Sasithorn Hannarong

1. Define the functions "less than" and "greater than" of two numerical arguments.
1.1 less than:
m is less than n if (sub m n) is less than zero. Also, less than ≡ not(greater or equal to)

LT := λaλb. NOT (GEQ b a)

Which
GEQ := λab. LEQ b a
LEQ := λmn. ISZERO (SUB m n)
ISZERO := λn. n (λx. FALSE) TRUE
SUB := λmn. n PRED m
PRED := λnfx. n (λgh. h (g f)) (λu. x) (λu. u)

1.2 greater than:
m is greater than n if (sub m n) is greater than zero. Also, greater than ≡ not(less than or equal to)

GT := λaλb. NOT (LEQ a b)

Which LEQ is defined in 1.1

2. Define the positive and negative integers using pairs of natural numbers.

we define natural number as (a, b) = -1^a * b.
Positive integer = +n = (0, n)
Negative integer = -n = (1, n)

We define pair as
pair => λx. λy. λf. ((f x) y)
first => λp. (p (λx. λy. x))
second => λp. (p (λx. λy. y))

Thus, +n = λx. pair 0 x
-n = λx. pair 1 x

3. Define addition and subtraction of integers
<u>addition</u>
m + n means add one to m, n times. For example, 5 + 3 means add one to five, three times.
The step of adding one to the previous number is called successor which is defined as
SUCC := λn.λf.λx.f (n f x)

Thus, addition can be defined as ADD := λmλn. n succ m

<u>subtraction</u>
m – n means take one off m, n times. For example, 5 -3 means take one off five, three times.
The step of taking one of is called predecessor (PRED n = n − 1) which is defined in 1.1 as
λn.λf.λx.n (λg.λh.h (g f)) (λu.x) (λu.u)

Thus, subtraction can be defined as SUB := λmλn. n PRED m
4. Define the division of positive integers recursively
we define division as a pair (m,n) = m/n
DIV := Y (λg.λq. λa. λb. LT a b (PAIR q a) (g (SUCC q) (SUB a b) b)) 0

Which
Y := λg. (λx. g (x x)) (λx. g (x x)) ≡ S (K (S I I)) (S (S (K S) K) (K (S I I)))
LT, PAIR, SUCC and SUB are defined in the previous questions

5. Define the function n! = n. (n - 1) … 1 recursively
(λx . (λy. x (y y)) (λy. x (y y))) (λ f. λ n.(ISZERO n) 1 (MULT n (f (PRED n))))

6. Define the rational numbers as pairs of integers
We define the rational numbers as Q = pair (m,n) = m/n ; m $\in$ Z, n $\in$ Z
Q = λm. λn. pair m n

7. Define functions for the addition, subtraction, multiplication and division of rationals.
Let $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers
We define pair m n = (m,n) = m / n and define x and y as rational numbers which
x := λa. λb. Pair a b and y := λc. λd. Pair c d

Addition is $\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$ so we can define the addition as a pair of ad+cb and bd
ADD := λx. λy. (pair ad+cb bd) x y

Subtraction is $\frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd}$ so we can define the addition as a pair of ad-cb and bd
SUB := λx. λy. (pair ad-cb bd) x y

Multiplication is $\frac{a}{b}*\frac{c}{d}=\frac{ac}{bd}$ so we can define the addition as a pair of ac and bd
MUL := λx. λy. (pair ac bd) x y

Division is $\frac{a}{b}/\frac{c}{d}=\frac{ad}{bc}$ so we can define the addition as a pair of ad and bc
DIV := λx. λy. (pair ad bc) x y

8. Define a data structure to represent a list of numbers.
A list can be defined as a pair of head and the rest
empty list: NIL := λx. TRUE
LIST := λx. λy. pair x y

which x is head and y is the rest of the list
HEAD := λp. p TRUE
REST := λp. p FALSE

9. Define a function which extracts the first element from a list.
HEAD := λp. p TRUE

10. Define a recursive function which counts the number of elements in a list.
LENGTH := Y (λg. λc. λx. NULL x c (g (SUCC c) (REST x))) 0