Limits and Sequences

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1 Basics

- **Sequence definition**. A sequence is a function from the natural to the real numbers.$a_i = a(i)$
- **Limit definition**. We say a_n has the limit c ,$\lim_{n\to\infty} a_n = c$, if and only if (iff) $\forall \epsilon > 0, \exists n_0$ such that $n \geq n_0 \Rightarrow \mid a_n - c \mid \leq \epsilon$.
- **Weierstrass theorem**. A monotonic bounded sequence of real numbers is convergent.
- The set of rationals is dense on the real numbers.
- **Continuity**. A function f is continuous on a set D iff $\forall c \in D, \lim_{x\to c} f(x) = f(c)$ or equivalently $\forall c \in D, \forall x_n \in D$ such that $\lim_{n\to\infty} x_n = c$ we have that
$$\lim_{n\to\infty} f(x_n) = f(c)$$
- B**olzano-Weierstrass**. Let a_i a bounded sequence on $\mathbb{R}^n$ then there exists a convergent subsequence of a_i.
- **Riemann Integral**. Let a function f on $[a, b]$ and let $x_1, ..., x_n$ a partition of $[a, b]$ along with $t_i \in [x_{i+1}, x_i]$.We say that a f is Riemann integrable on $[a, b]$ if for any partition along with any $t_i \in [x_{i+1}, x_i]$ as $max(x_{i+1} - x_i) \to 0$ $\sum_{i=1}^{n} f(t_i)(x_{i+1} - x_i) \to L$ and we call that limit the integral of the function on $[a, b]$, $L = \int_a^b f(x)dx$.

- A **Closed set** is be defined as a set which contains all its limit points. In a complete metric space, a closed set is a set which is closed under the limit operation.
  A is closed $\iff A = cl(A) \iff x$ limit point of A $\Rightarrow x \in A$

2 Problems

1. Let $n \geq 6$ be an integer. Show that
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$
2. Find $\lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}}$
3. Compute
$$\sqrt{7+\sqrt{7+...}}$$
4. Let x_i be a sequence such that $x_0 = 1$ and for $n \geq 0$
$$x_{n+1} = \ln(e^{x_n} - x_n).$$
   Show convergence and find the value for the series of x_i.
5. Let f be a real continuous function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x) = L$ (it may be infinite or finite).Prove that
$$\lim_{n\to\infty} \int_0^1 f(nx)dx = L$$
6. (**Cessaro-Stoltz**)Let x_i, y_i sequences such that $\lim_{n\to\infty} y_n = \infty$ $y_i$increasing and positive .If
$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L.$$
   Then the limit $\lim_{n\to\infty} \frac{x_n}{y_n}$ exists and it is equal to L.
7. If x_i a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 0$,then $\lim_{n\to\infty} \sqrt[n]{x_n} = L$

8. Let a_i be positive integers such that $\sum \frac{1}{a_n}$ converges.For each n , let b_n denote the number of positive integers i for which $a_i \leq n$.Prove that $\lim_{n\to\infty} \frac{b_n}{n} = 0$

9. For every real number x find a sequence of rational that converges to x.

10. Compute

$$\lim_{n\to\infty} \frac{1}{n^4} \sqrt[n]{\prod_{k=1}^{2n} (n^2 + k^2)}$$

11. Let a be an arbitrary real number.And let $a_1 = a$ and for $n > 1, a_n = \cos(a_{n-1})$.Prove that a_i converges and find the limit.

12. Study

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and

$$\sum_{0}^{\infty} \frac{1}{n!}$$

and prove irrationality for all powers of e and that the two sequences converge to the same limit.

13. Sub-additivity. Let $x_1, x_2, ...$ be a sequence of real numbers such that $x_{i+j} \leq x_i + x_j$ for all (not necessarily distinct) positive integers i and j. Then $\lim_{n\to\infty} x_n$ always exists, and is either a real number or $-\infty$.

14. Prove that the set of limit points of a sequence is closed.

15. Let $x_1, x_2, ..., x_{n^2+1}$ be a sequence of distinct reals.Then there exists either an increasing or decreasing $n + 1$ subsequence.

16. If you color $\binom{\mathbb{N}}{2}$ with finite colors then you can find an infinitely large monochromatic K_n.

17. Let x an irrational number and let a_n, b_n sequences of integers such that $\frac{a_n}{b_n} \to x$.Show that $b_n \to \infty$.

18. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there are no $x \in (a, b)$ with $f(x) = \acute{f}(x) = 0$. Prove that $f(a)f(b) = 0$ and give an example of such function on $[0, 1]$

19. Let
$$S_n = \sum_{1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right)$$
Prove that $S_n \to \frac{1}{4}$.

20. Find exists a function that is continuous only on the irrationals.

21. Prove that the sequence $\sqrt{7}, \sqrt{7 - \sqrt{7}}, \sqrt{7 - \sqrt{7 + \sqrt{7}}}$ converges and find its limit.

22. Prove that
$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + ...}}}} = 3$$

23. If a_i sequence with $a_{n+1} - a_n \to 0$ and there exists subsequences $a_{n_k} \to a, a_{x_k} \to b$ with $a < b$. Show that $\forall y \in [a, b]\ \exists$ subsequence $a_{m_k} : a_{m_k} \to y$.

24. Newton's Method,Picard's method.

25. Consider x_i a sequence given by
$$x_1 = 2, x_{n+1} = \frac{x_n + 1 + \sqrt{x^2 + 2x + 5}}{2}$$
Prove that the sequence $y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1}$ is convergent and find its limit.