Linear Algebra II

Ιακωβίδης Ιωάννης

13 Δεκεμβρίου 2017

1 Basics

If not specified $A, B \in M_n(\mathbb{C})$

- **Linear Transformations**. Let V and W be two vector spaces a function $T : V \to W$ is a linear transformation if $\forall x, y \in V$ and $\kappa, \lambda \in \mathbb{R}$ it is true that
$$T(\lambda x + \kappa y) = \lambda T(x) + \kappa T(y)$$
Given a linear Transformation and bases for $V, (v_i)_1^n$ and $W, (w_i)_1^m$ you can find unique matrix $A \in M_{m\times n}$ that describes the linear transformation in these bases. It's columns will be the images of the first base written to the second base. Or if $T(v_j) = \sum_{i=0}^m a_{ij} w_i$ then $A = \{a_{ij}\}$

- **Change of Bases**. Let A be the matrix that describes the linear transformation $T : V \to W$ in bases $(v_i)_1^n$ and $(w_i)_1^n$. Then the matrix B that describes the linear transformation in bases $(v_i')_1^n$ and $(w_i')_1^n$. Is $B = P_W A P_V^{-1}$ if $v_i' = P_V v_i$ and $w_i' = P_W w_i$.
Particularly if $V = W$ then $B = PAP^{-1}$.

- **Eigenvalues and Eigenvectors**. Eigenvalues of a matrix $A \in M_n(\mathbb{C})$ are complex numbers λ such that $\exists(x \neq 0) \in \mathbb{C}^n : Ax = \lambda x$ .The vectors that satisfy this relation are called eigenvectors of the eigenvalue λ.

- **Characteristic Polynomial**. The Characteristic Polynomial ϕ_A of a matrix A is defined by
$$\phi_A(x) = \det(Ix - A)$$
It's roots are the eigenvalues of A.

- **Eigenvalues, Determinant, Trace and Rank**. The sum of the eigenvalues,counted with algebraic multiplicities, is equal to the trace (sum of diagonal entries), and the product of the eigenvalues, counted with algebraic multiplicities, is the determinant. The multiplicity of zero as an eigenvalue is equal to the dimension of the kernel of the matrix.

- **Similarity** A and B are similar iff $\exists P \in GL_n(\mathbb{C}) : A = PBP^{-1}$. Or equivalently A and B represent the same linear transformation in two different bases. **Similar matrices have the same characteristic polynomial**.

- Let A and B real matrices and $A = P^{-1}BP$, where P a complex matrix . Then $A = Q^{-1}BQ$ for some real matrix Q.

- **Minimal Polynomial** The minimal polynomial μ_A of a matrix is the monic polynomial of the smallest degree for which $\mu_A(A) = 0$.
  - The roots of the minimal polynomial are the eigenvalues of the matrix.
  - The minimal polynomial divides every polynomial P that $P(A) = 0$.

- **Cayley-Hamilton Theorem. Every matrix is 'root' of its characteristic polynomial**. Let K be any field and let $A \in M_n(K)$. If
$$\det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
Then
$$A^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

- **Diagonalization** A matrix $A \in M_n(F)$ is diagonalizable in iff $\exists P \in GL_n(F)$ and diagonal matrix Λ such that $A = P\Lambda P^{-1}$.

- The following statements are equivalent
  - A is diagonalizable in $M_n(F)$.
  - There exists a bases of eigenvectors (in F^n).
  - For every eigenvalue it's geometric and algebraic multiplicity are equal.

  - The minimal polynomial of A is split over F, with pairwise distinct roots.
  - There is a polynomial $P \in F[x]$ which is split over F, with pairwise distinct roots and such that $P(A) = 0$.

- **Let $\lambda_1, \ldots, \lambda_k$ pairwise distinct eigenvalues of a linear transformation T. Then the λ_i-eigenspaces of T are in direct sum position.**

- **Gershgorin Discs**. Let $A = [a_{ij}] \in M_n(C)$ and $R_i = \sum_{j \neq i} |a_{ij}|$ then any eigenvalue of A belongs in the set

$$\bigcup_{i=1}^{n} \{z \in \mathbb{C} | |z - a_{ii}| \leq R_i\}$$

- **Theorem**. Let $A, B \in M_n(F)$ to diagonalizable matrices. A and B commute iff the are simultaneously diagonalizable.

- **Spectral Theorem** Any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries.

$$A = A^H \Rightarrow \exists U : A = U \Lambda U^H \quad and \quad U^{-1} = U^H$$

- **Triagonalization**. A matrix is triagonalizable means that it is similar to a upper-triangular matrix.

- **Theorem**. Let $A \in M_n(F)$ be a matrix. Then the following assertions are equivalent:
  - The characteristic polynomial of A is split over F.
  - A is triagonalizable in $M_n(F)$.

- **Jordan Theorem**. For any linear operator $A : V \to V$ over $\mathbb{C}$ there exists a basis which A can be written as a Jordan block diagonal matrix.

- **Positive Definite** A symmetric matrix $A \in M_n(\mathbb{R})$ is called positive definite iff $x^T A x > 0$ for all nonzero vectors $x \in \mathbb{R}^n$

- The following statements are equivalent (For a symmetric matrix $A \in M_n(\mathbb{R})$) :
  - A is positive definite.
  - All it's eigenvalues are positive.
  - All the pivots obtained without row exchanges or scalar multiplications of rows are positive.
  - All sub-determinants $\det(A_k) > 0$ $A_k \in M_k(\mathbb{R}), (A_k)_{ij} = (A)ij$
  - All A_k sub-matrices are positive definite.
- **Schur decomposition**. If A is a $n \times n$ square matrix with complex entries, then A can be expressed as

  $$A = QUQ^H$$

  where Q is a unitary matrix (so that its inverse Q^{-1} is also the conjugate transpose Q^H of Q), and U is an upper triangular matrix.
- The set of complex diagonalizable $n \times n$ matrices is dense in $M_n(\mathbb{C})$

2 Problems

1. Prove that if λ eigenvalue of A then $P(\lambda)$ eigenvalue of $P(A)$.
2. **Nilpotent** A matrix is nilpotent iff $\exists i$ such that $A^i = 0$.Prove a nilpotent matrix has all it's eigenvalues zero.
3. Let $A, B \in M_n(\mathbb{C})$ be matrices such that $AB - BA = A$. Prove that A is nilpotent.
4. Let $A, B \in M_2$ Prove that

   $$\det\left(I + \frac{2AB + 3BA}{5}\right) = \det\left(I + \frac{3AB + 2BA}{5}\right)$$

5. Let $A, B \in M_n(\mathbb{R})$ for n odd such that $2AB = (BA)^2 + I$ Show that $\det(I - AB) = 0$.

6. a) Prove that for every matrix $X \in M_2(\mathbb{C})$ there exists a matrix $Y \in M_2(\mathbb{C})$ such that $Y^3 = X^2$
b) Prove that there exists a matrix $A \in M_3(\mathbb{C})$ such that $Z^3 \neq A^2$ for all $\mathbb{Z} \in M_3(\mathbb{C})$

7. Let $M, N \in M_2(\mathbb{C})$ two non-zero matrices such that
$$M^2 = N^2 = 0 \quad MN + NM = I$$
. Prove that there is a invertable matrix A such that
$$M = A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A^{-1} \quad N = A \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} A^{-1}$$

8. Let $A, B \in M_2(\mathbb{R})$ be matrices with the property that
$$|\det(A + zB)| \leq 1$$
for any complex number $z : |z| = 1$. Prove that
$$(\det(A))^2 + (\det(B))^2 \leq 1$$

9. Define the sequence $A_1, A_2, \ldots$ of matrices by the following recurrence
$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \ldots)$$
where I_m is the $m \times m$ identity matrix A_n has $n + 1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ respectively.

10. Let k and n be positive integers. A sequence $(A_1, \ldots, A_k)$ of $n \times n$ real matrices is preferred by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_i A_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

11. Let V be a 10-dimensional real vector space and U_1, U_2 two linear subspaces such that $U_1 \subseteq U_2, \dim U_1 = 3, \dim U_2 = 6$. Let ε be the set of all linear maps $T : V \to V$ which have $T(U_1) \subseteq U_1, T(U_2) \subseteq U_2$. Calculate the dimension of ε. (again, all as real vector spaces)

12. Let N be the $n \times n$ matrix with all its elements equal to $1/n$, and let $A = (a_{ij})_{1 \le i,j \le n} \in M_{n,m}(\mathbb{R})$ be such that for some positive integer k, $A^k = N$. Prove that

$$\sum_{1 \le i,j \le n} a_{ij}^2 \ge 1$$

.

13. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example, $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$. The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n \to \infty} d_n$

14. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that $\det \begin{vmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{vmatrix} = n!$ for all $n \ge 0$. Show that u_n is an integer for all n.

15. For any square matrix A we define $\sin A$ by the usual power series.

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

Prove or disprove : $\exists 2 \times 2$ matrix $A \in M_2(\mathbb{R})$ such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}$$

16. For $n \ge 3$ find the eigenvalues (with their multiplicities) of the $n \times n$ matrix

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 1
\end{bmatrix}$$

17. Let A be a complex $n \times n$ matrix, an let $C(A)$ be the **commutant** of A; that is, the set of complex $n \times n$ matrices B such that $AB = BA$. (It is obviously a subspace of $M_{n\times n}$, the vector space of all complex $n \times n$ matrices). Prove that $\dim C(A) \geq n$.

18. Let S be the subspace of M_n(the vector space of all complex $n \times n$ matrices) generated by all matrices of the form $AB - BA$ with A and B in M_n

19. Whether m and n natural numbers, $m, n \geq 2$. Consider matrices, $A_1, A_2, ..., A_m \in M_n(R)$ not all nilpotent. Demonstrate that there is an integer number $k > 0$ such that $A^k{}_1 + A^k{}_2 + ... + A^k{}_m \neq O_n$

20. Given A, non-inverted matrices of order n with real elements, $n \geq 2$ and given A^*adjoin matrix A. Prove that $tr(A^*) \neq -1$ if and only if the matrix $I_n + A^*$ is invertible.

21. Let n and k be two natural numbers such that $n \geq 2$ and $1 \leq k \leq n-1$. Prove that if the matrix $A \in M_n(\mathbb{C})$ has exactly k minors of order $n-1$ equal to 0 , then $\det(A) \neq 0$.

22. Let $A \in M_n(\mathbb{C})$ be a matrix with rank$(A) = 1$. Prove that

$$det(I_n + A) = 1 + tr(A)$$

. Moreover,

$$det(\lambda I_n + A) = \lambda^n + \lambda^{n-1} \cdot B$$

for all complex numbers λ.

23. Let $A, B \in M_n(\mathbb{C})$ be two matrices such that $AB = A + B$. Prove that $rank(A) = rank(B)$.

24. Let T be a linear transformation on a finite dimensional F-vector space V and let $V = V_1 \oplus V_2$ be a decomposition of V into subspaces which are stable under T. Let P, P_1, P_2 be the minimal polynomials of $T, T|_{V_1}, T|_{V_2}$ respectively. Prove that P is the least common multiple of P_1 and P_2.

25. Let T be a linear transformation on a finite dimensional F-vector space V and let W be a subspace of V which is stable under T. Let T_1 be the restriction of T to W. Prove that $\phi_{T_1} | \phi_T$

26. Suppose that T is a linear transformation on some F-vector space V Then for any pairwise relatively prime polynomials $P_1, \ldots, P_k \in F[X]$ we have

$$\ker P(T) = \bigoplus_{i=1}^{k} \ker P_i(T)$$

where $P = P_1 P_2 \cdots P_k$.

27. Show that the above statement implies that

$$\mu_A(x) = \prod (x - \lambda_i), \lambda_i \neq \lambda_j \iff A \text{ is diagonalizable}$$

28. Let V be a finite dimensional vector space over a field F and let $T_1, T_2 : V \to V$ be linear transformations of V. Prove that if $T_1 \circ T_2 = T_2 \circ T_1$ then any eigenspace of T_2 is stable under T_1.

29. For $n \times n$ real matrix A, we define $e^A = \sum_0^\infty \frac{A^k}{k!}$. Prove or disprove that for all real polynomials p and square matrices A and B, $p(e^{AB})$ is nilpotent iff $p(e^{BA})$ is nilpotent.

30. Let V be a real vector space, and let $f, f_1, f_2 \ldots f_k$ be linear maps from V to R. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \cdots = f_k(x) = 0$. Prove that f is a linear combination of $(f_i)_1^k$.

31. Prove that a matrix is similar to its transpose.

32. Let A be a positive definite matrix find $\int_{-\infty}^{\infty} e^{-x^T A x} dx$

33. Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose entries are precisely the numbers $1, 2, ..., n^2$?