Chapitre 3

Analyse numérique matricielle

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Exercice 1

1. $$f(t, y) = 3t + y$$
$$|f(t, y_1) - f(t, y_2)| = |y_1 - y_2|$$
Donc f est 1-Lipschitzienne. D'après le cours f est Lipschitzienne donc le problème de Cauchy admet une et une seule solution.
2. $y(t) = 4e^t - 3t - 3$ vérifie bien l'équadiff $y' = 3t + y$ et $y(0) = 1$.
3. $$\begin{cases} y_{k+1} = y_k + hf(t_k, y_k) \\ y_0 = 1 \end{cases}$$
$$\begin{cases} y_{k+1} = y_k + h(3t_k + y_k) \\ y_0 = 1 \end{cases}$$
$$\begin{cases} y_{k+1} = (1 + h)y_k + 3ht_k \\ y_0 = 1 \end{cases}$$

Exercice 1

En général $y(t_k) \simeq y_k$
On $t_k = a + k \times h = 0 + k \times 0.1 \Longrightarrow 0.2 = t_2$
On cherche donc $y(0.2) \simeq y_2$

$$y_1 = (1 + h)y_0 + 3ht_0$$

$$y_1 = 1.1 \times 1 + 3 \times 0.1 \times 0 = 1.1$$

$$y_2 = (1 + h)y_1 + 3ht_1$$

$$y_2 = 1.1 \times 1.1 + 3 \times 0.1 \times 0.1 = 1.24$$

Exercice 1

$$\begin{cases} y_{k+1} = y_k + hf(t_k, y_{k+1}) \\ y_0 = 1 \end{cases}$$

$$\begin{cases} y_{k+1} = y_k + h(3t_k + y_{k+1}) \\ y_0 = 1 \end{cases}$$

$$\begin{cases} y_{k+1} = \frac{y_k + 3ht_k}{1-h} \\ y_0 = 1 \end{cases}$$

$$y_1 = \frac{1 + 3 \times 0.1 \times 0}{1 - 0.1} \simeq 1.11$$

$$y_2 = \frac{1.11 + 3 \times 0.1 \times 0.1}{1 - 0.1} \simeq 1.27$$

Exercice 1

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Exercice 1

$$\begin{cases} \overline{y}_i = y_i + hf(t_i, y_i) \\ y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_i + h, \overline{y}_i)\right] \end{cases}$$

$$\begin{cases} \overline{y}_i = y_i + h(3t_i + y_i) \\ y_{i+1} = y_i + \frac{h}{2}\left[(3t_i + y_i) + 3(t_i + h) + \overline{y}_i\right] \end{cases}$$

$$\begin{cases} \bar{y}_0 = 1 + 0.1(3 \times 0 + 1) \\ y_1 = 1 + 0.05\left[(3 \times 0 + 1) + 3(0 + 0.1) + \overline{y}_0\right] \end{cases}$$

$$\begin{cases} \bar{y}_0 = 1.1 \\ y_1 = 1 + 0.05\left[1 + 0.3 + 1.1\right] \simeq 1.12 \end{cases}$$

$$\begin{cases} \overline{y}_1 = 1.12 + 0.1(3 \times 0.1 + 1.12) \simeq 1.26 \\ y_2 = 1.12 + 0.05\left[(3 \times 0.1 + 1.12) + 3(0.1 + 0.1) + \overline{y}_1\right] \simeq 1.284 \end{cases}$$

$$y(0.2) = 1.28561$$

Exercice2

$$\begin{cases} y_{k+1} = y_k + hf(t_k, y_k) \\ y_0 = 1 \end{cases}$$

$$\begin{cases} y_{k+1} = y_k - hy_k = (1-h)y_k \\ y_0 = 1 \end{cases}$$

$$y_n = (1-h)^n$$

$$|y_n - y(t_n)| = |(1-h)^n - e^{-t_n}| = |(1-h)^n - e^{-nh}| = |(1-h)^n - (e^{-h})^n|$$

$$|y_n - y(t_n)| \leq n|(1-h) - e^{-h}|$$

Exercice2

Taylor-Lagrange :

$$|f(x) - f(x_0) - (x - x_0)f'(x_0)| \leq M_2 \frac{|x - x_0]^2}{2}$$

$$|f(h) - f(0) - hf'(0)| \leq M_2 \frac{h^2}{2}$$

$$f(t) = e^{-t}$$

$$|e^{-h} - 1 + h| \leq \frac{h^2}{2}$$

Finalement

$$|y_n - y(t_n)| \leq n|(1 - h) - e^{-h}| \leq n\frac{h^2}{2}$$

Or $h = \frac{b-a}{n} = \frac{10-0}{n} \Longrightarrow nh = 10$ donc $|y_n - y(t_n)| \leq \frac{50}{n}$

Exercice2

$$\begin{cases} y_{k+1} = y_k + hf(t_k, y_{k+1}) \\ y_0 = 1 \end{cases}$$

$$\begin{cases} y_{k+1} = y_k - hy_{k+1} \\ y_0 = 1 \end{cases}$$

$$\begin{cases} y_{k+1} = \frac{y_k}{1+h} \\ y_0 = 1 \end{cases}$$

$$y_n = \frac{1}{(1+h)^n}$$

$$|y_n - y(t_n)| = |(1+h)^{-n} - e^{-t_n}| = |(1+h)^{-n} - (e^h)^{-n}|$$

$$|y_n - y(t_n)| \leq n|(1+h) - e^h| \leq n\frac{h^2}{2} \leq \frac{50}{n}$$

Exercice 4

On pose $Y_1 = y$ et $Y_2 = y'$:

$$Y' = (Y_1', Y_2') = (y', y'') = (Y_2, -ty' - (1-t)y + 2)$$

$$Y' = (Y_2, -tY_2 - (1-t)Y_1 + 2)$$

$$Y' = F(t, Y)$$

Exercice 4

$$f(t,y) = \sin y + \sin t$$
$$|f(t,y) - f(t,z)| = |(\sin y + \sin t) - (\sin z + \sin t)| = |\sin y - \sin z|$$
$$|f(t,y) - f(t,z)| \leq |y - z|$$

$$y' = \sin y + \sin t \Longrightarrow y'' = y' \cos y + \cos t$$
$$|y''| \leq |y'| + 1 \leq 3$$