Four problems related to maximal green sequences

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

Kiyoshi Igusa, Department of Mathematics, Advisor

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

Ying Zhou

May, 2019

This dissertation, directed and approved by Ying Zhou's committee, has been accepted and approved by the Graduate Faculty of Brandeis University in partial fulfillment of the requirements for the degree of:

DOCTOR OF PHILOSOPHY

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Acknowledgments

First of all I would like to thank my advisor, Kiyoshi Igusa for his constant patience, dedication and support. I also would like to thank Gordana Todorov for being my de facto second advisor. I would like to thank Otto Kerner, Hugh Thomas, Mike Prest and Nathan Reading for their guidance. I also would like to thank Alex Martsinkovsky, Ralf Schiffler, Thomas Brustle, Dan Zacharia, Graham Leuschke, Shiping Liu and Nathan Reading for their hospitality during my visits.

I would like to thank my wonderful colleagues who worked and work in the same field as me, such as Charles Paquette, Hipolito Treffinger, Dylan Rupel, Steve Hermes, Alexander Garver, Kaveh Mousavand, Jacob Matherne, Emily Barnard, Shijie Zhu, Van Nguyen, Job Rock, Eric Hanson, Matt Garcia, Moses Kim and PJ Apruzzese. I also would like to thank Jordan Tirrell, Devin Murray, Joshua Eike, Daoyuan Han, Shahriar Mirzadeh, Tarakaram Gollamudi, Aaron Berkowitz, Nick Wadleigh, Cristobal Lemus-Vidales, Qing Liu, Anurag Rao and Mishel Skenderi. My life at Brandeis would have been boring without you all.

Special thanks go to Susan Parker, Becci Torrey and Keith Merrill for being such great course coordinators. I also would like to thank Alex Mitchell, Yuze He, Daniel Johnston, Prabu Gugagantha and many other undergrads I have taught, helped and chatted with for the great time we spent together.

Finally I would like to thank my family and other friends.

Abstract

Four problems related to maximal green sequences

A dissertation presented to the Faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Ying Zhou

Maximal green sequences were invented by Bernhard Keller and have a lot of applications in cluster algebra and particle physics. In this dissertation I will discuss four separate topics related to maximal green sequences. First of all I will discuss the problem of associated permutations of mutation sequences and establish a formula for the associated permutation in the case of A_n straight orientation which answers a question raised at a workshop in Snowbird, Utah in 2014. Secondly I will introduce the concept of m-maximal green sequences and discuss the problem of m-maximal green sequence-finiteness of path algebras of tame quivers which generalizes a result by Brustle, Dupont and Perotin. Then I will introduce two alternative definitions of m-maximal green sequences of hereditary algebras and show that they are both equivalent to the known ones which extends a result by Igusa. Finally I will discuss the problem of maximal green sequences in quivers with multiple edges and fully describe maximal green sequences of quivers with multiple edges which is the first step towards a proof of the conjecture that all acyclic quivers have finitely many maximal green sequences.

Contents

Chapter 1

Background

1.1 Notations and conventions

In this paper k is an algebraically closed field, all algebras will be finitely dimensional k-algebras. All modules are assumed to be right modules. The symbol $[n]$ is defined as the set $\{1, 2, \cdots, n\}$ which is consistent with how it is usually used in cluster theory. If $\mathcal{C}$ is an Abelian category then $K_0(\mathcal{C})$ is its Grothendieck group.

If $\mathcal{C}$ is a category, $M, N, L \in Ob\,\mathcal{C}, f \in Hom_{\mathcal{C}}(M, N), g \in Hom_{\mathcal{C}}(N, L)$ the composition is written as gf. Let Q be a quiver, let v be a path from $i \in Q_0$ to $j \in Q_0$, w be a path from j to $k \in Q_0$ then the composition is written as vw which is the opposite of how we denote morphism compositions.

When the category we are discussing is clearly $\mathcal{C}$ then (M, N) will be an abbreviation of $Hom_{\mathcal{C}}(M, N)$. $(M[>0], N)$ is the union of $(M[k], N)$ for all $k > 0$. If $\mathcal{P}$, $\mathcal{Q}$ are subcategories of $D^b(\Lambda)$ then $(\mathcal{P}, \mathcal{Q}) := \bigcup\limits_{M \in \mathcal{P}} \bigcup\limits_{N \in \mathcal{Q}} (M, N)$ and $(\mathcal{P}[>0], \mathcal{Q}) := \bigcup\limits_{k>0} (\mathcal{P}[k], \mathcal{Q})$. Let S be a set of objects in an additive category $\mathcal{C}$. $add(S)$ is defined as the set of all elements of $\mathcal{C}$ such that they are summands of finite direct sums of objects in S. $\mathcal{E}(X)$ is the extension closure of X.

The definition of green and red (vertices, mutations, sequences) are consistent with that of [28] and [6]. It is the exact opposite definition of green and red in [7].

1.2 Quivers and path algebras

In this section we will introduce the basics about quivers, path algebras, modules, c-vectors, Euler matrices and Euler-Ringel forms. Materials in this section will be used extensively in the rest of the dissertation.

1.2.1 Quivers

Definition 1.2.1. A *quiver* Q is a quadruple (Q_0, Q_1, s, t) with Q_0 and Q_1 sets and s, t: $Q_1 \to Q_0$. An element of Q_0 is a *vertex* of Q. An element of Q_1 is an *arrow* of Q. The map s maps each arrow to its source and t maps each arrow to its target.

Intuitively we can think of elements of Q_1 as oriented edges. Any arrow has a unique source and a unique target both of which are vertices. This is how we obtain the s and t maps. Unless necessary we generally omit the s and t and denote a quiver by $Q = (Q_0, Q_1)$.

Example 1.2.2. The following diagrams are all quivers.

$1 \longrightarrow 2 \qquad 1 \leftleftarrows 2 \longrightarrow 3 \qquad 1 \longrightarrow 2 \longrightarrow 3$

Now let's define opposite quivers.

Definition 1.2.3. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. The *opposite quiver* $Q^{op} := (Q_0^{op}, Q_1^{op}, s^{op}, t^{op})$ is a quiver such that $Q_0^{op} = Q_0$, $Q_1^{op} = Q_1$ and that for any $x \in Q_1^{op}$ we have $s^{op}(x) = t(x)$ and $t^{op}(x) = s(x)$.

That is, Q^{op} is formed by keeping all the vertices of Q and reversing all its arrows. Now we need to define subquivers.

Example 1.2.4. The opposite quiver of $Q : 1 \to 2 \to 3$ is $Q^{op} : 1 \leftarrow 2 \leftarrow 3$.

Definition 1.2.5. A *subquiver* $Q' = (Q'_0, Q'_1, s', t')$ in a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$, $s|_{Q'_1} = s'$ and $t|_{Q'_1} = t'$.

From now on we generally do not distinguish between s and s', t and t'.

Not all quivers are useful for the purpose of this paper. This is why we need to add restrictions. In order to do so we need to introduce several definitions.

Definition 1.2.6. An *oriented cycle* in a quiver Q is a subquiver $Q' = (Q'_0, Q'_1, s, t)$ such that $Q'_0 = \{v_0, v_1, \cdots, v_{k-1}\}, Q'_1 = \{a_0, a_1, \cdots, a_{k-1}\}, s(a_i) = v_i$ and $t(a_i) = v_{i+1}$. Here v_k is defined as v_0.

Definition 1.2.7. A *k-cycle* is an oriented cycle with k vertices.

Definition 1.2.8. A *loop* in a quiver Q is an arrow from a vertex to itself, that is, a 1-cycle.

Definition 1.2.9. A *cluster quiver* is a quiver without loops or 2-cycles.

In all but Chapter 4 and a part of Chapter 5 all quivers we discuss will be acyclic. Here is the definition of an acyclic quiver.

Definition 1.2.10. An *acyclic quiver* is a quiver without any oriented cycles.

1.2.2 Path Algebras

Definition 1.2.11. A *path* in a quiver Q is a sequence of vertices $\{v_0, \cdots, v_k\}$ and a sequence of arrows $\{a_0, \cdots, a_{k-1}\}$ if $k > 0$ such that $t(a_i) = v_{i+1}$, $s(a_i) = v_i$ for any $i = 0, 1, \cdots, k-1$. The source of the path is v_0 and the sink is v_k.

Paths with length 0 are known as *trivial paths*. A trivial path only has a single vertex v_0 and no arrows at all. All other paths are uniquely determined by their arrows.

Now we need to define multiplication of paths. In order to do so we need to define compatibility and concatenation.

Definition 1.2.12. Paths v, w are *compatible* if $t(v) = s(w)$.

Definition 1.2.13. The *concatenation* of compatible paths $v = \{a_0, \cdots, a_{k-1}\}$ and $w = \{b_0, \cdots, b_{l-1}\}$ is $vw = \{a_0, \cdots, a_{k-1}, b_0, \cdots, b_{l-1}\}$.

Definition 1.2.14. The *path algebra* of a quiver Q is a k-algebra generated by all the paths of the quiver. Multiplication of paths v and w is defined as the concatenation vw if they are compatible and 0 if they aren't,

From a homological point of view path algebras are very nice, namely they are *hereditary*. In other words their global dimensions are at most one.

Theorem 1.2.15. *The path algebra kQ of any acyclic quiver Q is hereditary. That is, for any $M, N \in modkQ$ for all $k > 1$ it is true that $Ext^k(M, N) = 0$.*

We mostly only discuss path algebras of acyclic quivers in this thesis because our results are only about finite dimensional algebras.

1.2.3 Modules in hereditary algebras

Now let's review some basic concepts about modules. In particular we will review the concepts of bricks and stones.

Definition 1.2.16. An indecomposable module M over an algebra Λ is *Schur* or a *brick* if its endomorphism ring $EndM$ is a division algebra. In particular if Λ is a k-algebra where k is an algebraically closed field then a brick is an indecomposable module such that $EndM = k$.

Definition 1.2.17. An indecomposable module M over an algebra Λ is *rigid*, *exceptional* or a *stone* if $Ext^1(M, M) = 0$.

Here is a well-known result about Schur and rigid modules in hereditary algebras.

Theorem 1.2.18. *[3]Let k be an algebraically closed field, let Λ be a finite-dimensional k-algebra. Then any rigid $\Lambda-$module M is Schur.*

1.2.4 Euler matrices and the Euler-Ringel form

Now let's define *Euler matrices* which will be very useful in the understanding of Chapter 5.

Definition 1.2.19. Let Q be an acyclic quiver. The *Euler matrix of* Q is defined as the matrix $E = (e_{ij})$ where $e_{ij} = \begin{cases} 1 & \text{if } i = j \\ -k & \text{if there are } k \text{ arrows from } i \text{ to } j \\ 0 & \text{in all other cases} \end{cases}$

Example 1.2.20. The Euler matrix E of the quiver $Q : 1 \to 2$ is $E = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Example 1.2.21. The Euler matrix E of the quiver $Q : \ 1 \longrightarrow 2 \rightrightarrows 3$ is $E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Using the Euler matrix we can define *Euler-Ringel forms*.

Definition 1.2.22. Let Q be an acyclic quiver and E be its Euler matrix. The Euler-Ringel form of Q (or kQ) is defined as $\langle x, y\rangle_Q := x^t E y$.

When which Q we are talking about isn't ambiguous we can just use $\langle x, y\rangle$ to refer to the Euler-Ringel form of Q.

1.2.5 Dimension vectors and roots

The concept of roots originated from Lie theory. Here we will introduce some basic terminologies. Basically a lot of concepts we define using modules can also be defined using their *dimension vectors* which we will define here.

Definition 1.2.23. The *dimension vector* of a module M in a finite dimensional k- algebra Λ with n indecomposable idempotents $e_1, \cdots, e_n$ is defined as $c_M := \{c_1, \cdots, c_n\} \in \mathbb{Z}^n$ where $c_i := dim_k M e_i$.

Now let's define real roots and imaginary roots. Before that we first need to define sign coherence in vectors.

Definition 1.2.24. A vector c in $\mathbb{Z}^n$ is *sign coherent* if it is nonzero and all its entries are either all nonpositive or all nonnegative.

The entries of a sign coherent vector are either all nonnegative or all nonpositive. In the former case we say it is *positive*. In the latter case we say it is *negative*.

Definition 1.2.25. Let Q be an acyclic quiver with n vertices. $c \in \mathbb{N}^n$ is a *real root* if it is sign coherent and $\langle c, c \rangle_Q = 1$.

Definition 1.2.26. Let Q be an acyclic quiver with n vertices. $c \in \mathbb{N}^n$ is an *imaginary root* if it is sign coherent and $\langle c, c \rangle_Q \leq 0$.

1.2.6 Cartan and Euler matrices

Now we can define Cartan matrices and provide another characterization of Euler matrices.

Definition 1.2.27. Let Q be an acyclic quiver. The *Cartan matrix* of Q is defined as $C := (c_{ij})$ where $c_{ij} = \#\{\text{paths from } j \text{ to } i\}$.

Since Q is acyclic for any vertex $i \in Q_0$ there exists one and only one path from i to itself, hence c_{ii} has to be 1.

Example 1.2.28. The Cartan matrix C of the quiver $Q: \ 1 \longrightarrow 2 \rightrightarrows 3$ is $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

It is easy to see that $C^t E = I$ in the example above. In fact this is something true in general.

Theorem 1.2.29. *Let Q be an acyclic quiver, E be its Euler matrix and C be its Cartan matrix. It is true that $C^t E = I$.*

Proof. Since Q is acyclic we can relabel elements of Q_0 as $1, 2, \cdots, n$ to ensure that for any i, j such that $i \geq j$ there are no arrows from i to j. Let $C = (c_{ij}), E = (e_{ij})$. Using the definition of Euler matrices for any $i \neq j$ there are $a_{ij} := -e_{ij}$ arrows from i to j. It is easy to see that C^t and E are both upper triangular with all diagonal entries equal to 1. Hence $C^t E$ is upper triangular with all diagonal entries equal to 1.

Now we need to prove that all other entries of $B := C^t E = (b_{ij})$ above the diagonal is 0. Let $n = |Q_0|$. To show that we only need to prove that $b_{1n} = 0$ assuming that $n > 1$. $b_{1n} = \Sigma_{i=1}^{n} c_{i1} e_{in} = c_{n1} - \Sigma_{i=1}^{n-1} c_{i1} a_{in}$. We know that c_{i1} is the amount of paths from 1 to $i$and a_{in} is the amount of arrows from i to n. Since any path w from 1 to n is non-trivial it can be uniquely decomposed as $w = w'l$ where l is the last arrow in it. Here w' can be trivial which means l is an arrow from 1 to n. It is clear that $c_{n1} = \Sigma_{i=1}^{n-1} c_{i1} a_{in}$. Hence $b_{1n} = 0$. Similarly we can show that any entry of B above the diagonal is 0. Hence $B = C^t E = I$. $\square$

Now we have the following crucial result about Euler-Ringel form which we will implicitly use extensively in Chapter 5.

Theorem 1.2.30. *[3] (Prop III.3.13) Let Q be an acyclic quiver. Let kQ be its path algebra. Then for any pair M, N of modules in $modkQ$ we have $\langle dimM, dimN \rangle = Hom(M, N) - Ext^1(M, N)$.*

1.3 Mutations, mutation sequences and the associated permutation

In this section we will introduce mutations of quivers and matrices, different kinds of mutation sequences including green sequences, maximal green sequences, reddening sequences and loop sequences. Stability conditions will be introduced. We will also define the associated permutation. Results in this section are mostly used in Chapters 2 and 5.

1.3.1 Mutation of quivers

The concept of maximal green sequences has many different equivalent definitions. We will use a simple definition using quiver mutations in this subsection. Later we will introduce other definitions.

Definition 1.3.1. Let Q be a cluster quiver. *Mutation* of Q at vertex k is defined in the following way:

1. For any pair of arrows $i \to k$ and $k \to j$ add an arrow $i \to j$.

2. Reverse all arrows starting from or ending up in k.

3. Delete all 2-cycles that are formed due to process (1) and (2).

Definition 1.3.2.
1. The *framed quiver* $\hat{Q}$ of Q is obtained from Q by adding a vertex i' and an arrow $i \to i'$ for every $i \in Q$.

2. The *coframed quiver* $\check{Q}$ of Q is obtained from Q by adding a vertex i' and an arrow $i' \to i$ for every $i \in Q$.

3. An *ice quiver* is a quiver Q where a possibly empty set, $F \subseteq Q_0$, consists of vertices that are not allowed to mutate.

An ice quiver (Q, F) can not mutate at elements of F, so we call them *frozen vertices.*

Definition 1.3.3. A non-frozen vertex i is *green* if and only if no arrow from a frozen vertex to i exists. Otherwise it is *red.*[28]

Example 1.3.4. In this graph below we did a mutation at 1 from the framed quiver of $Q : 1 \to 2$. After the mutation the vertex changed from being green to being red.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_1}"] & 1 & 2 \arrow[l] \arrow[d] \\ 1' & 2' & 1' \arrow[u] & 2' \end{tikzcd}$$

Definition 1.3.5. A *green sequence* is a sequence $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ such that for all $1 \leq t \leq N$ the vertex i_t is green in the partially mutated ice quiver $\hat{Q}(\mathbf{i}, t) = \mu_{i_{t-1}} \cdots \mu_2 \mu_1(\hat{Q})$.

Definition 1.3.6. A *maximal green sequence* is a green sequence such that $\hat{Q}(\mathbf{i}, N)$ does not have any green vertices.

Example 1.3.7. For quiver $1 \to 2$ here is one of its two maximal green sequences.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_1}"] & 1 & 2 \arrow[l] \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_2}"] & 1 \arrow[r] & 2 \\ 1' & 2' & 1' \arrow[u] & 2' & 1' \arrow[u] & 2' \arrow[u] \end{tikzcd}$$

We also need the definition of reddening sequences which are generalized versions of maximal green sequences in order to discuss the phenomenon of almost morphism finiteness in Chapter 3.

Definition 1.3.8. A *red-to-green sequence* or a *reddening sequence*, is a sequence $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ that transforms $\hat{Q}$ to a quiver $\hat{Q}(\mathbf{i}, N) = \mu_{i_N} \cdots \mu_2 \mu_1(\hat{Q})$ such that $\hat{Q}(\mathbf{i}, N)$ does not have any green vertices.[31]

1.3.2 Mutation of matrices

We can also use *c-vectors* for this purpose. To do so we need to reinterpret mutations of cluster quivers in terms of mutations of matrices. We recall that cluster quivers correspond to *exchange matrices* as defined below. For more details we recommend [18] and [19].

Definition 1.3.9. [18] An *exchange matrix* of a cluster quiver Q with n vertices is an $n \times n$ matrix such that b_{ij} is the number of arrows from i to j minus the number of arrows from j to i.

It is easy to see that exchange matrices of cluster quivers are always antisymmetric which is not true in the more general case of *valued quivers* which we won't discuss in this paper. Moreover there is a 1-1 correspondence between antisymmetric exchange matrices and cluster quivers.

Mutations of exchange matrices are defined here which exactly agree with mutations of cluster quivers.

Definition 1.3.10. [18] If we mutate an $n \times n$ exchange matrix $B = (b_{ij})$ at k we obtain

$B' = (b'_{ij})$ defined here. $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}|b_{kj}| & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} & \text{in all other cases} \end{cases}$

Each partially mutated ice quiver corresponds to an *extended exchange matrix* defined below.

Definition 1.3.11. The *extended exchange matrix* B' corresponding to a partially mutated ice quiver Q' is an $2n \times n$ matrix with the rows corresponding to vertices $\{1, 2, \cdots, n, 1', 2', \cdots n'\}$ while the columns corresponds to the vertices $\{1, 2, \cdots, n\}$. Here we use the number $n+i$ to represent i'. Here again b_{ij} is the number of arrows from i to j minus the number of arrows from j to i.

An extended exchange matrix B' has an upper and lower square submatrices, B and C respectively. The lower square matrix C is known as the *c-matrix.* Column vectors of an c-matrix are known as *c-vectors.* A c-vector is positive if all its entries are non-negative and at least one is positive. A c-vector is negative if all its entries are non-positive and at least one is negative. Here is an important result about c-vectors, namely *sign coherence.*

Definition 1.3.12. Let Q be an acyclic quiver and B its exchange matrix. A c-vector is *reachable* if it is a column vector in some C such that some $\begin{pmatrix} \tilde{B} \\ C \end{pmatrix}$ that can be obtained from $\begin{pmatrix} B \\ -I_n \end{pmatrix}$ through mutations.

Theorem 1.3.13. *[20][14][15] a reachable c-vector is either positive or negative.*

Moreover c-vectors can be completely described due to the following result by Chavez[12].

Theorem 1.3.14. *Let Q be an acyclic quiver. The set of c-vectors associated to Q is equal to the set of real Schur roots associated to Q and their opposites.*

Now we need to discuss the special case which we will use repeatedly, namely the case of A_n straight orientation.

Definition 1.3.15. Let Q be an acyclic quiver. A representation $(\{V_i\}_{i\in Q_0}, \{\phi_a\}_{a\in Q_1})$ of Q is *thin* if $dim(V_i) \leq 1$ for all $i \in Q_0$.

If Q is A_n straight orientation, namely quivers of the form $1 \to 2 \to 3 \to \cdots \to n$. Note that for quiver A_n all roots are real and Schur hence a real Schur root is just a root. In A_n in particular since all indecomposable representations are thin, the positive roots are $\beta_{ij} = e_j - e_i$ ($0 < i < j < n$, e_0 is defined as the zero vector). Hence we have the following lemma.

Lemma 1.3.16. *Any c-vector associated to A_n straight orientation is in the form of $\pm\beta_{ij}$.*

A mutation on vertex k is *green* if the c-vector c_k before the mutation is negative. A mutation on vertex k is *red* if the c-vector c_k before the mutation is positive. A *maximal green sequence* is a mutation sequence from $C = -I_n$ to a permuted version of I_n We can use a sequence of c-vectors to denote a maximal green sequence because we can use the c-vector corresponding to vertex k to represent mutation at vertex k which is possible since all c-matrices are invertible.

1.3.3 Stability conditions

Here is the definition of stability functions.

Definition 1.3.17. [5](Def 2.1) A *stability function* on an abelian category $\mathcal{C}$ is a group homomorphism $Z : K_0(\mathcal{C}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{C}$ the complex number $Z(E)$ lies in the strict upper half-plane $H = \{re^{i\pi\phi} : r > 0 \text{ and } 0 < \phi \leq 1\} \subseteq \mathbb{C}$.

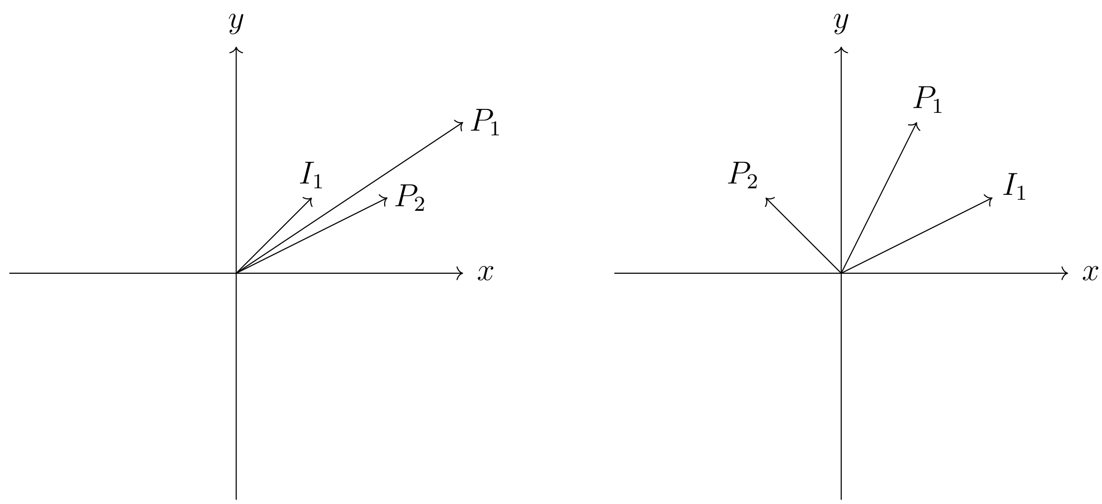
Now let's define the phase of a nonzero object.

Definition 1.3.18. [5] Let $Z : K_0(\mathcal{C}) \to \mathbb{C}$ be a stability function on an Abelian category $\mathcal{C}$. The *phase* of an object $0 \neq E \in \mathcal{C}$ is defined as $\phi(E) := \dfrac{1}{\pi} argZ(E)$.

Now we can define what it means for a nonzero object in an Abelian category to be semistable and stable.

Definition 1.3.19. [5](Def 2.2) Let $Z : K_0(\mathcal{C}) \to \mathbb{C}$ be a stability function on an Abelian category $\mathcal{C}$. An object $0 \neq E \in \mathcal{C}$ is said to be *semistable* (with respect to Z) if every subobject $0 \neq A \subseteq E$ satisfies $\phi(A) \leq \phi(E)$.

Definition 1.3.20. [5] Let $Z : K_0(\mathcal{C}) \to \mathbb{C}$ be a stability function on an Abelian category $\mathcal{C}$. An object $0 \neq E \in \mathcal{C}$ is said to be *stable* (with respect to Z) if every subobject $0 \neq A \subsetneq E$ satisfies $\phi(A) < \phi(E)$.



Example 1.3.21. Here are two stability functions of *modkQ* where Q is the quiver $1 \to 2$. From the picture on the left we can see that P_2, I_1 and P_1 are all stable. P_2 and I_1 are stable because they are simples. As for P_1 it is stable because its submodule P_2 satisfies $\phi(P_2) < \phi(P_1)$.
From the picture on the right we can observe that P_2 and I_1 are all stable. P_2 and I_1 are still stable because they are simples. As for P_1 it is unstable because its submodule P_2 satisfies $\phi(P_2) \geq \phi(P_1)$.

We will discuss stability conditions more in Chapter 4.

1.3.4 Permutations

All reddening sequences have associated permutations. When comparing the quivers obtained from transforming the same framed quiver using two different reddening sequences, it is easy to see that they are just one permutation away from each other: If you do a correct permutation of vertices (that means both rows and columns together) you can transform one such matrix into another. In particular any quiver obtained by using a reddening sequence to transform a framed quiver is one permutation away from the coframed quiver.

In this subsection if Q is a cluster quiver then $\hat{Q}, \breve{Q}$ are the framed and coframed quiver associated with cluster quiver Q respectively.

Here is the formal definition of such a permutation:

Definition 1.3.22. [6] A *permutation* from an ice quiver (Q, F) to (Q', F) is an isomorphism of quivers $Q \to Q'$ that preserve F.

We have a result from [6] which helps us define the permutation:

Theorem 1.3.23. *[6] Let Q be a cluster quiver and let Q' be a quiver that is a result of a reddening sequence on $\hat{Q}$, then Q' equals to a permutation of $\breve{Q}$. That is, for a reddening sequence $\mathbf{i} = (i_1, \cdots, i_N)$, for some $\rho \in S_n$ we have $\mu_{i_N} \cdots \mu_{i_1} \hat{Q} = \rho \breve{Q}$.*

Definition 1.3.24. [21] The *permutation of a reddening sequence* $\mathbf{i}$ is ρ for which $\mu_{i_N} \cdots \mu_{i_1} \hat{Q} = \rho \breve{Q}$.

Here are some simple examples that illustrate the concept of the permutation:

Example 1.3.25. Here is a maximal green sequence of $Q : 1 \to 2$.

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & 2 & & 1 & \longleftarrow & 2 & & 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow & \xrightarrow{\mu_1} & \uparrow & & \downarrow & \xrightarrow{\mu_2} & \uparrow & & \uparrow \\ 1' & & 2' & & 1' & & 2' & & 1' & & 2' \end{array}$$

It is clear that this sequence has permutation id.

Example 1.3.26. Here is another maximal green sequence of $Q : 1 \to 2$.

$$\begin{tikzcd}
1 \arrow[r] \arrow[dd, green] & 2 \arrow[dd, green] & 1 \arrow[dd, green] \arrow[ddr, green] & 2 \arrow[l] & 1 \arrow[r] & 2 \arrow[ddl, green] & 1 & 2 \arrow[l] \\
 & {} \arrow[r, "\mu_2"] & {} & {} \arrow[r, "\mu_1"] & {} & {} \arrow[r, "\mu_2"] & {} & \\
1' & 2' & 1' & 2' \arrow[uu, red] & 1' \arrow[uu, red] & 2' \arrow[uul, red] & 1' \arrow[uur, red] & 2' \arrow[uul, red]
\end{tikzcd}$$

It is clear that this sequence has permutation (12).

Example 1.3.27. Here is a reddening sequence of $Q : 1 \to 2$ that isn't a maximal green sequence.

$$\begin{tikzcd}
1 \arrow[r] \arrow[dd, green] & 2 \arrow[dd, green] & 1 & 2 \arrow[l] \arrow[dd, green] & 1 \arrow[r] & 2 & 1 & 2 \arrow[l] \arrow[dd, green] & 1 \arrow[r] & 2 \\
 & {} \arrow[r, "\mu_1"] & {} & {} \arrow[r, "\mu_2"] & {} & {} \arrow[r, "\mu_2"] & {} & {} \arrow[r, "\mu_2"] & {} & \\
1' & 2' & 1' \arrow[uu, red] & 2' & 1' \arrow[uu, red] & 2' \arrow[uu, red] & 1' \arrow[uu, red] & 2' & 1' \arrow[uu, red] & 2' \arrow[uu, red]
\end{tikzcd}$$

It is clear that this sequence has permutation *id*.

It is obvious that the result of $\mu_2\mu_1$ and $\mu_2\mu_1\mu_2$ are not identical, though they can be transformed into each other by a single permutation on vertices.

We can also define the associated permutation of sequences using extended exchange matrices.

Definition 1.3.28. The *matrix of a permutation*, $\sigma \in S_n$, is defined as the $n \times n$ matrix $P_\sigma = (\delta_{\sigma(i)j})$.

Definition 1.3.29.

1. For an $m \times n$ matrix $M = (M_1, \cdots, M_n)$ and a permutation $\sigma \in S_n$, if $C = (M_{\sigma(1)}, \cdots, M_{\sigma(n)})$ (or equivalently, $(c_{ij}) = (m_{i\sigma(j))})$, we denote this as $C = \sigma(M)$.

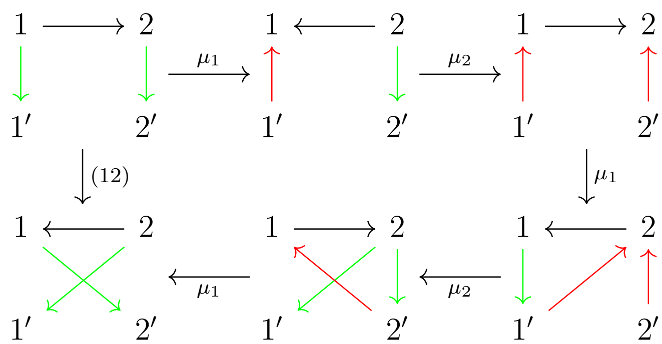
2. For an $n \times n$ matrix $A = (a_{ij})$ and a permutation $\sigma \in S_n$, if $D = (d_{ij}) = (a_{\sigma(i)\sigma(j)})$, we denote this as $D = \tilde{\sigma}(A)$.

It is easy to see that $C = \sigma(M)$ if and only if $C = MP_\sigma{}^{-1}$. $D = \tilde{\sigma}(A)$ if and only if $D = P_\sigma A P_\sigma{}^{-1}$.

Now let's define a new concept, namely *loop sequences* which is essential to the discussion about the permutation in Chapter 2.

Definition 1.3.30. A *loop sequence* w is a sequence of mutations $\mu_{i_k} \cdots \mu_{i_1}$ on an ice quiver (Q, F) such that $\mu_{i_k} \cdots \mu_{i_1}(Q) = \rho(Q)$ for some permutation ρ.

Example 1.3.31. Let Q be $1 \to 2$. Here is the loop sequence (1,2,1,2,1) with associated permutation (12).



Definition 1.3.32. For any loop sequence w the permutation ρ such that $w(\tilde{B}) = \rho(\tilde{B})$ is defined as *the associated permutation of the loop sequence* w.

In essence for all acyclic quivers, green-to-red sequences in general and maximal green sequences in particular do not have a natural definition of the permutation: The traditional one in essence is the permutation of an associated loop sequence: Take the reddening sequence and then do mutations at sinks only, go over all non-frozen vertices and return to the origin which constitutes the loop sequence we need.

1.4 Green sequences in bounded derived categories

In this section we will go over the basics about bounded derived categories, approximations, silting objects, simple-minded collections, torsion classes, t-structures and introduce the definition of numerous mutation sequences. This section mostly consists of background for Chapter 4 and Chapter 5.

1.4.1 Auslander-Reiten quivers

In this subsection we need to use Auslander-Reiten Theory. However I'm not going to talk about the entire Auslander-Reiten theory even though some parts of it are crucial to the understanding of Chapter 3. Here we will just introduce two concepts, namely irreducible morphisms and Auslander-Reiten quivers. For more information on Auslander-Reiten theory we refer the reader to Chapter IV of [3], [2] and [22].

Moreover we are not going to talk about what triangulated categories and bounded derived categories are in details. For those who want to read about them we recommend Daniel Murfet's notes [32][33][34] for introduction and [22] for its application in the theory of finite dimensional algebras. In particular [22] is a good source for Auslander-Reiten theory in bounded derived categories which we will use extensively here.

Definition 1.4.1. Let $\mathcal{C}$ be an Abelian or triangulated category, let M, N be objects of $\mathcal{C}$.

A morphism $f \in Hom(M, N)$ is *irreducible* if for any $L \in \mathcal{C}$ for any $g \in Hom(M, L), h \in Hom(N, L)$ such that $f = gh$ then either h is a split monomorphism or g is a split epimorphism.

The reason why we exclude cases where h is a section and g is a retraction is that these cases are simply trivial since it is very easy to decompose f using $M \xrightarrow{\begin{pmatrix} 1_M \\ 0 \end{pmatrix}} M \oplus K \xrightarrow{\begin{pmatrix} f & 0 \end{pmatrix}} N$ or $M \xrightarrow{\begin{pmatrix} f \\ 0 \end{pmatrix}} N \oplus K \xrightarrow{\begin{pmatrix} 1_N & 0 \end{pmatrix}} N$. Morally speaking an irreducible morphism can be understood as a morphism that can not be decomposed in any nontrivial way.

Example 1.4.2. Let Q be the quiver $1 \to 2 \to 3$. The inclusion $P_3 \hookrightarrow P_2$ is an irreducible morphism because the morphism does not factor through any other indecomposable module in a nontrivial way.

Example 1.4.3. Let Q be the quiver $1 \to 2 \to 3$. The inclusion $P_3 \hookrightarrow P_1$ is not an irreducible morphism because the morphism does factor through P_2 since $P_3 \hookrightarrow P_2 \hookrightarrow P_1$.

In order to define Auslander-Reiten quivers in Abelian categories we need several more concepts.

Definition 1.4.4. [3](Def A.3.3) The (Jacobian) *radical* of an additive k-category $\mathcal{C}$ is the two-sided ideal $rad_{\mathcal{C}}$ in $\mathcal{C}$ defined by the formula $rad_{\mathcal{C}}(X, Y) := \{h \in Hom_{\mathcal{C}}(X, Y) : 1_X - gh$ is invertible for any $g \in Hom_{\mathcal{C}}(Y, X)\}$.

Note that $rad_{\mathcal{C}}(X, Y)$ is a two-sided ideal. From now on when $\mathcal{C}$ is clear from the context we often omit it.

Definition 1.4.5. Let $\mathcal{C}$ be an additive k-category. Let $X, Y \in Ob\,\mathcal{C}$, let n be a positive integer, $X = X_0, Y = X_n$. We define $rad^n(X, Y) := \{f \in rad(X, Y) :$ for any $i \in [n]$ there exist $X_i \in Ob\,\mathcal{C}, f_i \in rad(X_{i-1}, X_i)$ such that $f = f_1 \cdots f_n\}$.

Note that $rad^n(X, Y)$ is a two-sided ideal. Hence we can define the space of irreducible morphisms as the following.

Definition 1.4.6. [3](IV.4) Let $\mathcal{C}$ be an additive k-category. Let $X, Y \in Ob\,\mathcal{C}$. The *space of irreducible morphisms* $Irr(X, Y)$ is defined as $rad(X, Y)/rad^2(X, Y)$.

Now we can define Auslander-Reiten quivers in Abelian categories.

Definition 1.4.7. Let $\mathcal{C}$ be an Abelian category. The *Auslander-Reiten (AR) quiver* $\Gamma(\mathcal{C})$ is the quiver with its vertices indecomposable objects of $\mathcal{C}$ and its arrows $[M] \to [N]$ vectors of a basis of $Irr(M, N)$ as a k-vector space.

Example 1.4.8. Here is the Auslander-Reiten quiver of $modkQ$ where Q is the quiver $1 \to 2$.

$$\begin{tikzcd} & P_1 \arrow[dr] & \\ P_2 \arrow[ur] & & I_1 \end{tikzcd}$$

There is an irreducible morphism from P_2 to P_1 and an irreducible morphism from P_1 to I_1. There are only 3 indecomposable objects in $modkQ$.

For triangulated categories the definition is fairly complex. So we refer interested readers to [22] (I.5.5) and [37].

1.4.2 Triangulated categories and bounded derived categories

Let's recall some basic facts about triangulated categories that we will use a lot in this paper. In a triangulated category $\mathcal{T}$ there exists an automorphism [1] known as the *translation functor*. $[n] := ([1])^n$ for any integer n.

Definition 1.4.9. The *cone* or *homotopy cokernel* of a morphism $A \xrightarrow{f} B \in \mathcal{T}$ is some $C \in \mathcal{T}$ such that there exists $g, h \in \mathcal{T}$ such that $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is a distinguished triangle.

Definition 1.4.10. The *homotopy kernel* of a morphism $A \xrightarrow{f} B \in \mathcal{T}$ is some $C \in \mathcal{T}$ such that there exists $g, h \in \mathcal{T}$ such that $C \xrightarrow{g} A \xrightarrow{f} B \xrightarrow{h} C[1]$ is a distinguished triangle.

Using axioms of triangulated categories any morphism $A \xrightarrow{f} B \in \mathcal{T}$ has a homotopy kernel and a homotopy cokernel.

Let's recall that bounded derived categories $D^b(\Lambda)$ are obtained by identifying homotopic chain maps in the category of chain complexes $C(\Lambda)$ and then formally invert all quasi-isomorphisms through localization. In bounded derived categories of hereditary algebras the indecomposable objects are of the form $M[i]$ where M is an indecomposable module and i is the amount of shifts we perform. In bounded derived categories it is true that $M, N \in mod\Lambda$

$$Hom_{D^b(\Lambda)}(M[i], N[j]) = \begin{cases} Ext_{\Lambda}^{j-i}(M,N) & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}.$$

Example 1.4.11. Let Q be $1 \longrightarrow 2 \longrightarrow 3$. Here is the Auslander-Reiten quiver of $D^b(kQ)$.

$$\begin{tikzcd}[column sep=small]
\cdots & I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] & \cdots \\
& \cdots & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \cdots & \\
\cdots & P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2] & \cdots
\end{tikzcd}$$

1.4.3 Approximations

According to [30] there are bijections between silting objects, t-structures, co-t-structures and simple-minded collections in a wide range of cases and such bijections respect mutations. In [11] more bijections are mentioned. Here we only need to cover three of them, namely silting objects, simple-minded collections and t-structures. To understand their mutations we must first introduce the concept of approximations.

Definition 1.4.12. Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. Let $M \in Ob\,\mathcal{C}, N \in Ob\mathcal{X}$ and $f \in Hom_{\mathcal{C}}(M, N)$.

1. f is a *left-$\mathcal{X}$ approximation* if for any $N' \in Ob\mathcal{X}$ and for any $q \in Hom_{\mathcal{C}}(M, N')$ we have q factors through f.

2. f is *left minimal* if for any $g \in End_{\mathcal{C}}N$ such that $g \circ f = f$ the morphism g is an isomorphism.

3. f is a *minimal left-$\mathcal{X}$ approximation* if it is both left minimal and is a left-$\mathcal{X}$ approximation.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "q"'] & N \arrow[d, dashed, "l"] \\
& N'
\end{tikzcd}$$

Definition 1.4.13. Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. Let $M \in Ob\,\mathcal{C}, N \in Ob\mathcal{X}$ and $f \in Hom_{\mathcal{C}}(M, N)$.

1. f is a *right-$\mathcal{X}$ approximation* if for any $M' \in Ob\mathcal{X}$ and for any $q \in Hom_{\mathcal{C}}(M', N)$ we have q factors through f.

2. f is *right minimal* if for any $g \in End_{\mathcal{C}}M$ such that $f \circ g = f$ the morphism g is an isomorphism.

3. f is a *minimal right-$\mathcal{X}$ approximation* if it is both right minimal and is a right-$\mathcal{X}$ approximation.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M' \arrow[u, dashed, "l"] \arrow[ur, "q"] & \end{tikzcd}$$

Example 1.4.14. Let $\mathcal{C}$ be $D^b(\Lambda)$ for some finite dimensional algebra Λ and let $\mathcal{X}$ be one of its full subcategories. If $M \in \mathcal{X}$ then 1_M is both a minimal left-$\mathcal{X}$ approximation and a minimal right-$\mathcal{X}$ approximation.

Example 1.4.15. Let Q be $1 \to 2$. Let $\mathcal{C}$ be $D^b(kQ)$. Let $M = P_2$ and $\mathcal{X} = add(P_1)$. The minimal left-$\mathcal{X}$ approximation is the canonical morphism $P_2 \to P_1$ induced by the inclusion $P_2 \to P_1$ in the module category.

Example 1.4.16. Let Q be $1 \to 2$. Let $\mathcal{C}$ be $D^b(kQ)$. Let $M = P_2$ and $\mathcal{X} = add(P_1)$. The minimal right-$\mathcal{X}$ approximation is the zero morphism because there is no other morphism from P_1 to P_2.

1.4.4 Silting objects

Now let's introduce silting objects.

Definition 1.4.17. Let Λ be an algebra with n primitive idempotents. A *silting object* T of $D^b(\Lambda)$ is an object such that T has n direct summands and $(T, T[m]) = 0$ for all $m > 0$. A *pre-silting object* is an object that only has to satisfy the second condition.

Example 1.4.18. Let's take A_3 straight orientation as an example.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

$\Lambda[i]$ is a silting object for any i. $T_1 = P_3 \oplus P_1 \oplus I_1[1]$ is also a silting object.

Now that we already have the definition of silting objects we can discuss their mutations.

Definition 1.4.19. A *forward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is the homotopy cokernel of the minimal left-$add(T/T_i)$ approximation of T_i.

A *backward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is homotopy kernel of the minimal right-$add(T/T_i)$ approximation of T_i.

Example 1.4.20. Again let's take A_3 straight orientation as an example.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

Λ is a silting object. When we do a forward mutation at P_3 we get $T' = S_2 \oplus P_2 \oplus P_1$. When we do a forward mutation at P_1 now we get $T'' = S_2 \oplus P_2 \oplus P_1[1]$. When we do another forward mutation at P_2 we get $T''' = S_2 \oplus P_3[1] \oplus P_1[1]$.

1.4.5 Simple-minded collections

Now let's introduce simple-minded collections. They are simple objects in some Abelian category known as hearts of t-structures.

Definition 1.4.21. Let Λ be an algebra with n primitive idempotents. A *simple-minded collection* $\{S_i\}_{i\in[n]}$ of $D^b(\Lambda)$ is an n-element set such that $(S_i[\geq 0], S_j) = 0$ for all $i \neq j$, $(S_i[> 0], S_i) = 0$ for all i and (S_i, S_i) is a division algebra.

Example 1.4.22. As usual our example is A_3 straight orientation.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. $\{P_3[1], P_2, I_1\}$ is also a simple-minded collection.

Definition 1.4.23. A *forward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S'_j\}$ where $S'_i = S_i[1]$ and S'_j $(j \neq i)$ is the homotopy cokernel of the minimal left-$add(S_i)$ approximation of $S_j[-1]$.
A *backward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S'_j\}$ where $S'_i = S_i[-1]$ and S'_j $(j \neq i)$ is the homotopy cokernel of the minimal left-$add(S_i[-1])$ approximation of S_j.

Example 1.4.24. The quiver here is A_3 straight orientation.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. When we do a forward mutation at P_3 we get $\{P_3[1], P_2, I_1\}$. When we do a forward mutation at P_2 now we get $\{S_2, P_2[1], P_1\}$. When we then do a forward mutation at P_1 we get $\{S_2, I_1, P_1[1]\}$.

Now we need to introduce two more results that are crucial to Chapter 5. Positive c-vectors are dimension vectors of elements of simple-minded collections. Such elements are all bricks. That is, all c-vectors are Schur. However we can indeed prove more. They are in fact real as well. In order to establish them we need two more lemmas.

Lemma 1.4.25. *[29] (Prop 6.4) Let Λ be a hereditary algebra and M, N be elements in* *a simple-minded collection of $D^b(\Lambda)$. Then $Ext^1(M,N)$ and $Ext^1(N,M)$ can not both be nonzero.*

Lemma 1.4.26. *[7] (Cor 3.3.2) Let Q be a quiver. Consider any maximal green sequence on any valued quiver Q. Then, at each step, the mutation is at a vertex of the mutated quiver Q_0 which is not the source of any arrow of infinite type.*

Lemma 1.4.27. *Let k be an algebraically closed field. Let Λ be a hereditary algebra over k. Then any c-vector c that appears in any maximal green sequence is a real Schur root.*

Proof. Since the simples of Λ are all exceptional if the lemma were incorrect then there must be some c-matrix in the maximal green sequence, C such that all columns of C are real Schur roots while one green mutation can somehow generate a root that isn't real. Here there can only be two cases, namely some mutation performed on $-v$ caused some $-w$ to be transformed into $-w' = -w - kv$ which isn't real, some mutation performed on $-v$ caused some $+w$ to be transformed into $w' = w - kv$ which isn't real. In the second case w' may be positive or negative.

For an arbitrary c-vector v let M_v be the brick such that v is the dimension vector of M_v. In this case $\langle v, v\rangle = 1 - dimExt^1(M_v, M_v)$. Hence v being real is equivalent to $\langle v, v\rangle = 1$.

Case 1: Assume that some mutation performed on $-v$ caused some $-w$ to be transformed into $-w' = -w - kv$ which isn't real. $\langle w', w'\rangle = \langle w, w\rangle + k\langle v, w\rangle + k\langle w, v\rangle + k^2\langle v, v\rangle$. Since $Hom(M_v, M_w) = Hom(M_w, M_v) = 0$ due to M_v, M_w being two elements in a simple-minded collection and v, w are both real $\langle w', w'\rangle = k^2+1-k\, dimExt^1(M_v, M_w)-k\, dimExt^1(M_w, M_v)$. Using properties of simple-minded collections $dimExt^1(M_w, M_v) = k$. Using Lemma 1.4.25 we can see that $Ext^1(M_v, M_w) = 0$, Hence $\langle w', w'\rangle = 1$. w' is real.

Case 2: Assume that some mutation performed on $-v$ caused some w to be transformed into $-w' = w - kv$ which isn't real. Using properties of simple-minded collections it is obvious that $Ext^1(M_v, M_w) = Hom(M_v, M_w) = 0$. Regardless of whether w' is positive or negative $\langle w', w'\rangle = \langle w, w\rangle - k\langle v, w\rangle - k\langle w, v\rangle + k^2\langle v, v\rangle = k^2 + 1 - k\, dimHom(M_w, M_v) + k\, Ext^1(M_w, M_v)$. Using properties of simple-minded collections $dimHom(M_w, M_v) = k$. Using Lemma 1.4.25 we can see that $Ext^1(M_w, M_v) = 0$, Hence $\langle w', w'\rangle = 1$. w' is real.

The assumption has been refuted. Any c-vector c that appears in any maximal green sequence is a real Schur root. □

In order to prove Theorem 5.1.1 we need to first prove a lemma.

Lemma 1.4.28. *If $-c_1, -c_2$ are negative c-vectors in c-matrix C' in an maximal green sequence, c_1 and c_2 are dimension vectors of indecomposable modules M_1 and M_2. If $dimExt^1(M_1, M_2) > 1$ then the mutation on C' must not be done on M_2.*

Proof. Assume that $-c_1$ is the i-th column and $-c_2$ is the j-th column. Using the definition of left mutations of simple-minded collections if $dimExt^1(M_1, M_2) > 1$ then the mutation on $-c_2$ would cause $-c_1$ to be transformed into $-c_1 - kc_2$ with $k > 1$ because which could only happen if there are multiple edges from i to j [30]. Due to Lemma 1.4.26 this was impossible.

□

1.4.6 t-structures

Here is the definition of $t-$structures.

Definition 1.4.29. A *t-structure* on $D^b(\Lambda)$ is a pair $(D^{\leq 0}, D^{\geq 0})$ such that the following holds.

1. For any $M \in D^b(\Lambda)$ there exists $M' \in D^{\leq 0}, M'' \in D^{\geq 0}$ such that $M' \to M \to M'' \to M'[1]$.

2. $D^{\leq 0}[1] \subseteq D^{\leq 0}$, $D^{\geq 0}[1] \supseteq D^{\geq 0}$.

3. $(D^{\leq 0}[1], D^{\geq 0}) = 0$.

Example 1.4.30. Let Λ be any finite dimensional algebra. The *standard t-structure* $(\cup_{m=0}^{\infty}\Lambda[m], \cup_{m=0}^{\infty}\Lambda[-m])$ is clearly a t-structure.

Now let's define hearts which will be very useful for a crucial proof in Chapter 4, namely the proof of Lemma 4.3.3.

Definition 1.4.31. The *heart* of a t-structure $(D^{\leq 0}, D^{\geq 0})$ is defined as $\mathcal{H} = D^{\leq 0} \cap D^{\geq 0}$.

Theorem 1.4.32. *[4] Hearts of t-structures are Abelian categories.*

Example 1.4.33. Let Λ be any finite dimensional algebra. The heart of the standard t-structure is $mod\Lambda$ itself which is of course Abelian.

Definition 1.4.34. [16](Def 1.5.3) Given an object $X \in \mathcal{C}$, then a Jordan-Hölder sequence or composition series for X is a finite filtration, i.e. a finite sequence of subobject inclusions into X, starting with the zero objects $0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow X_n = X$ such that at each stage i the quotient X_i/X_{i-1} (i.e. the coimage of the monomorphism $X_{i-1} \hookrightarrow X_i$) is a simple object of $\mathcal{C}$. If a Jordan-Hölder sequence for X exists at all, then X is said to be of finite length.

Definition 1.4.35. An Abelian category is a *length category* if all its objects have finite length.

Example 1.4.36. Let Q be the quiver $1 \to 2$. The Abelian category $modkQ$ is a length category because all its objects are finite direct sums of P_2, I_1 and P_1. P_2 and I_1 are simple and hence have length 1. P_1 has length 2. Hence any object in $modkQ$ has finite length.

Now let's introduce truncation functors associated with t-structures.

Lemma 1.4.37. *[4] Let $\mathcal{D}$ be a triangulated category. Let $(D^{\leq 0}, D^{\geq 0})$ be a t-structure on $\mathcal{D}$. The inclusion $\mathcal{D}^{\geq n} \to \mathcal{D}$ has a left adjoint $\tau_{\geq n}$. Similarly, the inclusion $\mathcal{D}^{\leq n} \to \mathcal{D}$ has a right adjoint $\tau_{\leq n}$. These are called truncation functors.*

Definition 1.4.38. [5] A t-structure $(D^{\leq 0}, D^{\geq 0})$ of the triangulated category $\mathcal{D}$ is *bounded* if $\mathcal{D} = \cup_{i,j}(D^{\leq i} \cap D^{\geq j})$.

For more information about mutations of t-structures we recommend [1], [11], [29] and [30].

1.4.7 Green sequences

The concepts we mentioned are related to each other due to the following result:

Theorem 1.4.39. *[30](Thm 6.1, 7.12) Let Λ be a finite-dimensional algebra over a field k. There are one-to-one correspondences between*

1. *Equivalence classes of silting objects in $K^b(proj\Lambda)$.*
2. *Equivalence classes of simple-minded collections in $D^b(mod\Lambda)$.*
3. *Bounded t-structures of $D^b(mod\Lambda)$ with length heart.*

Moreover such correspondences are preserved under mutations.

We will not explain the concepts of $K^b(proj\Lambda)$ and triangle equivalences because they are irrelevant to understanding of the problem. We refer interested readers to [22].

Theorem 1.4.40. *[22](3.3) If Λ has finite global dimension $K^b(proj\Lambda)$ is triangle equivalent to $D^b(mod\Lambda)$.*

Since we have maximal green sequences it is reasonable to look at the generalization of this concept, namely m-maximal green sequences. In order to do so we need to define the general concept of green and red sequences. In principle any forward mutation is considered green and any backward mutation red.

Definition 1.4.41.

1. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *green* if it contains only forward mutations.

2. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is *red* if it contains only backward mutations.

3. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is k-*red* if it contains k backward mutations.

4. Let Λ be a finite dimensional algebra of finite global dimension, a mutation sequence in $D^b(\Lambda)$ is k-*green* if it contains k forward mutations.

Note that a 0-red sequence is just a green one. A 0-green sequence is just a red one. Now we can introduce m-maximal green sequences. For the purpose of the proof in Chapter 3 it is much better to use silting objects.

Definition 1.4.42. An *m-maximal green sequence* is a green sequence of silting objects from Λ to $\Lambda[m]$.

It is easy to see that a 1-maximal green sequence is just a maximal green sequence.

Example 1.4.43. Again our example is A_3 straight orientation.

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

So $(P_1, P_2, P_3, P_1[1], P_2[1], P_3[1])$ is a 2-maximal green sequence, so is $(P_1, P_3, P_2, S_2, P_1[1], P_2[1], P_3[1])$ because they are both sequences of indecomposable objects forward mutations on which produce $\Lambda[2]$ from Λ.

1.5 Tame quivers and tame hereditary algebras

In this subsection we will review the basics about tame hereditary algebras, the components of their Auslander-Reiten quivers and the components of Auslander-Reiten quivers of their bounded derived categories for they are crucial to Chapter 3. For more details about tame algebras we would like to refer the readers to [13], [36] and [39].

1.5.1 Tame quivers

Definition 1.5.1. A *tame algebra* is a k-algebra such that for each dimension vector there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.

Definition 1.5.2. A *tame quiver* is a quiver such that its path algebra is a tame algebra.

Example 1.5.3. Here are some (connected) tame quivers, $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6.\tilde{E}_7, \tilde{E}_8$. The orientation of the edges can be arbitrary as long as the quiver remains acyclic in the case of $\tilde{A}_n$.

$\tilde{A}_n$

$$\begin{tikzcd}
& 2 \arrow[r] & \cdots \arrow[r] & i \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & & & n+1 \\
& i+1 \arrow[r] & \cdots \arrow[r] & n \arrow[ur] &
\end{tikzcd}$$

$\tilde{D}_n$

$$\begin{tikzcd}
1 \arrow[dr] & & & & & n \\
& 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[ur] \arrow[dr] & \\
2 \arrow[ur] & & & & & n+1
\end{tikzcd}$$

$\tilde{E}_6$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \\
& & 6 \arrow[d] & & \\
& & 7 & &
\end{tikzcd}$$

$\tilde{E}_7$

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] \arrow[d] & 5 \arrow[r] & 6 \arrow[r] & 7 \\
& & & 8 & & &
\end{tikzcd}$$

$$\tilde{E}_8 \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \\ & & 9 & & & & & \end{tikzcd}$$

1.5.2 Standard stable tubes

In this subsection we are going to discuss *standard stable tubes* because they are crucial to understanding of the proofs in Chapter 3. Before that we first need several more concepts.

Definition 1.5.4. A quiver Q is *locally finite* if for any vertex $x \in Q_0$ the amount of vertices adjacent to x is finite.

It is obvious that any finite quiver is locally finite.

Definition 1.5.5. [3] Let Γ be a locally finite quiver without loops and τ be a bijection whose domain and codomain are both subsets of Γ_0. (Γ, τ) (and often simply Γ) is a *translation quiver* if for every $x \in \Gamma_0$ such that τx exists and for any $y \in \Gamma_0$ such that there exists at least one arrow from y to x the number of arrows from y to x is equal to the number of arrows from τx to y.

Definition 1.5.6. Let (Γ, τ) be a translation quiver. A point $x \in \Gamma_0$ is a *projective point* if τx is undefined. A point $x \in \Gamma_0$ is a *injective point* if $\tau^{-1}x$ is undefined.

Translation quivers are relevant to Auslander-Reiten theory. Using them we can have another level of abstraction when convenient.

Lemma 1.5.7. *[3] The Auslander-Reiten quiver $\Gamma(mod\Lambda)$ of an algebra Λ is a translation quiver, the translation τ being defined for all points $[M]$ such that M is not a projective module by $\tau([M]) := [\tau M]$.*

Definition 1.5.8. Let Q be a quiver. Let $(\mathcal{T}, \tau)$ be a quiver with vertices of the form (i, j) where $j > 0$ and $i \in [r]$. Here r + 1 is considered to be 1. There is an arrow from $(i, j) \to (i, j + 1)$ for any i and an arrow from (i, j) to $(i - 1, j - 1)$ for any $j > 1$. Such a quiver is known as a *stable tube of rank r*.

Definition 1.5.9. If a stable tube has rank 1 it is *homogeneous*. Otherwise it is *nonhomogeneous*.

I'm going to introduce one example of nonhomogeneous and homogeneous standard stable tubes each. For more details we recommend Chapter X of [39].

Example 1.5.10. Here is a standard stable tube with rank 3.

$$\begin{tikzcd}
 & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & \\
M_{33} \arrow[ur] \arrow[dr] & & M_{13} \arrow[ur] \arrow[dr] & & M_{23} \arrow[ur] \arrow[dr] & & M_{33} \\
 & M_{12} \arrow[ur] \arrow[dr] & & M_{22} \arrow[ur] \arrow[dr] & & M_{32} \arrow[ur] \arrow[dr] & \\
M_1 \arrow[ur] & & M_2 \arrow[ur] & & M_3 \arrow[ur] & & M_1
\end{tikzcd}$$

M_{ik} is rigid iff $k \leq 2$.

Here $M_{ik} = \begin{array}{c} M_{i+k-1} \\ \\ \dots \\ \\ M_{i+1} \\ \\ M_i \end{array}$. we define the *quasi-length* of M_{ik} as k. The *quasi-top* of the module is defined as M_{i+k-1} and the *quasi-socle* M_i.

Now let's see a homogeneous tube.

Example 1.5.11. Here is a homogeneous standard stable tube.

$$\begin{tikzcd}
\dots \arrow[d, bend left] \\
M_3 \arrow[u, bend left] \arrow[d, bend left] \\
M_2 \arrow[u, bend left] \arrow[d, bend left] \\
M \arrow[u, bend left]
\end{tikzcd}$$

Note that no module in this tube is rigid.

Here $M_k = \begin{array}{c} M \\ \\ \dots \\ \\ M \end{array}$

Now we need to introduce what it means for a stable tube to be *standard*. Before doing so we need several new concepts.

Definition 1.5.12. [39] Let $\mathcal{C}$ be a component of the AR quiver $\Gamma(mod\Lambda)$ of some k-algebra Λ. For simplicity assume that $\Gamma(mod\Lambda)$ is without multiple arrows.

1. The *path category* $k\mathcal{C}$ of $\mathcal{C}$ is the k-category with objects points in $\mathcal{C}$ and morphisms from $x \in \mathcal{C}_0$ to $y \in \mathcal{C}_0$ the k-linear combinations of paths of $\mathcal{C}$ from x to y with coefficients in k.

2. The ideal $M_{\mathcal{C}}$ in the category $k\mathcal{C}$ is defined as follows. To every non-projective point $x \in \mathcal{C}_0$ corresponds a mesh in $\mathcal{C}$ of the form
$$\begin{tikzcd}
 & y_1 \arrow[ddr, "\alpha_1"] & \\
 & y_2 \arrow[dr, "\alpha_2"] & \\
\tau x \arrow[uur, "\alpha'_1"] \arrow[ur, "\alpha'_2"] \arrow[ddr, "\alpha'_t"'] & & x \\
 & \cdots & \\
 & y_t \arrow[uur, "\alpha_t"'] &
\end{tikzcd}$$
and to this mesh we associate an element m_x of $Hom_{k\mathcal{C}}(\tau x, x)$ called the *mesh element* defined by the formula $m_x = \Sigma_{i=1}^{t} \alpha'_i \alpha_i$. We denote by $M_{\mathcal{C}}$ the ideal of $k\mathcal{C}$ generated by all the mesh elements m_x where x ranges over all the non-projective points of $\mathcal{C}$.

3. The *mesh category* is the quotient k-category $k(\mathcal{C}) = k\mathcal{C}/M_{\mathcal{C}}$.

Definition 1.5.13. [39] Let $\mathcal{C}$ be a component of the AR quiver $\Gamma(mod\Lambda)$ of some k-algebra Λ. $\mathcal{C}$ is a *standard component* of $\Gamma(mod\Lambda)$ if there exists an equivalence of k-categories $k(\mathcal{C}) \approxeq ind\mathcal{C}$ where $ind\mathcal{C}$ is the full k-subcategory of $mod\Lambda$ whose objects are representatives of the isomorphism classes of the indecomposable modules in $\mathcal{C}$.

Example 1.5.14. Here is the AR quiver of $\mathcal{A} = modkQ$ where Q is the quiver $1 \to 2$ with arrows labelled.
$$\begin{tikzcd}
 & P_1 \arrow[dr, "b"] & \\
P_2 \arrow[ur, "a"] & & I_1
\end{tikzcd}$$
This AR quiver consists of only one component $\mathcal{C}$

which is standard. Here is why. The path category has objects P_2, P_1, I_1 and morphisms are generated by identity morphisms, a and b. Here $\mathcal{T}(P_2, P_1) = \mathcal{A}(P_2, P_1)$ and $\mathcal{T}(P_1, I_1) = \mathcal{A}(P_1, I_1)$. At the same time $\mathcal{T}(P_2, I_1)$ is generated by ba while $\mathcal{A}(P_2, I_1) = 0$. The mesh ideal is generated by ba, hence the mesh category is equivalent to $modkQ$. Hence $\mathcal{C}$ is a standard component.

1.5.3 Auslander-Reiten quivers of tame hereditary algebras

In this subsection we are going to discuss Auslander-Reiten quivers of basic tame hereditary algebras because information about them is slightly less well known.

Theorem 1.5.15. *[13] The Auslander-Reiten quiver of a tame path algebra consists of three parts, the preprojectives, the preinjectives and the regulars.*

Before we can describe Auslander-Reiten quivers of tame path algebras we have to define several special translation quivers.

Definition 1.5.16. $\mathbb{N}Q$ is defined as the quiver with vertices of the form (i, j) where $j \in Q_0$ and $i \in \mathbb{N}$. For any i for any arrow from j to k there is an arrow from $(i, j) \to (i, k)$ and an arrow from $(i + 1, k)$ to (i, j).

Definition 1.5.17. $-\mathbb{N}Q$ is defined as the quiver with vertices of the form (i, j) where $j \in Q_0$ and $i \in -\mathbb{N}$. For any i for any arrow from j to k there is an arrow from $(i, j) \to (i, k)$ and an arrow from (i, k) to $(i - 1, j)$.

Definition 1.5.18. $\mathbb{Z}Q$ is defined as the quiver with vertices of the form (i, j) where $j \in Q_0$ and $i \in \mathbb{Z}$. For any i for any arrow from j to k there is an arrow from $(i, j) \to (i, k)$ and an arrow from (i, k) to $(i-1, j)$.

Note that the number of arrows from i to j in Q is the same as the amount of arrows from (a, i) to (a, j) which is the same as the amount of arrows from (a, j) to $(a-1, i)$ in $\mathbb{Z}Q$.

Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras from [3] and [39].

Theorem 1.5.19.

1. *The AR quiver of kQ has one preprojective component which is isomorphic to $\mathbb{N}Q^{op}$*
2. *The AR quiver of kQ has one preinjective component which is isomorphic to $-\mathbb{N}Q^{op}$.*
3. *All preprojective and preinjective modules in kQ are rigid.*
4. *All but finitely many preprojectives and preinjectives are sincere.*
5. *There are infinitely many regular components, all of which are standard stable tubes $\mathbb{Z}A_\infty/(\tau^k)$.*
6. *All such tubes are pairwise orthogonal to each other. That is, if M, N are in different tubes then $Hom(M, N) = Hom(N, M) = 0$.*
7. *All but at most three tubes have $k = 1$. In this case we consider the component homogeneous.*
8. *All elements in a homogeneous tube are non-rigid, hence they and their shifts can not be summands of any silting object.*
9. *In a nonhomogeneous component $\mathbb{Z}A_\infty/(\tau^k)$ only indecomposables with quasi-length less than k are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.*

10. *Only finitely many regular indecomposable modules are rigid. Hence only finitely many regular indecomposables and their shifts can appear in an m-maximal green sequence.*

Now let's do an example of an AR quiver of a tame path algebra.

Example 1.5.20. The quiver is
$$\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 5 & 3 \arrow[l] \\ & 4 \arrow[u] & \end{tikzcd}.$$
Here is the preprojective component, $\mathcal{P}$.

$$\begin{tikzcd}
 & P_1 \arrow[ddr] & & \tau^{-1}P_1 \arrow[ddr] & & \cdots \\
 & P_2 \arrow[dr] & & \tau^{-1}P_2 \arrow[dr] & & \cdots \\
P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-1}P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-2}P_5 \cdots & \\
 & P_3 \arrow[ur] & & \tau^{-1}P_3 \arrow[ur] & & \cdots \\
 & P_4 \arrow[uur] & & \tau^{-1}P_4 \arrow[uur] & & \cdots
\end{tikzcd}$$

Here is the preinjective component, $\mathcal{Q}$.

$$
\begin{tikzcd}
\cdots & \tau I_1 \arrow[ddr] & & I_1 \\
\cdots & \tau I_2 \arrow[dr] & & I_2 \\
\cdots \tau I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
\cdots & \tau I_3 \arrow[ur] & & I_3 \\
\cdots & \tau I_4 \arrow[uur] & & I_4
\end{tikzcd}
$$

Here are the regular components. There are infinitely many homogeneous tubes and 3 nonhomogeneous ones. All objects in the homogeneous ones are non-rigid. The quasi-simple in the homogeneous tubes has dimension vector is (1,1,1,1,2). The quasi-simples in the three nonhomogeneous tubes have dimension vectors (1,1,0,0,1) and (0,0,1,1,1), (1,0,1,0,1) and (0,1,0,1,1), (1,0,0,1,1) and (0,1,1,0,1) respectively.

Finally let's discuss Auslander-Reiten quivers of $D^b(kQ)$. For a tame quiver Q there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are isomorphic to $\mathbb{Z}Q^{op}$. Let's label these components *transjective*. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$.

There are also infinitely many regular components. There are at most 3 nonhomogeneous tubes in $modkQ[m]$ for any m. There are also infinitely many homogeneous tubes in $modkQ[m]$ for any m. However since no module in a homogeneous tube is rigid they don't affect our problem.

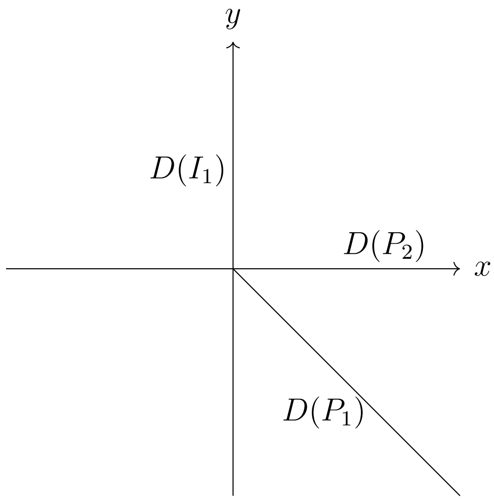
1.6 Wall-and-chamber structures

In this section we will discuss miscellaneous topics on the wall-and-chamber structures, namely picture groups and alternative definitions of maximal green sequences. This section is mostly relevant to Chapter 2 and Chapter 4. For more details we recommend [20][31][8][9][10][23] and [26].

1.6.1 Picture groups

We also need to use the concept of the picture groups in order to prove the formula in Chapter 2. For more details about picture groups we recommend [26].

Let's recall that for a quiver of finite type, any dimension vector of an indecomposable representation is referred to as a *root*. Let $(-,-)$ be the standard Euclidean product. Let $D(\beta) \subseteq \mathbb{R}^n$, $D(\beta) = \{x \in \mathbb{R}^n : (x, \beta) = 0,\ (x, \beta') \leq 0 \text{ when } \beta' \subseteq \beta\}$. Here $\beta' \subseteq \beta$ means the unique indecomposable representation of dimension vector β' is a subrepresentation of the unique indecomposable representation of dimension vector β. $D(\beta)$ for all these roots divide $\mathbb{R}^n$ into *compartments* (or *chambers*). The boundary of each compartment is the union of some $D(\beta)$ which we call *walls*.[26][24]



Example 1.6.1. The figure above is the wall-and-chamber structure of $Q : 1 \to 2$. $D(P_1), D(P_2)$ $D(I_1)$ are the walls and there are 5 chambers. Note that $D(P_1)$ is only half of the line $x + y = 0$.

Definition 1.6.2. [26] A *picture group* of a cluster quiver of finite type Q is a group $G(Q) = \langle S|R\rangle$ with S in bijection with the set of real Schur roots (the generator for β is $x(\beta)$) and R the set of relations $x(\beta_i)x(\beta_j) = \Pi x(\gamma_k)$ with γ_k running over all these real Schur roots which are linear combinations $\gamma_k = a_k\beta_i + b_k\beta_j$ with a_k/b_k increasing (going from $0/1$ where $\gamma_1 = \beta_j$ to $1/0$ where $\gamma_k = \beta_i$) for any pair (β_i, β_j) such that they are Hom-orthogonal and $Ext(\beta_i, \beta_j) = 0$.

Now we need to restrict the case to A_n straight orientation. In this case due to Lemma 1.3.16 the roots are in the form $\pm\beta_{ij}$ where $\beta_{ij} = e_j - e_i$ ($0 < i < j < n$, e_0 is defined as the zero vector). The root β_{ij} corresponds to the picture group generator x_{ij} which we will define right now.

We often simplify the notation of $x(\beta_{ij})$ to x_{ij} which we use interchangeably with $x(\beta_{ij})$. The picture group for A_n straight orientation is $G(A_n) = \{S|R\}$, $S = \{x_{ij}|0 \le i < j \le n\}$,

$R = \{x_{ij}x_{kl} = x_{kl}x_{ij} | [i, j] \cap [k, l] = \emptyset, [i, j]$ or $[k, l]$, i,j,k,l are distinct.$\} \cup \{x_{jk}x_{ij} = x_{ij}x_{ik}x_{jk} | 0 \leq i < j < k \leq n\}$.

Igusa and Todorov proved [26] that there exists a bijection between the set of maximal green sequences and the set $\mathcal{P}(c)$ of positive expressions of the Coxeter element of the picture group for any acyclic valued quiver of finite type which applies to A_n straight orientation.

1.6.2 Alternative definitions of maximal green sequences

In the following theorem by Kiyoshi Igusa multiple equivalent definition of maximal green sequences was introduced. The full version of Igusa's results includes more discussions about the wall-and-chamber structure which we will not discuss here. To understand more about the wall-and-chamber structure we suggest that the reader reads [20][31][8][9][10] and [23].

Definition 1.6.3. Let Λ be a finite dimensional algebra. $\Lambda-$modules $M_1, \cdots, M_m$ are a *backward Hom-orthogonal sequence* if $Hom_\Lambda(M_i, M_j) = 0$ for $i > j$.

Example 1.6.4. In the $Q : 1 \to 2$ example $\{P_2, P_1\}$ is a backward Hom-orthogonal sequence because $Hom(P_1, P_2) = 0$.

Example 1.6.5. In the $Q : 1 \to 2$ example $\{P_1, P_2\}$ is not a backward Hom-orthogonal sequence because $Hom(P_2, P_1) = k$.

Definition 1.6.6. Let Λ be a finite dimensional algebra. A backward Hom-orthogonal sequence $M_1, \cdots, M_m$ is *maximal* if no other modules can be inserted into the sequence preserving the property of backward Hom-orthogonality.

Example 1.6.7. In the $Q : 1 \to 2$ example $\{P_2, P_1\}$ is not a maximal backward Hom-orthogonal sequence even though it is a backward Hom-orthogonal sequence because it can be extended to P_2, P_1, I_1 without losing backward Hom-orthogonality.

Example 1.6.8. In the $Q : 1 \to 2$ example $\{I_1, P_2\}$ is a maximal backward Hom-orthogonal sequence because $Hom(P_2, I_1) = 0$ and that there is no place to fit any other module in the sequence. For example P_1 can not be inserted before P_2 because $Hom(P_2, P_1) \neq 0$. At the same time it can not be inserted after I_1 because $Hom(P_1, I_1) \neq 0$. Hence the sequence can not be extended to include P_1.

In fact $Q : 1 \to 2$ only has two maximal backward Hom-orthogonal sequences, namely I_1, P_2 and P_2, P_1, I_1. They are exactly the same as the two c-vectors in maximal green sequences of Q. Is this just a coincidence? No.

Definition 1.6.9. Let $\{M_1, \cdots, M_k\}$ be a fixed finite sequence of Schur objects in $mod\Lambda$. An *Harder-Narasimhan (HN) filtration* with respect to $\{M_i\}$ aka an HN filtration of an object X is k short exact sequences $0 \to X_i \to X_{i-1} \to X_{i-1}/X_i$ such that $X_{i-1}/X_i \in \mathcal{E}M_i$.

Definition 1.6.10. Let $\{M_1, \cdots, M_k\}$ be a fixed finite sequence of Schur objects in $mod\Lambda$. $\{M_1, \cdots, M_k\}$ is an *finite Harder-Narasimhan (HN) system* if any $X \in mod\Lambda$ has a unique HN filtration with respect to $\{M_i\}$.

Example 1.6.11. In the $Q : 1 \to 2$ example $\{I_1, P_2\}$ is a finite HN system because any module $X \in modkQ$ has a unique HN filtration. In particular the unique HN filtration of P_1 is $0 \to P_2 \to P_1 \to I_1 \to 0$.

Example 1.6.12. In the $Q : 1 \to 2$ example $\{P_2, P_1, I_1\}$ is a finite HN system because any module $X \in modkQ$ has a unique HN filtration.

There are only two finite HN systems of kQ where $Q : 1 \to 2$. They are exactly the same as the two maximal backward Hom-orthogonal sequences and the two maximal green sequences. This is in fact a general result.

Theorem 1.6.13. *[25] Let Λ be a finite dimensional hereditary algebra over a field K. Let $\beta_1, \cdots, \beta_m \in \mathbb{N}^n$ be any finite sequence of nonzero, nonnegative integer vectors. Then the following are equivalent.*

(a) There exist Λ-modules $M_m, \cdots, M_1$ with $\dim M_i = \beta_i$ which form a finite Harder-Narasimhan system for Λ.

(b) There exist a finite sequence of Schurian Λ-modules $\{M_1, \cdots, M_m\}$ with $\dim M_i = \beta_i$ such that $\{M_1, \cdots, M_m\}$ is a maximal backward Hom-orthogonal sequence.

(c) There is a maximal green sequence for Λ of length m whose ith mutation is at the c-vector β_i.

Such results have been generalized to the case of m-maximal green sequences in Chapter 4.

Chapter 2

Permutation

2.1 The general theory of permutations

2.1.1 Mutation systems

Let's define a natural setting of the theory of permutations which is completely combinatorial. Let $[n] = \{1, 2, \cdots, n\}$. A *mutation graph* is defined as a connected n-regular graph without loops or multiple edges. Let $T = (T_0, T_1)$ be a mutation graph. A *signed edge* of T is a triple (a, h, t) where $a \in T_1$, h and t are the two endpoints of a, defined to be the head and tail of the signed edge respectively. For every signed edge a^{ht} there is its inverse signed edge a^{th}. Let $\tilde{T}_1$ be the union of all signed edges of T. A *walk* is a path in $\tilde{T}_1$ such that the sources and targets of each signed edge are compatible. Walks on T are in the form $w = \Pi a_k^{i_{k-1} i_k}$.

For each vertex $x \in T_0$ we associate a set $N(x)$ which is the set that contains all vertices adjacent to x. Note that $|N(x)| = n$. For each $a^{ht} \in \tilde{T}_1$ we associate a bijection $f_{a^{ht}} : N(x) \to N(y)$ such that a^{ht} and a^{th} are inverses of each other for any $a \in T_1$. This bijection is called *mutation* as per [19]. The set of all f_a is denoted A, the *set of mutations*. The tuple (T, A) is called a *mutation system*. We can also define a natural bijection $f_w : N(x) \to N(y)$ associated with each walk $w = \Pi a_k^{i_{k-1}i_k}$, namely $f_w = f_{a_k}^{i_{k-1}i_k} \cdots f_{a_1}^{i_0 i_1}$.

Now let's define a bijection $j_x : [n] \to N(x)$ for each $x \in T_0$. This bijection is called the *fixed ordering* of the seed $N(x)$. Let J be the set of all fixed orderings which we call a *fixed ordering set*. The tuple (T, A, J) is called a *ordered mutation system*. Now for each bijection $j'_x : [n] \to N(x)$ we can define its *associated permutation relative to* J below:

Definition 2.1.1. For any (T, A, J) for any $x \in T_0$ for any bijection $g : [n] \to N(x)$ the *associated permutation relative to* J is defined as $\rho(g) = j_x^{-1} g$.

Now we can define what is the associated permutation of a mutation relative to J.

Definition 2.1.2. For any (T, A, J) for any $x, y \in T_0$ for any mutation $f_a : N(x) \to N(y)$ the *associated permutation relative to* J is defined as $\rho(f_a) = j_y^{-1} f_a j_x$.

In other words, $\rho(f_a)$ is the permutation such that the following diagram commutes:

$$\begin{tikzcd} {[n]} \arrow[r, "\rho(f_a)"] \arrow[d, "j_x"] & {[n]} \arrow[d, "j_y"] \\ N(x) \arrow[r, "f_a"] & N(y) \end{tikzcd}$$

Now we can define what it means to be the associated permutation relative to J of any walk $p = a_k^{i_k} \cdots a_1^{i_1}$, namely $\rho(p) = j_y^{-1} f_p j_x$. It is easy to see from the diagram below that $\rho(p) = \rho(a_k)^{i_k} \cdots \rho(a_1)^{i_1}$.

$$\begin{tikzcd}
{[n]} \arrow[r, "\rho(f_{a_1})^{i_1}"] \arrow[d, "j_{x_1}"] & {[n]} \arrow[r, "\rho(f_{a_2})^{i_2}"] \arrow[d, "j_{x_2}"] & \cdots {[n]} \arrow[d, "j_{x_k}"] \\
N(x_1) \arrow[r, "f_{a_1}^{i_1}"] & N(x_2) \arrow[r, "f_{a_2}^{i_2}"] & \cdots N(x_k)
\end{tikzcd}$$

We can also discuss the relation between the permutation of a walk p relative to different fixed orderings.

Theorem 2.1.3. *(Change of fixed ordering formula) For any mutation system (T, A) for any fixed ordering set $J_1.J_2$ for any $x, y \in T_0$ for any walk $p : x \to y$, let $\rho_1(p), \rho_2(p)$ be the associated permutation of p relative to J_1, J_2 respectively. Let $\tau_x = j_{1x}^{-1} j_{2x}$, $\tau_y = j_{1y}^{-1} j_{2y}$. Then $\rho_2(p) = \tau_y^{-1} \rho_1(p) \tau_x = j_{2y}^{-1} j_{1y} \rho_1(p) j_{1x}^{-1} j_{2x}$.*

The theorem can be verified easily by the diagram below.

$$\begin{tikzcd}
{[n]} \arrow[r, "\rho_2(p)"] \arrow[d, "\tau_x"] \arrow[dd, bend right, "j_{2x}"'] & {[n]} \arrow[d, "\tau_y"] \arrow[dd, bend left, "j_{2y}"] \\
{[n]} \arrow[r, "\rho_1(p)"] \arrow[d, "j_{1x}"] & {[n]} \arrow[d, "j_{1y}"] \\
N(x) \arrow[r, "f_a"] & N(y)
\end{tikzcd}$$

We can see that in essence the permutation of a reddening or loop sequence is just special cases of permutations of walks.

It is obvious that any (T, A, J) induces a map $g : \tilde{T_1} \to [n]$ that assigns a *fixed position* to each signed edge. We call the map g the *fixed position map*. It is also obvious that for any walk $w : x \to y$, we only need to fix j_x, j_y to have a fixed $\rho(w)$. $\rho(w)$ is independent of j_z for any $z \neq x, y$. Hence the definition of $\rho(w)$ can be done with any arbitrary choice of j_z for any $z \neq x, y$. In fact even not defining them is also fine since we can just define $\rho(w)$ as

$j_y^{-1} f_w j_x$.

2.2 The associated permutation in A_n

When the quiver is A_n straight orientation we can make much stronger claims. In fact there is a canonical permutation of any mutation sequence. Using the notations in Section 1.6.1 the formula of the associated permutation of reddening sequences and loop sequences is given as the following.

Theorem 2.2.1. *In A_n straight orientation, the permutation associated with a picture group element that transforms the framed quiver into the coframed quiver or the framed quiver itself is $\rho(\prod_k x_{i_k j_k}^{\delta_k}) = (\prod_k (i_k + 1, j_k))^{-1}$. Here $\delta_k \in \{+, -\}$.*

This formula works for any maximal green, reddening, loop sequences that starts from and ends up in the framed quiver. It also extended the definition of an associated permutation to the set of arbitrary finite sequences of mutations in A_n straight orientation. One interesting property of A_n is that the associated permutation of a mutation only depends on the c-vector but not which cluster-tilting object on which the mutation is conducted. This is a highly nontrivial fact: The associated permutation in the general case seems way less regular.

2.2.1 Forbidden Pairs-of-Walls Lemmas

Since we use picture groups and related structures to prove the theorem, we need to examine what kind of pairs of walls can not exist in any compartment. Before doing so we first need to discuss notations. Sometimes we abuse notations and use the root β to mean the wall $D(\beta)$ when the meaning is clear. We also use the notation $+\beta$ to mean the wall β is a part of the boundary of a compartment $\mathcal{U}$ and for any point $x \in \mathcal{U}$, $\langle x, \beta\rangle\ > 0$. Similarly we have the notation $-\beta$. For example $+\beta - \beta'$ means that β and β' are parts of the boundary of a compartment $\mathcal{U}$ and for any point $x \in \mathcal{U}$, $\langle x, \beta\rangle\ > 0$ and $\langle x, \beta'\rangle\ < 0$.

Lemma 2.2.2. *1. For any compartment for any short exact sequence of roots $0 \to s \to r \to q \to 0$ it is impossible to have pairs of such walls: $+s - r$ or $-q + r$.*

2. For quiver A_n for any compartment for any short exact sequence of roots $0 \to s \to r \to q \to 0$ it is impossible to have pairs of such walls: $-s - r$, $-r - q$, $+s + r$, or $+q + r$.

(1) is obvious since in either case $\langle x, s\rangle\ > 0$ but the root R is stable which is impossible. As for (2), the reason $-s - r$ can not appear is that in A_n when you cross $D(s)$ you are either going to have $+s-r$ or $+s-(r+s)$. The former is impossible due to (1). The latter is impossible due to Lemma 1.3.16 and the fact that the sum of a root and any of its subroots is no longer a root any more in A_n. The reason the other three cases can not happen is almost identical.

2.2.2 Proof of the formula

The basic idea in proving the theorem is below:

Since picture group elements freely commute with permutations, what we want to prove can be reduced to $\rho(\prod_k (i_k+1, j_k)x_{i_k j_k}) = id$. This property can further be reduced to proving that for all k, $(i_k+1, j_k)x_{i_k j_k}$ in some sense does not permute the c-vectors. This in turn can be reduced to $(i_k+1, j_k)x_{i_k j_k}$ as an operation on extended exchange matrices maintain certain properties defined below:

Definition 2.2.3. An $n \times n$ matrix $M \in M_n(\mathbb{Z})$ is *standard* if the following holds:

1. The diagonal entries are all nonzero.

2. All positive entries can only exist on the diagonal or above. and all negative entries can only exist on the diagonal or below.

3. All columns are in the form $\pm\beta_{ij}$.

It is easy to see that all columns of the form $-\beta_{ij}$ has to be the $(i+1)$-th column and all columns of the form β_{ij} has to be the j-th column since all other positions violate either Axiom 1 or 2. It is also trivial that the only results of a permutation of columns of $\pm I_n$ that are standard are $\pm I_n$ themselves.

Example 2.2.4. Here are several examples:

$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ are standard matrices because all three axioms hold.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ are not standard matrices since axioms 1 and 2 are violated.

The lemma to be proven that can almost immediately lead to the theorem is stated below:

Lemma 2.2.5. *In A_n straight orientation, $(i_k+1, j_k)x_{i_kj_k}$ or $(i_k+1, j_k)x_{i_kj_k}^{-1}$ transforms a standard matrix $\tilde{B}_{k-1}$ into a standard matrix $\tilde{B}_k$.*

Proof. We will only prove in the green case since the red case is almost identical to the green one. In this proof i_k is simplified as i and j_k is simplified as j.

Case 1: If $j-i=1$. Here we have a simple root and the associated permutation $(i+1, j)$ is identity. Hence the proof reduces to x_{ij} transforms a standard matrix to another standard one. x_{ij} merely flips the j-th column from $-e_j$ to e_j and may lengthen some $-\beta_{li}$ to $-\beta_{lj}$ for $l < i$ and shorten some β_{il} to β_{jl} for $l > j$ without changing which column they are in, but no other operation happens or violations of Lemma 2.2.2(1) $(-q+r)$ will happen, hence the resulting matrix is still standard.

Case 2: If $j-i>1$. Here we have an extra generator and the associated permutation is not identity. Now let's discuss what $(i+1, j)x_{ij}$ actually does on each column:

a) $l \le i$. Due to Lemma 1.3.16 all c-vectors have to be $\pm\beta_{ab}$ for some $0 \le a < b \le n$. So the only change that can ever happen is that $-\beta_{li}$ may be lengthen to $-\beta_{lj}$. Neither of these cause a c-matrix to violate standardness.

b) $i+1 < l < j$. Again due to Lemma 1.3.16 the only plausible situation is β_{il} was transformed into $-\beta_{lj}$. But this constitutes a $-q+r$ situation which violates Lemma 2.2.2(1).

c) $l = j$. We notice several facts:

(1).The c-vector c_j can not be negative. If this is the case we have a compartment with two walls $D(\beta_{ij})$ and $D(\beta_{j-1,m})$. This can not happen since these two roots can not be both stable due to [40] when $m > j$ or the $+r+s$ situation appears when $m = j$. Hence we can assume that the j-th column is β_{mj} for some $m < j$.

(2). It is true that m can not be less than i. Otherwise we have a $+s-r$ situation which violates Lemma 2.2.2(1).

(3). Also it is impossible for β_{mj} to remain itself after doing x_{ij} since otherwise we have a $-s-r$ situation which violates Lemma 2.2.2(2).

Hence the j-th column is positive, $m > i$ and $-\beta_{ij}$ actually add to the j-th column. So after $(i+1, j)x_{ij}$ is performed the i-th column is $-\beta_{im}$ and the j-th column is β_{ij}.

d) $l > j$. It is true that β_{il} can be shortened to β_{jl}. Other than that, the only plausible case is $l = j+1$ and the kth c-vector is $-\beta_{lm}$ for some $m > l$. In this case we can nor let $-\beta_{ij}$ add to the l-th column since otherwise $-q+r$ will be created after the mutation which violates Lemma 2.2.2(1).

Using similar methods we can see that $(i+1, j)x_{ij}$ transforms a standard matrix into another one.

□

The theorem can be proven below:

Proof. The identity matrix I_n is standard. Since $(i_k+1, j_k)x_{i_kj_k}$ or $(i_k+1, j_k)x^{-1}_{i_kj_k}$ transforms a standard matrix into another standard for all k, the result of transforming a standard matrix, I_n by $\prod_k(i_k+1, j_k)x^{\delta_k}_{i_kj_k}$ is standard. Since we get a permutation of columns of $\pm I_n$ at the end of this transformation and any permutation of columns of $\pm I_n$ that result in a standard matrix has to be the trivial permutation, $\rho(\prod_k(i_k+1, j_k)x^{\delta_k}_{i_kj_k}) = id$, hence the formula is correct.

□

2.2.3 The formula of associated permutation for any mutation sequences

Due to the theorem we can extend the definition of associated permutations to any arbitrary mutation sequence in A_n straight orientation which reduces to the existing definitions of the associated permutation of reddening and loop sequences due to the theorem above.

Definition 2.2.6. In A_n straight orientation, the *associated permutation of a mutation sequence* in correspondence to the picture group element $\prod_k x_{i_k j_k}^{\delta_k}$ acting on a c-matrix with associated permutation σ is defined as $\rho(\prod_k x_{i_k j_k}^{\delta_k}) = \sigma(\prod_k (i_k+1, j_k))^{-1}\sigma^{-1}$. Here $\delta_k \in \{+,-\}$.

In particular any mutation at a vertex with c-vector $\pm\beta_{ij}$ has an associated permutation $\sigma(i+1,j)\sigma^{-1}$ with σ the permutation of the c-matrix before the mutation. In the special case when $i+1=j$ which is when β_{ij} is a simple root the associated permutation is trivial.

Chapter 3

Tame path algebras are green sequence-finite

3.1 Introduction

In Brüstle-Dupont-Pérotin [6] and the paper by Brüstle, Hermes, Igusa and Todorov [7] it is proven that there are finitely maximal green sequences when the quiver is of finite, tame type or the quiver is mutation equivalent to a quiver of finite or tame type. Furthermore in [7] it is proven that any tame quiver has finitely many k-reddening sequences.

When we restrict our attention to the case where the algebra is basic, connected and hereditary it is a path algebra of a quiver [3]. In this chapter when we say an m-maximal green sequence of an algebra we mean an m-maximal green sequence of its path algebra. Here is the main theorem we have proven.

Theorem 3.1.1. *Any tame quiver has finitely many m-maximal green sequences.*

To prove this theorem we only need to prove that only finitely many indecomposable objects can appear as summands of silting objects that can appear in m-maximal green sequences of tame quivers Q. Since all indecomposable objects of a basic tame path algebra have to be transjective or regular, only finitely many rigid regular objects between Λ and $\Lambda[m]$ in $D_b(\Lambda)$, namely the modules on the nonhomogeneous tubes $\mathbb{Z}A_\infty/\langle\tau^k\rangle$ with no repeating composition factors and their shifts. Hence the problem is reduced to proving that only finitely many indecomposable transjective objects between Λ and $\Lambda[m]$ can appear in m-maximal green sequences.

To prove this theorem we need two lemmas.

Lemma 3.1.2. *For a tame quiver Q any silting object in $D^b(kQ)$ contains at most $n-2$ regular summands. In other words, at least 2 summands have to be transjective.*

Lemma 3.1.3. *For a tame quiver Q there is a uniform bound, depending only on Q and m, on the transjective degree of any transjective summand in any silting object in any m-maximal green sequence $D^b(kQ)$.*

It is easy to see why Lemma 3.1.3 implies the theorem. Here the *transjective degree* of an indecomposable transjective object $\tau^i P_j[k]$ is defined as $deg(\tau^i P_j[k]) = i$. The *maximal transjective degree* and *minimal transjective degree* of a silting object are defined as the highest/lowest transjective degree of its indecomposable transjective summands respectively.

In Section 2 we prove Lemma 3.1.2. In Section 3 we prove Lemma 3.1.3. In Section 4 we further generalize the theorem to arbitrary finite mutation sequences with finitely many forward/green or backward/red mutations.

3.2 Proof of Lemma 3.1.2

To prove Lemma 3.1.2 we need to understand regular components of Auslander-Reiten quivers of $D^b(kQ)$ for tame quivers. Regular components of Auslander-Reiten quivers of tame path algebras are all standard stable tubes with at most three tubes nonhomogeneous (see [13] and Chapter X of [39]). Note that no object on a homogeneous tube is rigid so no object there can appear in a silting object of $D^b(kQ)$. Hence we only need to discuss the nonhomogeneous tubes.

It is easy to see that in an indecomposable object in a standard stable tube $\mathcal{T}$ of size n, M and any of its shifts can not be in the same pre-silting object.

Definition 3.2.1. If $\{M_i\}_{i\in I}$ are a family of indecomposable objects of $D^b(kQ)$ and $\Pi_{i\in I}M_i[n_i]$ is not pre-silting for any $\{n_i\}_{i\in I}$ We say that $\{M_i\}_{i\in I}$ is *silting-incompatible.* Otherwise we say that it is *silting-compatible.*

From now on in this proof we identify $[n]$ with $\mathbb{Z}/n\mathbb{Z}$ and hence will no longer differentiate between 0 and n which we usually denote as n. It is also clear that it makes sense to define a cyclic order on $[n]$.

Definition 3.2.2. Let $a, b \in [n]$. The *interval* $[a, b]$ is defined as the following: $[a, b] :=$
$$\begin{cases} \{x|a \le x \le b, x \in [n]\} & \text{if } a \le b \\ \{x|a \le x \le n, x \in [n] \text{ or } 1 \le x \le b, x \in [n]\} & \text{if } a > b \end{cases}$$

Definition 3.2.3. Let $a_0, a_1 \cdots, a_{k-1}$ be elements of $[n]$. Define a_k The proposition $P(a_0, a_1, \cdots, a_{k-1})$ is defined as the statement that all a_i are distinct and that for any $i \in \{0, 1, \cdots, k-1\}$ for any $l \ne i, i+1$ it is true that a_l is not in the interval $[a_i, a_{i+1}]$.

For example $P(1,2,3)$ and $P(4,1,2)$ hold in [4] while $P(1,2,1)$, $P(2,1,3)$ and $P(4,3,1)$ do not.

Definition 3.2.4. Let M_i be the quasi-simples of the tube such that $\tau M_i = M_{i-1}$. a regular module in $D^b(kQ)$ *regular sincere* if its composition series contain all quasi-simples.

No indecomposable regular sincere modules or their shifts can appear as summands in any silting object because they are not rigid. (See Corollary X.2.7 of [39]).As for the remaining $n(n-1)$ indecomposable regular modules that are actually rigid we can unambiguously label them as M_{ij} if the quasi-top and quasi-socle of the object are M_j and M_i respectively. Note that $M_i = M_{ii}$. It is clear that $\tau M_{ij} = M_{i-1,j-1}$ and $\tau^{-1} M_{ij} = M_{i+1,j+1}$.

Now let's prove two easy lemmas on what can not appear in a pre-silting object in a regular component of the Auslander-Reiten quiver of $D^b(kQ)$.

Lemma 3.2.5. *1. If M and N are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Hom(M,N) \neq 0$ and $Ext^1(N,M) \neq 0$, then M and N are silting-incompatible.*

2. If $M_1, \cdots, M_k$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Ext^1(M_i, M_{i+1}) \neq 0$ for any $1 \leq i < k$ and $Ext^1(M_k, M_1) \neq 0$, then $\{M_i\}$ is silting-incompatible.

Proof. For (1) since $Hom(M,N) \neq 0$ if $i > j$ we have $Ext^{i-j}(M[i], N[j]) \neq 0$. Since $Ext^1(N,M) \neq 0$ if $i \leq j$ it is true that $Ext^{j-i+1}(N[j], M[i]) \neq 0$. Hence $M[i] \oplus N[j]$ is not pre-silting for any arbitrary i and j.

For (2) for arbitrary $n_1, \cdots n_k$ use the argument above it is easy to see that if $\oplus_{i=1}^{k} M_i[n_i]$ is pre-silting, then $n_2 > n_1$, $n_3 > n_2$, $\cdots, n_1 > n_k$ which is impossible. Hence $\{M_i\}$ is silting-incompatible. □

Lemma 3.2.6. *Any pre-silting object in a standard stable tube of size n contains at most $n-1$ summands.*

To prove this lemma we need the following lemma.

Lemma 3.2.7. *Any pre-silting object in a standard stable tube of size n can not be regular sincere.*

Proof. Assume that a pre-silting object $T = \oplus_{i=1}^{k} T_i$ in a standard stable tube of size n is regular sincere. Let $T_{l_1}, \cdots, T_{l_m}$ be a minimal set of indecomposable summands of T such that their direct sum $T' = \oplus_{i=1}^{m} T_{l_i}$ is regular sincere. Note that if M_{ij} and M_{kl} are both summands of T, $P(i,k,l,j)$ holds M_{kl} and M_{ij} can not both be summands of T' due to minimality. If $m = 1$ then T' is a regular sincere indecomposable regular object which contradicts the fact that T' is pre-silting. If $m > 1$ without loss of generality assume that $T_{l_1} = M_{1p}$ for some $p \neq n$. Any indecomposable object with its quasi-socle $M_i, 1 \leq i \leq p$ can not be a summand of T' either due to silting incompatibility or minimality. Hence there has to be a summand of T' with its quasi-socle $p+1$. Repeat this procedure it's easy to see that $T' = \oplus_{i=1}^{m} M_{(t_{i-1}+1)t_i}$ with $t_0 = t_m = n$. In this case by Lemma 3.2.5(2) the object can not be pre-silting. □

Now we can prove Lemma 3.2.6.

Proof. Since any pre-silting object in a standard stable tube of size n can not be regular sincere, without loss of generality it is a pre-silting object in the exact subcategory of $\mathcal{T}$

closed under extensions such that $M_1, \cdots M_{n-1}$ are the only simple objects. It is easy to see using the condition that the tube is standard stable which is a result of Theorem 1.5.19(5) that the category $add(\{M_{ij}\}_{1\leq i<j\leq n-1})$ is isomorphic to the module category of kA_{n-1} with straight orientation and as a result any pre-silting object with all indecomposable summands in it or its shifts has at most $n-1$ summands. □

Finally we can prove Lemma 3.1.2.

Proof. Due to Lemma 3.2.6 and [13] there are at most $n-2$ regular components in $D^b(kQ)$ when Q is a tame quiver. This is true for each type so this is true for all tame quivers. □

3.3 Proof of Lemma 3.1.3

To prove Lemma 3.1.3 we need to rephrase an argument in [6] using degrees.

Lemma 3.3.1. *([6], Lemma 10.1) Let H be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective H-module P, the H-modules $\tau^{-k}P$ and $\tau^k P[1]$ are sincere.*

Lemma 3.3.2. *([6]) Let Q be a tame quiver and M_1, M_2 two transjective modules of kQ. If $\{M_1, M_2\}$ is silting-compatible, then $|deg(M_1) - deg(M_2)| \leq N$*

Proof. If $k-l > N$ we need to prove that $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. If $i \leq j$ $Ext^{j-i+1}(\tau^l P_b[j], \tau^k P_a[i]) = Ext^1(\tau^l P_b, \tau^k P_a) = Hom(\tau^{k-1}P_a, \tau^l P_b) = Hom(P_a, \tau^{l-k+1}P_b) \neq 0$ since $\tau^{l-k+1}P_b$ is a sincere preprojective module. If $i > j$ $Ext^{i-j}(\tau^k P_a[i], \tau^l P_b[j]) = Ext^1(\tau^k P_a[1], \tau^l P_b) = Hom(\tau^{l-1}P_a, \tau^k P_b[1]) = Hom(P_a, \tau^{k-l+1}P_b[1]) \neq 0$ since $\tau^{k-l+1}P_b[1]$ is a sincere preinjective module. Hence $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. Exchange the objects if $k-l < -N$. Hence the lemma has been proven. □

Now we can prove Lemma 3.1.3 following a modified version of the argument in [6].

Proof. We only need to prove that there is a lower bound of minimal transjective degrees of silting objects that can appear in m-maximal green sequences. Assume that $\tau_k P_i[j]$ is in a silting object in an m-maximal green sequence of kQ. Note that due to Lemma 3.1.2 there are at least 2 transjective components in any silting object in $D^b(kQ)$. Note that each mutation on a transjective object T in $\mathcal{P}_i$ can result in a transjective object in $\mathcal{P}_{i+1}$, a transjective object in $\mathcal{P}_i$ with degree less than or equal to $deg(T)$ or a regular object in $\mathcal{R}_i$. Each mutation on a regular object T' in $\mathcal{R}_i$ can result in an object of $\mathcal{R}_i$, an object of $\mathcal{P}_{i+1}$ or an object of $\mathcal{R}_{i+1}$. Let L be the minimal transjective degree of a silting object. No green mutation within a component or green mutation from a regular component to another one can increase L. All other green mutations may increase L by at most N. However there are only n summands of a silting object, $m+1$ transjective components and m regular components so the amount of mutations that can increase L is finite. To reach $\Lambda[m]$ which is of degree 0 L has to be at least $-2mnN$. As a result no indecomposable transjective object in any silting object in an m-maximal green sequence can have a degree less than $-2mnN$. Similarly silting objects in m-maximal green sequences can not have maximal transjective degree higher than $2mnN$ or it can not start from Λ. $\square$

3.4 Almost morphism finiteness

Using the same method we can prove a stronger result.

Theorem 3.4.1. *If Q is a Dynkin or tame quiver and T_1, T_2 are silting objects of $D^b(kQ)$ then there are finitely many k-red and finitely many k-green mutation sequences from T_1 to T_2 for any k.*

Note that we only need to prove that part of the statement about k-red sequences. To prove the theorem we first need to prove the following lemma which is a generalization of Lemma 4.4.2 in [7].

Lemma 3.4.2. *1. Any k-red sequence from T_1 to T_2 can go through any silting object at most $r+1$ times.*

2. Any k-green sequence from T_1 to T_2 can go through any silting object at most $r+1$ times.

Proof. We only need to prove (1). It is clear from the definition of mutations that a green sequence can go through any silting object at most once. (See [11] and [30] for more details.) Let's define a *green arm* of a mutation sequence as a maximal subsequence of the mutation sequence that is green. Similarly we can define what is a *red arm*. Assume that an k-red sequence $\{T_i\}$ has n_r red arms and n_g green arms. $n_r \leq k$. $n_g \leq n_r+1$. Let n_1 be the number of red arms of length 1 and n_2 the number of red arms of length at least 2. It's clear that $n_r = n_1 + n_2$ and $n_1 + 2n_2 \leq k$. Note that any silting object on a red arm of length 1 is on a green arm. Hence $\{T_i\}$ can go through any silting object at most $n_g+n_2 \leq n_1+2n_2+1 \leq k+1$ times. □

It is easy to see that the bounds established in the lemma are optimal. Now we can prove the theorem. Note that the lemma above implies that in the Euclidean case if we can prove that for any k if there are finitely many rigid objects that ca$k$-redn appear as summands of silting objects in k-red sequences Theorem 3.4.1 will been proven.

Proof. As we said above we will only prove the part about k-red sequences. Assume that all indecomposable summands of T_1 and T_2 are between $\Lambda[i]$ and $\Lambda[j]$. Since there are only k red mutations, all indecomposable summands that appear in k-red sequences from T_1 to T_2

have to be between $\Lambda[i-k]$ and $\Lambda[j+k]$.

If Q is Dynkin there are only finitely many indecomposable objects between $\Lambda[i-k]$ and $\Lambda[j+k]$ and hence only finitely many silting objects can exist on an k-red sequence. Due to Lemma 3.4.2 there are finitely many k-red sequences.

From now on we assume that Q is Euclidean. There are only finitely many regular rigid indecomposable objects between $\Lambda[i-k]$ and $\Lambda[j+k]$ so the problem has been reduced to proving that only finitely many transjective indecomposable components can appear in silting objects in k-red sequences.

Let the minimal degree of T_2 be L. Note that a red mutation can increase the minimal degree of a silting object by at most N. Use an argument similar to that one used to prove Theorem 4.1.1 we can prove that no indecomposable transjective object with degree less than $L-2nN(2k+j-i)-kN$ can appear in any k-red sequences from T_1 to T_2. Similarly let the maximal degree of T_1 be U. No indecomposable transjective object with degree less than $U+2nN(2k+j-i)+kN$ can appear in any k-red sequence from T_1 to T_2. Hence there are only finitely many indecomposable transjective objects can appear in any k-red sequence from T_1 to T_2 and the theorem is proven. □

Note that the bounds of transjective degrees in the proofs of Theorem 3.1.1 and Theorem 3.4.1 above are very crude. In the future we will try to find better bounds.

Finally let's define a new term to characterize finite dimensional algebras that satisfy the conditions of Theorem 3.4.1.

Definition 3.4.3. 1. A finite dimensional algebra Λ of finite global dimension such that it has finitely many k-red sequences from any silting object T_1 to any silting object T_2 for any k is *almost morphism finite.*

2. A finite dimensional algebra Λ of finite global dimension such that it has finitely many green sequences from any silting object T_1 to any silting object T_2 for any m is *green sequence finite.*

Hence we can rephrase Theorem 3.4.1 as the following:

Theorem 3.4.4. *If Λ is the path algebra of a quiver of finite or tame type, then Λ is almost morphism finite.*

Note that the condition of an algebra being almost morphism finite is stronger than the condition that it is green sequence finite which is stronger than the condition that there are finitely many m-maximal green sequences for any m. An almost morphism finite algebra has finitely many k-red sequences for any k hence it has finitely many green-to-red sequences with k red mutations.

Chapter 4

Two alternative definitions of m-maximal green sequences

4.1 Introduction

In Chapter 1 we introduced a result by Igusa, namely Theorem 1.6.13 namely there are new alternative definitions of maximal green sequences. How the result can possibly be generalized to m-maximal green sequences in general is an interesting question that we have mostly solved.

Theorem 4.1.1. *(Theorem 4.5.1) Let Λ be a finite dimensional hereditary algebra. Let $\mathcal{T} = add(\cup_{i=0}^{m-1}(mod\,\Lambda)[i])$. Let $\{M_1, \cdots, M_n\}$ be a finite sequence of nonzero objects in $\mathcal{T}$. The following are equivalent:*

1. *The sequence is a maximal backward $Hom^{\leq 0}$-orthogonal sequence of Schurian objects $\{M_n\}$ in $\mathcal{T}$.*

2. *The sequence is a finite sequence in $\mathcal{T}$ that forms a finite HN system for $\mathcal{T}$.*

3. *The sequence is a sequence of simples from the simple-minded collection $\{S_1, \cdots, S_n\}$ to $\{S_1[m], \cdots, S_n[m]\}$. that is, it is an m-maximal green sequence..*

Results in this chapter can be generalized to arbitrary green sequences which we will show in an upcoming paper. Some proofs especially the proof of Lemma 4.4.5 will be modified to handle complexities resulting from arbitrariness of these green sequences.

In Section 2 we will discuss an alternative definition of the stability condition on module categoris. In Section 3 we will discuss maximal backward-$Hom^{\leq 0}$ orthogonal sequences. In Section 4 we will discuss Harder-Narasimhan filtrations. In Section 5 we will establish the fact that the two alternative definitions are equivalent to the original ones.

4.2 Alternative definition of the stability condition on module categories

This section is not used in the rest of the chapter. However it does provide new definitions of stability and semistability that are previously underdiscussed. In the future we might make it a special case of some weaker variant of the Bridgeland stability condition [5] in some upcoming paper. Moreover the ideas in this section are related to $Hom^{\leq 0}$-backward orthogonal sequences.

In general in a triangulated category the concepts of monomorphisms and epimorphisms are less important because there are almost no nontrivial ones. Instead the concept of homotopy kernels and homotopy cokernels are much more important.

Before we can generalize the idea of a maximal green sequence we first need to generalize the idea of a stability condition without always relying on monomorphisms and epimorphisms.

Theorem 4.2.1. *If Λ is a finite dimensional hereditary algebra for an indecomposable module M in a module category modΛ for a stability condition ϕ the following are equivalent:*

1. *M is stable. That is, for any proper submodule N of M it is true that $\phi(N) < \phi(M)$.*

2. *For any proper quotient module N of M it is true that $\phi(M) < \phi(N)$.*

3. *For any indecomposable stable module $N \ncong M$ such that $(M, N) \neq 0$ it is true that $\phi(M) < \phi(N)$.*

4. *For any indecomposable stable module $N \ncong M$ such that $(N, M) \neq 0$ it is true that $\phi(N) < \phi(M)$.*

Proof. (1)→(2) If N is a quotient module of M, we have the short exact sequence $0 \to \sum_{i=1}^{k} R_i \to M \to N \to 0$. Since M is stable $\phi(\sum_{i=1}^{k} R_i) < \phi(M)$ hence $\phi(N) > \phi(M)$.

(2)→(1) This proof is analogous to the proof of (1)→(2).

(1),(2)→(3) If M is a submodule of N or N is a quotient module of M then due to (1) and (2) the statement is trivially true. Assume that there exists neither monomorphisms nor epimorphisms from M to N. Assume that there exists $0 \neq f \in (M, N)$ it is easy to see that $Imf \in mod\Lambda$. Take one of its indecomposable summand, L. It is easy to see that L is a proper submodule of N and a proper quotient module of M at the same time. Since M, N are stable we have $\phi(M) < \phi(L) < \phi(N)$.

(1),(2)→(4) This proof is analogous to the proof of (1),(2)→(3).

(4)→(1) Use induction. If M is simple it is of course stable. Hence (4)→(1) is trivially true in the case of simples. Otherwise assume that (4)$\rightarrow$(1) is already true for all indecomposable modules with dimension less than $dim(M)$. If M satisfies condition (4) then for any of its stable submodule N we already have $\phi(N) < \phi(M)$ so we only need to focus on the non-stable ones. Assume that L is one of its minimal non-stable indecomposable proper submodules such that $\phi(L) \geq \phi(M)$. By induction since $\neg(1) \rightarrow \neg(4)$ holds for L there has to be an indecomposable stable module $N \ncong L$ such that $(N, L) \neq 0$ and $\phi(N) \geq \phi(L)$. Hence $\phi(N) \geq \phi(L) \geq \phi(M)$ and $(N, M) \neq 0$. Hence M does not satisfy condition (4) and we have reached a contradiction. As a result (4)$\rightarrow$(1) is proven.

(3)$\rightarrow$(2) This proof is analogous to the proof of (4)$\rightarrow$(1). Use induction. If M is simple (2) of course holds. Hence (3)$\rightarrow$(2) is trivially true in the case of simples. Otherwise assume that (3)$\rightarrow$(2) is already true for all indecomposable modules with dimension less than $dim(M)$. If M satisfies condition (3) then for any of its stable quotient submodule N we have $\phi(M) < \phi(N)$ so we only need to focus on the non-stable ones. Assume that L is one of its minimal non-stable indecomposable proper quotient modules such that $\phi(M) \geq \phi(L)$. By induction since $\neg(2) \rightarrow \neg(3)$ holds for L there has to be an indecomposable stable module $N \ncong L$ such that $(L, N) \neq 0$ and $\phi(L) \geq \phi(N)$. Hence $\phi(M) \geq \phi(L) \geq \phi(N)$ and $(M, N) \neq 0$. Hence M does not satisfy condition (3) and we have reached a contradiction. As a result (3)$\rightarrow$(2) is proven.

$\square$

We can obtain a similar result in the case of semistability.

Theorem 4.2.2. *If Λ is a finite dimensional algebra for an indecomposable module M in a module category modΛ for a stability condition ϕ the following are equivalent:*

1. *M is semistable.*

2. *For any proper quotient module N of M it is true that $\phi(M) \leq \phi(N)$.*

3. *For any indecomposable stable module N such that $(M, N) \neq 0$ it is true that $\phi(M) \leq \phi(N)$.*

4. *For any indecomposable stable module N such that $(N, M) \neq 0$ it is true that $\phi(N) \leq \phi(M)$.*

4.3 Maximal backward $Hom^{\leq 0}$ orthogonal sequences

In order to discuss maximal backward $Hom^{\leq 0}$ orthogonal sequences we first need to define them.

Definition 4.3.1. $M_1, M_2, \cdots, M_k$ is a *backward $Hom^{\leq 0}$ orthogonal sequence* of Schur objects if $(M_i[\geq 0], M_j) = 0$ for all $i > j$ and any M_i is non-zero.

Definition 4.3.2. $M_1, M_2, \cdots, M_k \in \mathcal{T}$ is a *maximal backward $Hom^{\leq 0}$ orthogonal sequence* of Schur objects on $\mathcal{T}$ if $(M_i[\geq 0], M_j) = 0$ for all $i > j$, all M_i are Schur and that for any other Schur object $M' \in \mathcal{T}$ if it is inserted anywhere in the sequence it will no longer be backward $Hom^{\leq 0}$ orthogonal.

Now let's prove a crucial lemma.

Lemma 4.3.3. *If H_1, H_2 are two hearts of t-structures $(C^{\leq 0}, C^{\geq 0})$ and $(C'^{\leq 0}, C'^{\geq 0})$ respectively, there exists a maximal backward $Hom^{\leq 0}$-orthogonal sequence from H_1 to H_2 then the first term of the sequence has to be a simple of H_1 that is not in H_2.*

Proof. Let's first assume that $M \in H_1[l]$ with $l \geq 0$ is the first term of the maximal backward $Hom^{\leq 0}$-orthogonal sequence. Let the truncation functors of $(C^{\leq 0}, C^{\geq 0})$ be $\tau_{\geq n}$ and $\tau_{\leq n}$ respectively. There exists some simple $S \in H_1$ such that $S[l]$ is a subobject of M in $H_1[l]$ which is Abelian. If there exists no non-initial term N in the sequence such that $(N, S) \neq 0$ (note that it is impossible to have $(N[i], S) \neq 0$ for positive i due to $N = \tau_{\leq 0} N$) then the sequence is not maximal because S can be inserted before M. Hence we assume that such an N exists, In this case $(N[l], M) = 0$ or the sequence wouldn't have been backward $Hom^{\leq 0}$-orthogonal. Let $N' = \tau_{\geq 0} N$ and $N'' = \tau_{<0} N$. So we have the canonical triangle $N'' \to N \to N' \to N''[1]$. Due to $(N[l], M) = 0$ it is obvious that $(N'[l], M) = 0$. Note that $M, N'[l], S[l] \in H_1[l]$. $(N'[l], S[l]) = 0$ since $S[l]$ is a subobject of M in an Abelian category. As a result $(N', S) = 0$ and hence $(N, S) = 0$ which contradicts the assumption that $(N, S) \neq 0$.

Now let's assume that $M \in \tau_{\geq -l}$ but not $\tau_{\geq -l+1}$. Let $M' = \tau_{\leq -l} M$. Let $S[l]$ be a subobject of M' in $H_1[l]$ which is Abelian. Note that $(S[l], M) = (S[l], M') \neq 0$ because $(S[l], M/M'[-1]) = 0$ since $M/M'[-1] \in \tau_{\geq -l+1}$. If there exists no non-initial term N in the sequence such that $(N, S) \neq 0$ (note that again it is impossible to have $(N[i], S) \neq 0$ for positive i due to $N = \tau_{\leq 0} N$) then the sequence is not maximal because S can be inserted before M. Hence we assume that such an N exists, In this case $(N[l], M) = 0$ or the sequence wouldn't have been backward $Hom_{\leq 0}$-orthogonal. Let $N' = \tau_{\geq 0} N$ and $N'' = \tau_{<0} N$. So we have the canonical triangle $N'' \to N \to N' \to N''[1]$. Due to $(N[l], M) = 0$ it is obvious that $(N'[l], M) = 0$ and $(N'[l], M') = 0$. Note that $M', N'[l], S[l] \in H_1[l]$. $(N'[l], S[l]) = 0$ since $S[l]$ is a subobject of M' in an Abelian category. As a result $(N', S) = 0$ and hence $(N, S) = 0$ which contradicts the assumption that $(N, S) \neq 0$.

$\square$

4.4 Harder-Narasimhan filtration

Now we need to introduce Harder-Narasimhan (HN) filtrations.

Definition 4.4.1. Let $\mathcal{T}$ be a subcategory of a triangulated category. $M_1, \cdots, M_k \in \mathcal{T}$ are a finite sequence of nonzero objects. An *HN filtration of object $X \in \mathcal{T}$ with respect to $\{M_i\}$* aka an HN filtration of an object X is the following diagram:

$$\begin{tikzcd}
0 = X_k \arrow[rr] & & X_{k-1} \arrow[rr] \arrow[dl] & & X_{k-2} \arrow[dl] & \cdots \arrow[r] & X_0 = X \arrow[dl] \\
& X_{k-1}/X_k \arrow[ul, dashed] & & X_{k-2}/X_{k-1} \arrow[ul, dashed] & \cdots & X_0/X_1 &
\end{tikzcd}$$

where X_{i-1}/X_i is a self-extension of M_i.

If an object has an HN filtration with respect to $\{M_i\}_{i \in [N]}$ then it makes sense to define its *lowest and highest indices* with respect to the filtration.

Definition 4.4.2. Let $\mathcal{T}$ be a subcategory of a triangulated category. $M_1, \cdots, M_k \in \mathcal{T}$ are a finite sequence of nonzero objects. If some object $X \in \mathcal{T}$ has an HN filtration

$$\begin{tikzcd}
0 = X_k \arrow[rr] & & X_{k-1} \arrow[rr] \arrow[dl] & & X_{k-2} \arrow[dl] & \cdots \arrow[r] & X_0 = X \arrow[dl] \\
& X_{k-1}/X_k \arrow[ul, dashed] & & X_{k-2}/X_{k-1} \arrow[ul, dashed] & \cdots & X_0/X_1 &
\end{tikzcd}$$

with respect to $M_1, \cdots, M_k$ then we can define the following:

1. The *lowest index* in the filtration is defined as the smallest i such that $X_{i-1}/X_i \neq 0$.

2. The *highest index* in the filtration is defined as the largest i such that $X_{i-1}/X_i \neq 0$.

If X has a unique HN filtration with respect to $M_1, \cdots, M_k$ then the lowest and highest indices of X are only dependent on $M_1, \cdots, M_k$. In this case we can refer to them as l_X and h_X without any ambiguity.

We may sometimes abuse notations and refer to the unique HN filtration of some X as $0 \to X_{h_X - 1} \to \cdots \to X_{l_X} = X$.

Definition 4.4.3. Let $\mathcal{T}$ be a subcategory of a triangulated category. $M_1, \cdots, M_k \in \mathcal{T}$ are a finite sequence of nonzero objects. $M_1, \cdots, M_k$ form a *finite HN system for* $\mathcal{T}$ if any object $X \in \mathcal{T}$ has a unique HN filtration with respect to $M_1, \cdots, M_k$.

Lemma 4.4.4. *Let Λ be a finite dimensional hereditary algebra. Let $\mathcal{T} = add(\cup_{i=0}^{m-1}(mod\,\Lambda)[i])$. Let $M_1, \cdots, M_n$ be a finite sequence of nonzero objects in $\mathcal{T}$. If any object Y in $\mathcal{T}$ accept a unique HN filtration $0 \to Y_m \to \cdots \to Y_1 = Y$ with $Y_i/Y_{i+1} \in \mathcal{E}(M_i)$, the following holds.*

1. *For any i it is true that M_i is indecomposable.*

2. *For any i it is true that M_i is Schur.*

Proof. For (1) assume that M_i is decomposable. Then $M_i = A \oplus B$ with $A \neq 0$ and $B \neq 0$. In this case A and B have nontrivial HN filtrations and adding them up we should obtain an HN filtration for M_i that isn't the canonical one which contradicts the fact that M_i has a unique HN filtration.

For (2) assume that M_i is not Schur. Without loss of generality we can assume that $M_i \in mod\Lambda$. Then there exists $f \in End(M_i)$ such that $Imf \neq M_i$. Take an indecomposable direct summand of Imf, Q. Since Q is a quotient module of M_i and Λ is hereditary $Ext^1(M_i/Q, M_i) \to Ext^1(M_i/Q, Q) \neq 0$ is a surjection. We have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & M_i/Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q \arrow[r] & M_i \arrow[r] & M_i/Q \arrow[r] & 0
\end{tikzcd}$$

Due to $0 \to M_i \to Q \oplus X \to M_i \to 0$ being a short exact sequence with is a consequence of a pushout diagram on the left and $M_i \rightarrowtail X$ being monomorphic, $Q \oplus X$ has two HN-filtrations, one containing the $i-$th entry only while the other definitely contain what is not in the i-th entry because Q can not only have the i-th entry which is clear because $dim_k(Q) < dim_k(M_i)$ as k-vector spaces.

$\square$

Lemma 4.4.5. *Let Λ be a finite dimensional hereditary algebra. Let $\mathcal{T} = add(\cup_{i=0}^{m-1}(mod\,\Lambda)[i])$. Let $M_1, \cdots, M_n$ be a finite sequence of nonzero objects in $\mathcal{T}$. If any object Y in $D^b(\Lambda)$ accept a unique HN filtration $0 \to Y_N \to \cdots \to Y_1 = Y$ with $Y_i/Y_{i+1} \in \mathcal{E}(M_i)$, the following holds.*

1. *For any $i > j$ it is true that $Hom(M_i, M_j) = 0$.*
2. *If $M_j = M_i[1]$ then $j > i$. Moreover if Y is any object of $\mathcal{T}$ such that $Y[1] \in \mathcal{T}$ then $l_Y \leq h_{Y[1]}$.*
3. *If $Y_i \neq 0$ for some $i \in [N]$ then $Hom(Y_i, Y) \neq 0$.*
4. *If $Y/Y_i \neq 0$ for some $1 < i \leq N$ then $Hom(Y, Y/Y_i) \neq 0$.*
5. *For any $i > j$ it is true that $Hom(M_i[1], M_j) = 0$.*
6. *For any $m > 0$ if $M_j = M_i[m]$ then $j > i$. Moreover if Y is any object of $\mathcal{T}$ such that $Y[m] \in \mathcal{T}$ then $l_Y \leq h_{Y[m]}$.*
7. *For any $m > 0$ for any $i > j$ it is true that $Hom(M_i[m], M_j) = 0$.*

Proof. (1) is true because otherwise we have completely different HN filtrations for $M_i \oplus M_j$, namely $M_i \overset{(1,0)^t}{\to} M_i \oplus M_j \overset{(0,1)}{\to} M_j \overset{0}{\to} M_i[1]$ and $M_i \overset{(1,f)^t}{\to} M_i \oplus M_j \overset{(f,-1)}{\to} M_j \overset{0}{\to} M_i[1]$ where f is a nontrivial morphism from M_i to M_j.

(2) is true because otherwise $M_i \to 0 \to M_i[1] \to M_i[1]$ will be an HN filtration of 0 and hence there will be at least two HN filtrations of 0. Similarly if the HN filtration of Y is strictly before the HN filtration of $Y[1]$ $0 \to Y \to Y[1] \to 0$ will be an HN filtration of 0.

Let $l \geq i$ be any number such that $Y_l/Y_{l+1} \neq 0$. Since $Y_i \neq 0$ such l must exist. (3) is true because $Y_i \to Y \to Y/Y_i \to Y_i[1]$ is a triangle. If $Hom(Y_i, Y) = 0$ the triangle splits and $Y/Y_i = Y \oplus Y_i[1]$. Since $Y_i[1]$ has a unique HN filtration Y/Y_i has two HN filtrations, one with the l-th entry 0 and one with a nontrivial l-th entry.

Let $l < i$ be any number such that $Y_l/Y_{l+1} \neq 0$. Since $Y/Y_i \neq 0$ such l must exist. (4) is true because $Y_i \to Y \to Y/Y_i \to Y_i[1]$ is a triangle. If $Hom(Y, Y/Y_i) = 0$ the triangle splits and $Y_i = Y \oplus Y_i[-1]$. Since $Y_i[-1]$ has a unique HN filtration Y_i has two HN filtrations, one with the l-th entry 0 and one with a nontrivial l-th entry.

Now let's prove (5). Assume that $i > j$ and $Hom(M_i[1], M_j) \neq 0$. Since $Hom(M_i, M_j[-1]) \neq 0$ and that $M_j \in \mathcal{T}$ it is clear that $M_j[-1] \in \mathcal{T}$ and hence has a unique HN filtration. Let's first assume that the highest term of the HN filtration of $M_j[-1]$ is a self-extension of M_i. If this is not the case assume that the highest term is a self-extension of $M_{i'}$. If $i' \leq j$ then it is clear that $Hom(M_i, M_j[-1]) = 0$ since the Hom from M_i to all terms in the HN filtration of $M_j[-1]$ is 0 due to (1). Hence $i' > j$ and we can simply use i' instead of i since $Hom(M_{i'}, M_j[-1]) \neq 0$. So we can indeed assume that the highest entry of the HN filtration of $M_j[-1]$ is a self-extension of M_i. Let such an entry be X_i. Since $Hom(M_i[1], M_j) \neq 0$ and $M_i \in \mathcal{T}$ we can see that $M_i[1] \in \mathcal{T}$. As a result $X_i[1] \in \mathcal{T}$. Let h be the highest nontrivial index of the HN filtration of $X_i[1]$ then h is not higher than j or the highest nontrivial index of $M_j[-1]/X_i$ both of which are lower than i due to (4) since $M_j[-1] \to M_j[-1]/X_i \to X_i[1] \to M_j$ is a triangle. Apply (4) again to $X_i[1]$ as a self-extension of $M_i[1]$ the highest nontrivial index of the HN filtration of $M_i[1]$ is lower than i. Apply (2) to M_i and we can reach a contradiction.

As for (6), since (2) is already proven let's assume that the result has been proven for all positive integers below m and use induction. For the first claim it is clear that $M_i[1]$ can not be any M_l or the induction hypothesis would be violated. It is also clear that $M_i[1] \in \mathcal{T}$ since $M_i, M_i[m] \in \mathcal{T}$. Take the lowest nonzero entry Y_k of the HN filtration of $M_i[1]$. If $k = i$ then $Hom(M_i[1], M_i) \neq 0$ which can not happen. If $k > i$ then we can apply the induction hypothesis to the HN filtration of $M_i[1]$ and $M_i[m]$ and show that this is false. Hence $k < i$. In this case $Hom(M_i[1], Y_k) \neq 0$. Hence $Hom(M_i[1], M_k) \neq 0$ which is impossible due to (5). Now let's prove the second claim. Here since $Y, Y[m] \in \mathcal{T}$ so does $Y[1]$. Let the lowest entry of the HN filtration of $Y[1]$ be $N_k \in \mathcal{E}(M_k)$ and let $0 \to Y_j \to \cdots Y_l = Y$ be the HN filtration of Y. It is easy to see that $k < l$ due to the induction hypothesis applied to $Y[1]$ and $Y[k]$. Hence $Hom(Y[1], M_k) \neq 0$. Hence for some $h > k$ we have $Hom(M_h[1], M_k) \neq 0$ which is impossible due to (5).

Finally we need to prove (7). Assume that the result is true for any positive integer below m which is legit because (5) is already proven. Assume that $i > j$ and $Hom(M_i[m], M_j) \neq 0$. Since $Hom(M_i, M_j[-m]) \neq 0$ and $M_j \in \mathcal{T}$ it is clear that $M_j[-m] \in \mathcal{T}$. Let's first assume that the highest entry of the HN filtration of $M_j[-m]$ is a self-extension of M_i. If this is not the case assume that the highest term is a self-extension of $M_{i'}$. If $i' \leq j$ then it is clear that $Hom(M_i, M_j[-m]) = 0$ since the Hom from M_i to all terms in the HN filtration of $M_j[-m]$ is 0 due to (1). Hence $i' > j$ and we can simply use i' instead of i since $Hom(M_{i'}, M_j[-m]) \neq 0$. So we can indeed assume that the highest entry of the HN filtration of $M_j[-m]$ is a self-extension of M_i and let such an entry be X_i. Since $Hom(M_i[1], M_j[1-m]) \neq 0$ and $M_i \in \mathcal{T}$ we can see that $M_i[1] \in \mathcal{T}$. As a result $X_i[1] \in \mathcal{T}$. Let h be the highest nontrivial index of the HN filtration of $X_i[1]$ then h is no higher than the highest nontrivial index of $M_j[-m]/X_i$ or $Hom(M_h, M_j[1-m]) \neq 0$ in which case $h \leq j$ by induction since $M_j[-m] \to M_j[-m]/X_i \to X_i[1] \to M_j[1-m]$ is a triangle. Hence in both cases $h < i$. Apply (4) again to $X_i[1]$ as a self-extension of $M_i[1]$ the highest nontrivial index of the HN filtration of $M_i[1]$ is lower than i. Apply (2) to M_i and we can reach a contradiction.

□

4.5 Equivalence of the definitions

Theorem 4.5.1. *Let Λ be a finite dimensional hereditary algebra. Let $\mathcal{T} = add(\cup_{i=0}^{m-1}(mod\,\Lambda)[i])$. Let $M_1, \cdots, M_n$ be a finite sequence of nonzero objects in $\mathcal{T}$. The following are equivalent:*

1. *The sequence is a maximal backward $Hom^{\leq 0}$-orthogonal sequence of Schurian objects $\{M_n\}$ in $\mathcal{T}$.*
2. *The sequence is a finite sequence in $\mathcal{T}$ that forms a finite HN system for $\mathcal{T}$.*
3. *The sequence is a sequence of simples from the simple-minded collection $\{S_1, \cdots, S_n\}$ to $\{S_1[m], \cdots, S_n[m]\}$, that is, it is an m-maximal green sequence.*

Proof. (1)→(3) By applying Lemma 4.3.3 repeatedly it is easy to see that any maximal backward $Hom^{\leq 0}$-orthogonal sequence of Schurian objects $\{M_n\}$ on $\mathcal{T}$ is also a sequence of simples in hearts of t-structures related to each other by a finite sequence of forward mutations. Hence (1) implies (3).

(2)→(1) Schurness has been proved in Lemma 4.4.4. Backward $Hom^{\leq 0}$ orthogonality has already been proven in Lemma 4.4.5. Maximality holds because of Lemma 4.4.5(3) and (4) due to the reasoning below. Since if the sequence is not maximal then there exists some M.such that it can be inserted in the backward $Hom^{\leq 0}$ orthogonal sequence. However such an M must have a unique HN filtration. Hence there exists some $i \leq j$ such that $(M_j, M) \neq 0$

and $(M, M_i) \neq 0$. In this case M can not be inserted in the backward $Hom^{\leq 0}$ orthogonal sequence which proves its maximality.

(3)→(2) This is obvious because using truncation functors we can easily show that any object in $\mathcal{T}$ can be written uniquely as an HN filtration. $\square$

Chapter 5

Quivers with multiple edges

5.1 Introduction

Maximal green sequences (MGSs) were invented by Bernhard Keller [28]. Brustle-Dupont-Perotin [6] and the paper by the first author together with Brustle, Hermes and Todorov [7] have proven that there are finitely maximal green sequences when the quiver is of finite, tame type or the quiver is mutation equivalent to a quiver of finite or tame types. Furthermore in [7] it is proven that any tame quiver has finitely many k-reddening sequences.

However the situation is still pretty much uncharted in the wild case other than cases where the quiver has three vertices which was proven in [6] which contains a proof highly dependent on the quiver only having three vertices. Despite the fact that the wild case is still unknown in general we can indeed solve it for many easy cases. For example for quivers such as the k-Kronecker quiver and $1 \Rrightarrow 2 \Rrightarrow 3$ things are really simple due to Lemma 1.4.26.

In this chapter we will generalize the results and introduce three theorems that can significantly simplify understanding of maximal green sequences in simply-laced quivers with multiple edges.

We can completely describe MGSs of ME-ful quivers using MGSs of their ME-free versions defined below.

Theorem 5.1.1. *(Theorem 5.3.12) MGSs of an acyclic quiver Q are a subset of the set of Q-ME-free MGSs of its ME-free version, Q'.*

Theorem 5.1.2. *(Theorem 5.3.13) Let Q be an ME-ful acyclic quiver and Q' be its ME-free version. The MGSs of Q are exactly the Q-ME-free MGSs $(C_0, C_1, \cdots C_m)$ of Q' such that for any multiple edge from i to j in Q for any c-matrix C_i in the MGS such that there exists a negative c-vector with support containing i the mutation on C_i in the MGS isn't done on any negative c-vector with support containing j.*

In other words to understand MGSs of an acyclic quiver Q we only need to understand the MGSs of its ME-free version which makes multiple edges largely irrelevant in understanding MGSs of acyclic quivers.

We can obtain the following crucial corollaries in the acyclic case:

Corollary 5.1.3. *(Corollary 5.3.14)The following statements are true:*

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*
2. *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*
3. *No minimally MGS-infinite quiver can contain multiple edges.*
4. *Any two ME-equivalent quivers are MGS-equivalent to each other.*

If the quiver isn't necessarily acyclic we still have the following result:

Theorem 5.1.4. *(Theorem 5.4.3)Assume that $(\tilde{Q}, \breve{Q})$ are k-partition of Q for some $k > 1$ any MGS of Q is an MGS of $\tilde{Q} \cup \breve{Q}$.*

In Section 2 we will discuss MGS-finiteness in general. In Section 3 we will prove Theorems 5.1.1 and 5.1.2. In Section 4 we will prove Theorem 5.1.4.

5.2 MGS-finiteness

In this section let's review the basics about what kind of quivers have finitely many maximal green sequences.

Definition 5.2.1. A quiver Q is *MGS-finite* if Q has finitely many maximal green sequences. Any quiver that isn't MGS-finite is *MGS-infinite*.

Here are some results that are either already known or easily proven about MGS-finiteness of quivers.

Theorem 5.2.2. *[6]Any acyclic quiver Q of finite type or tame type as well as any acyclic quiver Q of wild type with three vertices are MGS-finite.*

Theorem 5.2.3. *[7] (Thm 2) If the quiver Q is mutation equivalent to an acyclic quiver of tame type, then Q has only finitely many maximal green sequences.*

Theorem 5.2.4. *[7] (Thm 2.2.4, Rotation Lemma) Let B be a skew-symmetric matrix, let (k_0, k_1, k_{m-1}) be a reddening sequence for B with associated permutation σ and let $B_0 = \mu_{k_0} B$. Then the last c-matrix in the mutation sequence $(k_1, k_2, \cdots k_{m-1}, \sigma^{-1}(k_0))$ is $-P_\sigma$, i.e. this is a reddening sequence for B_0 with the same permutation as the reddening sequence for B. Furthermore, this new reddening sequence has exactly r red mutations. In particular, a maximal green sequence for B gives a maximal green sequence for B_0.*

Theorem 5.2.5. *Any quiver Q mutation equivalent to an acyclic quiver of finite or tame type is MGS-finite.*

Proof. Due to Theorem 5.2.3 the result is already proven in the mutation-equivalent to tame type case. For the mutation-equivalent to finite type case using Theorem 5.2.4 it is obvious that any MGS in such a quiver must be an k-reddening sequence of an acyclic quiver of finite type for a fixed k. There are only finitely many such sequences because a k-reddening sequence can only repeat a cluster $k+1$ times due to Lemma 3.4.2 and in an acyclic quiver of finite type there are only finitely many cluster-tilting objects and hence finitely many clusters. □

Lemma 5.2.6. *If Q is a quiver that isn't connected, Q^1, Q^2, $\cdots$ Q^n are its connected components. Each Q^i is MGS-finite if and only if Q is MGS-finite.*

Proof. Any MGS of Q is essentially formed from taking an MGS w_i of Q^i for each i and then put these mutations together such that the order of elements in each w_i is preserved.

Since we can obtain all MGSs of Q^i by deleting all c-vectors not supported on Q^i_0 from all MGSs of Q it is easy to see that if Q is MGS-finite so is Q^i for any i.

On the other hand if all Q^is are MGS-finite it is easy to see that so is Q because the set of admissible c-vectors of Q is the union of admissible c-vectors in MGSs of Q^i all of which are finite.

□

There is also an unrelated result about MGS-finiteness we proved which we will include here.

Definition 5.2.7. A quiver is of *finite green mutation type* if there are finitely many exchange matrices along its maximal green sequences.

It is easy to see that any quiver that has finitely many maximal green sequences is of finite green mutation type.

Lemma 5.2.8. *If the coframed quiver $\breve{Q}$ of a quiver Q is of finite green mutation type, Q has finitely many maximal green sequences.*

Proof. For a quiver Q with $|Q_0| = n$, let $Q' = \breve{Q}$ be its coframed quiver and Q'' be the coframed quiver of Q'. Let's label the extra vertices of Q' as $1', \cdots, n'$. Note that any maximal green sequence $w = (w_1, \cdots, w_k)$ of Q can be extended into a maximal green sequence of Q', $w' = (w_1, \cdots, w_k, 1', 2', \cdots, n')$. Note that any extended exchange matrix that appears in any maximal green sequence of Q is an exchange matrix in some maximal green sequence of Q'. Since Q' is of finite green mutation type, there are only finitely many exchange matrices in all maximal green sequences of Q'. Hence there are only finitely many extended exchange matrices in any maximal green sequence of Q. Since extended exchange matrices can not be repeated in a maximal green sequence, Q has finitely many maximal green sequences. $\square$

Here is an easy corollary of the lemma above:

Corollary 5.2.9. *If all quivers are of finite green mutation type, all quivers have finitely many maximal green sequences.*

Lemma 5.2.8 and Corollary 5.2.9 also hold for valued quivers which we won't discuss in this paper. The proofs don't change when generalized to valued quivers.

5.3 The acyclic case

Now we need some basic definitions in order to describe and prove the results.

Definition 5.3.1. A quiver with at least one multiple edge is *ME-ful*. Otherwise it is *ME-free*.

Definition 5.3.2. A *multiple edges-free (ME-free)* version of a quiver Q is produced by removing all multiple edges from Q while retaining single edges and vertices.

For example the ME-free version of the m-Kronecker quiver for any m is the quiver $A_1 \times A_1$, namely the quiver with two vertices and no arrows.

In this section we will use the fact that a path in the semi-invariant picture of Q is also a path in the semi-invariant picture of its ME-free version, Q'. Since the definition of whether a path is green and generic differ in semi-invariant pictures of different quivers we will use the concept of *strong genetic green paths* to exclude problematic cases.

Definition 5.3.3. Let Q be an ME-ful quiver, Q' be its ME-free version. A path in the semi-invariant pictures of Q and Q' is *strong generic green* if it is a generic green path in both pictures.

Definition 5.3.4. Let Q be an ME-ful quiver.

1. A c-vector in Q is *ME-free* if it is ME-free if considered as a dimension vector of Q. Any c-vector in Q that isn't ME-free is *ME-ful*.

2. An MGS in Q is *ME-free* if all its c-vectors are ME-free. An MGS of Q that isn't ME-free is *ME-ful*.

3. A generic green path in the semi-invariant picture of Q is *ME-free* if it crosses no wall corresponding to an ME-ful c-vector. A generic green path in the semi-invariant picture of Q that isn't ME-free is *ME-ful*.

4. A module of kQ is *ME-free/ME-ful* if its c-vector is ME-free/ME-ful.

Note that if an MGS is ME-free all c-vectors in all c-matrices in it including those that aren't mutated must be ME-free.

If Q is an ME-ful quiver and Q' is its ME-free version it does not technically make sense to discuss ME-fulness of any module of kQ'. Here we are going to use the same definition we used in defining ME-fulness of vectors and MGSs of Q.

Definition 5.3.5. Let Q be an ME-ful quiver and let Q' be its ME-free version.

1. A c-vector in Q' is *Q-ME-free* if it is ME-free if considered as a dimension vector of Q. Any c-vector in Q' that isn't Q-ME-free is *Q-ME-ful*.

2. An MGS in Q' is *Q-ME-free* if all its c-vectors are Q-ME-free. An MGS of Q' that isn't Q-ME-free is *Q-ME-ful*.

3. A generic green path in the semi-invariant picture of Q' is *Q-ME-free* if it crosses no wall corresponding to a Q-ME-ful c-vector. A generic green path in the semi-invariant picture of Q' that isn't Q-ME-free is *Q-ME-ful*.

4. A strongly generic green path in the semi-invariant picture of Q' is *strongly Q-ME-free* if it is Q-ME-free and does not cross any wall corresponding to a Q-ME-ful c-vector in the semi-invariant picture of Q. A generic green path in the semi-invariant picture of Q' that isn't strongly Q-ME-free is *weakly Q-ME-ful*.

5. A module of kQ' is *Q-ME-free/Q-ME-ful* if its c-vector is Q-ME-free/Q-ME-ful.

We will sometimes abuse the notations and use the term Q-ME-free for c-vectors/MGSs of Q. In this case they are just ME-free c-vectors/MGSs.

Definition 5.3.6. If Q and Q' have the same number of vertices, a GS w of kQ *is equivalent to* a GS w' of kQ' if w and w' mutates on the same sequence of c-vectors and start from the same c-matrix up to permutations.

Using the equivalence it makes sense to identify certain MGSs of Q and Q'. It is in this sense that we claim and prove that all MGSs of an ME-ful quiver Q are MGSs of its ME-free version, Q'.

In order to state a corollary we also need three more definitions.

Definition 5.3.7. The *skeleton* of a quiver Q is produced by replacing all multiple edges from Q by single edges with the sources and targets unchanged.

For example the ME-free version of the m-Kronecker quiver for any m is the quiver A_2.

Definition 5.3.8. Q and Q' are quivers. If they have the same ME-free version and the same skeleton then they are *ME-equivalent*.

Definition 5.3.9. If every MGS of Q corresponds to some MGS of Q' and vice versa then Q and Q' are MGS-equivalent.

Lemma 5.3.10. *Let Q be a quiver and Q' be its ME-free version. The following holds:*

1. *The set of Q-ME-free c-vectors of Q and Q' coincide.*

2. *If Q is an ME-ful quiver then for any positive Q-ME-ful vector $c \in \mathbb{R}^n$ then $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$.*

3. *If Q is an ME-ful quiver. Then any of the Q-ME-ful c-vectors can not be a dimension vector of an exceptional module for Q'. Any of the Q-ME-ful c-vectors of Q' can not be a dimension vector of an exceptional module for Q.*

Proof. For (1) let the Euler matrices of Q, Q'' be $E = e_{ij}, E' = (e'_{ij})$ respectively. Then we have $\langle c, c\rangle_{kQ} = \langle c, c\rangle_{kQ'}$ because whenever e_{ij}, e'_{ij} differ $c_i = 0$ or $c_j = 0$ leaving the term related to (i, j) being 0. Hence the set of Q-ME-free c-vectors of Q, Q' corresponding to exceptional modules coincide.

For (2) assume that such a vector c exists. We have $\langle c, c\rangle_{kQ} = \langle c, c\rangle_{kQ'} = 1$. However the Euler matrix $E = (e_{ij})$ of Q and the Euler matrix $E' = (e'_{ij})$ of Q' differ in the sense that there exists some pair $(i, j) \in [n]$ such that $c_i \neq 0, c_j > 0$ and $0 = e'_{ij} > -2 \geq e_{ij}$. Since for any $k, l \in [n]$ we have $e'_{kl} \geq e_{kl}$ it is easy to see that $\langle c, c\rangle_{kQ'} > \langle c, c\rangle_{kQ}$ and that $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$.

(3) is a consequence of (2) since $\langle M, M\rangle_{kQ}$ and $\langle M, M\rangle_{kQ'}$ can not both be 1. □

Lemma 5.3.11. *Let Q be an ME-ful quiver. Any MGS of an ME-ful quiver Q must not contain any Q-ME-ful c-vector of Q' or any vector c which is an imaginary root of Q'.*

Proof. Due to Lemma 5.3.10(3) we only need to prove the second part. In that case $\langle c, c\rangle_{kQ} \leq \langle c, c\rangle_{kQ'} < 1$. Hence c is not a c-vector of Q. □

Now we can easily establish the following theorem.

Theorem 5.3.12. *MGSs of an acyclic quiver Q are a subset of the set of Q-ME-free MGSs of its ME-free version, namely Q'.*

Proof. If the statement is incorrect along an MGS of Q pick the first c-matrix that isn't shared by Q' assuming that such an MGS exists.

In this case either at least one c-vector is Q-ME-ful or none is. If some c-vector is

Q-ME-ful it must be formed by extending one Q-ME-free exceptional module by another Q-ME-free exceptional module in Q (i.e. $dimExt_{kQ}(A,B) = 1$ because it can not be larger due to Lemma 1.4.28. Let's label the indecomposable module formed by the extension M. We need $Ext_{kQ}(B,A) = 0$ so that $\langle M, M\rangle_{kQ} = 1$) while in Q' there are no such extensions (i.e. $dimExt_{kQ'}(A,B) = 0$). However this is impossible because A, B are rigid, Hom-orthogonal and indecomposable because $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$ which causes $\langle M, M\rangle_{kQ'}$ to be at least 3 which is impossible because $\langle M, M\rangle_{kQ'} = \langle A, A\rangle_{kQ'} + \langle A, B\rangle_{kQ'} + \langle B, A\rangle_{kQ'} + \langle B, B\rangle_{kQ'} = 2 - dimExt_{kQ'}(A,B) - dimExt_{kQ'}(B,A)$ is at most 2.

If no c-vector is Q-ME-ful then in kQ, kQ' the relevant Hom and Ext groups shouldn't differ because neither of them involve the multiple edges that are absent in kQ' . As a result that can't happen either.

Hence the c-matrices corresponding to Q, Q' in the MGS are all the same. Any MGS of Q must be an MGS of Q' with the same c-matrices. Since all the c-matrices of the two quivers are the same they have the same associated permutation.

□

Now we can prove a stronger result.

Theorem 5.3.13. *Let Q be an ME-ful acyclic quiver and Q' be its ME-free version. The MGSs of Q are exactly the Q-ME-free MGSs $(C_0, C_1, \cdots C_m)$ of Q' such that for any multiple edge from i to j in Q for any c-matrix C_i in the MGS such that there exists a negative c-vector with support containing i it is true that the mutation on C_i in the MGS isn't done on any negative c-vector with support containing j.*

Proof. Let's compare $\langle M, N\rangle_{kQ}$ and $\langle M, N\rangle_{kQ'}$. They differ if and only if there exists some multiple edge from i to j such that i is in the support of M and j is in the support of

N. In this case since $Hom_{kQ}(M,N) = Hom_{kQ'}(M,N) = Ext_{kQ'}(M,N) = 0$ it is true that $dimExt_{kQ}(M,N) > 0$. Repeating the argument in Theorem 5.3.12 we can show that this is the only possible scenario for a Q-ME-free MGSs of Q' to not be identical to an MGS in Q. □

Corollary 5.3.14. *The following statements are true:*

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*

2. *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*

3. *No minimally MGS-infinite quiver can contain multiple edges.*

4. *Any two ME-equivalent quivers are MGS-equivalent to each other.*

Proof. Only (4) needs to be proven even though it is still obvious. For ME-equivalent quivers Q and Q' the conditions of Theorem 5.3.13 are identical which is why the number of MGS are identical. □

Example 5.3.15. The maximal green sequences of Q :
$$\begin{tikzcd} 1 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$
are maximal green sequences of its ME-free version $Q' : 1 \to 2 \to 3$ that has no c-vector with support containing $\{1,3\}$ and satisfies the conditions in Theorem 5.3.13 with respect to the arrow $1 \rightrightarrows 3$. It's easy to see that Q is MGS-finite. In fact it has 3 MGSs.

Example 5.3.16. The maximal green sequences of $Q : 1 \longrightarrow 2 \rightrightarrows 3 \longrightarrow 4$ are some maximal green sequences of its ME-free version $Q' : 1 \to 2 \quad 3 \to 4$ that has no c-vector with support containing $\{2,3\}$ and satisfies the conditions in Theorem 5.3.13 with respect to the arrow $2 \rightrightarrows 3$. It's easy to see that Q is MGS-finite because A_2 is.

Now we can provide a much shorter proof to the fact that all acyclic quivers with three vertices are MGS-finite which was originally proven in [6].

Corollary 5.3.17. *Any acyclic quiver with at most three vertices is MGS-finite.*

Proof. Due to the theorem we only need to show that any ME-free acyclic quiver with at most three vertices is MGS-finite. Such a quiver is either of finite or tame type and is hence MGS-finite. $\square$

5.4 The general case

In the general case the theorem above isn't correct. We can show that using the following counterexample. The quiver Q here is

$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \\ & 2 \arrow[ur] & \end{tikzcd}.$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_2}{\rightarrow} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_1}{\rightarrow} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_3}{\rightarrow} \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \overset{\mu_1}{\rightarrow} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \overset{\mu_2}{\rightarrow} \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Here we have a maximal green sequence with at least one ME-full c-vector. Moreover it is easy to see that if we replace the double edge by triple edge and obtain Q' :

$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll, shift left=2] \arrow[ll] \arrow[ll, shift right=2] \\ & 2 \arrow[ur] & \end{tikzcd}$$

(2,1,3,1,2) is not an MGS of the quiver Q' nor is it an MGS of the ME-free version or skeleton of Q.

However we can still perform quiver cutting in more limited situations. Let's first introduce a concept.

Definition 5.4.1. A *k-edge* is a tuple (i, j) where $i, j \in [n]$ and $k|b_{ij}, k|b_{ji}$.

Definition 5.4.2. Let Q be a quiver possibly having oriented cycles, let k be an integer greater than 1. Assume that $Q_0 = \tilde{Q}_0 + \breve{Q}_0$, $P = Q]_{\tilde{Q}_0}, R = Q]_{\breve{Q}_0}$. If for all $i \in \tilde{Q}_0, j \in \breve{Q}_0$ $k|b_{ij}$ and $k|b_{ji}$ we say Q is *k-partible* and (P, R) is a *k-partition* of Q.

Theorem 5.4.3. *Assume that $(\tilde{Q}, \breve{Q})$ are k-partition of Q for some $k > 1$ then any MGS of Q is an MGS of $\tilde{Q} \cup \breve{Q}$.*

Proof. The property that for any $i \in Q_1$ and $j \in Q_2$ $k|c_{ij}$ is preserved by mutation. Hence any mutation that cause any c-vector to cross bot has to violate the Sink before Source Theorem in [7]. □

Corollary 5.4.4. *Under the conditions of the theorem above, if $\tilde{Q}$ and $\breve{Q}$ are MGS-finite then so is Q.*

Proof. If $\tilde{Q}$ and $\breve{Q}$ are MGS-finite so is $\tilde{Q} \cup \breve{Q}$. As a result so is Q due to the theorem. □

Example 5.4.5. Q :

$$\begin{tikzcd} 1 \arrow[dr] & & & 5 \arrow[dd] \\ & 2 \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & 4 \arrow[ur] & \\ 3 \arrow[uu] & & & 6 \arrow[ul] \end{tikzcd}$$

is a quiver with oriented cycles. Due to the theorem we can cut the $2 \rightrightarrows 4$ arrow. After cutting this arrow it is easy to see that Q is MGS-finite.

Example 5.4.6. Q :
$$\begin{tikzcd}
 & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[dr] & \\
1 \arrow[ur] & & & 4 \arrow[dl] \\
 & 6 \arrow[ul] & 5 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] &
\end{tikzcd}$$
is another quiver with oriented cycles.

Due to the theorem we can cut the $2 \rightrightarrows 3$ and $6 \leftleftarrows\!\!\leftleftarrows 5$ arrows. After cutting these arrows it is easy to see that Q is MGS-finite.

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