Linear Algebra Part I

Ιακωβίδης Ιωάννης

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1 Basics

- **(Inverses Commute)** We say a matrix A is invertible if $\exists A^{-1}$: $AA^{-1} = A^{-1}A = I$.
- **Determinant**. $det : M_{nxn} \to \mathbb{R}$. $\det(A) = |A| = \sum_{\forall \sigma(i) \in S_n} \epsilon(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} ... a_{n,\sigma(n)}$.

  $$\det(A) \neq 0 \iff \exists A^{-1}$$

  Or equivalently the determinant is a multi-linear and alternating map of the columns of a matrix with $\det(I) = 1$.
- The determinant is a multiplicative function

  $$\det(AB) = \det(A)\det(B)$$
- **Elimination**. Every matrix A can be decomposed to two matrices U, L where the one is upper diagonal and the other is lower diagonal with diagonal elements equal to one. Or $A = LU$.
- **Rank Nullity Theorem**. Let a matrix $A : V \to V$

  $$rank(A) + \dim Ker(A) = \dim(V)$$
- **The determinant of a matrix is a polynomial function of its entries.**
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  - Any spanning set of a vectorspace V can be cut down to a basis.
  - Any linearly independent set can be extended to a basis.
- The set of invertible matrices is dense in the set of all matrices.

2 Problems

1. Find the matrices $A \in M_{nxn}(\mathbb{R})$ that $\forall B \in M_{nxn}(\mathbb{R})$ $AB = BA$

2. Let A,B,C,D in $M_n(\mathbb{R}), n > 2$ with $AD - BC = O$ and $AC - BD = I$. Show that $DA - CB = 0, CA - DB = I$.

3. Suppose A, B, C, D are nxn matrices, satisfying the conditions that AB^T and CD^T are symmetric and $AD^T - BC^T = I$. Prove that $A^T D - C^T B = I$.

4. Let
$$V_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \dots & {x_n}^{n-1} \end{bmatrix}$$
Compute $\det(V_n)$.

5. Show that there only one polynomial of n degree passing throw any n+1 points (x_i, y_i) with $x_i \neq x_j \forall i \neq j$.

6. Consider a natural number $n \geq 1$ and a continuous real valued function f defined on the interval $[a, b]$. Show that there is only one polynomial function p of degree $\leq n$ such that
$$p(a) = f(a)$$
and
$$\int_a^b (f(x) - p(x))q(x)dx = 0$$
, $\forall q$ polynomial function of degree $\leq n - 1$.

7. Find the determinant of a circulant matrix.
$$C_n = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_2 & c_3 & c_4 & \dots & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_1 & c_2 & \dots & c_{n-1} \end{bmatrix}$$

8. Prove Rank Nullity Theorem.

9. Prove $rank(A) = rank(B) \Leftrightarrow \exists X, Z \in GL(n) : A = XBZ$.

10. Let V be a vector space over a field F and let V_1, V_2 be subspaces of V. Prove that the union of V_1, V_2 is a subspace of V if and only if $V_1 \subset V_2$ or $V_2 \subset V_1$.

11. Let $T_1, T_2 : V \to W$ be linear transformations.Prove that

$$|rank(T_1) - rank(T_2)| \leq rank(T_1 + T_2) \leq rank(T_1) + rank(T_2)$$

12. Let A, B, C, D be nxn matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$$

13. Let A be a square matrix with real entries. Show that $\det(A^2 + I) \geq 0$.

14. $rank(A) = \max\{n : \exists A_{nxn} \in GL(n)$ submatrix of A $\}$.

15. Prove **Cramer's Rule**.Let system $Ax = b$ with $\det(A) \neq 0$ then

$$x_i = \frac{\det(A_1, A_2 ..., b, ...A_n)}{\det A}$$

16. Prove **Expansion in minors**.

$$det(A) = \sum (-1)^{i+1} |M_{1,i}|$$

Mi, j the matrix if that you get if you delete the i'th

17. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2 + y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

18. Let A, B be $n \times n$ matrices, such that $rank(AB - BA) = 1$. Prove that $(AB - BA)^2$ is zero matrix.

19. Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ such that $T^m x = 0, T^{m-1}x \neq 0$ for some positive integer m. Show that $x, Tx, ..., T^{m-1}x$ are linearly independent.

20. Let $A \in M_2(\mathbb{C})$ so that $tr(A) = \varepsilon, det(A) = \varepsilon^2$, where ε is n order unit root,$n \geq 4$. Prove that $\sum_{k=1}^{n} \varepsilon^k det(A - \varepsilon^k I) = 0$.

21. Let A, B, C be nxn real matrices that pairwise commute and $ABC = O$. Show that $\det(A^3 + B^3 + C^3)\det(A + B + C) \geq 0$

22. (**Grassman formula**) If U and W are subspaces of a finite dimensional vector space, $\dim U + \dim W = \dim(U \cup W) + \dim(U + W)$.

23. (**Frobenius inequality**)

$$rankBC + rankAB \leq rankABC + rankB$$

24. (**Sylvester inequality**)

$$rankA + rankB \leq rankAB + n$$

where n is the number of columns of the matrix A and also the number of rows of the matrix B.

25. Let A and B be two nxn real matrices that commute. Suppose that $det(A + B) \geq 0$. Prove that $\det\left(A^k + B^k\right) \geq 0 \; \forall k \geq 1$

26. Let A be real skew-symmetric square matrix (i.e., $A^T = -A$). Prove that $det(I + tA^2) \geq 0$ for all real t.

27. Prove that $SO(n, \mathbb{Z})$. The set of integer matrices with integral inverses is exactly the same as the set of integer matrices with determinant $1, -1$.

28. Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

29. Let $A = (a_{ij})$ be real nxn matrix satisfying

$$a_{ii} > \sum_{i \neq j} a_{ij}$$

Show that A is invertible.

30. (Oddtown). In a town with n people, m clubs have been formed. Every club have an odd number of members, and every two clubs have an even number of members in common. Prove that $m \leq n$.

31. (Fisher's inequality). Let k be a positive integer. In a town with n people, m clubs have been formed. Every two clubs share exactly k members. Prove that $m \leq n$.

32. Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.