

# Maximal green sequences of quivers with multiple edges

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# Outline

- 1 Motivation: Kronecker quiver
- 2 Background
  - Simple-minded collections
  - Two results about  $c$ -vectors
  - MGS-finiteness
- 3 The acyclic case
  - Definitions
  - Results
  - Examples
- 4 The general case
  - Limitation of the theorems in the acyclic case
  - Result
  - Examples

# Abbreviations

In this talk GS refers to green sequence, MGS refers to maximal green sequence, ME refers to multiple edges.

# Motivation: Kronecker quiver

Let's explain this phenomenon using the example of Kronecker quiver,  $Q : 1 \rightrightarrows 2$ . It has one maximal green sequence.

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If we cut the double edge we obtain  $Q' : 1 \rightarrow 2$  which we say is the *multiple edge-free (ME-free) version* of  $Q$ . There are two maximal green sequences.

# Comparison of MGS

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} & \xrightarrow{\mu_1} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} & \xrightarrow{\mu_2} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{array}$$

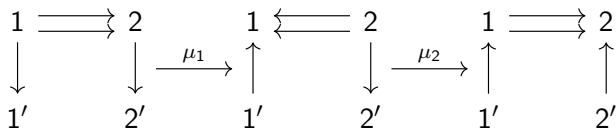
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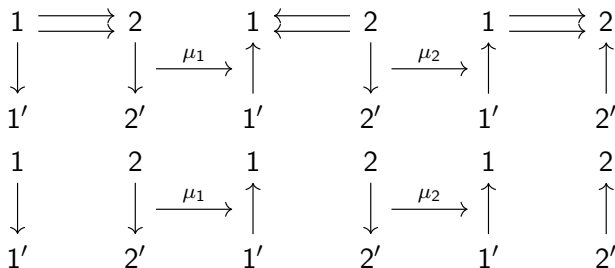
They have the same mutations and the same  $c$ -matrices.



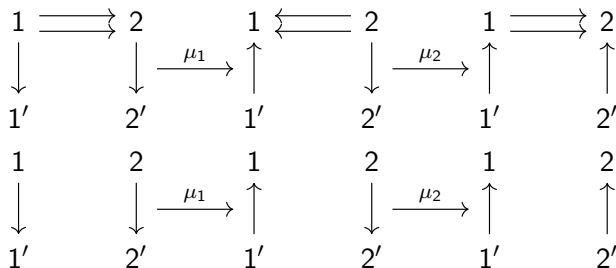
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# Question

- Let  $Q$  be a quiver which has multiple edges. Let  $Q'$  be its ME-free version, what can we say about MGSs of  $Q$  and  $Q'$ ?

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# Simple-minded collections

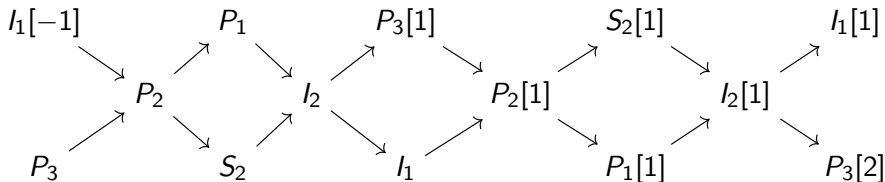
## Simple-minded collections

### Definition

Let  $\Lambda$  be an algebra with  $n$  primitive idempotents. A *simple-minded collection*  $\{S_i\}_{i \in [n]}$  of  $D^b(\Lambda)$  is an  $n$ -element set such that  $(S_i[\geq 0], S_j) = 0$  for all  $i \neq j$ ,  $(S_i[> 0], S_i) = 0$  for all  $i$ ,  $(S_i, S_i)$  is a division algebra and that  $\{S_i\}_{i \in [n]}$  generates  $D^b(\Lambda)$ .

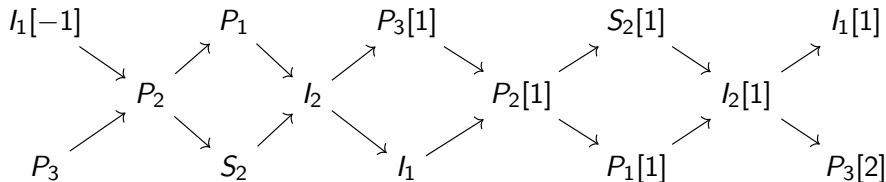
# Simple-minded collections

Ex:  $D^b(A_3)$



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$\{l_1, S_2, P_3\}$  is a simple-minded collection.  $\{P_3[1], P_2, l_1\}$  is also a simple-minded collection.



# Simple-minded collections

## Approximations

### Definition

Let  $\mathcal{C}$  be a category and  $\mathcal{X}$  be one of its subcategories. If  $M \in \text{Ob}\mathcal{C}$ ,  $N \in \text{Ob}\mathcal{X}$ , a morphism  $f \in \text{Hom}_{\mathcal{C}}(M, N)$  is a *minimal left- $\mathcal{X}$  approximation* if for any  $g \in \text{End}_{\mathcal{C}} N$  such that  $g \circ f = f \circ g$  is an isomorphism and for any  $N' \in \text{Ob}\mathcal{X}$  for any  $q \in \text{Hom}_{\mathcal{C}}(M, N')$  we have  $q$  factors through  $f$ .

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$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow & \nearrow q & \\ M' & & \end{array}$$

The diagram illustrates the factorization property of a minimal right- $\mathcal{X}$  approximation. It shows a commutative triangle where a morphism  $q$  from  $M'$  to  $N$  factors through the morphism  $f$  from  $M$  to  $N$ . A dashed vertical arrow labeled  $i$  indicates the inclusion of  $M'$  into  $M$ .

# Simple-minded collections

## Mutations

### Definition

A *forward mutation* on the element  $S_i$  of the simple-minded collection  $\{S_j\}$  is  $\{S'_j\}$  where  $S'_i = S_i[1]$  and  $S'_j$  ( $j \neq i$ ) is the cone/homotopy cokernel of the minimal left- $\text{add}(S_i)$  approximation of  $S_j[-1]$ .

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A *backward mutation* on the element  $S_i$  of the simple-minded collection  $\{S_j\}$  is  $\{S'_j\}$  where  $S'_i = S_i[-1]$  and  $S'_j$  ( $j \neq i$ ) is the cone/homotopy cokernel of the minimal left- $\text{add}(S_i[-1])$  approximation of  $S_j$ .

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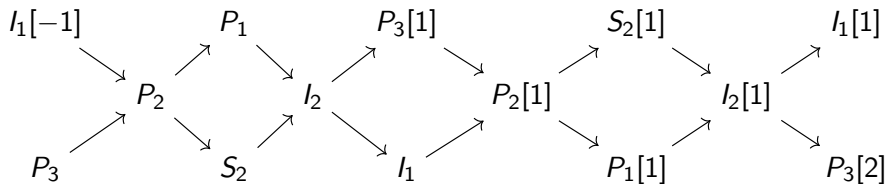
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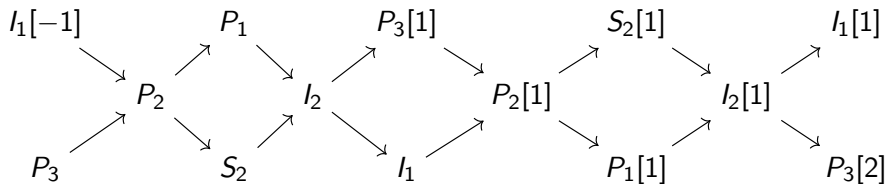
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# Simple-minded collections

Ex:  $D^b(A_3)$



$\{l_1, S_2, P_3\}$  is a simple-minded collection. When we do a forward mutation at  $P_3$  we get  $\{P_3[1], P_2, l_1\}$ . When we do a forward mutation at  $P_2$  now we get  $\{S_2, P_2[1], P_1\}$ . When we then do a forward mutation at  $P_1$  we get  $\{S_2, l_1, P_1[1]\}$ .

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# $c$ -vectors

Positive  $c$ -vectors are dimension vectors of elements of simple-minded collections. Such elements are all bricks. That is, all  $c$ -vectors are Schur. However we can indeed prove more. They are in fact real as well.

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## Lemma

*Let  $k$  be an algebraically closed field. Let  $\Lambda$  be a hereditary algebra over  $k$ . Then any  $c$ -vector  $c$  that appears in any MGS is a real Schur root.*

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The crucial fact we need to use is that for any  $c$ -vector in an MGS  $\langle c, c \rangle = 1$ .

# $c$ -vectors

## Lemma

*If  $-c_1, -c_2$  are negative  $c$ -vectors in  $C$ -matrix  $C'$  in an MGS,  $c_1$  and  $c_2$  are dimension vectors of exceptional modules  $M_1$  and  $M_2$ . If  $\dim \text{Ext}^1(M_1, M_2) > 1$  then the mutation on  $C'$  must not be done on  $M_2$ .*

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# Definition: MGS-finiteness

## Definition

A quiver  $Q$  is *MGS-finite* if  $Q$  has finitely many maximal green sequences. Any quiver that isn't MGS-finite is *MGS-infinite*.



# Which quivers are MGS-finite?

- Acyclic quivers of finite or tame type.  
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(Brustle-Hermes-Igusa-Todorov) [2]
- Quivers with a full subquiver with no MGS. (Muller)[14]

# Are all quivers MGS-finite

## Conjecture

All acyclic quivers are MGS-finite.

# Roadmap to the proof

- Tackle multiple edges.

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We will finish the first step.

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# ME-ful quivers

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A quiver with at least one multiple edge is *ME-ful*. Otherwise it is *ME-free*.

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For example the  $m$ -Kronecker quiver for any  $m > 1$  is ME-ful. On the other hand  $A_2$  is ME-free.

# ME-free versions

## Definition

A *multiple edges-free (ME-free)* version of a quiver  $Q$  is produced by removing all multiple edges from  $Q$  while retaining single edges and vertices.

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For example the ME-free version of the  $m$ -Kronecker quiver for any  $m > 1$  is the quiver  $A_1 \times A_1$ , namely the quiver with two vertices and no arrows.

# ME-ful/ME-free

## Definition

Let  $Q$  be an ME-ful quiver.

- 1 A  $c$ -vector in  $Q$  is *ME-free* if its support is ME-free. Any  $c$ -vector in  $Q$  that isn't ME-free is *ME-ful*.
- 2 An MGS in  $Q$  is *ME-free* if all its  $c$ -vectors are ME-free. An MGS of  $Q$  that isn't ME-free is *ME-ful*.
- 3 A module of  $kQ$  is *ME-free/ME-ful* if its  $c$ -vector is ME-free/ME-ful.



# $Q$ -ME-ful/ $Q$ -ME-free

## Definition

Let  $Q$  be an ME-ful quiver and let  $Q'$  be its ME-free version.

- 1 A  $c$ -vector in  $Q'$  is  $Q$ -ME-free if it is ME-free when considered as a dimension vector of  $Q$ . Any  $c$ -vector in  $Q'$  that isn't  $Q$ -ME-free is  $Q$ -ME-ful.

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- 2 An MGS in  $Q'$  is *Q-ME-free* if all its  $c$ -vectors are *Q-ME-free*. An MGS of  $Q'$  that isn't *Q-ME-free* is *Q-ME-ful*.

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- 3 A module of  $kQ'$  is *Q-ME-free/Q-ME-ful* if its  $c$ -vector is *Q-ME-free/Q-ME-ful*.

# Equivalence of GSs

## Definition

If  $Q$  and  $Q'$  have the same number of vertices, a GS  $w$  of  $kQ$  is *equivalent* to a GS  $w'$  of  $kQ'$  if  $w$  and  $w'$  mutates on the same sequence of  $c$ -vectors and start from the same  $c$ -matrix up to permutations.

# Skeletons

## Definition

The *skeleton* of a quiver  $Q$  is produced by replacing all multiple edges from  $Q$  by single edges with the sources and targets unchanged.

For example the ME-free version of the  $m$ -Kronecker quiver for any  $m$  is the quiver  $A_2$ .

# ME-equivalence and MGS-equivalence

## Definition

$Q$  and  $Q'$  are quivers. If they have the same ME-free version and the same skeleton then they are *ME-equivalent*.

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## Definition

If every MGS of  $Q$  corresponds to some MGS of  $Q'$  and vice versa then  $Q$  and  $Q'$  are MGS-equivalent.

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# Lemmas

## Lemma

*Let  $Q$  be a quiver and  $Q'$  be its ME-free version. The following holds:*

- ① *The set of  $Q$ -ME-free  $c$ -vectors of  $Q$  and  $Q'$  coincide.*
- ② *If  $Q$  is an ME-ful quiver then for any positive  $Q$ -ME-ful vector  $c \in \mathbb{R}^n$   $\langle M, M \rangle_{kQ} - \langle M, M \rangle_{kQ'} \leq -2$ .*
- ③ *If  $Q$  is an ME-ful quiver. Then any of the  $Q$ -ME-ful  $c$ -vectors can not be a dimension vector of an exceptional module for  $Q'$ . Any of the  $Q$ -ME-ful  $c$ -vectors of  $Q'$  can not be a dimension vector of an exceptional module for  $Q$ .*

# Lemmas

## Lemma

*Let  $Q$  be an ME-ful quiver. Any MGS of an ME-ful quiver  $Q$  must not contain any  $Q$ -ME-ful  $c$ -vector of  $Q'$  or any vector  $c$  which is an imaginary root of  $Q'$ .*

# Theorems in the acyclic case

## Theorem

*MGSs of an acyclic quiver  $Q$  are a subset of the set of  $Q$ -ME-free MGSs of its ME-free version,  $Q'$ .*

## Theorem

*Let  $Q$  be an ME-ful acyclic quiver and  $Q'$  be its ME-free version. The MGSs of  $Q$  are exactly the  $Q$ -ME-free MGSs  $(C_0, C_1, \dots, C_m)$  of  $Q'$  such that for any multiple edge from  $i$  to  $j$  in  $Q$  for any  $C$ -matrix  $C_i$  in the MGS such that there exists a negative  $c$ -vector with support containing  $i$  the mutation on  $C_i$  in the MGS isn't done on any negative  $c$ -vector with support containing  $j$ .*

# Corollaries

## Corollary

*The following statements are true:*

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- ② *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*
- ③ *No minimally MGS-infinite quiver can contain multiple edges.*
- ④ *Any two ME-equivalent quivers are MGS-equivalent to each other.*

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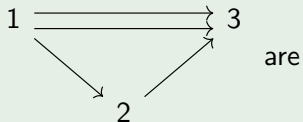
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# Examples

## Example

The maximal green sequences of  $Q$  :



are

maximal green sequences of its ME-free version  $Q' : 1 \rightarrow 2 \rightarrow 3$  that has no  $c$ -vector with support containing  $\{1, 3\}$  and satisfies the conditions in the second theorem above with respect to the arrow  $1 \rightrightarrows 3$ . It's easy to see that  $Q$  is MGS-finite. In fact it has 3 MGSs.

# Examples

## Example

The maximal green sequences of

$Q : 1 \longrightarrow 2 \rightrightarrows 3 \longrightarrow 4$  are some maximal green

sequences of its ME-free version

$Q' : 1 \longrightarrow 2 \qquad 3 \longrightarrow 4$  that has no  $c$ -vector with

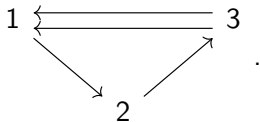
support containing  $\{2, 3\}$  and satisfies the conditions in the second theorem above with respect to the arrow  $2 \rightrightarrows 3$ . It's easy to see that  $Q$  is MGS-finite because  $A_2$  is.

# Outline

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  - MGS-finiteness
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  - Results
  - Examples
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  - Result
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# A counterexample

In the general case the theorems above aren't true. We can show that using the following counterexample. The quiver  $Q$  here is



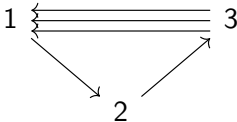
# A counterexample

$$\begin{array}{ccc}
 \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \xrightarrow{\mu_2} & \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \xrightarrow{\mu_1} & \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} & \xrightarrow{\mu_3} \\
 \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} & \xrightarrow{\mu_1} & \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & \xrightarrow{\mu_2} & \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
 \end{array}$$

# A counterexample

Here we have a maximal green sequence with at least one ME-full  $c$ -vector. Moreover it is easy to see that if we replace the double

edge by triple edge and obtain  $Q'$  :



$(2,1,3,1,2)$  is not an MGS of the quiver  $Q'$  nor is it an MGS of the ME-free version or skeleton of  $Q$ .

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  - Examples

# $k$ -edges

## Definition

A  $k$ -edge is a tuple  $(i, j)$  where  $i, j \in [n]$  and  $k | b_{ij}, k | b_{ji}$ .

## Definition

Let  $Q$  be a quiver possibly having oriented cycles, let  $k$  be an integer greater than 1. Assume that  $Q_0 = \tilde{Q}_0 + \check{Q}_0$ ,  $P = Q|_{\tilde{Q}_0}, R = Q|_{\check{Q}_0}$ . If for all  $i \in \tilde{Q}_0, j \in \check{Q}_0$   $k | b_{ij}$  and  $k | b_{ji}$  we say  $Q$  is  $k$ -partible and  $(\tilde{Q}, \check{Q})$  is a  $k$ -partition of  $Q$ .



# Theorem in the general case

## Theorem

*Assume that  $(\tilde{Q}, \check{Q})$  are  $k$ -partition of  $Q$  for some  $k > 1$  any MGS of  $Q$  is an MGS of  $\tilde{Q} \cup \check{Q}$ .*

# Theorem in the general case

## Theorem

*Assume that  $(\tilde{Q}, \check{Q})$  are  $k$ -partition of  $Q$  for some  $k > 1$  any MGS of  $Q$  is an MGS of  $\tilde{Q} \cup \check{Q}$ .*

## Corollary

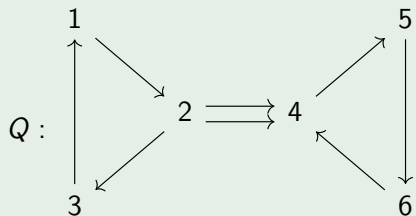
*Under the conditions of the theorem above, if  $\tilde{Q}$  and  $\check{Q}$  are MGS-finite so is  $Q$ .*

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  - Results
  - Examples
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  - Limitation of the theorems in the acyclic case
  - Result
  - **Examples**

# Examples

## Example

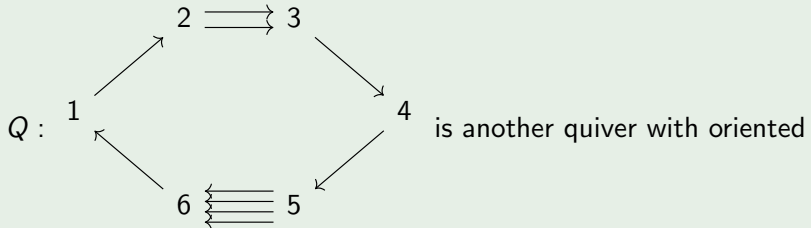


is a quiver with oriented cycles.

Due to the theorem we can cut the  $2 \rightrightarrows 4$  arrow. After cutting this arrow it is easy to see that  $Q$  is MGS-finite.

# Examples

## Example



cycles. Due to the theorem we can cut the  $2 \rightrightarrows 3$  and  $6 \rightrightarrows 5$  arrows. After cutting these arrows it is easy to see that  $Q$  is MGS-finite.

# Summary

- We can completely describe MGSs of an ME-full quiver using MGSs of its ME-free version.

# Summary





- We can completely describe MGSs of an ME-full quiver using MGSs of its ME-free version.
- Any two ME-equivalent quivers are MGS-equivalent to each other.

# Summary





- We can completely describe MGSs of an ME-full quiver using MGSs of its ME-free version.
- Any two ME-equivalent quivers are MGS-equivalent to each other.
- Removal of multiple edges is possible in more limited cases when the quiver isn't acyclic.







# For Further Reading I

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-  Gregory Muller, *The existence of a maximal green sequence is not invariant under quiver mutation*, Combinatorics, Volume 23, Issue 2 (2016) :P2.47.