Maximal green sequences of quivers with multiple edges

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Outline

Abbreviations

In this talk GS refers to green sequence, MGS refers to maximal green sequence, ME refers to multiple edges.

Motivation: Kronecker quiver

Let's explain this phenomenon using the example of Kronecker quiver, $Q: \quad 1 \rightrightarrows 2$. It has one maximal green sequence.

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Let's explain this phenomenon using the example of Kronecker quiver, $Q: \; 1 \rightrightarrows 2$. It has one maximal green sequence. If we cut the double edge we obtain $Q': \; 1 \quad 2$ which we say is the *multiple edge-free (ME-free) version* of Q. There are two maximal green sequences.

Comparison of MGS

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\mu_1} \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\mu_2} \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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They have the same mutations and the same c-matrices.

Comparison of MGS

$$
\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & 2 \arrow[dd] & & 1 \arrow[from=dd] & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[dd] & & 1 \arrow[r, shift left] \arrow[r, shift right] \arrow[from=dd] & 2 \arrow[from=dd] \\
 & {} \arrow[r, "\mu_1"] & {} & & {} \arrow[r, "\mu_2"] & {} & & \\
1' & 2' & & 1' & 2' & & 1' & 2'
\end{tikzcd}
$$

Comparison of MGS

$$
\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] \arrow[dd] & 2 \arrow[dd] & 1 & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[dd] & 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \\
 & {} \arrow[r, "\mu_1"] & {} & {} \arrow[r, "\mu_2"] & {} & \\
1' & 2' & 1' \arrow[uu] & 2' & 1' \arrow[uu] & 2' \arrow[uu]
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \arrow[dd] & 2 \arrow[dd] & 1 & 2 \arrow[dd] & 1 & 2 \\
 & {} \arrow[r, "\mu_1"] & {} & {} \arrow[r, "\mu_2"] & {} & \\
1' & 2' & 1' \arrow[uu] & 2' & 1' \arrow[uu] & 2' \arrow[uu]
\end{tikzcd}
$$

Comparison of MGS

$$
\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & 2 \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_1}"] & 1 & 2 \arrow[l, shift left] \arrow[l, shift right] \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_2}"] & 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \\
1' & 2' & 1' \arrow[u] & 2' & 1' \arrow[u] & 2' \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_1}"] & 1 & 2 \arrow[d] \arrow[r, phantom, "\xrightarrow{\mu_2}"] & 1 & 2 \\
1' & 2' & 1' \arrow[u] & 2' & 1' \arrow[u] & 2' \arrow[u]
\end{tikzcd}
$$

They have the same mutations and the same c-matrices.

Question

- Let Q be a quiver which has multiple edges. Let Q' be its ME-free version, what can we say about MGSs of Q and Q'?

Outline

Simple-minded collections

Simple-minded collections

Definition

Let Λ be an algebra with n primitive idempotents. A *simple-minded collection* $\{S_i\}_{i\in[n]}$ of $D^b(\Lambda)$ is an n-element set such that $(S_i[\geq 0], S_j) = 0$ for all $i \neq j$, $(S_i[> 0], S_i) = 0$ for all i, (S_i, S_i) is a division algebra and that $\{S_i\}_{i\in[n]}$ generates $D^b(\Lambda)$.

Simple-minded collections

Ex: $D^b(A_3)$

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

Simple-minded collections
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$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. $\{P_3[1], P_2, I_1\}$ is also a simple-minded collection.

Simple-minded collections

Approximations

Definition

Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $M \in Ob\mathcal{C}, N \in Ob\mathcal{X}$, a morphism $f \in Hom_{\mathcal{C}}(M, N)$ is a *minimal left-$\mathcal{X}$ approximation* if for any $g \in End_{\mathcal{C}}N$ such that $g \circ f = f$ g is an isomorphism and for any $N' \in Ob\mathcal{X}$ for any $q \in Hom_{\mathcal{C}}(M, N')$ we have q factors through f.

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$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "q"'] & N \arrow[d, dashed, "l"] \\ & N' \end{tikzcd}$$

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$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M' \arrow[u, dashed, "l"] \arrow[ur, "q"] & \end{tikzcd}$$

Simple-minded collections

Mutations

Definition

A *forward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S'_j\}$ where $S'_i = S_i[1]$ and S'_j $(j \neq i)$ is the cone/homotopy cokernel of the minimal left-$add(S_i)$ approximation of $S_j[-1]$.

Simple-minded collections
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A *backward mutation* on the element S_i of the simple-minded collection $\{S_j\}$ is $\{S_j'\}$ where $S_i' = S_i[-1]$ and S_j' ($j \neq i$) is the cone/homotopy cokernel of the minimal left-$add(S_i[-1])$ approximation of S_j.

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\end{tikzcd}
$$

$\{I_1, S_2, P_3\}$ is a simple-minded collection. When we do a forward mutation at P_3 we get $\{P_3[1], P_2, I_1\}$. When we do a forward mutation at P_2 now we get $\{S_2, P_2[1], P_1\}$. When we then do a forward mutation at P_1 we get $\{S_2, I_1, P_1[1]\}$.

Outline

c-vectors

Positive c-vectors are dimension vectors of elements of simple-minded collections. Such elements are all bricks. That is, all c-vectors are Schur. However we can indeed prove more. They are in fact real as well.

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Lemma

Let k be an algebraically closed field. Let Λ be a hereditary algebra over k. Then any c-vector c that appears in any MGS is a real Schur root.

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Lemma

Let k be an algebraically closed field. Let Λ be a hereditary algebra over k. Then any c-vector c that appears in any MGS is a real Schur root.

The crucial fact we need to use is that for any c-vector in an MGS $\langle c, c\rangle = 1$.

c-vectors

Lemma

If $-c_1, -c_2$ are negative c-vectors in C-matrix C' in an MGS, c_1 and c_2 are dimension vectors of exceptional modules M_1 and M_2. If $\dim \mathrm{Ext}^1(M_1, M_2) > 1$ then the mutation on C' must not be done on M_2.

Outline

Definition: MGS-finiteness

Definition

A quiver Q is *MGS-finite* if Q has finitely many maximal green sequences. Any quiver that isn't MGS-finite is *MGS-infinite*.

Which quivers are MGS-finite?

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- Quivers mutation equivalent to acyclic quivers of tame type.
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- Quivers with a full subquiver with no MGS. (Muller)[14]

Are all quivers MGS-finite

Conjecture

All acyclic quivers are MGS-finite.

Roadmap to the proof

- Tackle multiple edges.

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We will finish the first step.

Outline

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A quiver with at least one multiple edge is *ME-ful*. Otherwise it is *ME-free*.

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For example the m-Kronecker quiver for any $m > 1$ is ME-ful. On the other hand A_2 is ME-free.

ME-free versions

Definition

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For example the ME-free version of the m-Kronecker quiver for any $m > 1$ is the quiver $A_1 \times A_1$, namely the quiver with two vertices and no arrows.

ME-ful/ME-free

Definition

Let Q be an ME-ful quiver.

1. A c-vector in Q is *ME-free* if its support is ME-free. Any c-vector in Q that isn't ME-free is *ME-ful*.
2. An MGS in Q is *ME-free* if all its c-vectors are ME-free. An MGS of Q that isn't ME-free is *ME-ful*.
3. A module of kQ is *ME-free/ME-ful* if its c-vector is ME-free/ME-ful.

Q-ME-ful/Q-ME-free

Definition

Let Q be an ME-ful quiver and let Q' be its ME-free version.

1. A c-vector in Q' is *Q-ME-free* if it is ME-free when considered as a dimension vector of Q. Any c-vector in Q' that isn't Q-ME-free is *Q-ME-ful*.

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3. A module of kQ' is *Q-ME-free/Q-ME-ful* if its c-vector is Q-ME-free/Q-ME-ful.

Equivalence of GSs

Definition

If Q and Q' have the same number of vertices, a GS w of kQ *is equivalent to* a GS w' of kQ' if w and w' mutates on the same sequence of c-vectors and start from the same c-matrix up to permutations.

Skeletons

Definition

The *skeleton* of a quiver Q is produced by replacing all multiple edges from Q by single edges with the sources and targets unchanged.

For example the ME-free version of the m-Kronecker quiver for any m is the quiver A_2.

ME-equivalence and MGS-equivalence

Definition

Q and Q' are quivers. If they have the same ME-free version and the same skeleton then they are *ME-equivalent*.

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Definition

If every MGS of Q corresponds to some MGS of Q' and vice versa then Q and Q' are MGS-equivalent.

Outline

Lemmas

Lemma

Let Q be a quiver and Q' be its ME-free version. The following holds:

1. *The set of Q-ME-free c-vectors of Q and Q' coincide.*
2. *If Q is an ME-ful quiver then for any positive Q-ME-ful vector $c \in \mathbb{R}^n$ $\langle M, M\rangle_{kQ} - \langle M, M\rangle_{kQ'} \leq -2$.*
3. *If Q is an ME-ful quiver. Then any of the Q-ME-ful c-vectors can not be a dimension vector of an exceptional module for Q'. Any of the Q-ME-ful c-vectors of Q' can not be a dimension vector of an exceptional module for Q.*

Lemmas

Lemma

Let Q be an ME-ful quiver. Any MGS of an ME-ful quiver Q must not contain any Q-ME-ful c-vector of Q' or any vector c which is an imaginary root of Q'.

Theorems in the acyclic case

Theorem

MGSs of an acyclic quiver Q are a subset of the set of Q-ME-free MGSs of its ME-free version, Q'.

Theorem

Let Q be an ME-ful acyclic quiver and Q' be its ME-free version. The MGSs of Q are exactly the Q-ME-free MGSs $(C_0, C_1, \cdots C_m)$ of Q' such that for any multiple edge from i to j in Q for any C-matrix C_i in the MGS such that there exists a negative c-vector with support containing i the mutation on C_i in the MGS isn't done on any negative c-vector with support containing j.

Corollaries

Corollary

The following statements are true:

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*

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3. *No minimally MGS-infinite quiver can contain multiple edges.*

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The following statements are true:

1. *The number of maximal green sequences of a quiver Q is no greater than that of its ME-free version.*
2. *All quivers with an MGS-finite ME-free version must themselves be MGS-finite.*
3. *No minimally MGS-infinite quiver can contain multiple edges.*
4. *Any two ME-equivalent quivers are MGS-equivalent to each other.*

Outline

Examples

Example

The maximal green sequences of Q :

$$\begin{tikzcd} 1 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & 3 \\ & 2 \arrow[ur] & \end{tikzcd}$$

are maximal green sequences of its ME-free version $Q' : 1 \to 2 \to 3$ that has no c-vector with support containing $\{1,3\}$ and satisfies the conditions in the second theorem above with respect to the arrow $1 \rightrightarrows 3$. It's easy to see that Q is MGS-finite. In fact it has 3 MGSs.

Examples

Example

The maximal green sequences of

$Q: \quad 1 \longrightarrow 2 \rightrightarrows 3 \longrightarrow 4$ are some maximal green

sequences of its ME-free version

$Q': \quad 1 \longrightarrow 2 \qquad 3 \longrightarrow 4$ that has no c-vector with

support containing $\{2,3\}$ and satisfies the conditions in the second theorem above with respect to the arrow $2 \rightrightarrows 3$. It's easy to see that Q is MGS-finite because A_2 is.

Outline

A counterexample

In the general case the theorems above aren't true. We can show that using the following counterexample. The quiver Q here is

$$\begin{tikzcd}
1 \arrow[dr] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \\
& 2 \arrow[ur] &
\end{tikzcd}.$$

A counterexample

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_2}{\rightarrow} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_1}{\rightarrow} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \overset{\mu_3}{\rightarrow}$$

$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \overset{\mu_1}{\rightarrow} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \overset{\mu_2}{\rightarrow} \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

A counterexample

Here we have a maximal green sequence with at least one ME-full c-vector. Moreover it is easy to see that if we replace the double edge by triple edge and obtain Q' :

$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll, shift left=2] \arrow[ll] \arrow[ll, shift right=2] \\ & 2 \arrow[ur] & \end{tikzcd}$$

(2,1,3,1,2) is not an MGS of the quiver Q' nor is it an MGS of the ME-free version or skeleton of Q.

Outline

1. Motivation: Kronecker quiver
2. Background
   - Simple-minded collections
   - Two results about c-vectors
   - MGS-finiteness
3. The acyclic case
   - Definitions
   - Results
   - Examples
4. The general case
   - Limitation of the theorems in the acyclic case
   - Result
   - Examples

k-edges

Definition

A *k-edge* is a tuple (i, j) where $i, j \in [n]$ and $k | b_{ij}, k | b_{ji}$.

Definition

Let Q be a quiver possibly having oriented cycles, let k be an integer greater than 1. Assume that $Q_0 = \tilde{Q}_0 + \check{Q}_0$, $P = Q]_{\tilde{Q}_0}, R = Q]_{\check{Q}_0}$. If for all $i \in \tilde{Q}_0, j \in \check{Q}_0$ $k | b_{ij}$ and $k | b_{ji}$ we say Q is *k-partible* and $(\tilde{Q}, \check{Q})$ is a *k-partition* of Q.

Theorem in the general case

Theorem

Assume that $(\tilde{Q}, \check{Q})$ are k-partition of Q for some $k > 1$ any MGS of Q is an MGS of $\tilde{Q} \cup \check{Q}$.

Theorem in the general case

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Assume that $(\tilde{Q}, \check{Q})$ are k-partition of Q for some $k > 1$ any MGS of Q is an MGS of $\tilde{Q} \cup \check{Q}$.

Corollary

Under the conditions of the theorem above, if $\tilde{Q}$ and $\check{Q}$ are MGS-finite so is Q.

Outline

Examples

Example

$$Q: \quad \begin{tikzcd} 1 \arrow[dr] & & & 5 \arrow[dd] \\ & 2 \arrow[dl] \arrow[r, shift left] \arrow[r, shift right] & 4 \arrow[ur] & \\ 3 \arrow[uu] & & & 6 \arrow[ul] \end{tikzcd}$$

is a quiver with oriented cycles.

Due to the theorem we can cut the $2 \rightrightarrows 4$ arrow. After cutting this arrow it is easy to see that Q is MGS-finite.

Examples

Example

$$Q: \begin{tikzcd} & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[dr] & \\ 1 \arrow[ur] & & & 4 \arrow[dl] \\ & 6 \arrow[ul] & 5 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \end{tikzcd}$$

is another quiver with oriented cycles. Due to the theorem we can cut the $2 \rightrightarrows 3$ and $6 \xleftarrow{\ \ } \!\!\!\!\!\! \leftleftarrows \!\! \leftleftarrows 5$ arrows. After cutting these arrows it is easy to see that Q is MGS-finite.

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- Removal of multiple edges is possible in more limited cases when the quiver isn't acyclic.

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