Four problems related to maximal green sequences

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

Kiyoshi Igusa, Department of Mathematics, Advisor

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Ying Zhou

May, 2019

This dissertation, directed and approved by Ying Zhou's committee, has been accepted and approved by the Graduate Faculty of Brandeis University in partial fulfillment of the requirements for the degree of:

DOCTOR OF PHILOSOPHY

Eric Chasalow, Dean of Arts and Sciences

Dissertation Committee:

Kiyoshi Igusa, Department of Mathematics, Chair

Olivier Bernardi, Department of Mathematics

Gordana Todorov, Department of Mathematics, Northeastern University

Abstract

Four problems related to maximal green sequences

A dissertation presented to the Faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Ying Zhou

Maximal green sequences were invented by Bernhard Keller and have a lot of applications in cluster algebra and particle physics. In this dissertation I will discuss four separate topics related to maximal green sequences. First of all I will discuss the problem of associated permutations of mutation sequences and establish a formula for the associated permutation in the case of A_n straight orientation which answers a question by Muller. Secondly I will introduce the concept of m-maximal green sequences and discuss the problem of m-maximal green sequence-finiteness of path algebras of tame quivers. Then I will introduce two alternative definitions of green mutation sequences and show that they are both equivalent to the known ones which extends a result by Igusa. Finally I will discuss the problem of maximal green sequences in quivers with multiple edges and fully describe maximal green sequences of quivers with multiple edges which is the first step towards a proof of the conjecture that all acyclic quivers have finitely many maximal green sequences.

Contents

Abstract **iv**

1 Background **1**
1.1 Notations 1
1.2 Quivers and path algebras 2
1.3 Mutations, mutation sequences and the associated permutation 4
1.4 Bounded derived categories 12
1.5 Tame quivers and tame hereditary algebras 21
1.6 Wall-and-chamber Structures 27
1.7 Quiver folding 30

2 Permutation **34**
2.1 The general theory of permutations 34
2.2 The associated permutation in A_n 37

3 Tame hereditary algebras are green sequence-finite **43**
3.1 Introduction 43
3.2 Proof of Lemma 3.1.2 45
3.3 Proof of Lemma 3.1.3 47
3.4 Almost morphism finiteness 49

4 Two Alternative Definitions of Green Sequences **52**
4.1 Alternative definition of the stability condition on module categories 53
4.2 Maximal backward-$Hom^{\leq 0}$ orthogonal sequences 55
4.3 Harder-Narasimhan filtration 56
4.4 Equivalence of the definitions 59

5 Quivers with multiple edges **61**
5.1 Introduction 61
5.2 MGS-finiteness 63
5.3 The acyclic case 65

5.4 The general case 72

Chapter 1

Background

1.1 Notations

In this paper k is an algebraically closed field, all algebras will be finitely dimensional k-algebras. When the category I are discussing is clearly $\mathcal{C}$ (M, N) will be an abbreviation of $Hom_{\mathcal{C}}(M, N)$. $(M[>0], N)$ is the union of $(M[k], N)$ for all $k > 0$. If $\mathcal{P}, \mathcal{Q}$ are subcategories $D^b(\Lambda)$ then $(\mathcal{P}, \mathcal{Q}) := \bigcup_{M \in \mathcal{P}} \bigcup_{N \in \mathcal{Q}} (M, N)$ and $(\mathcal{P}[>0], \mathcal{Q}) := \bigcup_{k>0} (\mathcal{P}[k], \mathcal{Q})$. Let S be a set of objects in an additive category $\mathcal{C}$. $add(S)$ is defined as the set of all elements of $\mathcal{C}$ such that they are finite direct sums of objects in S.

The definition of green and red are consistent with that of [22] and [5]. It is the exact opposite definition of green and red in [6].

1.2 Quivers and path algebras

1.2.1 Quivers

Definition 1.2.1. A *quiver* Q is a quadruple (Q_0, Q_1, s, t) with Q_0 and Q_1 sets and s, t: $Q_1 \to Q_0$. An element of Q_0 is a *vertex* of Q. An element of Q_1 is an *arrow* of Q. s maps each arrow to its smyce and t maps each arrow to its target.

Intuitively I can think of elements of Q_1 as oriented edges. Any arrow has a unique smyce and a unique target both of which are vertices. This is how I obtain the s and t maps. Unless necessary I generally omit the s and t and denote a quiver by $Q = (Q_0, Q_1)$.

Example 1.2.2. $1 \longrightarrow 2 \qquad 1 \leftleftarrows 2 \longrightarrow 3 \qquad 1 \longrightarrow 2 \longrightarrow 3$

These three are quivers.

Now I need to define subquivers.

Definition 1.2.3. A *subquiver* $Q' = (Q'_0, Q'_1, s', t')$ in a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, $s|_{Q'_1} = s'$ and $t|_{Q'_1} = t'$.

From now on I generally do not distinguish betIen s and s', t and t'.

Not all quivers are useful for the purpose of this paper. This is why I need to add restrictions. In order to do so I need to introduce several definitions.

Definition 1.2.4. An *oriented cycle* in a quiver Q is a subquiver $Q' = (Q'_0, Q'_1, s, t)$ such that $Q'_0 = \{v_0, v_1, \cdots, v_{k-1}\}, Q'_1 = \{a_0, a_1, \cdots, a_{k-1}\}, s(a_i) = v_i, t(a_i) = v_{i+1}$. Here v_k is defined as v_0.

Definition 1.2.5. A *k-cycle* is an oriented cycle with k vertices.

Definition 1.2.6. A *loop* in a quiver Q is an arrow from a vertex to itself, that is, a 1-cycle.

Definition 1.2.7. A *cluster quiver* is a quiver without loops or 2-cycles.

In all but Chapter 4 and a part of Chapter 5 all quivers I discuss will be acyclic. Here is the definition of an acyclic quiver.

Definition 1.2.8. An *acyclic quiver* is a quiver without any oriented cycles.

1.2.2 Path Algebras

Definition 1.2.9. A *path* in a quiver Q is a sequence of vertices $\{v_0, \cdots, v_k\}$ and a sequence of arrows $\{a_0, \cdots, a_{k-1}\}$ if $k > 0$ such that $t(a_i) = v_{i+1}$, $s(a_i) = v_i$ for any $i = 0, 1, \cdots, k-1$. The smyce of the path is v_0 and the sink is v_k.

Paths with length 0 are known as *trivial paths.* A trivial path only has a single vertex v_0 and no arrows at all. All other paths are uniquely determined by their arrows.

Now I need to define multiplication of paths. In order to do so I need to define compatibility and concatenation.

Definition 1.2.10. Paths v, w are *compatible* if $t(v) = s(w)$.

Definition 1.2.11. The *concatenation* of compatible paths $v = \{a_0, \cdots, a_{k-1}\}$ and $w = \{b_0, \cdots, b_{l-1}\}$ is $vw = \{a_0, \cdots, a_{k-1}, b_0, \cdots, b_{l-1}\}$.

Definition 1.2.12. The *path algebra* of a quiver Q is a k-algebra generated by all the paths of the quiver. Multiplication of paths v and w is defined as the concatenation if they are compatible and 0 if they aren't, I only discuss path algebras of acyclic quivers in this thesis because my results are only about finite dimensional algebras.

1.3 Mutations, mutation sequences and the associated permutation

In this section I will introduce mutations of quivers and matrices, different kinds of mutation sequences including green sequences, maximal green sequences, reddening sequences and loop sequences. I will also define the associated permutation. Results in this section are mostly used in Chapters 2 and 5.

1.3.1 Mutation of quivers

The concept of maximal green sequences has many different equivalent definitions. I will use a simple definition using quiver mutations in this subsection. Later I will introduce other definitions. Mutations of cluster quivers at vertex k are defined in the following way:

1. For any pair of arrows $i \to k$ and $k \to j$ add an arrow $i \to j$.

2. Reverse all arrows starting from or ending up in k.

3. Delete all 2-cycles that are formed due to process (1) and (2).

Definition 1.3.1. 1. The *framed quiver* $\hat{Q}$ of Q is obtained from Q by adding a vertex i' and an arrow $i \to i'$ for every $i \in Q$.

2. The *coframed quiver* $\breve{Q}$ of Q is obtained from Q by adding a vertex i' and an arrow $i' \to i$ for every $i \in Q$.

3. An *ice quiver* is a quiver Q where a possibly empty set, $F \subseteq Q_0$, consists of vertices that can not mutate.

An ice quiver (Q, F) can not mutate at elements of F, so I call them *frozen vertices*.

Definition 1.3.2. A non-frozen vertex i is *green* if and only if no arrow from a frozen vertex to i exists. Otherwise it is *red*.[22]

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[r, "\mu_1"] \arrow[d] & 1 & 2 \arrow[l] \arrow[d] \\ 1' & 2' & 1' \arrow[u] & 2' \end{tikzcd}$$

Definition 1.3.3. A *green sequence* is a sequence $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ such that for all $1 \leq t \leq N$ the vertex i_t is green in the partially mutated ice quiver $\hat{Q}(\mathbf{i}, t) = \mu_{i_{t-1}} \cdots \mu_2\mu_1(\hat{Q})$.

Definition 1.3.4. A *maximal green sequence* is a green sequence such that $\hat{Q}(\mathbf{i}, N)$ does not have any green vertices.

Example 1.3.5. For quiver $1 \to 2$ here is one of its two maximal green sequences.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[r, "\mu_1"] \arrow[d] & 1 & 2 \arrow[l] \arrow[r, "\mu_2"] \arrow[d] & 1 \arrow[r] & 2 \\ 1' & 2' & 1' \arrow[u] & 2' & 1' \arrow[u] & 2' \arrow[u] \end{tikzcd}$$

I also need the definition of reddening sequences which are generalized versions of maximal green sequences in order to discuss the phenomenon of almost morphism finiteness in Chapter 3.

Definition 1.3.6. A *red-to-green sequence* or a *reddening sequence*, is a sequence $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ that transforms $\hat{Q}$ to a quiver $\hat{Q}(\mathbf{i}, N) = \mu_{i_N} \cdots \mu_2\mu_1(\hat{Q})$ such that $\hat{Q}(\mathbf{i}, N)$ does not have any green vertices.[25]

Now let's define a new concept, namely *loop sequences* which is essential to the discussion about the permutation in Chapter 2.

Definition 1.3.7. A *loop sequence* w is a sequence of mutations $\mu_{i_k} \cdots \mu_{i_1}$ on an ice quiver (Q, F) such that $\mu_{i_k} \cdots \mu_{i_1}(Q) = \rho(Q)$ for some permutation ρ.

1.3.2 Mutation of matrices

I can also use *c-vectors* for this purpose. To do so I need to reinterpret mutations of cluster quivers in terms of mutations of matrices. I recall that cluster quivers correspond to *exchange matrices* as defined below. For more details I recommend [11] and [12].

Definition 1.3.8. [11] An *exchange matrix* of a cluster quiver Q with n vertices is an $n \times n$ matrix such that b_{ij} is the number of arrows from i to j minus the number of arrows from j to i.

It is easy to see that exchange matrices of cluster quivers are always antisymmetric which is not true in the more general case of *valued quivers* which I won't discuss in this paper. Moreover there is a 1-1 correspondence betIen antisymmetric exchange matrices and cluster quivers.

Mutations of exchange matrices are defined here which exactly agree with mutations of cluster quivers.

Definition 1.3.9. [11] If I mutate an $n \times n$ exchange matrix $B = (b_{ij})$ at k I obtain $B' = (b'_{ij})$ defined here. $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}|b_{kj}| & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} & \text{in all other cases} \end{cases}$

Each partially mutated ice quiver corresponds to an *extended exchange matrix* defined below.

Definition 1.3.10. The *extended exchange matrix* B' corresponding to a partially mutated ice quiver Q' is an $2n \times n$ matrix with the rows corresponding to vertices $\{1, 2, \cdots, n, 1', 2', \cdots n'\}$ while the columns corresponds to the vertices $\{1, 2, \cdots, n\}$. Here I use the number $n + i$ to represent i'. b_{ij} is the number of arrows from i to j minus the number of arrows from j to i.

An extended exchange matrix B' has an upper and lower square submatrices, B and C respectively. The lower square matrix C is known as the *C-matrix.* Column vectors of an C-matrix are known as *c-vectors.* A c-vector is positive if all its entries are non-negative and at least one is positive. A c-vector is negative if all its entries are non-positive and at least one is negative. Due to [13] a c-vector is either positive or negative which is known as *sign coherence.*

A mutation on vertex k is *green* if the c-vector c_k before the mutation is negative. A mutation on vertex k is *red* if the c-vector c_k before the mutation is positive. A *maximal green sequence* is a mutation sequence from $C = -I_n$ to a permuted version of I_n We can use a sequence of c-vectors to denote a maximal green sequence because we can use the c-vector corresponding to vertex k to represent mutation at vertex k.

Now we need to introduce two more results that are crucial to 5. The proof requires some knowledge of simple minded collections. For readers unfamiliar with this concept who want to understand the proofs the next section should be read first. Positive c-vectors are dimension vectors of elements of simple-minded collections. Such elements are all bricks. That is, all c-vectors are Schur. However we can indeed prove more. They are in fact real as well.

Lemma 1.3.11. *Let k be an algebraically closed field. Let Λ be a hereditary algebra over k. Then any c-vector c that appears in any MGS is a real Schur root.*

Proof. Since the simples of Λ are all exceptional if the lemma were incorrect then there must be some c-matrix in the MGS, C such that all columns of C are real Schur roots while one green mutation can somehow generate a root that isn't real. Here there can only be two cases, namely some mutation performed on $-v$ caused some $-w$ to be transformed into $-w' = -w - kv$ which isn't real, some mutation performed on $-v$ caused some $+w$ to be transformed into $w' = w - kv$ which isn't real. In the second case w' may be positive or negative.

For an arbitrary c-vector v let M_v be the brick such that v is the dimension vector of M_v. In this case $\langle v, v\rangle = 1 - dimExt^1(M_v, M_v)$. Hence v being real is equivalent to $\langle v, v\rangle = 1$.

Case 1: Assume that some mutation performed on $-v$ caused some $-w$ to be transformed into $-w' = -w - kv$ which isn't real. $\langle w', w'\rangle = \langle w, w\rangle + k\langle v, w\rangle + k\langle w, v\rangle + k^2\langle v, v\rangle$. Since $Hom(M_v, M_w) = Hom(M_w, M_v) = 0$ due to M_v, M_w being two elements in a simple-minded collection and v, w are both real $\langle w', w'\rangle = k^2 + 1 - k\, dimExt^1(M_v, M_w) - k\, dimExt^1(M_w, M_v)$.

Using properties of simple-minded collections $dimExt^1(M_w, M_v) = k$. Using Prop 6.4 in [23] we can see that $Ext^1(M_v, M_w) = 0$, Hence $\langle w', w' \rangle = 1$. w' is real.

Case 2: Assume that some mutation performed on $-v$ caused some w to be transformed into $-w' = w - kv$ which isn't real. Using properties of simple-minded collections it is obvious that $Ext^1(M_v, M_w) = Hom(M_v, M_w) = 0$. Regardless of whether w' is positive or negative $\langle w', w' \rangle = \langle w, w \rangle - k\langle v, w \rangle 0k\langle w, v \rangle + k^2\langle v, v \rangle = k^2 + 1 - k\, dimHom(M_w, M_v) + k\, Ext^1(M_w, M_v)$. Using properties of simple-minded collections $dimHom(M_w, M_v) = k$. Using Prop 6.4 in [23] we can see that $Ext^1(M_w, M_v) = 0$, Hence $\langle w', w' \rangle = 1$. w' is real.

The assumption has been refuted. Any c-vector c that appears in any MGS is a real Schur root. □

In order to prove 5.1.1 we need to first prove a lemma.

Lemma 1.3.12. *If $-c_1, -c_2$ are negative c-vectors in C-matrix C' in an MGS, c_1 and c_2 are dimension vectors of indecomposable modules M_1 and M_2. If $dimExt^1(M_1, M_2) > 1$ then the mutation on C' must not be done on M_2.*

Proof. Assume that $-c_1$ is the i-th column and $-c_2$ is the j-th colu,n. According to [24] using the definition of left mutations of simple-minded collections if $dimExt^1(M_1, M_2) > 1$ then the mutation on $-c_2$ would cause $-c_1$ to be transformed into $-c_1 - kc_2$ with $k > 1$ because which could only happen if there are multiple edges from i to j. Due to [6] this was impossible.

□

1.3.3 Permutations

All reddening sequences have associated permutations. When comparing the quivers obtained from transforming the same framed quiver using two different reddening sequences, it is easy to see that they are just one permutation away from each other: If you do a correct permutation of vertices (that means both rows and columns together) you can transform one such matrix into another. In particular any quiver obtained by using a reddening sequence to transform a framed quiver is one permutation away from the coframed quiver.

Here is the formal definition of such a permutation:

Definition 1.3.13. A *permutation* from an ice quiver (Q, F) to (Q', F) is an isomorphism of quivers $Q \to Q'$ that preserve F.[5]

I have a result from [5] which helps us define the permutation:

Theorem 1.3.14. *Let Q be a cluster quiver and let Q' be a quiver that is a result of a reddening sequence on $\hat{Q}$, then Q' equals to a permutation of $\check{Q}$[5]*
Due to the proposition, for a reddening sequence $\mathbf{i} = (i_1, \cdots, i_N)$, for some $\rho \in S_n$ I have $\mu_{i_N} \cdots \mu_{i_1} \hat{Q} = \rho\check{Q}$.

Definition 1.3.15. The *permutation of a reddening sequence* $\mathbf{i}$ is ρ for which $\mu_{i_N} \cdots \mu_{i_1} \hat{Q} = \rho\check{Q}$.[14]

Here is one of the simplest examples of the concept of the permutation:

$$
\begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \arrow[d] \arrow[r, "\mu_1"] & 1 & 2 \arrow[l] \arrow[d] \arrow[r, "\mu_2"] & 1 \arrow[r] & 2 \\
1' \arrow[d, "\mu_2"] & 2' & 1' \arrow[u] & 2' & 1' \arrow[u] & 2' \arrow[u] \\
1 \arrow[d] \arrow[dr] & 2 \arrow[l] \arrow[r, "\mu_1"] & 1 \arrow[r] & 2 \arrow[dl] \arrow[r, "\mu_2"] & 1 & 2 \arrow[l] \arrow[u, "(12)"'] \\
1' & 2' \arrow[u] & 1' \arrow[u] & 2' \arrow[ul] & 1' \arrow[ur] & 2' \arrow[ul]
\end{tikzcd}
$$

It is obvious that the result of $\mu_2\mu_1$ and $\mu_2\mu_1\mu_2$ are not identical, though they can be transformed into each other by a single permutation on vertices.

The associated permutation is not only intuitive when defined on reddening sequences. In the case of loop sequences they are even more intuitive. Using the c-vector theorem from [33] I can see that the only nontrivial effect loop sequences can have on an extended matrix is a permutation. Hence loop sequences are equivalent to permutations on an extended matrix.

I can also define the associated permutation of sequences using extended exchange matrices.

Definition 1.3.16. The *matrix of a permutation*, $\sigma \in S_n$, is defined as the $n \times n$ matrix $P_\sigma = (\delta_{\sigma(i)j})$.

Definition 1.3.17. (1)For an $m \times n$ matrix $M = (M_1, \cdots, M_n)$ and a permutation $\sigma \in S_n$, if $C = (M_{\sigma(1)}, \cdots, M_{\sigma(n)})$ (or equivalently, $(c_{ij}) = (m_{i\sigma(j)})$), I denote this as $C = \sigma(M)$.
(2)For an $n \times n$ matrix $A = (a_{ij})$ and a permutation $\sigma \in S_n$, if $D = (d_{ij}) = (a_{\sigma(i)\sigma(j)})$, I denote this as $D = \tilde{\sigma}(A)$.

It is easy to see that $C = \sigma(M)$ if and only if $C = MP_\sigma{}^{-1}$. $D = \tilde{\sigma}(A)$ if and only if

$D = P_\sigma A {P_\sigma}^{-1}$.

Definition 1.3.18. For any loop sequence w the permutation ρ such that $w(\tilde{B}) = \rho(\tilde{B})$ is defined as *the associated permutation of the loop sequence w.*

In essence for all acyclic quivers, green-to-red sequences in general and maximal green sequences in particular do not have a natural definition of the permutation: The traditional one in essence is the permutation of an associated loop sequence: Take the reddening sequence and then do mutations at sinks only, go over all non-frozen vertices and return to the origin which constitutes the loop sequence I need.

1.4 Bounded derived categories

In this section I will go over the basics about bounded derived categories, approximations, silting objects, simple-minded collections, torsion classes, t-structures and introduce the definition of numerous mutation sequences. This section mostly consists of background for chapters 4 and 5.

1.4.1 Bounded derived categories

In this subsection we need to use Auslander-Reiten Theory. However I'm not going to talk about the entire Auslander-Reiten theory even though some parts of it are crucial to the understanding of Chapter 3. For Auslander-Reiten theory I refer the reader to Chapter IV of [3] and [2].

I'm not going to talk about what triangulated categories and bounded derived categories are in details. For those who want to read about them I recommend Daniel Murfet's notes [26][27][28] for introduction and [15] for its application in the theory of finite dimensional algebras. In particular [15] is a good source for Auslander-Reiten theory in bounded derived categories which we will use extensively here.

Let's recall that bounded derived categories $D^b(\Lambda)$ are obtained by identifying homotopic chain maps in the category of chain complexes $C(\Lambda)$ and then formally invert all quasi-isomorphisms through localization. In bounded derived categories of hereditary algebras the indecomposable objects are of the form $M[i]$ where M is an indecomposable module and i is the amount of shifts I perform. In bounded derived categories it is true that $M, N \in mod\Lambda$

$$Hom_{D_b(\Lambda)}(M[i], N[j]) = \begin{cases} Ext_\Lambda^{j-i}(M,N) & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}.$$

Example 1.4.1. Let Q be $1 \longrightarrow 2 \longrightarrow 3$. Here is the Auslander-Reiten quiver of $D^b(kQ)$.

$$\begin{tikzcd}
\cdots & I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] & \cdots \\
& \cdots & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \cdots & \\
\cdots & P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2] & \cdots
\end{tikzcd}$$

1.4.2 Approximations

According to [24] there are bijections between silting objects, t-structures, co-t-structures and simple-minded collections in a wide range of cases and such bijections respect mutations. In [8] more bijections are mentioned. Here I only need to cover three of them, namely silting objects, simple-minded collections and t-structures. To understand their mutations I must first introduce the concept of approximations.

Definition 1.4.2. Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $M \in Ob\mathcal{C}, N \in Ob\mathcal{X}$, a morphism $f \in Hom_{\mathcal{C}}(M, N)$ is a *minimal left-$\mathcal{X}$ approximation* if for any $g \in End_{\mathcal{C}}N$ such that $g \circ f = f$ g is an isomorphism and for any $N' \in Ob\mathcal{X}$ for any $q \in Hom_{\mathcal{C}}(M, N')$ I have q factors through f.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "q"'] & N \arrow[d, dashed, "l"] \\ & N' \end{tikzcd}$$

Definition 1.4.3. Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $N \in Ob\mathcal{C}, M \in Ob\mathcal{X}$, A morphism $f \in Hom_{\mathcal{C}}(M, N)$ is a *minimal right-$\mathcal{X}$ approximation* if for any $g \in End_{\mathcal{C}}M$ such that $f \circ g = f$ g is an isomorphism and for any $M' \in Ob\mathcal{X}$ for any $q \in Hom_{\mathcal{C}}(M', N)$ I have q factors through f.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M' \arrow[u, dashed, "l"] \arrow[ur, "q"] & \end{tikzcd}$$

Example 1.4.4. Let $\mathcal{C}$ be $D^b(\Lambda)$ for some finite dimensional algebra Λ and let $\mathcal{X}$ be one of its full subcategories. If $M \in \mathcal{X}$ then 1_M is both a minimal left-$\mathcal{X}$ approximation and a minimal right-$\mathcal{X}$ approximation.

Example 1.4.5. Let Q be A_2 straight orientation. Let $\mathcal{C}$ be $D^b(kQ)$. Let $M = P_2$ and $\mathcal{X} = add(P_1)$. The minimal left-$\mathcal{X}$ approximation is the canonical morphism $P_2 \to P_1$ induced by the inclusion $P_2 \to P_1$ in the module category.

Example 1.4.6. Let Q be A_2 straight orientation. Let $\mathcal{C}$ be $D^b(kQ)$. Let $M = P_2$ and $\mathcal{X} = add(P_1)$. The minimal right-$\mathcal{X}$ approximation is the zero morphism because there is no other morphism from P_1 to P_2.

1.4.3 Silting objects

Now let's introduce silting objects. I can think of indecomposable summands of them as indecomposable projectives in an Abelian category.

Definition 1.4.7. Let Λ be an algebra with n primitive idempotents. A *silting object* T of $D^b(\Lambda)$ is an object such that T has n direct summands and $(T, T[m]) = 0$ for all $m > 0$. A *pre-silting object* is an object that only has to satisfy the second condition.

Example 1.4.8. Let's take A_3 straight orientation as an example.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

$\Lambda[i]$ is a silting object for any i. $T_1 = P_3[1] \oplus P_1[1] \oplus I_1[1]$ is also a silting object.

Now that I already have the definition of silting objects I can discuss their mutations.

Definition 1.4.9. A *forward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is the cone/homotopy cokernel of the minimal left-$add(T/T_i)$ approximation of T_i.

A *backward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is homotopy kernel/ [-1] of the cone/ of the minimal right-$add(T/T_i)$ approximation of T_i.

Example 1.4.10. Again let's take A_3 straight orientation as an example.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

Λ is a silting object. When I do a forward mutation at P_3 I get $T' = S_2 \oplus P_2 \oplus P_1$. When I do a forward mutation at P_1 now I get $T'' = S_2 \oplus P_2 \oplus P_1[1]$. When I do another forward mutation at P_2 I get $T''' = S_2 \oplus P_3[1] \oplus P_1[1]$.

1.4.4 Simple-minded collections

Now let's introduce simple-minded collections. They are simple objects in some Abelian category known as hearts of t-structures.

Definition 1.4.11. Let Λ be an algebra with n primitive idempotents. A *simple-minded collection* $\{S_i\}_{i\in[n]}$ of $D^b(\Lambda)$ is an n-element set such that $(S_i[\geq 0], S_j) = 0$ for all $i \neq j$, $(S_i[> 0], S_i) = 0$ for all i, (S_i, S_i) is a division algebra.

Example 1.4.12. As usual my example is A_3 straight orientation.

$$
\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}
$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. $\{P_3[1], P_2, I_1\}$ is also a simple-minded collection.

Definition 1.4.13. A *forward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S'_j\}$ where $S'_i = S_i[1]$ and S'_j $(j \neq i)$ is the cone/homotopy cokernel of the minimal left-$add(S_i)$ approximation of $S_j[-1]$.
A *backward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S'_j\}$ where $S'_i = S_i[-1]$ and S'_j $(j \neq i)$ is the cone/homotopy cokernel of the minimal left-$add(S_i[-1])$ approximation of S_j.

Example 1.4.14. The quiver here is A_3 straight orientation.

$$
\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}
$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. When I do a forward mutation at P_3 I get $\{P_3[1], P_2, I_1\}$. When I do a forward mutation at P_2 now I get $\{S_2, P_2[1], P_1\}$. When I then do a forward mutation at P_1 I get $\{S_2, I_1, P_1[1]\}$.

1.4.5 t-structures

Here is the definition of $t-$structures.

Definition 1.4.15. A *t-structure* on $D^b(\Lambda)$ is a pair $(D^{\leq 0}, D^{\geq 0})$ such that the following holds.

1. For any $M \in D^b(\Lambda)$ there exists $M' \in D^{\leq 0}, M'' \in D^{\geq 0})$ such that $M' \to M \to M'' \to M'[1]$.
2. $D^{\leq 0}[1] \subseteq D^{\leq 0}$, $D^{\geq 0}[1] \supseteq D^{\geq 0}$.
3. $(D^{\leq 0}[1], D^{\geq 0}) = 0$

Example 1.4.16. Let Λ be any finite dimensional algebra. $(\cup_{m=1}^{\infty}\Lambda[m], \cup_{m=0}^{\infty}\Lambda[-m])$ is clearly a t-structure.

Now let's define hearts which will be very useful for a crucial proof in 4, namely the proof of Lemma 4.2.2.

Definition 1.4.17. The *heart* of a t-structure $(D^{\leq 0}, D^{\geq 0})$ is defined as $\mathcal{H} = D^{\leq 0} \cap D^{\geq 0}$

Theorem 1.4.18. *[4] Hearts of t-structures are Abelian categories.*

t-structures can be mutated just like silting objects and simple-minded collections. In order to do so I first need to define torsion pairs in Abelian categories.

Definition 1.4.19. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ in an Abelian category $\mathcal{C}$ is a pair of two subcategories such that the following holds.

1. $Hom(\mathcal{T}, \mathcal{F}) = 0$

2. For any $M \in \mathcal{C}\, \exists T \in \mathcal{T}$, $F \in \mathcal{F}$ such that $0 \to T \to M \to F \to 0$ is a short exact sequence.

3. If for $M \in \mathcal{C}$ $Hom(M, \mathcal{F}) = 0$, $M \in \mathcal{T}$.

4. If for $M \in \mathcal{C}$ $Hom(\mathcal{T}, M) = 0$, $M \in \mathcal{F}$.

Example 1.4.20. Let $\mathcal{C}$ be $modkQ$ with Q being A_2 straight orientation if I take $\mathcal{T} = add(P_2)$ and $\mathcal{F} = add(I_1)$ I can see that the pair $(\mathcal{T}, \mathcal{F})$ satisfies the conditions above and is hence a torsion pair in $\mathcal{C}$.

Example 1.4.21. Let $\mathcal{C}$ be $modkQ$ with Q being A_2 straight orientation if I take $\mathcal{T} = add(P_1, I_1)$ and $\mathcal{F} = add(P_2)$ I can see that the pair $(\mathcal{T}, \mathcal{F})$ satisfies the conditions above and is hence a torsion pair in $\mathcal{C}$.

Now it is possible to define mutations of t-structures.

Definition 1.4.22. [24]Let Λ be a finite dimensional algebra, let $D^b(\Lambda)$ be the bounded derived category of Λ. Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure of $D^b(\Lambda)$. Let $\mathcal{A}$ be its heart. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. The *left mutation* or *forward mutation* $\mu_i^+(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) = (\mathcal{C}'^{\leq 0}, \mathcal{C}^{\geq 0})$ where $\mathcal{C}'^{\leq 0} = \{M \in \mathcal{C} | H^m(M) = 0 \text{ for } m > 0 \text{ and } H^0(M) \in \mathcal{T}\}$, $\mathcal{C}'^{\geq 0} = \{M \in \mathcal{C} | H^m(M) = 0 \text{ for } m < -1 \text{ and } H^{-1}(M) \in \mathcal{F}\}$. Similarly we can define *right mutations* (or *backward mutations*).

1.4.6 Green sequences

Since I have maximal green sequences it is reasonable to look at the generalization of this concept, namely m-maximal green sequences. In order to do so I need to define the general concept of green and red sequences. In principle any forward mutation is considered green and any backward mutation red.

Definition 1.4.23. 1. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *green* if it contains only forward mutations.

2. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *red* if it contains only backward mutations.

3. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *k-red* if it contains k backward mutations.

4. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *k-green* if it contains k forward mutations.

Note that a 0-red sequence is just a green one. A 0-green sequence is just a red one. Now I can introduce m-maximal green sequences. For the purpose of the proof in Chapter 3 it is much better to use silting objects.

Definition 1.4.24. An *m-maximal green sequence* is a green sequence of silting objects from Λ to $\Lambda[m]$.

It is easy to see that a 1-maximal green sequence is just a maximal green sequence.

Example 1.4.25. Again my example is A_3 straight orientation.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

So $(P_1, P_2, P_3, P_1[1], P_2[1], P_3[1])$ is a 2-maximal green sequence, so is $(P_1, P_3, P_2, S_2, P_1[1], P_2[1], P_3[1])$ because they are both sequences of indecomposable objects forward mutations on which produce $\Lambda[2]$ from Λ.

1.5 Tame quivers and tame hereditary algebras

In this subsection I will review the basics about tame hereditary algebras, the components of their Auslander-Reiten quivers and the components of Auslander-Reiten quivers of their bounded derived categories for they are crucial to Chapter 3. For more details about tame algebras I would like to refer the readers to [9], [30] and [32].

1.5.1 Tame quivers

Definition 1.5.1. A *tame algebra* is a k-algebra such that for each dimension there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.

Definition 1.5.2. A *tame quiver* is a quiver such that its path algebra is a tame algebra.

Example 1.5.3. Here are all the (connected) tame quivers, $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6.\tilde{E}_7, \tilde{E}_8$.

$\tilde{A}_n$

$$\begin{tikzcd}
 & 2 \arrow[r] & \cdots \arrow[r] & i \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & & & n+1 \\
 & i+1 \arrow[r] & \cdots \arrow[r] & n \arrow[ur] &
\end{tikzcd}$$

$\tilde{D}_n$

$$\begin{tikzcd}
1 \arrow[dr] & & & & & n \\
 & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] & & & & & n+1
\end{tikzcd}$$

$\tilde{E}_6$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \\
 & & 6 \arrow[d] & & \\
 & & 7 & &
\end{tikzcd}$$

$\tilde{E}_7$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] \arrow[d] & 5 \arrow[r] & 6 \arrow[r] & 7 \\
 & & & 8 & & &
\end{tikzcd}$$

$\tilde{E}_8$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \\
 & & 9 & & & & &
\end{tikzcd}$$

1.5.2 Auslander-Reiten quivers of tame hereditary algebras

In this subsection I'm going to discuss Auslander-Reiten quivers of basic tame hereditary algebras because information about them is slightly less well known.

Theorem 1.5.4. *The Auslander-Reiten quiver of a tame path algebra consists of three parts, the preprojectives, the preinjectives and the regulars.*

Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras.

Theorem 1.5.5.

1. *The AR quiver of kQ has one preprojective component which is isomorphic to $\mathbb{N}Q^{op}$*
2. *The AR quiver of kQ has one preinjective component which is isomorphic to $-\mathbb{N}Q^{op}$.*
3. *All preprojective and preinjective modules in kQ are rigid.*
4. *All but finitely many preprojectives and preinjectives are sincere.*
5. *There are infinitely many regular components, all of which are standard tubes $\mathbb{Z}A_\infty/(\tau^k)$.*
6. *All but at most three tubes have $k = 1$. In this case I consider the component homogeneous.*
7. *All elements in a homogeneous tube are non-rigid, hence they and their shifts can not be summands of any silting object.*
8. *In a nonhomogeneous component $\mathbb{Z}A_\infty/(\tau^k)$ only indecomposables with quasi-length less than k are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.*

9. *Only finitely many regular indecomposable modules are rigid. Hence only finitely many regular indecomposables and their shifts can appear in an m-maximal green sequence.*

I'm going to introduce one example of nonhomogeneous and homogeneous standard stable tubes each. For more details I recommend Chapter X of [32].

Example 1.5.6. Here is a standard stable tube with rank 3.

$$\begin{tikzcd}
 & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & \\
M_{33} \arrow[ur] \arrow[dr] & & M_{13} \arrow[ur] \arrow[dr] & & M_{23} \arrow[ur] \arrow[dr] & & M_{33} \\
 & M_{12} \arrow[ur] \arrow[dr] & & M_{22} \arrow[ur] \arrow[dr] & & M_{32} \arrow[ur] \arrow[dr] & \\
M_1 \arrow[ur] & & M_2 \arrow[ur] & & M_3 \arrow[ur] & & M_1
\end{tikzcd}$$

M_{ik} is rigid iff $k \leq 2$.

Here $M_{ik} = \begin{matrix} M_{i+k-1} \\ \cdots \\ M_{i+1} \\ M_i \end{matrix}$. I define the *quasi-length* of M_{ik} as k.

Now let's see a homogeneous tube.

Example 1.5.7. Here is a homogeneous standard stable tube.

$$\begin{tikzcd}
\cdots \arrow[d, bend left] \\
M_3 \arrow[u, bend left] \arrow[d, bend left] \\
M_2 \arrow[u, bend left] \arrow[d, bend left] \\
M \arrow[u, bend left]
\end{tikzcd}$$

Note that no module in this tube is rigid.

Here $M_k = \begin{array}{c} M \\ \dots \\ M \end{array}$

Now let's do an example of an AR quiver of a tame path algebra.

Example 1.5.8. The quiver is $\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 5 & 3 \arrow[l] \\ & 4 \arrow[u] & \end{tikzcd}$. Here is the preprojective component, $\mathcal{P}$.

$$\begin{tikzcd}
 & P_1 \arrow[ddr] & & \tau^{-1}P_1 \arrow[ddr] & \cdots \\
 & P_2 \arrow[dr] & & \tau^{-1}P_2 \arrow[dr] & \cdots \\
P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-1}P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-2}P_5 \cdots \\
 & P_3 \arrow[ur] & & \tau^{-1}P_3 \arrow[ur] & \cdots \\
 & P_4 \arrow[uur] & & \tau^{-1}P_4 \arrow[uur] & \cdots
\end{tikzcd}$$

Here is the preinjective component, $\mathcal{Q}$.

$$\begin{tikzcd}
\cdots & \tau I_1 \arrow[ddr] & & I_1 \\
\cdots & \tau I_2 \arrow[dr] & & I_2 \\
\cdots \tau I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
\cdots & \tau I_3 \arrow[ur] & & I_3 \\
\cdots & \tau I_4 \arrow[uur] & & I_4
\end{tikzcd}$$

Here are the regular components. There are infinitely many homogeneous tubes and 3 nonhomogeneous ones. All objects in the homogeneous ones are non-rigid. The quasi-simple in the homogeneous tubes has dimension vector is (1,1,1,1,2). The quasi-simples in the three nonhomogeneous tubes have dimension vectors (1,1,0,0,1) and (0,0,1,1,1), (1,0,1,0,1) and (0,1,0,1,1), (1,0,0,1,1) and (0,1,1,0,1) respectively.

Finally let's discuss Auslander-Reiten quivers of $D^b(kQ)$. For a tame quiver Q there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are isomorphic to $\mathbb{Z}Q^{op}$. Let's label these components *transjective*. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$.

There are also infinitely many regular components. There are at most 3 nonhomogeneous tubes in $modkQ[m]$ for any m. There are also infinitely many homogeneous tubes in $modkQ[m]$ for any m. However since no module in a homogeneous tube is rigid they don't affect my problem.

1.6 Wall-and-chamber Structures

In this section I will discuss the basics of the wall-and-chamber structure, picture groups and alternative definitions of maximal green sequences. This section is mostly relevant to chapters 2 and 4.

1.6.1 Picture groups

I also need to use the concept of the picture groups in order to prove the formula in 2.

For a quiver of finite type, any dimension vector of an indecomposable representation, which I also refer to as a *root*. Let $D(\beta) \subseteq \mathbb{R}^n$, $D(\beta) = \{x \in \mathbb{R}^n :< x, \beta >= 0, < x, \beta' > \leq 0$ when $\beta' \subseteq \beta\}$. Here $\beta' \subseteq \beta$ means the unique indecomposable representation of dimension vector β' is a subrepresentation of the unique indecomposable representation of dimension vector β. $D(\beta)$ for all these roots divide $\mathbb{R}^n$ into *compartments*. The boundary of each compartment is the union of some $D(\beta)$ which I call *walls*.[19][17] Sometimes my abuse notation and use the root β to mean the wall $D(\beta)$ when the meaning is clear. I also use the notation

$+\beta$ to mean the wall β is a part of the boundary of a compartment $\mathcal{U}$ and for any point $x \in \mathcal{U}$, $< x, \beta > > 0$. Similarly I have the notation $-\beta$. For example $+\beta - \beta'$ means that β and β' are parts of the boundary of a compartment $\mathcal{U}$ and for any point $x \in \mathcal{U}$, $< x, \beta > > 0$ and $< x, \beta' > < 0$.

In A_n in particular since all indecomposable representations are thin, the roots are $\beta_{ij} = e_j - e_i$ ($0 < i < j < n$, e_0 is defined as the zero vector).

Definition 1.6.1. A *picture group* of a cluster quiver of finite type Q is a group $G(Q) =< S|R >$ with S in bijection with the set of real Schur roots (the generator for β is $x(\beta)$) and R the set of relations $x(\beta_i)x(\beta_j) = \Pi x(\gamma_k)$ with γ_k running over all these real Schur roots which are linear combinations $\gamma_k = a_k\beta_i + b_k\beta_j$ with a_k/b_k increasing (going from 0/1 where $\gamma_1 = \beta_j$ to 1/0 where $\gamma_k = \beta_i$) for any pair (β_i, β_j) such that they are Hom-orthogonal and $Ext(\beta_i, \beta_j) = 0$. [19]

Note that for quiver A_n all roots are real and Schur hence a real Schur root is just a root. Also I often simplify the notation of $x(\beta_{ij})$ to x_{ij} which I use interchangeably with $x(\beta_{ij})$. The picture group for A_n straight orientation is $G(A_n) = \{S|R\}$, $S = \{x_{ij}|0 \leq i < j \leq n\}$, $R = \{x_{ij}x_{kl} = x_{kl}x_{ij}|[i, j] \cap [k, l] = \emptyset, [i, j]$ or $[k, l]$, i,j,k,l are distinct.$\} \cup \{x_{jk}x_{ij} = x_{ij}x_{ik}x_{jk}|0 \leq i < j < k \leq n\}$.

Todorov proved with Igusa [19] that there exists a bijection betIen the set of maximal green sequences and the set $\mathcal{P}(c)$ of positive expressions of the Coxeter element of the picture group for any acyclic valued quiver of finite type which applies to A_n straight orientation.

1.6.2 Alternative definitions of maximal green sequences

In the following theorem by Kiyoshi Igusa multiple equivalent definition of maximal green sequences was introduced. To understand more about the wall-and-chamber structure I suggest that the reader reads [16], [13] or [7].

Theorem 1.6.2. *[18] Let Λ be a finite dimensional hereditary algebra over a field K. Let $\beta_1, \cdots, \beta_m \in \mathbb{N}^n$ be any finite sequence of nonzero, nonnegative integer vectors. Then the following are equivalent.*

(a) There is a nonlinear stability function $Z_t : K_0\Lambda \to \mathbb{C}$ which is green and has exactly m semistable pairs (M_i, t_i) with $t_1 < t_2 < \cdots < t_m$ so that $\dim M_i = \beta_i$ for all i.

(b) There is a generic green path $\gamma : \mathbb{R} \to \mathbb{R}^n$ which crosses the walls $D(M_i)$, $i = 1, \cdots, m$ in that order, and no other walls, so that $\dim M_i = \beta_i$ for all i.

(c) There exist Λ-modules $M_m, \cdots, M_1$ with $\dim M_i = \beta_i$ which form a finite Harder-Narasimhan system for Λ.

(d) There exist Schurian Λ-modules $M_1, \cdots, M_m$ with $\dim M_i = \beta_i$ so that

(1) $Hom_\Lambda(M_i, M_j) = 0$ for $i > j$.

(2) No other modules can be inserted into the sequence preserving (1).

(e) There is a maximal green sequence for Λ of length m whose ith mutation is at the c-vector β_i.

1.7 Quiver folding

In this section we will introduce the theory of quiver folding. Folding theory has been in folklore for a while. Since it will be useful for proving a result in Chapter 5 I'm going to discuss it here in details.

Definition 1.7.1. Let B be an $n \times n$ exchange matrix and let $\rho \in S_n$ be a permutation. If $\rho(B) = B$ then ρ is a *symmetry* of B. The group of symmetries of B is the *Symmetry Group of B* which I denote as *Sym* B. An exchange matrix with a non-trivial symmetry group is a *symmetric exchange matrix.* Any nontrivial subgroup of B is a *Symmetry Subgroup of B.* The symmetry group of a valued quiver is defined as the symmetry group of the exchange matrix of the valued quiver.

A symmetry subgroup G acts on the extended exchange matrices $\tilde{B}$ in the obvious way, namely for some $\rho \in G$ $\rho\tilde{B} := \rho(\tilde{B})$. I can also define right group actions of elements of G on the set of c-vectors similarly, namely for any $v = (v_i) \in \mathbb{Z}^n$, $v\rho := (v_{\rho(i)})$. Let $C(\mathcal{A})$ be the set of c-vectors of a cluster algebra of geometric type $\mathcal{A}(B)$. Orbits of $c \in C(\mathcal{A})$ is denoted as Gc. It is easy to see that all orbits are finite. $Fc := \Sigma_{c' \in Gc} c'$ is the *folded version of c.*

Now I need to fold vertices first. For any nontrivial subgroup of S_n let n' be the set of orbits of the canonical group action of S_n on $[n]$. Hence there exists maps from $[n]$ to $[n']$. Pick some surjection f from $[n]$ to $[n']$ such that f maps each orbit to one element of $[n']$. f is a *vertices folding map.* I sometimes abuse notations and identify $f(i)$ and Gi when I do not need to specify f.

Definition 1.7.2. (1)For any valued quiver Q and its symmetry subgroup G, the *folded version of Q with respect to G* is defined as below:
For each valued arrow $i \overset{(d_{ij},d_{ji})}{\longrightarrow} j$ it is replaced by $Gi \overset{(d_{ij}|Gi|,d_{ji}|Gj|)}{\longrightarrow} Gj$.
(2)For any symmetric exchange matrix B and its symmetry subgroup G, the *folded version of B with respect to G* is defined as $\tilde{B} = (b'_{kl})$ where $b'_{GiGj} = b_{ij}|Gj|$.
[31][6]

Definition 1.7.3. For any symmetry subgroup G of B any extended exchange matrix $\tilde{B} = (B', C')$ such that B' is symmetric is a *symmetric extended exchange matrix with respect to G* if for any $i \in [n]$ $|Gc_i| = |G_i|$ and for any $\rho \in G$, $Gc_{\rho(i)} = Gc_i$.

Any symmetric extended exchange matrix can be folded.

Definition 1.7.4. For any symmetry subgroup G of B any symmetric extended exchange matrix $\tilde{B} = (B', C')'$ with respect to G. Then for any vertices folding map f $F\tilde{B} := (FB', FC')'$ is the *folded version of* $\tilde{B}$ with $FC' = (\tilde{c}_1, \cdots, \tilde{c}_n)$ defined below: $\tilde{c}_i = \Sigma_{g\in f^{-1}(i)}\tilde{c}_g/|Gg|$ where $\tilde{c}_{gi} := \Sigma_{k\in f^{-1}(i)}c_{gk}$.

It is easy to see that if $c_i = e_i$ then $C = I_n$ then $C' = I_{n'}$ and if $C = -I_n$ then $C' = -I_{n'}$. Hence a framed valued quiver is folded into a framed valued quiver and a coframed valued quiver is folded into a coframed valued quiver. Also positive c-vectors are folded into positive ones and negative c-vectors are folded into negative ones.

Definition 1.7.5. For any symmetry subgroup G of B a mutation sequence $w = \Pi_{i=m}^{1} \mu_{k_i}$ starting from an extended exchange matrix $\tilde{B} = (B', C')'$ with B symmetric is *symmetric with respect to* a symmetry subgroup G if the following holds:
1.w is in the form $w = \Pi_{i=m}^{1} \Pi_{j \in Gk_i} \mu_j$, which roughly means that vertices in any orbit is "mutated together".
2.$\Pi_{i=m'}^{1} \mu_{k_i} \tilde{B}$ is symmetric.

Definition 1.7.6. For any symmetry subgroup G of a symmetric extended exchange matrix $\tilde{B}$ for any symmetric mutation sequence $w = \Pi_{i=m}^{1} \Pi_{j \in Gk_i} \mu_j$, *the folded version of* w is defined as $Gw := \Pi_{i=m}^{1} \mu'_{Gk_i}$.

It is easy to see that folding symmetric reddening sequences results in reddening sequences. Also folding symmetric green sequences results in green sequences. Folding symmetric maximal green sequences results in maximal green sequences.

Theorem 1.7.7. *For any symmetry subgroup G of a symmetric extended exchange matrix $\tilde{B}$ for any symmetric mutation sequence w, $F \circ w = Gw \circ F$.*

Proof. Let's assume that the length of Gw is 1. When the theorem has been proven in this particular case the rest is clear from induction. Hence let's assume $w = \Pi_{j \in Gi} \mu_j$ and $Gw = \mu'_{Gi}$. Note that $b_{jk} = 0$ for any $j, k \in Gi$ since otherwise j and k would not be in the same orbit. It is also clear from symmetry that the set of vertices in $[n]]$ that is a smyce of any valued arrow with some $j \in Gi$ its target is independent of the choice of j, which I denote as $P^-(Gi)$. Similarly I can define $P^+(Gi)$. The set of vertices in $[n] \backslash Gi$ that is not connected to any $j \in Gi$ is denoted as $I(Gi)$. Using the invariance lemmas it is easy to see that for any $j \in I(i)$ mutations at any element of Gi does not affect the j-th row and the j-th column at all.

Assume that I do mutations on a symmetric extended exchange matrix $\tilde{B} = (B', C')'$. Let $B = (b_{jk})$, $C = (c_{jk})$. Hence $b'_{GjGk} = b_{ij}|Gj|$. $c'_{GjGk} = \Sigma_{l\in Gjk\in Gm} c_{lm}$. Let $\tilde{C} = \mu'_{Gi}(C) = (c'_{j'k'})$ $w(\tilde{B}) = (\tilde{b}_{jk})$ $w(\tilde{C}) = (\tilde{c}_{jk})$ $F\tilde{B} = (\hat{B}', \hat{C}')'$ $F\circ w(\tilde{B}) = (B'_1, C'_1)'$ $Gw\circ F(\tilde{B}) = (B'_2, C'_2)'$. $B_1 = (b^1_{jk})$ $C_1 = (c^1_{jk})$ $B_2 = (b^2_{jk})$ $C_2 = (b^1_{jk})$.

Let's first calculate the left hand side. If j or $k \in I(Gi)$ it is clear that $\tilde{b}_{jk} = b_{jk}$. If $j \in i$ or $k = i$ $\tilde{b}_{jk} = -b_{jk}$ due to the fact that $b_{jj'} = 0$ for any $j, j' \in Gi$. Otherwise it is easy to see that $\tilde{b}_{jk} = b_{jk} + |Gi|sp(b_{ji}, b_{ik})$ due to two facts: First of all, before a mutation at $l \in Gi$, the l-th row and column of the exchange matrix can not be affected by all previous mutations since betIen elements of Gi there are no connections. Secondly for all $l \in Gi$, b_{jl} and b_{lk} are independent of l. Similarly, if $k \in Gi$ $\tilde{c}_{jk} = -c_{jk}$. Otherwise $\tilde{c}_{jk} = c_{jk} + \Sigma_{l\in Gi} sp(c_{jl}, b_{ik})$. Hence $b^1_{GjGk} = -b_{jk}|Gk|$ if j or k is in Gi. Otherwise $b^1_{GjGk} = b_{jk}|Gk| + |Gi||Gk|sp(b_{ji}, b_{ik})$. If $k \in Gi$ $c^1_{GjGk} = -|Gk|\bar{c}_{GjGk}$. Otherwise $c^1_{GjGk} = \bar{c}_{GjGk} + \Sigma_{l\in Gi} sp(\bar{c}_{GjGl}, b_{ik}) = \bar{c}_{GjGk} + |Gi|sp(\bar{c}_{GjGi}, b_{ik})$.

Now let's calculate the right hand side. $\hat{b}_{GjGk} = |Gk|b_{jk}$. $\hat{c}_{GjGk} = |Gk|\bar{c}_{GjGk}$. If j or $k \in Gi$ $b^2_{GjGk} = -b_{jk}|Gk|$. Otherwise $b^2_{GjGk} = b_{jk}|Gk| + sp(|Gi|b_{ji}, |Gk|b_{ik}) = b_{jk}|Gk| + |Gi||Gk|sp(b_{ji}, b_{ik})$. If j or $k \in Gi$ $c^2_{GjGk} = -|Gk|\bar{c}_{GjGk}$. Otherwise $c^2_{GjGk} = |Gk|\bar{c}_{GjGk} + sp(|Gi|\bar{c}_{GjGi}, b_{ik}|Gk|) = |Gk|\bar{c}_{GjGk} + |Gi||Gk|sp(\bar{c}_{GjGi}, b_{ik})$.

Hence $F \circ w = Gw \circ F$ has been proven.

$\square$

Chapter 2

Permutation

2.1 The general theory of permutations

2.1.1 Mutation systems

Let's define a natural setting of the theory of permutations which is completely combinatorial. Let $[n] = \{1, 2, \cdots, n\}$. A *mutation graph* is defined as a connected n-regular graph without loops. Let $T = (T_0, T_1)$ be a mutation graph. A *signed edge* of T is a triple (a, h, t) where $a \in T_1$, h and t are the two endpoints of a, defined to be the head and tail of the signed edge respectively. Let $\tilde{T}_1$ be the union of all signed edges of T. A *walk* is a path in $\tilde{T}_1$ such that the smyces and targets of each signed edge are compatible. Walks on T are in the form $w = \Pi a_k^{i_{k-1} i_k}$.

For each vertex $x \in T_0$ I associate a set $N(x)$ which is the set that contains all vertices adjacent to x. Note that $|N(x)| = n$. For each $a_{ht} \in \tilde{T}_1$ I associate a bijection $f_{a_{ht}} : N(x) \to N(y)$ such that a_{ht} and a_{th} are inverses of each other for any $a \in T_1$. This bijection is called *mutation* as per [**?**]. The set of all f_a is denoted A, the *set of muta-*

tions. The tuple (T, A) is called a *mutation system*. I can also define a natural bijection $f_w : N(x) \to N(y)$ associated with each walk $w = \Pi a_k^{i_{k-1}i_k}$, namely $f_w = f_{a_k}^{i_{k-1}i_k} \cdots f_{a_1}^{i_0 i_1}$.

Now let's define a bijection $j_x : [n] \to N(x)$ for each $x \in T_0$. This bijection is called the *fixed ordering* of the seed $N(x)$. Let J be the set of all fixed orderings which I call a *fixed ordering set*. The tuple (T, A, J) is called a *ordered mutation system*. Now for each bijection $j'_x : [n] \to N(x)$ I can define its *associated permutation relative to* J below:

Definition 2.1.1. For any (T, A, J) for any $x \in T_0$ for any bijection $g : [n] \to N(x)$ the *associated permutation relative to* J is defined as $\rho(g) = j_x^{-1} g$.

Now I can define what is the associated permutation of a mutation relative to J.

Definition 2.1.2. For any (T, A, J) for any $x, y \in T_0$ for any mutation $f_a : N(x) \to N(y)$ the *associated permutation relative to* J is defined as $\rho(f_a) = j_y^{-1} f_a j_x$.

In other words, $\rho(f_a)$ is the permutation such that the following diagram commutes:

$$\begin{tikzcd}
[n] \arrow[r, "\rho(f_a)"] \arrow[d, "j_x"] & [n] \arrow[d, "j_y"] \\
N(x) \arrow[r, "f_a"] & N(y)
\end{tikzcd}$$

Now I can define what it means to be the associated permutation relative to J of any walk $p = a_k^{i_k} \cdots a_1^{i_1}$, namely $\rho(p) = j_y^{-1} f_p j_x$. It is easy to see from the diagram below that $\rho(p) = \rho(a_k)^{i_k} \cdots \rho(a_1)^{i_1}$.

$$\begin{tikzcd}
[n] \arrow[r, "\rho(f_{a_1})^{i_1}"] \arrow[d, "j_{x_1}"] & [n] \arrow[r, "\rho(f_{a_2})^{i_2}"] \arrow[d, "j_{x_2}"] & \cdots [n] \arrow[d, "j_{x_k}"] \\
N(x_1) \arrow[r, "f_{a_1}^{i_1}"] & N(x_2) \arrow[r, "f_{a_2}^{i_2}"] & \cdots N(x_k)
\end{tikzcd}$$

I can also discuss the relation betIen the permutation of a walk p relative to different fixed orderings.

Theorem 2.1.3. *(Change of fixed ordering formula) For any mutation system (T, A) for any fixed ordering set $J_1.J_2$ for any $x, y \in T_0$ for any walk $p : x \to y$, let $\rho_1(p), \rho_2(p)$ be the associated permutation of p relative to J_1, J_2 respectively. Let $\tau_x = j_{1x}^{-1} j_{2x}$, $\tau_y = j_{1y}^{-1} j_{2y}$. Then $\rho_2(p) = \tau_y^{-1} \rho_1(p) \tau_x = j_{2y}^{-1} j_{1y} \rho_1(p) j_{1x}^{-1} j_{2x}$.*

The theorem can be verified easily by the diagram below.

$$\begin{tikzcd}
[n] \arrow[r, "\rho_2(p)"] \arrow[d, "\tau_x"] \arrow[dd, bend right=60, "j_{2x}"'] & [n] \arrow[d, "\tau_y"] \arrow[dd, bend left=60, "j_{2y}"] \\
[n] \arrow[r, "\rho_1(p)"] \arrow[d, "j_{1x}"] & [n] \arrow[d, "j_{1y}"] \\
N(x) \arrow[r, "f_a"] & N(y)
\end{tikzcd}$$

I can see that in essence the permutation of a reddening or loop sequence is just special cases of permutations of walks.

It is obvious that any (T, A, J) induces a map $g : \tilde{T}_1 \to [n]$ that assigns a *fixed position* to each signed edge. I call the map g the *fixed position map*. It is also obvious that for any walk $w : x \to y$, I only need to fix j_x, j_y to have a fixed $\rho(w)$. $\rho(w)$ is independent of j_z for any $z \neq x, y$. Hence the definition of $\rho(w)$ can be done with any arbitrary choice of j_z for any $z \neq x, y$. In fact even not defining them is also fine since I can just define $\rho(w)$ as $j_y^{-1} f_w j_x$.

2.2 The associated permutation in A_n

When the quiver is A_n straight orientation I can make much stronger claims. In fact there is a canonical permutation of any mutation sequence. The formula of the associated permutation of reddening sequences and loop sequences is given as the following.

Theorem 2.2.1. *In A_n straight orientation, the permutation associated with a picture group element that transforms the framed quiver into the coframed quiver or the framed quiver itself is $\rho(\prod_k x_{i_k j_k}^{\delta_k}) = (\prod_k (i_k+1, j_k))^{-1}$. Here $\delta_k \in \{+, -\}$.*

This formula works for any maximal green, reddening, loop sequences that starts from and ends up in the framed quiver. It also extended the definition of an associated permutation to the set of arbitrary finite sequences of mutations in A_n straight orientation. One interesting property of A_n is that the associated permutation of a mutation only depends on the c-vector but not which cluster-tilting object on which the mutation is conducted. This is a highly nontrivial fact: The associated permutation in the general case seems way less regular.

2.2.1 Forbidden Pairs-of-Walls Lemmas

Since I use picture groups and related structures to prove the theorem, I need to examine what kind of pairs of walls can not exist in any compartment.

Lemma 2.2.2. *(First Forbidden Pairs-of-Walls Lemma) For any compartment for any short exact sequence of roots* $0 \to s \to r \to q \to 0$ *it is impossible to have pairs of such walls:* $+s-r$ *or* $-q+r$*.*

(Second Forbidden Pairs-of-Walls Lemma) For quiver A_n *for any compartment for any short exact sequence of roots* $0 \to s \to r \to q \to 0$ *it is impossible to have pairs of such walls:* $-s-r$*,* $-r-q$*,* $+s+r$*, or* $+q+r$*.*

The first lemma is obvious since in either case $< x, s > > 0$ but the root R is stable which is impossible. As for the second lemma, the reason $-s-r$ can not appear is that in A_n when you cross $D(s)$ you are either going to have $+s-r$ or $+s-(r+s)$. The former is impossible due to Forbidden Pairs-of-Walls Lemma. The latter is impossible due to c-vector theorem and the fact that the sum of a root and any of its subroots is no longer a root any more in A_n. The reason the other three cases can not happen is almost identical.

2.2.2 Proof of the formula

The basic idea in proving the theorem is below:

Since picture group elements freely commute with permutations, what I want to prove can be reduced to $\rho(\prod_k (i_k+1, j_k)x_{i_k j_k}) = id$. This property can further be reduced to proving that for all k, $(i_k+1, j_k)x_{i_k j_k}$ in some sense does not permute the c-vectors. This in turn can be reduced to $(i_k+1, j_k)x_{i_k j_k}$ as an operation on extended exchange matrices maintain certain properties defined below:

Definition 2.2.3. An $n \times n$ matrix $M \in M_n(\mathbb{Z})$ is *standard* if the following holds:

1. The diagonal entries are all nonzero.

2. All positive entries can only exist on the diagonal or above. and all negative entries can only exist on the diagonal or below.

3. All columns are in the form $\pm\beta_{ij}$.

It is easy to see that all columns of the form $-\beta_{ij}$ has to be the $(i+1)$-th column and all columns of the form β_{ij} has to be the j-th column since all other positions violate either Axiom 1 or 2. It is also trivial that the only results of a permutation of columns of $\pm I_n$ that are standard are $\pm I_n$ themselves.

Example 2.2.4. Here are several examples:

$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ are standard matrices because all three axioms hold.
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ are not standard matrices since axioms 1 and 2 are violated.

The lemma to be proven that can almost immediately lead to the theorem is stated below:

Lemma 2.2.5. *In A_n straight orientation, $(i_k+1, j_k)x_{i_kj_k}$ or $(i_k+1, j_k)x_{i_kj_k}^{-1}$ transforms a standard matrix $\tilde{B}_{k-1}$ into a standard matrix $\tilde{B}_k$.*

Proof. I will only prove in the green case since the red case is almost identical to the green one. In this proof i_k is simplified as i and j_k is simplified as j.

Case 1: If $j - i = 1$. Here I have a simple root and the associated permutation $(i+1, j)$ is identity. Hence the proof reduces to x_{ij} transforms a standard matrix to another standard one. x_{ij} merely flips the j-th column from $-e_j$ to e_j and may lengthen some $-\beta_{li}$ to $-\beta_{lj}$ for $l < i$ and shorten some β_{il} to β_{jl} for $l > j$ without changing which column they are in, but no other operation happens or violations of First Forbidden Pairs-of-Wall Theorem $(-q+r)$ will happen, hence the resulting matrix is still standard.

Case 2: If $j - i > 1$. Here I have an extra generator and the associated permutation is not identity. Now let's discuss what $(i+1, j)x_{ij}$ actually does on each column:

a) $l \leq i$. Due to c-vector theorem [?] all c-vectors have to be $\pm\beta_{ab}$ for some $0 \leq a < b \leq n$. So the only change that can ever happen is that $-\beta_{li}$ may be lengthen to $-\beta_{lj}$. Neither of these cause a C-matrix to violate standardness.

b) $i + 1 < l < j$. Again due to c-vector theorem in [?] the only plausible situation is β_{il} was transformed into $-\beta_{lj}$. But this constitutes a $-q + r$ situation which violates the Forbidden Pairs-of-Wall Lemma.

c) $l = j$. I notice several facts:

(1).The c-vector c_j can not be negative. If this is the case I have a compartment with two walls $D(\beta_{ij})$ and $D(\beta_{j-1,m})$. This can not happen since these two roots can not be both stable due to [33] when $m > j$ or the $+r + s$ situation appears when $m = j$. Hence I can assume that the j-th column is β_{mj} for some $m < j$.

(2). m can not be less than i. Otherwise I have a $+s - r$ situation which violates the First Forbidden Pairs-of-Wall Lemma.

(3). Also it is impossible for β_{mj} to remain itself after doing x_{ij} since otherwise I have a $-s - r$ situation which violates the Second Forbidden Pairs-of-Wall Lemma.

Hence the j-th column is positive, $m > i$ and $-\beta_{ij}$ actually add to the j-th column. So after $(i+1, j)x_{ij}$ is performed the i-th column is $-\beta_{im}$ and the j-th column is β_{ij}.

d) $l > j$. β_{il} can be shortened to β_{jl}. Other than that, the only plausible case is $l = j+1$ and the kth c-vector is $-\beta_{lm}$ for some $m > l$. In this case I can nor let $-\beta_{ij}$ add to the l-th column since otherwise $-q+r$ will be created after the mutation which violates the Forbidden Pairs-of-Wall Lemma.

Using similar methods I can see that $(i+1, j)x_{ij}$ transforms a standard matrix into another one.

$\square$

The theorem can be proven below:

Proof. The identity matrix I_n is standard. Since $(i_k+1, j_k)x_{i_k j_k}$ or $(i_k+1, j_k)x^{-1}_{i_k j_k}$ transforms a standard matrix into another standard for all k, the result of transforming a standard matrix, I_n by $\prod_k (i_k+1, j_k)x^{\delta_k}_{i_k j_k}$ is standard. Since I get a permutation of columns of $\pm I_n$ at the end of this transformation and any permutation of columns of $\pm I_n$ that result in a standard matrix has to be the trivial permutation, $\rho(\prod_k (i_k+1, j_k)x^{\delta_k}_{i_k j_k}) = id$, hence the formula is correct.

$\square$

2.2.3 The formula of associated permutation for any mutation sequences

Due to the theorem I can extend the definition of associated permutations to any arbitrary mutation sequence in A_n straight orientation which reduces to the existing definitions of the associated permutation of reddening and loop sequences due to the theorem above.

Definition 2.2.6. In A_n straight orientation, the *associated permutation of a mutation se-*

quence in correspondence to the picture group element $\prod_k x_{i_k j_k}^{\delta_k}$ acting on a c-matrix with associated permutation σ is defined as $\rho(\prod_k x_{i_k j_k}^{\delta_k}) = \sigma(\prod_k (i_k+1, j_k))^{-1}\sigma^{-1}$. Here $\delta_k \in \{+,-\}$.

In particular any mutation at a vertex with c-vector $\pm\beta_{ij}$ has an associated permutation $\sigma(i+1,j)\sigma^{-1}$ with σ the permutation of the c-matrix before the mutation. In the special case when $i+1=j$ which is when β_{ij} is a simple root the associated permutation is trivial.

Chapter 3

Tame hereditary algebras are green sequence-finite

3.1 Introduction

In Brüstle-Dupont-Pérotin [5] and the paper by Igusa together with Brüstle, Hermes and Todorov [6] it is proven that there are finitely maximal green sequences when the quiver is of finite, tame type or the quiver is mutation equivalent to a quiver of finite or tame type. Furthermore in [6] it is proven that any tame quiver has finitely many k-reddening sequences.

When I restrict my attention to the case where the algebra is basic, connected and hereditary it is a path algebra of a quiver [3]. From now on when I say an m-maximal green sequence of an algebra I mean an m-maximal green sequence of its path algebra. Here is the main theorem I have proven.

Theorem 3.1.1. *Any tame quiver has finitely many m-maximal green sequences.*

To prove this theorem I only need to prove that only finitely many indecomposable objects can appear as summands of silting objects that can appear in m-maximal green sequences of tame quivers Q. Since all indecomposable objects of a basic tame path algebra has to be transjective or regular, only finitely many rigid regular objects betIen Λ and $\Lambda[m]$ in $D_b(\Lambda)$, namely the modules on the nonhomogeneous tubes $\mathbb{Z}A_\infty / < \tau^k >$ with no repeating composition factors and their shifts. Hence the problem is reduced to proving that only finitely many indecomposable transjective objects betIen Λ and $\Lambda[m]$ can appear in m-maximal green sequences.

To prove this theorem I need two lemmas.

Lemma 3.1.2. *For a tame quiver Q any silting object in $D^b(kQ)$ contains at most $n-2$ regular summands. In other words, at least 2 summands have to be transjective.*

Lemma 3.1.3. *For a tame quiver Q there is a uniform bound, depending only on Q and m, on the transjective degree of any transjective summand in any silting object in any m-maximal green sequence $D^b(kQ)$.*

It is easy to see why 3.1.3 implies the theorem. Here the *transjective degree* of an indecomposable transjective object $\tau^i P_j[k]$ is defined as $deg(\tau^i P_j[k]) = i$. The *maximal transjective degree* and *minimal transjective degree* of a silting object are defined as the highest/loIst transjective degree of its indecomposable transjective summands respectively.

In Section 2 I prove 3.1.2. In Section 3 I prove 3.1.3. In Section 4 I further generalize the theorem to arbitrary finite mutation sequences with finitely many forward/green or backward/red mutations.

3.2 Proof of Lemma 3.1.2

To prove 3.1.2 I need to understand regular components of Auslander-Reiten quivers of $D^b(kQ)$ for tame quivers. Regular components of Auslander-Reiten quivers of tame path algebras are all standard stable tubes with at most three tubes nonhomogeneous (see [9] and Chapter X of [32]). Note that no object on a homogeneous tube is rigid so no object there can appear in a silting object of $D^b(kQ)$. Hence I only need to discuss the nonhomogeneous tubes.

It is easy to see that in an indecomposable object in a standard stable tube $\mathcal{T}$ of size n, M and any of its shifts can not be in the same pre-silting object. Hence for each indecomposable object I can consider them just regular modules in different degrees. If $\{M_i\}_{i\in I}$ are a family of indecomposable objects of $D^b(kQ)$ and $\Pi_{i\in I}M_i[n_i]$ is not pre-silting for any $\{n_i\}_{i\in I}$ I say that $\{M_i\}_{i\in I}$ is *silting-incompatible*. Otherwise I say that it is *silting-compatible*. From now on in this proof I identify $[n]$ with $\mathbb{Z}/n$ and hence will no longer differentiate betIen 0 and n which I usually denote as n. It is also clear that it makes sense to define $[a_1, a_2, \cdots, a_k]$ on $[n]$ in the sense of cyclic orders. Ex: $[1, 2, 3]$ and $[2, 3, 1]$ hold in $[4]$. Let M_i be the regular simples of the tube such that $\tau M_i = M_{i-1}$. I call a regular module in $D^b(kQ)$ *regular sincere* if its composition series contain all regular simples. All indecomposable regular sincere modules and their shifts can not appear as summands in any silting object because they are not rigid. (See Corollary X.2.7 of [32]).As for the remaining $n(n-1)$ indecomposable objects I can unambiguously label them as M_{ij} if the regular top and regular socle of the object are M_j and M_i respectively. Note that $M_i = M_{ii}$. It is clear that $\tau M_{ij} = M_{i-1,j-1}$ and $\tau^{-1}M_{ij} = M_{i+1,j+1}$.

Now let's prove two easy lemmas on what can not appear in a pre-silting object in a regular component of the Auslander-Reiten quiver of $D^b(kQ)$.

Lemma 3.2.1. *1. If M and N are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Hom(M, N) \neq 0$ and $Ext^1(N, M) \neq 0$, then M and N are silting-incompatible.*

2. If $M_1, \cdots, M_k$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Ext^1(M_i, M_{i+1}) \neq 0$ for any $1 \leq i < k$ and $Ext^1(M_k, M_1) \neq 0$, then $\{M_i\}$ is silting-incompatible.

Proof. For (1) since $Hom(M, N) \neq 0$, $Ext^{i-j}(M[i], N[j]) \neq 0$ if $i > j$. Since $Ext^1(N, M) \neq 0$ $Ext^{j-i+1}(N[j], M[i]) \neq 0$ if $i \leq j$. Hence $M[i] \oplus N[j]$ is not pre-silting for any arbitrary i and j.

For (2) for arbitrary $n_1, \cdots n_k$ use the argument above it is easy to see that if $\oplus_{i=1}^k M_i[n_i]$ is pre-silting, then $n_2 > n_1$, $n_3 > n_2$, $\cdots, n_1 > n_k$ which is impossible. Hence $\{M_i\}$ is silting-incompatible. □

Lemma 3.2.2. *Any pre-silting object in a standard stable tube of size n contains at most $n - 1$ summands.*

To prove this lemma I need the following lemma.

Lemma 3.2.3. *Any pre-silting object in a standard stable tube of size n can not be regular sincere.*

Proof. Assume that a pre-silting object $T = \oplus_{i=1}^k T_i$ in a standard stable tube of size n is regular sincere. Let $T_{l_1}, \cdots, T_{l_m}$ be a minimal set of indecomposable summands of T

such that their direct sum $T' = \oplus_{i=1}^{m} T_{l_i}$ is regular sincere. Note that if M_{ij} and M_{kl} are both summands of T, $[i, k, l, j]$ holds M_{kl} and M_{ij} can not both be summands of T' due to minimality. If $m = 1$ T' is a regular sincere indecomposable regular object which contradicts the fact that T' is pre-silting. If $m > 1$ without loss of generality assume that $T_{l_1} = M_{1p}$ for some $p \neq n$. Any indecomposable object with its regular socle $M_i, 1 \leq i \leq p$ can not be a summand of T' either due to silting incompatibility or minimality. Hence there has to be a summand of T' with its regular socle $p + 1$. Repeat this procedure it's easy to see that $T' = \oplus_{i=1}^{m} M_{(t_{i-1}+1)t_i}$ with $t_0 = t_m = n$. In this case by Lemma 2.3 the object can not be pre-silting. □

Now I can prove 3.2.2.

Proof. Since any pre-silting object in a standard stable tube of size n can not be regular sincere, without loss of generality it is a pre-silting object in the exact subcategory of $\mathcal{T}$ closed under extensions such that $M_1, \cdots M_{n-1}$ are the only simple objects. This category is isomorphic to the module category of KA_{n-1} with linear orientation and as a result any pre-silting object in it has at most $n - 1$ summands. □

Finally I can prove 3.1.2.

Proof. Due to 3.2.2 and [9] there are at most $n - 2$ regular components in $D^b(kQ)$ when Q is a tame quiver. This is true for each type so this is true for all tame quivers. □

3.3 Proof of Lemma 3.1.3

To prove 3.1.3 I need to rephrase an argument in [5] using degrees.

Lemma 3.3.1. *([5], Lemma 10.1) Let H be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective H-module P, the H-modules $\tau^{-k}P$ and $\tau^k P[1]$ are sincere.*

Lemma 3.3.2. *([5]) Let Q be a tame quiver and M_1, M_2 two transjective modules of kQ. If $\{M_1, M_2\}$ is silting-compatible, then $|deg(M_1) - deg(M_2)| \leq N$*

Proof. If $k - l > N$ I need to prove that $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. If $i \leq j$ $Ext^{j-i+1}(\tau^l P_b[j], \tau^k P_a[i]) = Ext^1(\tau^l P_b, \tau^k P_a) = Hom(\tau^{k-1} P_a, \tau^l P_b) = Hom(P_a, \tau^{l-k+1} P_b) \neq 0$ since $\tau^{l-k+1} P_b$ is a sincere preprojective module. If $i > j$ $Ext^{i-j}(\tau^k P_a[i], \tau^l P_b[j]) = Ext^1(\tau^k P_a[1], \tau^l P_b) = Hom(\tau^{l-1} P_a, \tau^k P_b[1]) = Hom(P_a, \tau^{k-l+1} P_b[1]) \neq 0$ since $\tau^{k-l+1} P_b[1]$ is a sincere preinjective module. Hence $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. Exchange the objects if $k - l < -N$. Hence the lemma has been proven. □

Now I can prove 3.1.3 following a modified version of the argument in [5].

Proof. I only need to prove that there is a loIr bound of minimal transjective degrees of silting objects that can appear in m-maximal green sequences. Assume that $\tau_k P_i[j]$ is in a silting object in an m-maximal green sequence of kQ. Note that due to 3.1.2 there are at least 2 transjective components in any silting object in $D^b(kQ)$. Note that each mutation on a transjective object T in $\mathcal{P}_i$ can result in a transjective object in $\mathcal{P}_{i+1}$, a transjective object in $\mathcal{P}_i$ with degree less than or equal to $deg(T)$ or a regular object in $\mathcal{R}_i$. Each mutation on a regular object T' in $\mathcal{R}_i$ can result in an object of $\mathcal{R}_i$, an object of $\mathcal{P}_{i+1}$ or an object of $\mathcal{R}_{i+1}$. Let L be the minimal transjective degree of a silting object. No green mutation within a component or green mutation from a regular component to another one can increase L. All other green mutations may increase L by at most N. HoIver there are only n summands of a silting object, $m+1$ transjective components and m regular components so the amount of mutations that can increase L is finite. To reach $\Lambda[m]$ which is of degree 0 L has to be at least $-2mnN$. As a result no indecomposable transjective object in any silting object in an m-maximal green sequence can have a degree less than $-2mnN$. Similarly silting objects in m-maximal green sequences can not have maximal transjective degree higher than $2mnN$ or it can not start from Λ. □

3.4 Almost morphism finiteness

Using the same method I can prove a stronger result.

Theorem 3.4.1. *If Q is a Dynkin or tame quiver and T_1, T_2 are silting objects of $D^b(kQ)$ then there are finitely many k-red and finitely many k-green mutation sequences from T_1 to T_2 for any k.*

Note that I only need to prove that part of the statement about k-red sequences. To prove the theorem I first need to prove the following lemma which is a generalization of Lemma 4.4.2 in [6].

Lemma 3.4.2.

1. *Any k-red sequence from T_1 to T_2 can go through any silting object at most $r+1$ times.*
2. *Any k-green sequence from T_1 to T_2 can go through any silting object at most $r+1$ times.*

Proof. I only need to prove (1). It is clear from the definition of mutations that a green sequence can go through any silting object at most once. (See [8] and [24] for more details.) Let's define a *green arm* of a mutation sequence as a maximal subsequence of the mutation sequence that is green. Similarly I can define what is a *red arm*. Assume that an k-red sequence $\{T_i\}$ has n_r red arms and n_g green arms. $n_r \leq k$. $n_g \leq n_r + 1$. Let n_1 be the number of red arms of length 1 and n_2 the number of red arms of length at least 2. It's clear that $n_r = n_1 + n_2$ and $n_1 + 2n_2 \leq k$. Note that any silting object on a red arm of length 1 is on a green arm. Hence $\{T_i\}$ can go through any silting object at most $n_g + n_2 \leq n_1 + 2n_2 + 1 \leq k + 1$ times. □

It is easy to see that the bounds established in the lemma are optimal. Now I can prove the theorem. Note that the lemma above implies that in the Euclidean case if I can prove that for any k if there are finitely many rigid objects that ca$k$-redn appear as summands of silting objects in k-red sequences Theorem 3.4.1 will been proven.

Proof. As I said above I will only prove the part about k-red sequences. Assume that all indecomposable summands of T_1 and T_2 are betIen $\Lambda[i]$ and $\Lambda[j]$. Since there are only k red mutations, all indecomposable summands that appear in k-red sequences from T_1 to T_2 have to be betIen $\Lambda[i-k]$ and $\Lambda[j+k]$.

If Q is Dynkin there are only finitely many indecomposable objects betIen $\Lambda[i-k]$ and $\Lambda[j+k]$ and hence only finitely many silting objects can exist on an k-red sequence. Due to Lemma **??** there are finitely many k-red sequences. From now on I assume that Q is Euclidean. There are only finitely many regular rigid indecomposable objects betIen $\Lambda[i-k]$ and $\Lambda[j+k]$ so the problem has been reduced to proving that only finitely many transjective indecomposable components can appear in silting objects in k-red sequences.

Let the minimal degree of T_2 be L. Note that a red mutation can increase the minimal degree of a silting object by at most N. Use an argument similar to that one used to prove Theorem 3.1.1 I can prove that no indecomposable transjective object with degree less than $L - 2nN(2k + j - i) - kN$ can appear in any k-red sequences from T_1 to T_2. Similarly let the maximal degree of T_1 be U. No indecomposable transjective object with degree less than

$U + 2nN(2k + j - i) + kN$ can appear in any k-red sequence from T_1 to T_2. Hence there are only finitely many indecomposable transjective objects can appear in any k-red sequence from T_1 to T_2 and the theorem is proven. □

Note that the bounds of transjective degrees in the proofs of theorem 3.1.1 and 3.4.1 above are very crude. In the future I will try to find better bounds.

Finally let's define a new term to characterize finite dimensional algebras that satisfy the conditions of Theorem 3.4.1.

Definition 3.4.3.

1. A finite dimensional algebra Λ of finite global dimension such that it has finitely many k-red sequences from any silting object T_1 to any silting object T_2 for any k is *almost morphism finite*.

2. A finite dimensional algebra Λ of finite global dimension such that it has finitely many green sequences from any silting object T_1 to any silting object T_2 for any m is *green sequence finite*.

Hence I can rephrase Theorem 3.4.1 as the following:

Theorem 3.4.4. *If Λ is the path algebra of a quiver of finite or tame type, then Λ is almost morphism finite.*

Note that the condition of an algebra being almost morphism finite is stronger than the condition that it is green sequence finite which is stronger than the condition that there are finitely many m-maximal green sequences for any m. An almost morphism finite algebra has finitely many k-red sequences for any k hence it has finitely many green-to-red sequences with k red mutations.

Chapter 4

Two Alternative Definitions of Green Sequences

In the introduction I introduced a result by Igusa, namely 1.6.2 there are new alternative definitions of maximal green sequences. How the result can possibly be generalized to green sequences in general is and interesting question that I have mostly solved.

In section 2 I will discuss an alternative definition of the stability condition on module categoris. In section 3 I will discuss maximal backward-$Hom^{\leq 0}$ orthogonal sequences. In section 4 I will discuss Harder-Narasimhan filtrations. In section 5 I will establish the fact that the two alternative definitions are equivalent to the original ones.

4.1 Alternative definition of the stability condition on module categories

In general in a triangulated category the concepts of monomorphisms and epimorphisms are less important because there are almost no nontrivial ones. Instead the concept of homotopy kernels and homotopy cokernels are much more important.

Before I can generalize the idea of a maximal green sequence I first need to generalize the idea of a stability condition without always relying on monomorphisms and epimorphisms.

Theorem 4.1.1. *If Λ is a finite dimensional algebra for an indecomposable module M in a module category $mod\Lambda$ for a stability condition ϕ the following are equivalent:*

1. *M is stable.*
2. *For any quotient module N of M $\phi(M) < \phi(N)$.*
3. *For any indecomposable stable module $N \neq M$ such that $(M, N) \neq 0$ $\phi(M) < \phi(N)$.*
4. *For any indecomposable stable module $N \neq M$ such that $(N, M) \neq 0$ $\phi(N) < \phi(M)$.*

Proof. (1)→(2) If N is a quotient module of M, I have the short exact sequence $0 \to \sum_{i=1}^{k} R_i \to M \to N \to 0$. Since M is stable $\phi(R_i) < \phi(M)$ hence $\phi(N) < \phi(M)$.

(2)→(1) This proof is analogous to the proof of (1)→(2).

(1),(2)→(3) If M, N are indecomposable stable modules and $0 \neq f \in (M, N)$ it is easy to see that $Imf \in mod\Lambda$. Take one of its indecomposable summand, L. It is easy to see that L is a submodule of N and a quotient module of M at the same time. Since M, N are stable I have $\phi(M) < \phi(L) < \phi(N)$. The proof of the second statement of (3) can be achieved by swapping M and N.

(1),(2)→(4) This proof is analogous to the proof of (1),(2)→(3).

(4)→(1) Use induction. If M is simple it is of cmyse stable. Otherwise assume that (4)→(1) is already true for all indecomposable modules with dimension less than $dim(M)$. If M satisfies condition (4) for any of its stable submodule L I have $\phi(L) < \phi(M)$ so I only need to focus on the unstable ones. Assume that L is one of its minimal unstable indecomposable submodules such that $\phi(N) \geq \phi(M)$. By induction there has to be an indecomposable stable submodule of N, L such that $\phi(L) \geq \phi(N)$. Hence $\phi(L) \geq \phi(N) \geq \phi(M)$ and $(L, M) \neq 0$. Hence M does not satisfy condition (4) and I have reached a contradiction. As a result (4)→(1) is proven.

(3)→(2) This proof is analogous to the proof of (4)→(1).

□

I can obtain a similar result in the case of semistability.

my

Theorem 4.1.2. *If Λ is a finite dimensional algebra for an indecomposable module M in a module category modΛ for a stability condition ϕ the following are equivalent:*

1. *M is semistable.*
2. *For any quotient module N of M $\phi(M) \leq \phi(N)$.*
3. *For any indecomposable stable module N such that $(M, N) \neq 0$ $\phi(M) \leq \phi(N)$.*
4. *For any indecomposable stable module N such that $(N, M) \neq 0$ $\phi(N) \leq \phi(M)$.*

4.2 Maximal backward-$Hom^{\leq 0}$ orthogonal sequences

Definition 4.2.1. $M_1, M_2, \cdots, M_k$ is a backward $Hom^{\leq 0}$ orthogonal sequence of Schur objects if $(M_i[\geq 0], M_j) = 0$ for all $i > j$.

my goal is to prove that if $M_1, M_2, \cdots, M_k$ is a maximal $Hom^{\leq 0}$-backward orthogonal sequence of Schur objects I can have $E_0(M_1, \cdots, M_k) = \mathcal{T}$.

Lemma 4.2.2. *If H_1, H_2 are two hearts of t-structures, there exists a backward maximal Hom-≤ 0-orthogonal sequence from H_1 to H_2 then the first term of the sequence has to be a simple of H_1 that is not in H_2.*

Proof. Let's first assume that $M \in H_1[l]$ with $l \geq 0$ is the first term. There exists some simple $S \in H_1$ such that $S[l]$ is a subobject of M in $H_1[l]$ which is Abelian. If there exists no non-initial term N in the sequence such that $(N, S) \neq 0$ (note that it is impossible to have $(N[i], S) \neq 0$ for positive i due to $N = \tau^{\leq 0}N$) then the sequence is not maximal because S can be inserted before M. Hence I assume that such an N exists, In this case $(N[l], M) = 0$ or the sequence would have no longer been backward Hom-$\leq$ 0-orthogonal. Let $N' = (\tau^{\geq 0}N$ and $N'' = \tau^{<0}N$. So I have the canonical triangle $N'' \to N \to N' \to N''[1]$. Due to $(N[l], M) = 0$ it is obvious that $(N'[l], M) = 0$. Note that $M, N'[l], S[l] \in H_1[l]$. $(N'[l], S[l]) = 0$ since $S[l]$ is a subobject of M in an Abelian category. As a result $(N', S) = 0$ and hence $(N, S) = 0$ which contradicts the assumption that $(N, S) \neq 0$.

Now let's assume that $M \in \tau^{\geq -l}$ but not $\tau^{\geq -l+1}$. Let $M' = \tau^{\leq -l}M$. Let $S[l]$ be a subobject of M' in $H_1[l]$ which is Abelian. Note that $(S[l], M) = (S[l], M') \neq 0$ because $(S[l], M/M'[1]) = 0$ which is a consequence of . If there exists no non-initial term N in the sequence such that $(N, S) \neq 0$ (note that again it is impossible to have $(N[i], S) \neq 0$ for positive i due to $N = \tau^{\leq 0}N$) then the sequence is not maximal because S can be inserted before M. Hence I assume that such an N exists, In this case $(N[l], M) = 0$ or the sequence would have no longer been backward Hom-≤ 0-orthogonal. Let $N' = \tau^{\geq 0}N$ and $N'' = \tau^{<0}N$. So I have the canonical triangle $N'' \to N \to N' \to N''[1]$. Due to $(N[l], M) = 0$ it is obvious that $(N'[l], M) = 0$ and $(N'[l], M') = 0$. Note that $M', N'[l], S[l] \in H_1[l]$. $(N'[l], S[l]) = 0$ since $S[l]$ is a subobject of M' in an Abelian category. As a result $(N', S) = 0$ and hence $(N, S) = 0$ which contradicts the assumption that $(N, S) \neq 0$.

□

4.3 Harder-Narasimhan filtration

The fact that a category accepts a unique HN filtration is a very strong condition. In this case I can define the *degree* of any object in $D^b(\Lambda)$ as the composition length of the object when decomposed using the HN filtration.

Lemma 4.3.1. *If any object Y in $D^b(\Lambda)$ accept a unique HN filtration $0 \to Y_m \to \cdots \to Y_1 = Y$ with $Y_i/Y_{i+1} \in \mathcal{E}(M_i)$, the following holds.*

1. *For any i M_i is indecomposable.*

2. *For any i M_i is Schur.*

Proof. For (1) assume that M_i is decomposable. Then $M_i = A \oplus B$ with $A \neq 0$ and $B \neq 0$. In this case A and B have nontrivial HN filtrations with entries other than the i-th entry and adding them up I should obtain an HN filtration for M_i with entries other than the i-th entry which contradicts the fact that M_i has a unique HN filtration.

For (2) assume that M_i is not Schur. Then there exists $f \in End(M_i)$ such that $Imf \neq M_i$. Take an indecomposable direct summand of Imf, N. I have the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \longrightarrow & X & \longrightarrow & M_i/Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Q & \longrightarrow & M_i & \longrightarrow & M_i/Q & \longrightarrow & 0 \end{array}$$

Due to $0 \to M_i \to Q \oplus X \to M_i \to 0$ being a short exact sequence, $Q \oplus X$ has two HN-filtrations, one containing the $i-$th entry only while the other definitely contain what is not in the i-th entry because Q can not only have the i-th entry.

$\square$

Lemma 4.3.2. *If any object Y in $D^b(\Lambda)$ accept a unique HN filtration $0 \to Y_m \to \cdots \to Y_1 = Y$ with $Y_i/Y_{i+1} \in \mathcal{E}(M_i)$, the following holds.*

1. *$Hom(M_i, M_j) = 0$ for any $i > j$.*

2. *If $M_j = M_i[1]$ then $j > i$. Moreover if Y is any object and the loIst nonzero entry of the HN filtration of Y has index can not be higher than the highest nonzero entry of the HN filtration of $Y[1]$.*

3. *If $Y_i \neq 0$ $Hom(Y_i, Y) \neq 0$ for any i.*

4. *If $Y/Y_i \neq 0$ $Hom(Y, Y/Y_i) \neq 0$ for any $i > 1$.*

5. *$Hom(M_i[1], M_j) = 0$ for any $i > j$.*

6. *If $M_j = M_i[m]$ where $m > 0$ then $j > i$. Moreover if Y is any object and the loIst nonzero entry of the HN filtration of Y has index can not be higher than the highest nonzero entry of the HN filtration of $Y[m]$ for any positive m.*

7. *$Hom(M_i[m], M_j) = 0$ where $m > 0$ for any $i > j$.*

Proof. (1) is true because otherwise I have completely different HN filtrations for $M_i \oplus M_j$, namely $M_i \stackrel{(1,0)^t}{\to} M_i \oplus M_j \stackrel{(0,1)}{\to} M_j \stackrel{0}{\to} M_i[1]$ and $M_i \stackrel{(1,f)^t}{\to} M_i \oplus M_j \stackrel{(f,-1)}{\to} M_j \stackrel{0}{\to} M_i[1]$.

(2) is true because otherwise $M_i \to 0 \to M_i[1] \to M_i[1]$ will be an HN filtration of 0 and hence there will be at least two HN filtrations of 0. Similarly if the HN filtration of Y is strictly before the HN filtration of $Y[1]$ $0 \to Y \to Y[1] \to 0$ will be an HN filtration of 0.

(3) is true because $Y_i \to Y \to Y/Y_i \to Y_i[1]$ is a triangle. If $Hom(Y_i, Y) = 0$ the triangle splits and $Y/Y_i = Y \oplus Y_i[1]$. Since $Y_i[1]$ has a unique HN filtration Y/Y_i has two HN filtrations, one with the m-th entry 0 and one with a nontrivial m-th entry.

(4) is true because $Y_i \to Y \to Y/Y_i \to Y_i[1]$ is a triangle. If $Hom(Y, Y/Y_i) = 0$ the triangle splits and $Y_i = Y \oplus Y_i[-1]$. Since $Y_i[-1]$ has a unique HN filtration Y_i has two HN filtrations, one with the first entry 0 and one with a nontrivial first entry.

Now let's prove (5). Assume that $i > j$ and $Hom(M_i[1], M_j) \neq 0$. Let's first assume that the highest term of the HN filtration of $M_j[-1]$ is a self-extension of M_i. If this is not the case assume that the highest term is a self-extension of $M_{i'}$. If $i' \leq j$ then it is clear that $Hom(M_i, M_j[-1]) = 0$ since the Hom from M_i to all terms in the HN filtration of $M_j[-1]$ is 0 due to (1). Hence $i' > j$ and I can simply use i' instead of i since $Hom(M_{i'}, M_j[-1]) \neq 0$. So I can indeed assume that he highest entry of the HN filtration of $M_j[-1]$ is a self-extension of M_i. Let such an entry be X_i. Let h be the highest nontrivial index of the HN filtration of $X_i[1]$ then h is loIr than j or the highest nontrivial index of $M_j[-1]/X_i$ both of which are loIr than i due to (4). Apply (4) again to $X_i[1]$ as a self-extension of $M_i[1]$ the highest nontrivial index of the HN filtration of $M_i[1]$ is loIr than i. Apply (2) to M_i and I can reach a contradiction.

As for (6), since (2) is already proven let's assume that the result has been proven for all positive integers below m and use induction. For the first claim it is clear that $M_i[1]$ can not be any M_l or the induction hypothesis would be violated. Take the loIst nonzero entry Y_k of the HN filtration of $M_i[1]$. If $k = i$ then $Hom(M_i[1], M_i) \neq 0$ which can not happen. If $k > i$ then I can apply the induction hypothesis to the HN filtration of $M_i[1]$ and $M_i[m]$ and show that this is false. Hence $k < i$. In this case $Hom(M_i[1], Y_k) \neq 0$. Hence $Hom(M_i[1], M_k) \neq 0$ which is impossible due to (5). Now let's prove the second claim. Let the loIst entry of the HN filtration of $Y[1]$ be $N_k \in \mathcal{E}(M_k)$ and let $0 \to Y_j \to \cdots Y_l \to 0$ be the HN filtration of Y. It is easy to see that $k < l$ due to the induction hypothesis applied to $Y[1]$ and $Y[k]$. Hence $Hom(Y[1], M_k) \neq 0$. Hence for some $h > k$ $Hom(M_h[1], M_k) \neq 0$ which is impossible due to (5).

Finally I need to prove (7). Assume that the result is true for any positive integer below m which is legit because (5) is already proven. Assume that $i > j$ and $Hom(M_i[m], M_j) \neq 0$. Let's first assume that the highest entry of the HN filtration of $M_j[-m]$ is a self-extension of M_i. If this is not the case assume that the highest term is a self-extension of $M_{i'}$. If $i' \leq j$ then it is clear that $Hom(M_i, M_j[-m]) = 0$ since the Hom from M_i to all terms in the HN filtration of $M_j[-m]$ is 0 due to (1). Hence $i' > j$ and I can simply use i' instead of i since $Hom(M_{i'}, M_j[-m]) \neq 0$. So I can indeed assume that he highest entry of the HN filtration of $M_j[-m]$ is a self-extension of M_i and let such an entry be X_i. Let h be the highest nontrivial index of the HN filtration of $M_i[1]$ then h is loIr than the highest nontrivial index of $M_j[-m]/X_i$ or $Hom(M_h, M_j[1-m]) \neq 0$ in which case $h < j$ by induction. Hence in both cases $h < i$. Apply (2) to M_i and I can reach a contradiction.

$\square$

4.4 Equivalence of the definitions

Theorem 4.4.1. *The following are equivalent.*

1. *The sequence is a maximal sequence of backward $Hom^{\leq 0}$-orthogonal Schurian objects $\{M_n\}$ on $\mathcal{T}$.*

2. *The sequence is a finite c-green sequence on $\mathcal{T}$.*

3. *The sequence is a green sequence from a simple-minded collection $\{X_i\}$ to another one $\{Y_i\}$.*

Proof. (1)→(3) This has been proven in Lemma 4.2.2.

(2)→(1) This has already been proven in Lemma 4.3.2. In particular maximality holds because of 4.3.2(3) and (4).

(3)→(2) This is obvious because using truncation functors I can easily show that any object in $\cup_{i=0}^{k} modkQ[i]$ can be written uniquely as an HN filtration. □

It is easy to see this theorem can be easily generalized to arbitrary green sequences.

Theorem 4.4.2. *Let Λ be a finite dimensional algebra. Let $(C^{\leq 0}, C^{\geq 0}), (C'^{\leq 0}, C'^{\geq 0})$ be two t-structures such that there exists at least one green sequence from $(C^{\leq 0}, C^{\geq 0})$ to $(C'^{\leq 0}, C'^{\geq 0})$. Let $\mathcal{T} = C^{\leq 0} \cap C'^{\geq 0})$. Let $M_1, \cdots, M_n$ be a finite sequence in $\mathcal{T}$. The following are equivalent.*

1. *The sequence is a maximal sequence of backward $Hom^{\leq 0}$-orthogonal Schurian objects $\{M_n\}$ on $\mathcal{T}$.*

2. *The sequence is a finite c-green sequence on $\mathcal{T}$.*

3. *The sequence is a green sequence from a simple-minded collection $\{X_i\}$ to another one $\{Y_i\}$.*

Chapter 5

Quivers with multiple edges

5.1 Introduction

Maximal green sequences (MGSs) were invented by Bernhard Keller [22]. Brustle-Dupont-Perotin [5] and the paper by the first author together with Brustle, Hermes and Todorov [6] have proven that there are finitely maximal green sequences when the quiver is of finite, tame type or the quiver is mutation equivalent to a quiver of finite or tame types. Furthermore in [6] it is proven that any tame quiver has finitely many k-reddening sequences.

However the situation is still pretty much uncharted in the wild case other than cases where the quiver has three vertices which was proven in [5] which contains a proof highly dependent on the quiver only having three vertices. Despite the fact that the wild case is still unknown in general we can indeed solve it for many easy cases. For example for quivers such as the k-Kronecker quiver and $1 \Rrightarrow 2 \Rrightarrow 3$ things are really simple due to the Target before Source Theorem in [6].

In this chapter we will generalize the results and introduce three theorems that can significantly simplify understanding of maximal green sequences in simply-laced quivers with multiple edges.

We can completely describe MGSs of ME-ful quivers using MGSs of their ME-free versions.

Theorem 5.1.1. *(Theorem 5.3.12) MGSs of an acyclic quiver Q are a subset of the set of Q-ME-free MGSs of its ME-free version, Q'.*

Theorem 5.1.2. *(Theorem 5.3.13) Let Q be an ME-ful acyclic quiver and Q' be its ME-free version. The MGSs of Q are exactly the Q-ME-free MGSs $(C_0, C_1, \cdots C_m)$ of Q' such that for any multiple edge from i to j in Q for any C-matrix C_i in the MGS such that there exists a negative c-vector with support containing i the mutation on C_i in the MGS isn't done on any negative c-vector with support containing j.*

In other words to understand MGSs of an acyclic quiver Q we only need to understand the MGSs of its ME-free version which makes multiple edges largely irrelevant in understanding MGSs of acyclic quivers. We can obtain the following crucial corollaries in the acyclic case:

Corollary 5.1.3. *(Corollary 5.3.14)The following statements are true:*

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*

2. *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*

3. *No minimally MGS-infinite quiver can contain multiple edges.*

4. *Any two ME-equivalent quivers are MGS-equivalent to each other.*

If the quiver isn't necessarily acyclic we still have the following result:

Theorem 5.1.4. *(Theorem 5.4.3)Assume that* $(\tilde{Q}, \breve{Q})$ *are* k*-partition of* Q *for some* $k > 1$ *any MGS of* Q *is an MGS of* $\tilde{Q} \cup \breve{Q}$*.*

In Section 2 we will provide the background required to understand the rest of the paper. In Section 3 we will prove Theorems 5.1.1 and 5.1.2. In Section 4 we will prove Theorem 5.1.4.

5.2 MGS-finiteness

In this section let's review the basics about what kind of quivers have finitely many maximal green sequences.

Definition 5.2.1. A quiver Q is *MGS-finite* if Q has finitely many maximal green sequences. Any quiver that isn't MGS-finite is *MGS-infinite*.

Here are some results that are either already known or easily proven about MGS-finiteness of quivers.

Theorem 5.2.2. *[5]Any acyclic quiver* Q *of finite type or tame type as well as any acyclic quiver* Q *of wild type with three vertices are MGS-finite.*

Theorem 5.2.3. *Any quiver* Q *mutation equivalent to an acyclic quiver of finite or tame type is MGS-finite.*

Proof. Due to [6] the result is already proven in the mutation-equivalent to tame type case. For the mutation-equivalent to finite type case using the Rotation Lemma in [6] it is obvious that any MGS in such a quiver must be an k-reddening sequence of an acyclic quiver of finite type for a fixed k. There are only finitely many such sequences because a k-reddening sequence can only repeat a cluster $k+1$ times as shown in [6] and [21] and in an acyclic quiver of finite type there are only finitely many cluster-tilting objects and hence clusters. $\square$

Lemma 5.2.4. *If Q is a quiver that isn't connected, Q^1, Q^2, $\cdots$ Q^n are its connected components. Each Q^i is MGS-finite if and only if Q is MGS-finite.*

Proof. Any MGS of Q is essentially formed from taking an MGS w_i of Q^i for each i and then put these mutations together such that the order of elements in each w_i is preserved.

Since we can obtain all MGSs of Q^i by deleting all c-vectors not supported on Q^i_0 from all MGSs of Q it is easy to see that if Q is MGS-finite so is Q^i for any i.

On the other hand if all Q^is are MGS-finite it is easy to see that so is Q because the set of admissible c-vectors of Q is the union of admissible c-vectors in MGSs of Q^i all of which are finite.

□

There is also an unrelated result about MGS-finiteness I proved which I will include here.

Definition 5.2.5. A valued quiver is of *finite green mutation type* if there are finitely many exchange matrices along its maximal green sequences.

It is easy to see that any valued-quiver that has finitely many maximal green sequences is of finite green mutation type.

Lemma 5.2.6. *If the coframed quiver $\check{Q}$ of a valued quiver Q is of finite green mutation type, Q has finitely many maximal green sequences.*

Proof. For a valued quiver Q with $|Q_0| = n$, let $Q' = \check{Q}$ be its coframed quiver and Q'' be the coframed quiver of Q'. Let's label the extra vertices of Q' as $1', \cdots, n'$. Note that any maximal green sequence $w = (w_1, \cdots, w_k)$ of Q can be extended into a maximal green sequence of Q', $w' = (w_1, \cdots, w_k, 1', 2', \cdots, n')$. Note that any extended exchange matrix that appears in any maximal green sequence of Q is an exchange matrix in some maximal green sequence of Q'. Since Q' is of finite green mutation type, there are only finitely many exchange matrices in all maximal green sequences of Q'. Hence there are only finitely many extended exchange matrices in any maximal green sequence of Q. Since extended exchange matrices can not be repeated in a maximal green sequence, Q has finitely many maximal green sequences.

$\square$

Here is an easy corollary of the lemma above:

Corollary 5.2.7. *If all valued quivers are of finite green mutation type, all valued quivers have finitely many maximal green sequences.*

5.3 The acyclic case

Now we need some basic definitions in order to describe and prove the results.

Definition 5.3.1. A quiver with at least one multiple edge is *ME-ful*. Otherwise it is *ME-free*.

Definition 5.3.2. A *multiple edges-free (ME-free)* version of a quiver Q is produced by removing all multiple edges from Q while retaining single edges and vertices.

For example the ME-free version of the m-Kronecker quiver for any m is the quiver $A_1 \times A_1$, namely the quiver with two vertices and no arrows.

In this section we will use the fact that a path in the semi-invariant picture of Q is also a path in the semi-invariant picture of its ME-free version, Q'. Since the definition of whether a path is green and generic differ in semi-invariant pictures of different quivers we will use the concept of *strong genetic green paths* to exclude problematic cases.

Definition 5.3.3. Let Q be an ME-ful quiver, Q' be its ME-free version. A path in the semi-invariant pictures of Q and Q' is *strong generic green* if it is a generic green path in both pictures.

Definition 5.3.4. Let Q be an ME-ful quiver.

1. A c-vector in Q is *ME-free* if it is ME-free if considered as a dimensioHowevern vector of Q. Any c-vector in Q that isn't ME-free is *ME-ful*.

2. An MGS in Q is *ME-free* if all its c-vectors are ME-free. An MGS of Q that isn't ME-free is *ME-ful*.

3. A generic green path in the semi-invariant picture of Q is *ME-free* if it crosses no wall corresponding to an ME-ful c-vector. A generic green path in the semi-invariant picture of Q that isn't ME-free is *ME-ful*.

4. A module of kQ is *ME-free/ME-ful* if its c-vector is ME-free/ME-ful.

Note that if an MGS is ME-free all c-vectors in all c-matrices in it including those that aren't mutated must be ME-free.

If Q is an ME-ful quiver and Q' is its ME-free version it does not technically make sense to discuss ME-fulness of any module of kQ'. Here we are going to use the same definition we used in defining ME-fulness of vectors and MGSs of Q.

Definition 5.3.5. Let Q be an ME-ful quiver and let Q' be its ME-free version.

1. A c-vector in Q' is *Q-ME-free* if it is ME-free if considered as a dimension vector of Q. Any c-vector in Q' that isn't Q-ME-free is *Q-ME-ful*.

2. An MGS in Q' is *Q-ME-free* if all its c-vectors are Q-ME-free. An MGS of Q' that isn't Q-ME-free is *Q-ME-ful*.

3. A generic green path in the semi-invariant picture of Q' is *Q-ME-free* if it crosses no wall corresponding to a Q-ME-ful c-vector. A generic green path in the semi-invariant picture of Q' that isn't Q-ME-free is *Q-ME-ful*.

4. A strongly generic green path in the semi-invariant picture of Q' is *strongly Q-ME-free* if it is Q-ME-free and does not cross any wall corresponding to a Q-ME-ful c-vector in the semi-invariant picture of Q. A generic green path in the semi-invariant picture of Q' that isn't strongly Q-ME-free is *weakly Q-ME-ful*.

5. A module of kQ' is *Q-ME-free/Q-ME-ful* if its c-vector is Q-ME-free/Q-ME-ful.

We will sometimes abuse the notations and use the term Q-ME-free for c-vectors/MGSs of Q. In this case they are just ME-free c-vectors/MGSs.

Definition 5.3.6. If Q and Q' have the same number of vertices, a GS w of kQ *is equivalent to* a GS w' of kQ' if w and w' mutates on the same sequence of c-vectors and start from the same c-matrix up to permutations.

Using the equivalence it makes sense to identify certain MGSs of Q and Q'. It is in this sense that we claim and prove that all MGSs of an ME-ful quiver Q are MGSs of its ME-free version, Q'. In order to state a corollary we also need three more definitions.

Definition 5.3.7. The *skeleton* of a quiver Q is produced by replacing all multiple edges from Q by single edges with the sources and targets unchanged.

For example the ME-free version of the m-Kronecker quiver for any m is the quiver A_2.

Definition 5.3.8. Q and Q' are quivers. If they have the same ME-free version and the same skeleton then they are *ME-equivalent.*

Definition 5.3.9. If every MGS of Q corresponds to some MGS of Q' and vice versa then Q and Q' are MGS-equivalent.

Lemma 5.3.10. *Let Q be a quiver and Q' be its ME-free version. The following holds:*

1. *The set of Q-ME-free c-vectors of Q and Q' coincide.*
2. *If Q is an ME-ful quiver then for any positive Q-ME-ful vector $c \in \mathbb{R}^n$ $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$.*
3. *If Q is an ME-ful quiver. Then any of the Q-ME-ful c-vectors can not be a dimension vector of an indecomposable rigid module for Q'. Any of the Q-ME-ful c-vectors of Q' can not be a dimension vector of an indecomposable rigid module for Q.*

Proof. For (1).Let the Euler matrices of Q, Q'' be $E = e_{ij}, E' = (e'_{ij})$ respectively. $\langle c, c\rangle_{kQ} = \langle c, c\rangle_{kQ'}$ because whenever e_{ij}, e'_{ij} differ $c_i = 0$ or $c_j = 0$ leaving the term related to (i, j) 0. Hence the set of Q-ME-free c-vectors of Q, Q' corresponding to indecomposable rigid modules coincide.

For (2) Assume that such a vector, c exists. $\langle c, c\rangle_{kQ} = \langle c, c\rangle_{kQ'} = 1$. However the Euler matrix $E = (e_{ij})$ of Q and the Euler matrix $E' = (e'_{ij})$ of Q' differ in the sense that there exists some pair $(i, j) \in [n]$ such that $c_i \neq 0, c_j > 0$ and $0 = e'_{ij} > -2 \geq e_{ij}$. Since for any $k, l \in [n]$ $e'_{kl} \geq e_{kl}$ it is easy to see that $\langle c, c\rangle_{kQ'} > \langle c, c\rangle_{kQ}$ and that $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$.

(3) is a consequence of (2) since $\langle M, M\rangle_{kQ}$ and $\langle M, M\rangle_{kQ'}$ can not both be 1. $\square$

Lemma 5.3.11. *Let Q be an ME-ful quiver. Any MGS of an ME-ful quiver Q must not contain any Q-ME-ful c-vector of Q' or any vector c which is an imaginary root of Q'.*

Proof. Due to 5.3.10(3) we only need to prove the second part. In that case $\langle c, c\rangle_{kQ} \leq \langle c, c\rangle_{kQ'} < 1$. Hence c is not a c-vector of Q. □

Now we can easily establish the following theorem.

Theorem 5.3.12. *MGSs of an acyclic quiver Q are a subset of the set of Q-ME-free MGSs of its ME-free version, Q'.*

Proof. If the statement is incorrect along an MGS of Q pick the first C-matrix that isn't shared by Q' assuming that such an MGS exists.

In this case either at least one c-vector is Q-ME-ful or none is. If some c-vector is Q-ME-ful it must be formed by extending one Q-ME-free indecomposable rigid module by another Q-ME-free indecomposable rigid module in Q (i.e. $dimExt_{kQ}(A, B) = 1$ because it can not be larger due to Lemma 1.3.12. Let's label the indecomposable module formed by the extension M. We need $Ext_{kQ}(B, A) = 0$ so that $\langle M, M\rangle_{kQ} = 1$) while in Q' there are no such extensions (i.e. $dimExt_{kQ'}(A, B) = 0$). However this is impossible because A, B are rigid, Hom-orthogonal and indecomposable because $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$ which causes $\langle M, M\rangle_{kQ'}$ to be at least 3 which is impossible because $\langle M, M\rangle_{kQ'} = \langle A, A\rangle_{kQ'} + \langle A, B\rangle_{kQ'} + \langle B, A\rangle_{kQ'} + \langle B, B\rangle_{kQ'} = 2 - dimExt_{kQ'}(A, B) - dimExt_{kQ'}(B, A)$ is at most 2.

If no c-vector is Q-ME-ful then in kQ, kQ' the relevant Hom and Ext groups shouldn't differ because neither of them involve the multiple edges that are absent in kQ' . As a result that can't happen either.

Hence the C-matrices corresponding to Q, Q' in the MGS are all the same. Any MGS of Q must be an MGS of Q' with the same C-matrices. Since all the C-matrices of the two quivers are the same they have the same associated permutation.

□

Now we can prove a stronger result.

Theorem 5.3.13. *Let Q be an ME-ful acyclic quiver and Q' be its ME-free version. The MGSs of Q are exactly the Q-ME-free MGSs $(C_0, C_1, \cdots C_m)$ of Q' such that for any multiple edge from i to j in Q for any C-matrix C_i in the MGS such that there exists a negative c-vector with support containing i the mutation on C_i in the MGS isn't done on any negative c-vector with support containing j.*

Proof. Let's compare $\langle M, N\rangle_{kQ}$ and $\langle M, N\rangle_{kQ'}$. They differ if and only if there exists some multiple edge from i to j such that i is in the support of M and j is in the support of N. In this case since $Hom_{kQ}(M,N) = Hom_{kQ'}(M,N) = Ext_{kQ'}(M,N) = 0$ $dimExt_{kQ}(M,N) > 0$. Repeating the argument in 5.3.13 we can show that this is the only possible scenario for a Q-ME-free MGSs of Q' to not be identical to an MGS in Q. □

Corollary 5.3.14. *The following statements are true:*

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*

2. *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*

3. *No minimally MGS-infinite quiver can contain multiple edges.*

4. *Any two ME-equivalent quivers are MGS-equivalent to each other.*

Proof. Only (4) needs to be proven even though it is still obvious. For ME-equivalent quivers Q and Q' the conditions of 5.3.13 are identical which is why the number of MGS are identical. $\square$

Example 5.3.15. The maximal green sequences of Q :
$$\begin{tikzcd} 1 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$
are some maximal green sequences of its ME-free version $Q' : 1 \to 2 \to 3$ that has no c-vector with support containing $\{1, 3\}$ that remain ME-free in Q. It's easy to see that Q is MGS-finite. In fact it has 3 MGSs.

Example 5.3.16. The maximal green sequences of Q :
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[r] & 4 \end{tikzcd}$$
are some maximal green sequences of its ME-free version Q' :
$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[r] & 4 \end{tikzcd}$$
that has no c-vector with support containing $\{2, 3\}$ that remain ME-free in Q. It's easy to see that Q is MGS-finite because A_2 is.

Now we can provide a much shorter proof to the fact that all acyclic quivers with three vertices are MGS-finite which was originally proven in [5].

Corollary 5.3.17. *Any acyclic quiver with at most three vertices is MGS-finite.*

Proof. Due to the theorem we only need to show that any ME-free acyclic quiver with at most three vertices is MGS-finite. Such a quiver is either of finite or tame type and is hence MGS-finite. $\square$

5.4 The general case

In the general case the theorem above isn't correct. We can show that using the following counterexample. The quiver Q here is
$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \\ & 2 \arrow[ur] & \end{tikzcd}.$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\mu_2} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\mu_3} \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\mu_2}$$
$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Here we have a maximal green sequence with at least one ME-full c-vector. Moreover it is easy to see that if we replace the double edge by triple edge and obtain Q' :
$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll] \arrow[ll, shift left] \arrow[ll, shift right] \\ & 2 \arrow[ur] & \end{tikzcd}$$
(2,1,3,1,2) is not a c-vector of the quiver Q' nor is it an MGS of the ME-free version or skeleton of Q.

However we can still perform quiver cutting in more limited situations. Let's first introduce a concept.

Definition 5.4.1. A *k-edge* is a tuple (i, j) where $i, j \in [n]$ and $k|b_{ij}, k|b_{ji}$.

Definition 5.4.2. Let Q be a quiver possibly having oriented cycles, let k be an integer greater than 1. Assume that $Q_0 = \tilde{Q}_0 + \breve{Q}_0$, $P = Q]_{\tilde{Q}_0}, R = Q]_{\breve{Q}_0}$. If for all $i \in \tilde{Q}_0, j \in \breve{Q}_0$ $k|b_{ij}$ and $k|b_{ji}$ we say Q is *k-partible* and $(\tilde{Q}, \breve{Q})$ is a *k-partition* of Q.

Theorem 5.4.3. *Assume that $(\tilde{Q}, \breve{Q})$ are k-partition of Q for some $k > 1$ any MGS of Q is an MGS of $\tilde{Q} \cup \breve{Q}$.*

Proof. The property that for any $i \in Q_1$ and $j \in Q_2$ $k|c_{ij}$ is preserved by mutation. Hence any mutation that cause any c-vector to cross bot has to violate the Sink before Source Theorem. $\square$

Corollary 5.4.4. *Under the conditions of the theorem above, if $\tilde{Q}$ and $\breve{Q}$ are MGS-finite so is Q.*

Proof. If $\tilde{Q}$ and $\breve{Q}$ are MGS-finite so is $\tilde{Q} \cup \breve{Q}$. As a result so is Q due to the theorem. $\square$

Example 5.4.5. Q :
$$\begin{tikzcd} 1 \arrow[dr] & & & 5 \arrow[dd] \\ & 2 \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & 4 \arrow[ur] & \\ 3 \arrow[uu] & & & 6 \arrow[ul] \end{tikzcd}$$
is a quiver with oriented cycles. Due to the theorem we can cut the $2 \rightrightarrows 4$ arrow. After cutting this arrow it is easy to see that Q is MGS-finite.

Example 5.4.6. Q :
$$\begin{tikzcd} & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[dr] & \\ 1 \arrow[ur] & & & 4 \arrow[dl] \\ & 6 \arrow[ul] & 5 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \end{tikzcd}$$
is another quiver with oriented cycles.

Due to the theorem we can cut the $2 \rightrightarrows 3$ and $6 \leftleftarrows\!\leftleftarrows 5$ arrows. After cutting these arrows it is easy to see that Q is MGS-finite.

Bibliography

[1] Takuma Aihara and Osamu Iyama, *Silting mutation in triangulated categories*, J London Math Soc (2012) 85 (3): 633-668.

[2] Maurice Auslander, Idun Reiten and Sverre O. Smalo, *Representation Theory of Artin Algebras*, Cambridge University Press, Aug 21, 1997.

[3] Ibrahim Assem, Daniel Simson and Andrzej Skowronski, *Elements of the Representation Theory of Associative Algebras, Volume 1, Techniques of Representation Theory*, London Mathematical Society Student Texts, 2006.

[4] A.A. Beilinson, J. Bernstein, Pierre Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque 100, 1983. (in French)

[5] Thomas Brüstle, Grégoire Dupont and Matthieu Pérotin, *On Maximal Green Sequences*, Int Math Res Notices (2014), 4547-4586.

[6] Thomas Brüstle, Stephen Hermes, Kiyoshi Igusa and Gordana Todorov, *Semi-invariant pictures and two conjectures on maximal green sequences*, J Algebra **473** (2017): 80-109.

[7] Thomas Brüstle, David Smith and Hipolito Treffinger, *Stability conditions, τ-tilting Theory and Maximal Green Sequences*, arXiv:1705.08227 [math.RT], 2017.

[8] Thomas Brüstle and Dong Yang, *Ordered Exchange Graphs*, Advances in Representation Theory of Algebras, 135–193, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich, 2013.

[9] Vlastimil Dlab and Claus Michael Ringel, *Indecomposable representations of graphs and algebras*, Memoirs of AMS, Vol. 173, 1976.

[10] Anna Felikson, Michael Shapiro, Pavel Tumarkin, *Cluster algebras and triangulated orbifolds*, Advances in Mathematics, Volume 231, Issue 5, (2012) 2953-3002.

[11] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15 (2002), 497-529.

[12] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras IV: Coefficients*, Compositio Math. 143 (2007) 112-164.

[13] Mark Gross, Paul Hacking, Sean Keel and Maxim Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. 31 (2018), 497-608.

[14] Alexander Garver and Gregg Musiker, *On Maximal Green Sequences For Type A Quivers*, G. J Algebr Comb (2017) 45: 553-599.

[15] Dieter Happel, *Triangulated Categories in the Representation of Finite Dimensional Algebras*, London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, 1988. .

[16] Kiyoshi Igusa, Kent Orr, Gordana Todorov and Jerzy Weyman, *Modulated semi-invariants*, arXiv:1507.03051 [math.RT].

[17] Kiyoshi Igusa, Kent Orr, Gordana Todorov and Jerzy Weyman, *Picture groups of finite type and cohomology in type A_n*, unpublished preprint, 2014.

[18] Kiyoshi Igusa, *Linearity of stability conditions*, arXiv:1706.06986.

[19] Kiyoshi Igusa and Gordana Todorov, *Picture groups and maximal green sequences*, unpublished preprint 2014.

[20] Osamu Iyama and Yuji Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, Y. Invent. math. (2008) 172: 117-168.

[21] Kiyoshi Igusa and Ying Zhou, *Tame Hereditary Algebras have finitely many m-Maximal Green Sequences*, arXiv:1706.09118.

[22] Bernhard Keller, *Quiver mutation and quantum dilogarithm identities*, Representations of Algebras and Related Topics, Editors A. Skowronski and K. Yamagata, EMS Series of Congress Reports, European Mathematical Society (2011): 85-116.

[23] Alastair King, Yu Qiu, *Exchange graphs and Ext quivers*, Advances in Mathematics, Volume 285, 1106-1154, 2015.

[24] Steffen Koenig and Dong Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras*, Doc. Math. 19 (2014), 403-438.

[25] Gregory Muller, *The existence of a maximal green sequence is not invariant under quiver mutation*, Combinatorics, Volume 23, Issue 2 (2016) :P2.47.

[26] Daniel Murfet, Derived Categories Part I.

[27] Daniel Murfet, Derived Categories Part II.

[28] Daniel Murfet, Triangulated Categories Part I.

[29] Tomoki Nakanishi and Andrei Zelevinsky, *On tropical dualities in cluster algebras*, Contemporary Mathematics 565, 217-226.

[30] Claus Michael Ringel, *Tame Algebras and Integral Quadratic Forms*, Volume 1099 of *Lecture Notes in Mathematics*, Springer-Verlag, 1984

[31] Ibrahim Saleh, *Exchange Maps of Cluster Algebras*, arXiv:1011.0894 [math.RT], 2014.

[32] Daniel Simson and Andrzej Skowronski, *Elements of the Representation Theory of Associative Algebras, Volume 2, Tubes and Concealed Algebras of Euclidean Type*, London Mathematical Society Student Texts, 2006.

[33] David Speyer, Hugh Thomas, *Acyclic cluster algebras revisited*, In: Buan A., Reiten I., Solberg . (eds) *Algebras, Quivers and Representations. Abel Symposia*, vol 8. Springer, Berlin, Heidelberg.