Sums and Series

Ιακωβίδης Ιωάννης

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1 Basics

Let $a_1, a_2, \ldots$ be a sequence of real numbers.

- **p-series**. The series $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

- **Cauchy Sequence**. We say that a sequence is a Cauchy sequence iff

$$\forall \epsilon > 0 \exists N_0 : \forall m, n > N |a_m - a_n| < \epsilon.$$

  In a complete metric space every sequence by definition converges iff it is a Cauchy sequence. $\mathbb{R}^n$ along with the vector metric is a complete metric space.

- **Absolute convergence**. Let a_i be a complex sequence if $\sum |a_n|$ converges then we say that the series of the sequence a_i is converging absolutely and as a consequence the $\sum \epsilon_n a_n$ for every sequence ϵ_i such that $|\epsilon_i| = 1, \forall i \in \mathbb{N}$

- **Convergence Tests**.
  - **Ratio**. If $\lim_{n \to \infty} \frac{a_n + 1}{a_n} = c$.Then for $c < 1$ the series converge and for $c > 1$ the series diverge.
  - **Root**. If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = c$.Then for $c < 1$ the series converge and for $c > 1$ the series diverge.

- **Integral**. If $(a_n)_{n\geq 1} = f(n)$ be decreasing sequence of positive numbers then the series$\sum_N^\infty a_n$ and the integral $\int_N^\infty f(x)dx$ both converge and diverge simultaneously. In fact

$$\int_N^\infty f(x)dx \leq \sum_1^\infty a_n \leq f(N) + \int_N^\infty f(x)dx$$

- **Cauchy Condensation**. If $a_n \geq a_{n+1} \geq 0, \forall n$ then the series $\sum_0^\infty a_n$ converges if and only if the "condensed" series $\sum_0^\infty 2^n a_{2^n}$.
- **Dirichlet criterion** .Let $(a_n)_{n\geq 1}$ be a real sequence and $(b_n)_{n\geq 1}$ be a sequence of complex numbers satisfying $a_n \geq a_{n+1}$,$a_n \to 0$ and $\exists M : \sum_{n=1}^m b_n \leq M, \forall m$. Then $\sum a_n b_n$ converges.

• **Root of Unity Filter**. Let $f(x) = \sum_0^\infty a_n x^n$ and k be positive integer if $\omega = e^{\frac{2\pi i}{k}}$ then

$$\sum_{n=m(modk)}^\infty a_n x_n = a_m x^m + a_{m+k} x^{m+k} + \cdots = \frac{1}{k}\sum_{j=1}^k \omega^{-jm} f(\omega^j x)$$

• **Power Series** is an infinite series of the form $\sum a_n x^n$. There is always a number R ,radius of convergence, such that it converges $x < R$ and diverges if $x > R$. $R = \liminf |a_n|^{-\frac{1}{n}}$ or $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ if this limit exists.

• **Partial Summation**. Let (a_n) and (b_n) with $n' \in \mathbb{N}$ with $B_n = \sum_{i=0} nb_i$. Then

$$\sum_{i=0}^n a_i b_i = a_n b_n - \sum_0^{n-1} B_n(a_{n+1} - a_n)$$

• **Taylor Theorem** Let $k \geq 1$ be an integer and let the function $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at a point $a \in R$ .Then there exist a function $h_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^k(a)}{k!}(x-a)^k + h_k(x-a)^k$$

with $h_k \to 0$ as $x \to a$.

- **Uniform Convergence**. We say that a sequence of functions f_n converges uniformly on a function f in a set E iff

$$\forall \epsilon > 0 \exists N_\epsilon : n > N_\epsilon \Rightarrow |f_n(x) - f(x)| < \epsilon, \forall x \in E$$

  - If f_n converges uniformly to f and f_n are continuous then f is continuous.
  - If f_n Riemann integrable functions defined on a compact interval I which uniformly converge with limit f. Then f is also Riemann integrable and it's integral can be computed as the limit of the integrals of f_n
$$\int_I f = \lim_{n\to\infty} \int_I f_n.$$
  - If f_n a sequence of differentiable functions on $[a, b]$ such that $\lim_{n\to\infty} f_n(x_0)$ exists and is finite for some $x_0 \in [a, b]$ and f_n' converge uniformly, then f_n converge uniformly to a function f on $[a, b]$ and $f' = \lim_{n\to\infty} f_n', \forall x \in [a, b]$.

- **Weierstrass M-Test**. Suppose that $(f_n)_{n>0}$ a sequence of functions defined at an interval E and that $|f_n| \leq M_n, \forall x \in E$. Then if $\sum_0^\infty M_n$ converges then f_n converge uniformly.

2 Problems

1. Let $f : [0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \log_3(3^x - x), \forall x \in [0, \infty)$

   a Considering the sequence $(x_n)_{n\geq 0}$, where $x_0 = \frac{1}{2}$ and $x_{n+1} = f(x_n)$ evaluate $\sum x_n$

   b Calculate

$$\lim_{x\to 0}(x^{2017}[(x - f(x))\ln 3 - \sum_{k=1}^{2016} k^{-1}x^k 3^{-kx}]^{-1})$$

2. Does $\sum_{n=1}^\infty \frac{|\sin(n)|}{n}$ converge ?

3. Let $(a_n)_{n=1}^{\infty}$ be a sequence with $a_n \in \{0,1\}$ for every n. Let $F : (-1,1) \to \mathbb{R}$ be defined by

$$F(x) = \sum a_n x^n.$$

If $F(\frac{1}{2}) \in \mathbb{Q}$. Show that F is a quotient of two polynomial in $\mathbb{Z}[x]$.

4. Compute

$$\sum_{k=0}^{\infty} \arctan\Big(\frac{2}{(2k+1)^2}\Big).$$

5. Find the power series expansion of $(\arcsin x)^2$

6. Find a sequence $(a_n)_{n=1}^{\infty}$ such that $\sum a_n$ converges and that $\sum (a_n)^3$ diverges.

7. Compute $\sum_{k\geq 0} \left(\begin{array}{c} 1000 \\ 3k \end{array} \right)$

8. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$.

9. Find the function of the power series

$$\frac{x^3}{3!} + \frac{x^9}{9!} + \frac{x^{15}}{15!} + \dots$$

10. Compute $\sum_{n=0}^{m} \cos(n\phi + \theta)$ $\phi, \theta \in \mathbb{R}$.

11. Evaluate in closed form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{n!m!}{(m+n+2)!}$$

12. Let p_i the sequence of primes. Prove that $\sum \frac{1}{p_n}$ diverges.

13. Show that if $\sum a_i^2$ and $\sum b_i^2$ converge then so does $\sum (a_i - b_i)^p$ converge for every $p \geq 2$.

14. Compute

$$\sum_{k=0}^{n} \frac{\sin^3(3^k)}{3^k}$$

15. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

16. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

17. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

18. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n) c(n+2).$$

19. Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

20. Compute the sum of the series

$$1 + \frac{1}{2} x + \frac{1}{2} \frac{3}{4} x^2 + \frac{1}{2} \frac{3}{4} \frac{5}{6} x^3 + \ldots$$

21. Give an example

a) of a series for which the root test succeeds and the ratio test fails

b) a function whose Maclaurin series converges everywhere but represents the function at only one point.

c) a convergent trigonometric series that is not a Fourier series.

22. Find a_{2013} where

$$a_n = \sum_{n=0}^{6^n} (-1)^n \begin{pmatrix} 6^n - k \\ k \end{pmatrix}, n = 0, 1, 2, \ldots$$

23. Prove that

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}$$

24. Let $f : [1, \infty) \to (0, \infty)$ be a non-increasing function such that

$$\limsup_{n \to \infty} \frac{f(2^{n+1})}{f(2^n)} < \frac{1}{2}.$$

Prove that

$$\int_1^{\infty} f(x) dx < \infty.$$

25. For every $k \in \mathbb{Z}$ prove that

$$a_k = \sum_{j=1}^{\infty} \frac{j^k}{j!} \notin \mathbb{Q}.$$

26. Determine the value of

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \cdot \ln\left(1 + \frac{1}{2n}\right) \cdot \ln\left(1 + \frac{1}{2n+1}\right).$$

27. Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

28. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!\,2^n} \cos\left(\frac{\pi n - 1}{2}\right).$$

29. (Riemann) Let $(a_n)_{n\geq 1}$ be a sequence.

    1. If $\sum_{n\geq 1} a_n$ is absolutely convergent with limit S then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the sum $\sum_{n\geq 1} a_{\sigma(n)} = S$ as well.
    2. If $\sum_{n\geq 1} a_n$ is conditionally convergent but not absolutely. Then $\forall M \in \mathbb{R}\ \exists \sigma_M : \sum_{n\geq 1} a_{\sigma_M(n)} = M$

30.
$$\sum_{n=1}^{\infty} n^{-n} = \int_0^1 x^{-x}\, dx.$$

31. Let $(a_n)_{n\geq 0}$ be a strictly decreasing sequence of positive numbers. Show that for every z complex number with $|z| < 1$ the series

$$\sum_{n=0}^{\infty} a_n z^n$$

is not zero.

32. Prove the linearity of expected value.

33.
$$\sum_{n=0}^{\infty} \frac{2^n}{1 + x^{2^n}}$$

Prove that it converges for $x > 1$ and find its value.

34. Prove that $\sum_{d|n} \phi(d) = n$.

35. Catalan Numbers Find the number of valid strings of parentheses of n pairs.For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

36. Stirling Numbers(of the second kind) Find the number of ways of partitioning n distinct objects to k non-empty sets

37. Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

38. Find the value of the Poisson Integral

$$\int_0^{\pi} \ln\big(1 - 2x\cos\theta + x^2\big)\, d\theta$$

39. Complex Sequences. Let $z_1, z_2, ...$ be a sequence of complex numbers converging to 0. Is there a sequence of signs $e_n = \pm 1$, for which the series $\sum_1^{\infty} e_n z^n$ converges.