Inequalities

Ιακωβίδης Ιωάννης

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1 Basics

- **Cauchy-Schwarz**

  1. Discrete Form.Let $a_1, ..., a_n$ and $b_1, ..., b_n$ real numbers then the following is true:

  $$({a_1}^2 + ... + {a_n}^2)({b_1}^2 + ... + {b_n}^2) \geq (a_1 b_1 + ... + a_n b_n)^2$$

  With the equality present iff $\frac{a_1}{b_1} = ... = \frac{a_n}{b_n}$.

  2. Integral Form.Let f, g be real valued functions defined in an interval $[a, b]$. Then

  $$\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)\right) \geq \left(\int_a^b f(x)g(x)dx\right)^2$$

  With the equality present iff $f(x) = \lambda g(x) \forall x \in [a, b]$

- **AM-GM** For $x_1, x_2, ..., x_n$ positive real numbers.We have

  $$\frac{\sum_{i=1}^n x_i}{n} \geq \sqrt[n]{\prod_{i=1}^n x_i}$$

  With the equality to be satisfied only when all the variables are equal.

- **Convex set**.A set of points in the plane is said to be convex if the line segment joining any two points in the set lies entirely within the set.

- **Convexity**. A function f defined on an interval (which may be open, closed or infinite on either end) is said to be convex if the set $\{(x, y) \in \mathbb{R}^{\not\vDash} : y \geq f(x)\}$ is convex. Or equivalently f is convex $[a, b]$ iff

  $$\forall\lambda \in [0, 1], \forall x, y \in [a, b] f(x)(1 - \lambda) + f(y)\lambda \geq f((1 - \lambda)x + \lambda y)$$

  . If f is differentiable this is also equivalent to f is increasing.

- **Jensen**. Let f be a convex function on an interval I let $p_1, .., p_n$ non-negative weights that sum up to 1 .Then for all $x_1, ..., x_n \in I$ we have

  $$f(x_1)p_1 + f(x_2)p_2 + ... + f(x_n)p_n \geq f(x_1p_1 + ... + x_np_n)$$

  .

- f is concave iff $-f$ is convex.

- **Smoothing principle** states that if you have a quantity and it's becoming smaller as you move two variables closer together(while keeping some constraint) then it is minimized if you make all the variables equal to each other .

- **Chebyshev** Let f,g be two increasing functions in an interval $[a, b]$ then

  $$\int_a^b f(x)dx \int_a^b g(x)dx \leq (b - a) \int_a^b f(x)g(x)dx$$

  or

  $$E(f) * E(g) \leq E(fg)$$

- **Hölder**

  - Discrete Form Let $a_1, \ldots, a_n, b_1, \ldots, b_n, p, q$ be non-negative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then

    $$\sqrt[p]{\sum_1^n {a_k}^p} \sqrt[q]{\sum_1^n {b_k}^q} \geq \sum_1^n a_k b_k$$

  - Integral Form

    $$\sqrt[p]{\int_a^b |f|^p} \sqrt[q]{\int_a^b |g|^q} \geq \int_a^b |fg|$$

    with the equality if they are collinear .

- **Minkowski**
  - Discrete Form Let $a_1, \dots, a_n, b_1, \dots, b_n, p, q$ be non-negative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\sqrt[p]{\sum_1^n a_k^p} + \sqrt[p]{\sum_1^n b_k^p} \geq \sqrt[p]{\sum_1^n (a_k + b_k)^p}$$

  - Integral Form

$$\sqrt[p]{\int_a^b |f|^p} + \sqrt[p]{\int_a^b |g|^p} \geq \sqrt[n]{\int_a^b |f + g|^p}$$

  with the equality if they are collinear .

2 Problems

1. Show that $\log\left(1 + \frac{1}{x}\right) > \frac{1}{x+1} \forall x > 0$.
2. Find 3 proofs of AM-GM and Cauchy-Schwarz.
3. (SEEMOUS 2011 Problem 1.) For a given integer $n \geq 1$, let $f : [0, 1] \to \mathbb{R}$ be a non-decreasing function. Prove that

$$\int_0^1 f(x)dx \leq (n+1) \int_0^1 x^n f(x)dx.$$

   Find all non-decreasing continuous functions for which equality holds.
4. (SEEMOUS 2013 Problem 3.) Find the maximum value of

$$\int_0^1 |f'(x)|^2 \, |f(x)| \frac{1}{\sqrt{x}} dx$$

   over all continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and

$$\int_0^1 |f'(x)|^2 dx \leq 1.$$

5. For $p > 0$, the Gamma function is defined as

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$$

You may use without proof that $\Gamma(p+1) = p\Gamma(p)$ for every $p > 0$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(i) Prove that $\Gamma(p+\frac{1}{2}) < \Gamma(p)\sqrt{p}$ for every $p > 0$.
(ii) Prove that for every $n \in \mathbb{N}$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \binom{2n}{n}\frac{1}{2^{2n}} < \frac{1}{\sqrt{\pi n}}.$$

6. If f is twice-differentiable then f is convex iff $f'' \geq 0$

7. If f is continuous on D then f is convex on D iff

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) \quad \forall x, y \in D$$

8. Let f be a convex function and let x, y if $x \to x', y \to y'$. while keeping the sum constant and bringing the variables closer together then

$$f(x) + f(y) \geq f(x') + f(y')$$

9. Let $x_1, ..., x_n$ be real numbers. Find the value of x that minimizes

$$|x - x_1| + |x - x_2| + ... + |x - x_n|.$$

10. If f is convex on $[a, b]$ then f is continuous on (a,b).

11. For $n \geq 2$ and $a_1, ..., a_n \in (0, \infty)$ such that

$$(a_1 + ... + a_n)(\frac{1}{a_1} + ... + \frac{1}{a_n}) \leq (n + \frac{1}{2})^2$$

Show thar $4\min(a_1, ...a_n) \geq \max(a_1, ..., a_n.)$

12. Let $a_0, ..., a_n$ be real numbers in $(0, \pi/2)$ such that

$$\sum_{i=0}^{n} \tan(a_i - \pi/4) \geq n - 1$$

Prove that $\prod_{i=0}^{n} \tan(a_i) \geq n^{n+1}$.

13. Let $a_1, ..., a_n$ $n \geq 3$ be real numbers such that

$$a_1 + .. + a_n \geq n, {a_1}^2 + ... + {a_n}^2 \geq n^2.$$

Prove that $\max(a_1, ..., a_n) \geq 2$.

14. Let n be a positive integer and let real numbers $a_1, ..., a_n, b_1, ..., b_n$ such that $a_i + b_i > 0, i = 1, .., n$ Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} (a_i + b_i)}$$

15. Prove that

$$\int_0^{\pi} e^{\sin x} dx > \pi e^{\frac{2}{\pi}}.$$

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying $f(0) = 0$ and $|f'(x)| \leq f(x)$. Show that f is constant.

17. Suppose all roots of the polynomial $x^n + a_{n-1}x^{n-1} + ... + a_0$ are real.Then the roots are contained in the interval with the endpoints

$$-\frac{a_{n-1}}{n} \pm \frac{n-1}{n} \sqrt{a_{n-1}^2 - \frac{2n}{n-1} a_{n-2}}.$$

18. Let $p(x)$ be a non-constant polynomial with only real roots.We have $(p'(x))^2 \geq p(x)p''(x) \forall x \in \mathbb{R}$.

19. Let $f : [0, 1] \to \mathbb{R}^+$ be continuous decreasing function. Show that

$$\int_0^1 f(x)dx \int_0^1 xf(x)^2 dx \leq \int_0^1 xf(x)dx \int_0^1 f(x)^2 dx.$$

20. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^{\alpha} f(x)\, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

21. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

22. (Carleman's Inequality.) Let $\{a_n\}_{n=1}^{\infty}$ be sequence of positive numbers with $\sum_{n=1}^{\infty} a_n < \infty$. Prove that

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} \leq e\sum_{n=1}^{\infty} a_n.$$