Combinatorics

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1 Enumerative

- Useful Formulas
$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} = (-1)^k \left(\!\!\binom{n}{k}\!\!\right)$$

- **Generating Functions**. The ordinary generating function of a set S is defined as
$$\sum_{r \in S} x^r$$
if we allow sets to have repeats , a multiset is a set that allows repeats , then we must count the number of times each element occurs as the coefficient:
$$\sum_{r \in S} (\text{nof r's in S}) x^r$$
$$\Big(\sum_{n\geq 0} a_n x^n\Big)\Big(\sum_{n\geq 0} b_n x^n\Big) = \sum_{n\geq 0} c_n x^n, c_n = \sum_{0}^{n} a_i b_{n-i}$$
$$\Big(\sum_{n\geq 0} \frac{a_n}{n!} x^n\Big)\Big(\sum_{n\geq 0} \frac{b_n}{n!} x^n\Big) = \sum_{n\geq 0} \frac{c_n}{n!} x^n, c_n = \sum_{0}^{n} \binom{n}{i} a_i b_{n-i}$$
If $F(x)$ and $G(x)$ are elements of $\mathbb{C}[[x]]$ satisfying $F(x)G(x) = 1$, then we write $G(x) = F(x)^{-1}$. $F(x)^{-1}$ exists (in which case it is unique) if and only if $a_0 \neq 0$.

- A **combinatorial class** of objects is a set A with a size function $|\cdot| : A \to \mathbb{N}$ s.t. the number of elements of weight n is finite for all n. Let a_n to be the number of elements of weight n. Then
$$A(z) = \sum_{0}^{\infty} a_n z^n$$

- **Fundamental Bijection**. Let a permutation w in circle notation $\hat{w}$ is defined by deleting the parenthesis and reading it a one line notation. We have that $\hat{} : S_n \to S_n$ is a bijection. And if w has k circles then $\hat{w}$ has k left to right maxima (records).

- **Stirling number $s(n,k)$ of the first kind**. Let $c(n,k)$ be the number of permutations of an n-element set with exactly k cycles. Then

$$s(n,k) = (-1)^{n-k} c(n,k)$$

$$\sum x^{\text{n of cycles}} = \sum_{k=1}^{n} c(n,k) x^k = x(x+1)(x+2)\dots(x+n-1)$$

- **The Stirling number of the second kind** $S(n,k)$ is the number of partitions of an n-element set into exactly k non-empty subsets.

- **Gamma and Betta functions**

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, n > 0, m > 0$$

Properties

$$\Gamma(n+1) = n\Gamma(n)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}!\right) = \frac{1}{2}\sqrt{\pi}$$

- **Rota's Twelvefold Way**

2 Problems

1. Find the number of domino tilings of a $2 \times n$ and $3 \times n$ rectangle.

2. **Catalan Numbers**.
   - The number of ways to arrange n pairs of matching parentheses.
   - The number of ways a convex polygon of $n+2$ sides can be split into n triangles by $n-1$ non intersecting diagonals.
   - The number of rooted binary trees with exactly $n+1$ leaves.
   - The number of paths with $2n$ steps on a rectangular grid from $(0,0)$ to (n,n) that do not cross above the main diagonal.

3. The number of permutations $w \in S_n$ of type $(c_1, \dots, c_n)$ is equal to

$$\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}$$

4. Find the number of permutations that are fixed under the fundamental bijection.

5. Let n be a positive integer and let $a_k = \dfrac{1}{\binom{n}{k}}, b_k = 2^{k-n},\ (k = 1..n)$.

   Show that $\sum_{k=1}^{n} \dfrac{a_k - b_k}{k} = 0$.

6. Prove
$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k} \sum \cos^n \frac{j\pi}{k} \cos \frac{nj\pi}{k}$$

7. Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n, n, n) starting from $(0, 0, 0)$ in exactly $3n$ jumps.

8. Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that
$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$
   for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity
$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$
   does not depend on n, and find its value.

3 Extremal

- **Bounds**
$$k! \geq \left(\frac{k}{e}\right)^k, \forall k \in \mathbb{N}$$
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

- **Ramsey Number**. $R(k)$ is the minimum number n such that given any coloring of the edges of K_n in two colors there can be found at least one monochromatic K_k.
$$R(k) = \min\{n : c : E(K_n) \to \{0, 1\} \quad \exists MC(K_k)\}$$

- **Ramsey Theorem** (finite) Any given number of colors, c, and any given integers $n_1, \ldots n_c$, there is a number, $R(n_1, \ldots, n_c)$, such that if the edges of a complete graph of order $R(n_1, ..., n_c)$ are colored with c different colors, then for some i between 1 and c, it must contain a complete subgraph of order n_i whose edges are all color i.

  One will find monochromatic cliques in any edge labeling (with colors) of a sufficiently large complete graph $R(n_1, \ldots, n_c) < \infty, \forall n_i \in \mathbb{N}$.

- **Infinite Ramsey** If $c : \binom{\mathbb{N}}{2} \to [r]$ then there exists an infinite MC clique.
- **Upper bounds**
$$R(p,q) \leq \binom{p+q-2}{p-1}$$
and $R(k) \leq \binom{2k}{k} \sim 4^k$
- **Lower bounds**
$$R(k) > \sqrt{2}^k$$
- **Happy Ending Problem** Given n points in $\mathbb{R}^2$ no 3 collinear, for n sufficiently large. There are k of them that are in convex position.
- **Ramsey for m-sets** For all k, m if n is sufficiently large then there exists a MC complete m-set.
- **Van Der Warden's Theorem** Given $c : \mathbb{N} \to \{0, 1\}$ there exist a k-term arithmetic progression.
- **Extremal number of a Graph** If H is a graph and $n \in \mathbb{N}$ then
$$ex(n,H) = \{\max E(n) : \exists G = \{n, E(n)\} \text{ with } H \not\subset G\}$$
- **Mantel**
$$ex(n, \triangle) = \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$$
- **Turan's Theorem**. Let G be any graph with n vertices, such that G is K_{r+1} -free. Then the number of edges in G is at most
$$ex(n, K_{r+1}) \approx (1 - \frac{1}{r})\frac{n^2}{2}$$
- **Turan's Theorem** (Second Formulation). Among the n-vertex simple graphs with no $(r+1)$-cliques, $T(n,r)$ has the maximum number of edges.
- If the graph G on n vertices has no 4-cycles, then
$$|E| \leq \frac{n}{4}(1 + \sqrt{4n+3})$$
- We say that F is an **Anti-chain** if there are not $A, B \in F$ such that $A \subset B$.
- **Sterner Theorem** The size of the largest anti-chain of an n-set is $\binom{n}{\frac{n}{2}}$

- **MYLB lemma** Let F be an anti-chain then

$$\sum_{A \in F} \frac{1}{\binom{n}{|A|}} \leq 1$$

- A family of sets is **intersecting** if for every two elements the intersection is non empty.
- **Erdos-Ko-Rado** The largest size of an intersecting k-family in an n-set is $\binom{n-1}{k-1}$ when $n \geq 2k$.
- **Erdos-Renyi Random Graph** G(n, p) , a graph is constructed by connecting nodes randomly. Each edge is included in the graph with probability p independent from every other edge.
- For any $k \geq 2$. There exists a graph G with $x(G) \geq k$ and $g(G) \geq k$.
- An **induced subgraph** of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset.

4 Problems

1. The maximum density of a sum free subset of $\mathbb{N}$ is $\frac{1}{2}$
2. Every set A of n nonzero integers contains a sum-free subset of size $|A| > \frac{n}{3}$
3. The numbers $1, 2, ..., n^2$ are placed randomly in a $n \times n$ table. Prove that there are two adjacent cells such that the numbers in them differ by at least one.
4. Prove that
$$R(k,t) \leq R(k-1,t) + R(k,t-1)$$
and that for even $R(k-1,t)$ and $R(k,t-1)$ it can be strengthened to $R(k,t) \leq R(k-1,t) + R(k,t-1) - 1$.
5. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $\frac{n}{2017}$ persons in S have exactly two friends in S.
6. Let $n \geq k$ be positive integers, and let $\mathscr{F}$ be a family of finite sets with the following properties: (i) $\mathscr{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements; (ii) for any two sets $A, B \in \mathscr{F}$, their union $A \cup B$ also belongs to $\mathscr{F}$. Prove that $\mathscr{F}$ contains at least three sets with at least n elements.
7. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/2b$.

8. Suppose that $C_1, C_2, \ldots, C_n, n \geq 2$ circles of radius 1 in the plane such that no two of them are tangent and the subset of the plane formed by the union of these circles is connected. Let S be the set of points that belong to at least two circles prove that $|S| \geq n$

9. Let $A \in \mathbb{N}^\star$, $|A| = n$ $A = \{a_1, \ldots, a_n\}$. Given any $x \in \mathbb{N}$. How many $S \subset [n]$ can we have such that
$$\sum_{i \in S} a_i = x.$$

10. Prove that $\forall k, l$ $\exists n_0(k, l)$ such that if A is an l-intersecting family of k subsets of $[n]$ then $|A| \leq \binom{n-l}{k-l}$ for any $n \geq n_0(k, l)$.

11. **Cayley's Formula** There are n^{n-2} different labeled trees on n vertices .

12. Every family of at most 2^{d-1} d-sets is 2-colorable, that is $m(d) > 2^{d-1}$

5 Complex Combinatorics

- Lemma If p is a prime number let $a_i \in \mathbb{Q}, i = 0, \ldots, p-1$ with $\sum_0^{p-1} a_i \epsilon^i = 0$ where $\epsilon = e^{\frac{2\pi}{p}}$ then $a_0 = a_1 = \cdots = a_{p-1}$.