves our claim.

14. (a) This result is due to David Wagner and can be found in [262].

    (b) Let P be the antichain of n elements. Then P is a forest, so the result of part (a) applies. On the other hand, $L(P) = S_n$, since the restriction that if $p_i <_P p_j$, then $i < j$ as integers is vacuous. So $W(P,x) = A_n(x)/x$, and the claim is proved.

15. Results in this exercise are due to Petter Brändén [69].

    (a) We claim that in this case, $W(P,x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} x^k$. Indeed, let $p = p_1 p_2 \cdots p_n \in L(P)$. It follows from the structure of P that all descents of p must be formed by one element from each chain. That is, if $p_j > p_{j+1}$, then p_j ("the descent top") is one of the n elements of $[m+n]$ that are larger than n, and p_{j+1} ("the descent bottom") is one of the m elements of $[m]$. If p is to have i descents, then any i-element subset of $[m+n]-[m]$ and any i-element subset of $[m]$ can play the role of descent tops and descent bottoms in exactly one (increasing) order, proving the claim.

    (b) We claim that $W_{P_{m,n},x} = \sum_{k=1}^{\min(m,n)} \binom{m}{k}\binom{n}{k} x^k$. Indeed, the only permutation that is not in $L(P_{m,n})$ but is in $L(P)$ where P is as in part (a) is the increasing permutation.

    (c) See [69] for a proof of the result that if M is a positive integer, then for m and n such that $\min(m,n)$ is sufficiently large, the polynomial $W(P_{m,n},x)$ has more than M non-real zeros.

Note that the Stanley–Neggers conjecture is still open for *naturally labeled posets*, that is, for posets in which if $x > y$, then the label of x is larger than the label of y.

16. First, we claim that $a_n = na_{n-1} + (n-1)a_{n-2}$, with $a_0 = a_1 = 1$. Indeed, we can insert $n+1$ into any position of any permutation counted by a_{n-1} except immediately on the right of n. Furthermore, in any permutation counted by a_{n-2}, we can replace any entry i by the string $i\,(n+1)\,(i+1)$. This recurrence relation leads to the generating function $A(x) = e^{-x}/(1-x)^2$.

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