Fundamental Theorems

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1 Basics

- **Dirichlet Approximation Theorem** Let $n \geq 1$ be any integer. Then there exists a rational number $\frac{p}{q}$ such that $1 \leq q \leq n$ and $\left| x - \frac{p}{q} \right| < \frac{1}{nq}$.

- **Change of variables** Let $f : \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}$ and let

$$x(y_1, \ldots, y_n) = (x_1(y_1, \ldots, y_n), \ldots, x_n(y_1, \ldots, y_n))$$

be a bijection $\mathbb{D} \to \mathbb{T}$. Then

$$\int_D f(x_1, \ldots, x_n) dx_1 ... dx_n = \int_T f(x_1(y_1, \ldots, y_n), \ldots, x_n(y_1, \ldots, y_n))|J| dy_1 \ldots dy_n$$

Where $|J| = \det\left(\frac{\partial x_i}{\partial y_j}\right)$ the jacobian determinant.
Particularly Polar Coordinates

- **Wronskian** For n real- or complex-valued functions $f_1, \ldots, f_n$, which are $n-1$ times differentiable on an interval I, the Wronskian $W(f_1, \ldots, f_n)$ as a function on I is defined by

$$W(f_1, \ldots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}, \qquad x \in I.$$

the functions f_i are linearly dependent, then so are the columns of the Wronskian as differentiation is a linear operation, so the Wronskian vanishes. Thus, the Wronskian can be used to show that a set of differentiable functions is linearly independent on an interval by showing that it does not vanish identically.

- **Fubini** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a piecewise continuous function such that
$$\int_a^b \int_c^d |f(x,y)|\, dx\, dy < \infty$$
Then
$$\int_a^b \int_c^d f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dy\, dx$$
The limits of integration may be finite or infinite. Useful particular cases are
$$\sum_m \sum_n f(m,n) = \sum_n \sum_m f(m,n)$$
$$\int_a^b \sum_n f(n,x)\, dx = \sum_n \int_a^b f(n,x)\, dx$$

- **Tonneli** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a positive piecewise continuous function Then
$$\int_a^b \int_c^d f(x,y) dx dy = \int_c^d \int_a^b f(x,y) dy dx$$
The limits of integration may be finite or infinite.

- **Absolute convergence**
  - Series The series of a complex valued sequence is said to converge absolutely iff $\sum_{n=0}^{\infty} ||a_n|| = L \in \mathbb{R}$
  - Integrals The integral of a complex function f is said to be absolutely convergent iff $\int_A ||f|| = L \in \mathbb{R}$

- **Riemann** Let $(a_n)_{n\geq 1}$ be a sequence.
If $\sum_{n\geq 1} a_n$ is absolutely convergent with limit S then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the sum $\sum_{n\geq 1} a_{\sigma(n)} = S$ as well.

- **Compactness** Means closed and bounded in $\mathbb{R}^n$ but it comes as two equivalent forms 1. Sequential compactness a set is sequentially compact iff every sequence in it has a convergent subsequence in it 2. For every cover there is a finite subcover. The image of a continuous function of a compact set is compact.

- **Completness** A metric space M is complete if every Cauchy sequence converges in it.

- **Uniform Convergence**. We say that a sequence of functions f_n converges uniformly on a function f in a set E iff

$$\forall \epsilon > 0 \exists N_\epsilon : n > N_\epsilon \Rightarrow |f_n(x) - f(x)| < \epsilon, \forall x \in E$$

  - If f_n converges uniformly to f and f_n are continuous then f is continuous.
  - If f_n Riemann integrable functions defined on a compact interval I which uniformly converge with limit f. Then f is also Riemann integrable and it's integral can be computed as the limit of the integrals of f_n
$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$
  - If f_n a sequence of differentiable functions on $[a, b]$ such that $\lim_{n\to\infty} f_n(x_0)$ exists and is finite for some $x_0 \in [a, b]$ and f_n' converge uniformly, then f_n converge uniformly to a function f on $[a, b]$ and $f' = \lim_{n\to\infty} f_n', \forall x \in [a, b]$.

- **Weierstrass M-Test**. Suppose that $(f_n)_{n>0}$ a sequence of functions defined at an interval E and that $|f_n| \leq M_n, \forall x \in E$. Then if $\sum_0^\infty M_n$ converges then the series of f_n converge uniformly.

- Suppose that K is compact and

  - f_n a sequence of continuous functions in K
  - f_n converges point-wise in a function f.
  - $f_n \geq f_{n+1} \forall x \in K$

  Then $f_n \to f$ uniformly on K.

- **Uniform Convergence For Integrals** For an improper integral

$$I(p) = \int_0^\infty f(x, p) dx$$

We say that $I(p)$ converges uniformly on $[p_1, p_2]$ if for every $\epsilon > 0$ and $p \in [p_1, p_2]$ there is a constant $A \geq a$, independent of p, such that $x \geq A$ implies

$$|\int_a^\infty f(x,p)dx - \int_a^A f(x,p)dx| < \epsilon$$

- **M-Test for integrals** If $f(x,p)$ is integrable on every finite interval and there is an integrable fuction $\phi(x)$ on $[0, \infty]$ such that

$$|f(x,p)| \leq \phi(x), \forall x \geq a, p \in [p_1, p_2]$$

then $I(p)$ converges uniformly on $[p_1, p_2]$.

- **Dini Test**. If

$$\Big|\int_a^A f(x,p)dx\Big|$$

is uniformly bounded in A and p, $g(x,p)$ is monotonic in x and $g(x,p) \to 0$ uniformly in p as $x \to \infty$, then

$$\int_a^\infty f(x,p)g(x,p)dx$$

converges uniformly on $[p_1, p_2]$.
As an immediate consequence of the Dini test, if $\int_a^\infty f(x)dx$ converges, then

$$\int_a^\infty e^{-xp} f(x)dx \quad \int_a^\infty e^{-x^2 p} f(x)dx$$

are uniformly convergent for $p > 0$.

- **Leibniz's Rule**. If f and its partial derivative f_p are continuous on the rectangle $R = \{(x,p) : x \in [a,b], p \in [p_1, p_2]\}$, then $I(p)$ is differentiable in (p_1, p_2) and

$$I'(p) = \int_a^b f_p(x,p)dx$$

- **Weierstrass Approximation Theorem** Let f a continuous function on an interval $[a,b]$ then

$$\forall \epsilon > 0 \quad \exists P(x) \in \mathbb{R}[x] : |f(x) - P(x)| < \epsilon, \forall x \in [a,b]$$

Or with other words any continuous function in an interval can be uniformly approximated by polynomials.

- **Brower Fixed Point**. Every continuous function $f : \mathbb{B}^n \to \mathbb{B}^n$ of an n-dimensional ball to itself has a fixed point ,$x \in \mathbb{B}^n : f(x) = x$.

- Dominated Convergence Let f_n sequence of measurable functions on a measurable space S . Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n| \leq g(x), \forall x \in S$$

$$\int_S g(x)dx < \infty$$

  Then

$$\lim_{n\to\infty} \int_S f_n dx = \int_S f(x)dx$$

- **Kronecer lemma** Every additive group of real numbers is either cyclic or dense.

- **Dense** Given any interval I we say that a subset X of I is dense (in I) if $\forall a \in I$ and $\forall \epsilon > 0$ $X \cup (a - \epsilon, a + \epsilon) \neq \emptyset$.
  This is equivalent with $\forall a \in I \quad \exists x_n \in X : x_n \to a$

- **Nested intervals** If K_n is a sequence of non-empty compact sets such that $K_n \supset K_{n+1}$ and if $diam(K_n) \to 0$ Then $\cap_{n=1}^{\infty} K_n$ consists of exactly one point.

- **Descartes rule of signs** The rule states that if the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Multiple roots of the same value are counted separately.

2 Problems

1. Gaussian integral Prove

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}.$$

2. Show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy = \zeta(2)$$

And find its value by change of variables $u = \frac{y+x}{2} \quad v = \frac{y-x}{2}$.

3. Compute:

- $$\int_0^1 \frac{\arctan(x)}{x\sqrt{1-x^2}} dx$$

- $$\int_0^1 \frac{x^5 - 1}{\ln x}$$

- $$\int_0^{\frac{\pi}{2}} \ln\left(\frac{a + b\sin x}{a - b\sin x}\right) \frac{dx}{\sin x}$$

- $$\int_0^1 \frac{\ln(x+1)}{x^2+1}$$

4. Show that

$$\int_0^{\infty} \ln\left(\frac{e^x + 1}{e^x - 1}\right) dx = \frac{\pi^2}{4}$$

5. Compute the integral

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx.$$

Use the answer to prove Wallis formula

$$\lim_{n\to\infty} \left[\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)}\right]^2 \frac{1}{n} = \pi$$

6. (**Stirling's Formula**)

$$n! \sim \sqrt{2\pi} n^n \sqrt{n} e^{-n}$$

7. Show that

$$\sum_0^{\infty} \left| \binom{\frac{1}{2}}{i} \right| < \infty.$$

8. Does the series converge

$$\sum_{2}^{\infty} \frac{1}{\ln(n!)}$$

9. (Frullani's Integral) Prove the following formula:

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) - \lim_{x\to\infty} f(x) \ln\frac{b}{a}$$

Where f is continuously differentiable and the integral $\int_A^\infty \frac{f(x)}{x} dx$ is defined for all $A > 0$.

10. Compute the sum

$$\sum_{n=1}^{\infty} \frac{\cos n}{1+n^2}$$

11. Compute

$$\int_1^\infty \frac{\arctan(2x)}{x^2\sqrt{x^2-1}} dx$$

12. Compute the Fresnel integrals

$$I = \int_0^\infty \cos x^2 \text{ and } J = \int_0^\infty \sin x^2 dx$$

13. Let $|x| < 1$. Prove that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t} \ln(1-t) dt.$$

14. Solve the Cauchy functional equation for continuous functions

$$f(x+y) = f(x) + f(y).$$

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nonzero function, satisfying the equation

$$f(x+y) = f(x)f(y), \forall x, y \in \mathbb{R}$$

Prove that there exists $c > 0$ such that $f(x) = c^x$

16. Find all continuous functions $f \in C(\mathbb{R})$ such that

$$f(2x - y) + f(2y - x) + 2f(x + y) = 9f(x) + 9f(y)$$

17. Let f continuous function with

$$f(2x^2 - 1) = 2xf(x)$$

Prove that $f(x) = 0, \forall x \in [-1, 1]$

18. (Sperner's Lemma in 2 dimensions) Consider any triangulation of the 2-dimensional simplex, i.e., a triangle. Suppose that all the vertices have been colored from 3 colors, with the following properties.

    - The 3 vertices of the large outermost triangle all receive distinct colors.
    - Along any outer edge, the only colors which appear are of the colors of the far endpoints of that outer edge.

    Then, somewhere in the triangulation, there is an elementary triangle with vertices of all 3 colors.

19. (**Contraction maps**) Let X be a closed subset of $\mathbb{R}^n$ (or in general a complete metric space) and $f : X \to X$ a function with the property that $||f(x) - f(y)|| \leq c||x - y||$ for any $x, y \in X$, where $0 < c < 1$ is a constant. Then f has a unique fixed point in X.

20. Peano's Theorem There exists a continuous surjection $\phi : [0, 1] \to [0, 1] \times [0, 1]$.

21. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function and let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t)dt \ \ (x \in (0, 1], \, n = 0, 1, 2, ...).$$

    Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in (0, 1]$.

22. Find

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \frac{x \ln(1 + x/n)}{1 + x} dx.$$

23. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}dx\right) = \int_0^1 f(x)\,dx,$$

where $f(x) = \frac{\arctan x}{x}$ if $x \in (0,1]$ and $f(0) = 1$.

24. a) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n x^k dx,$$

where $k \in \mathbb{N}$.
b) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x) dx,$$

where $f : [0,1] \to \mathbb{R}$ is a continuous function.

25. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \left(\frac{(2n+1)!}{(n!)^2}\right)^2 \int_0^1 \int_0^1 (xy(1-x)(1-y))^n f(x,y) dx dy.$$

26. (i) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x) dx = 0.$$

(ii) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) dx = f(1),$$

and

$$\lim_{n\to\infty} n \int_0^1 x^n f(x^n) dx = \int_0^1 f(x) dx.$$

(iii) Let $f, g : [0,1] \to \mathbb{R}$ be two continuous functions. Show that

$$\lim_{n\to\infty} \frac{\int_0^1 x^n f(x) dx}{\int_0^1 x^n g(x) dx} = \frac{f(1)}{g(1)},$$

and

$$\lim_{n\to\infty} \frac{\int_0^1 x^n f(x^n)dx}{\int_0^1 x^n g(x^n)dx} = \frac{\int_0^1 f(x)dx}{\int_0^1 g(x)dx}.$$

(iv) Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L.$$

27. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f(\sqrt[n]{x_1 \cdots x_n}) dx_1 \cdots dx_n = f\Big(\frac{1}{e}\Big).$$

28. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\Big(\frac{x_1 + \cdots + x_n}{n}\Big) dx_1 \cdots dx_n = f\Big(\frac{1}{2}\Big).$$

29. (Stone-Weierstrass Theorem) If X is any compact space, let A be a subalgebra of the algebra $C(X)$ over the reals R with binary operations $+$ and $\times$. Then, if A contains the constant functions and separates the points of X (i.e., for any two distinct points x and y of X, there is some function f in A such that $f(x) \neq f(y)$), A is dense in $C(X)$ equipped with the uniform norm.

30. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function of period $T > 0$. Show that:
(i) $\lim_{x\to\infty} \frac{1}{x} \int_0^x f(x)dx = \frac{1}{T} \int_0^T f(x)dx$
(ii) $\lim_{n\to\infty} \int_a^b f(nx)dx = \frac{(b-a)}{T} \int_a^b f(t)dt$
(iii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all real numbers x. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \int_0^1 f(x)dx \int_0^1 g(x)dx$$

31. Let $F_o = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate $\lim_{x\to\infty} \frac{n! F_n(1)}{\ln n}$

32. Let f, g be continuous functions such that

$$\int_{\mathbb{R}} |f(x)|\,dx \int_{\mathbb{R}} |g(x)|\,dx < \infty.$$

Prove that

$$\int_{\mathbb{R}} f(x)\,dx \int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} (f * g)(x)\,dx,$$

in 2 different ways. Note that

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy.$$

33. The Chebyshev polynomials T_n are defined by

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, 2, 3, ...$$

- What are the domain and range of these functions?
- $T_0(x) = 1$ and $T_1(x) = x$. Express T_2 explicitly as a quadratic polynomial and T_3 as a cubic polynomial.
- Show that, for $n \geq 1$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

.

- What are the zeros of T_n? At what numbers does T_n have local maximum and minimum values?
- Based on your observations from parts (g) and (h), how are the zeros of T_n related to the zeros of T_{n+1}? What are the x-coordinates of the maximum and minimum values?
- What can you say about $\int_{-1}^{1} T_n(x)\,dx$ when n is odd and when n is even?
- The family of functions $f(x) = \cos(c \arccos x)$ are defined even when c is not an integer (but then f is not a polynomial). Describe how the graph of f changes as c increases.
- Show that T_n is a sequence of orthogonal polynomials with respect to the weight $\frac{1}{\sqrt{1-x^2}}$ on the interval [-1,1].

- Given any polynomial with degree n and leading coefficient 1.

$$f(x) = \frac{1}{2^{n-1}} T_n$$

is the one of which the maximal absolute value on the interval $[-1, 1]$ is minimal.

34. Let $A_1, A_2, \ldots, A_n$ be points in the plane. Prove that on any segment of length l is a point M such that

$$(MA_1) \cdot (MA_2) \cdots (MA_n) \geq 2\left(\frac{l}{4}\right)^n$$

35. Prove the identity

$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k} \sum_{j=1}^{k} \cos^n(\frac{j\pi}{k}) \cos \frac{(nj\pi)}{k}.$$

36. Let K be a positive real number, and let $f : [0, 1] \to \mathbb{R}$ be a differentiable function whose derivative satisfies $|f'(x)| < K, \forall x \in [0, 1]$. Prove that

$$\left| \int_0^1 f(x)dx - \sum_0^n \frac{f(\frac{i}{n})}{n} \right| \leq \frac{K}{n}.$$

37. Let S be a set and P all it's subsets. Let $f : P \to P$ be a fuction such that $X \subset Y \Rightarrow f(X) \subset f(Y)$. Show that there is some $K \subset S$ such that $f(K) = K$.

38. Does there exist an uncountable set of subsets of $\mathbb{N}$ such that any two distinct subsets have finite intersection?